On some equations concerning quantum electrodynamics coupled to quantum gravity, the gravitational contributions to the gauge couplings and quantum effects in the theory of gravitation: mathematical connections with some sector of String Theory and Number Theory

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Abstract

This paper is principally a review, a thesis, of principal results obtained from various authoritative theoretical physicists and mathematicians in some sectors of theoretical physics and mathematics.
In this paper in the **Section 1**, we have described some equations concerning the quantum electrodynamics coupled to quantum gravity. In the **Section 2,** we have described some equations concerning the gravitational contributions to the running of gauge couplings. In the **Section 3**, we have described some equations concerning some quantum effects in the theory of gravitation. In the **Section 4**, we have described some equations concerning the supersymmetric Yang-Mills theory applied in string theory and some lemmas and equations concerning various gauge fields in any non-trivial quantum field theory for the pure Yang-Mills Lagrangian. Furthermore, in conclusion, in the **Section 5**, we have described various possible mathematical connections between the argument above mentioned and some sectors of Number Theory and String Theory, principally with some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, some Ramanujan's identities concerning π and the zeta strings.

1. On some equations concerning the quantum electrodynamics coupled to quantum gravity. [1]

The key equations that govern the behaviour of the coupling constants in quantum field theory are the renormalisation group Callan-Symanzik equations. If we let g denote a generic coupling constant, then the value of g at energy scale E, the running coupling constant $g(E)$, is determined by

$$E\frac{dg(E)}{dE}=\beta(E,g), \quad (1)$$

where $\beta(E,g)$ is the renormalisation group β-function. Asymptotic freedom is signaled by $g(E) \to 0$ as $E \to \infty$, requiring $\beta < 0$ in this limit.
With regard the Einstein gravity with a cosmological constant coupled to quantum electrodynamics in four spacetime dimensions, a standard calculation shows that the effective action to one-loop order is given by

$$\Gamma^{(1)} = \frac{1}{2} \ln \det \Delta^i_j - \ln \det Q_{\alpha\beta} - \ln \det \left(i\gamma^\mu \partial_\mu + e\gamma^\mu \overline{A}_\mu - im \right). \quad (2)$$

The last term (with $\overline{A}_\mu$ the background gauge field) is the result of performing a functional integral over the Dirac field. The middle term is the contribution from the ghost fields required to remove the unphysical degrees of freedom of the gravity and electromagnetic fields. The first term is the result of integrating over the spacetime metric and electromagnetic fields. For operator Δ^i_j the heat kernel $K^i_j(x,x';\tau)$ is defined by

$$-\frac{\partial}{\partial \tau} K^i_j(x,x';\tau) = \Delta^i_k K^k_j(x,x';\tau) \quad (3)$$

with boundary condition $K^i_j(x,x';\tau=0) = \delta^i_j \delta(x,x')$. τ is called the proper time. The Green function $G^i_j(x,x')$ for the operator Δ^i_j is

$$\Delta^i_k G^k_j(x,x') = \delta^i_j \delta(x,x'). \quad (4)$$

It follows that the Green function and heat kernel are related by

$$G^i_j(x,x') = \int_0^\infty d\tau K^i_j(x,x';\tau). \quad (5)$$

The importance of the heat kernel for quantum field theory arises from the existence of an asymptotic expansion as $\tau \to 0$:

$$K^i_j(x,x;\tau) \approx (4\pi\tau)^{-n/2} \sum_{r=0}^{\infty} \tau^r E_{r\,j}^{\ i}(x) \quad (6)$$

where n is the spacetime dimension (chosen as 4 here) and the heat kernel coefficients $E_{r\,j}^{\ i}(x)$ depend only locally on the details of coefficients of the differential operator Δ^i_j. The divergent part of the effective action, as well as the Green function, can be related to the heat kernel coefficients. Formally

$$L_\Delta = \frac{1}{2} \ln \det \Delta^i_j = -\frac{1}{2} \int d^n x \int_0^\infty \frac{d\tau}{\tau} tr K^i_j(x,x;\tau). \quad (7)$$

The one-loop effective action (2) is then given by

$$\Gamma^{(1)} = L_\Delta - 2L_Q - 2L_{Dirac}. \quad (8)$$

As with the Green function (4) the divergent part of (7) comes from the $\tau \approx 0$ limit of the proper time integral. The divergent part of L_Δ is

$$\operatorname{divp}\left(L_\Delta\right) = -\frac{1}{32\pi^2}\int d^4x\left\{\frac{1}{2}E_c^4 trE_0 + E_c^2 trE_1 + trE_2 \ln E_c^2\right\}. \quad (9)$$

The lower limit on the proper time integration can be kept as $\tau = 0$ and the divergent part of the effective action L_Δ contains a simple pole as $\varepsilon \to 0$ given by

$$\operatorname{divp}\left(L_\Delta\right) = \frac{1}{16\pi^2\varepsilon}\int d^4x trE_2 . \quad (10)$$

The general form of $\Delta^i_{\ j}$ is

$$\Delta^i_{\ j} = \left(A^{\alpha\beta}\right)^i_j \partial_\alpha \partial_\beta + \left(B^\alpha\right)^i_j \partial_\alpha + \left(C\right)^i_j \quad (11)$$

for coefficients $\left(A^{\alpha\beta}\right)^i_j$, $\left(B^\alpha\right)^i_j$ and $\left(C\right)^i_j$ that depend on the spacetime coordinates through the background field. Normal coordinates are introduced at x' with $x^\mu = x'^\mu + y^\mu$ and all of the coefficients in (11) are expanded about $y^\mu = 0$. This gives

$$\left(A^{\alpha\beta}\right)^i_j = \left(A_0^{\alpha\beta}\right)^i_j + \sum_{n=1}^{\infty}\left(A^{\alpha\beta}{}_{\mu_1\ldots\mu_n}\right)^i_j y^{\mu_1}\ldots y^{\mu_n} \quad (12)$$

with similar expansions for $\left(B^\alpha\right)^i_j$ and $\left(C\right)^i_j$. The Green function is Fourier expanded as usual,

$$G^i_j\left(x,x'\right) = \int \frac{1}{\left(2\pi\right)^n} d^n p e^{ip\cdot y} G^i_j\left(p\right), \quad (13)$$

except that the Fourier coefficient $G^i_j\left(p\right)$ can also have a dependence on the origin of the coordinate system x' that is not indicated explicitly. If

$$G^i_j\left(p\right) = G_0{}^i_j\left(p\right) + G_1{}^i_j\left(p\right) + G_2{}^i_j\left(p\right) + \ldots \quad (14)$$

where $G_r{}^i_j\left(p\right)$ is of order p^{-2-r} as $p \to \infty$ it is easy to see that to calculate the pole part of $G^i_j\left(x,x\right)$ as $n \to 4$ only terms up to and including $G_2{}^i_j\left(p\right)$ are needed.

The gravity and gauge field contributions result in

$$\operatorname{tr} E_1 = \kappa^2\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{1}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\overline{F}^2 + \left(12 + 8\xi^2 + 3\nu + \nu\zeta^2\right)\Lambda . \ (15)$$

The overall result for the quadratically divergent part of the complete one-loop effective action (8) that involves $\overline{F}^2$ is

$$\Gamma^{(1)}_{quad} = -\frac{\kappa^2 E_c^2}{32\pi^2}\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{5}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\int d^4x\bar{F}^2 . \quad (16)$$

$$\Gamma^{(1)}_{quad} = -\frac{1}{16}\frac{\kappa^2 E_c^2}{2\pi^2}\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{5}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\int d^4x\bar{F}^2 . \quad (16b)$$

If $\xi \to 0$, $\zeta \to 0$, $\omega \to 1$ are taken to obtain the gauge condition independent result, the non-zero result

$$\Gamma^{(1)}_{quad} = -\frac{\kappa^2 E_c^2}{128\pi^2}\int d^4x\bar{F}^2 \quad (17)$$

is found.

The net result for the divergent part of the effective action that involves $\bar{F}^2$ and therefore contributes to charge renormalization is

$$\mathrm{divp}\left(\Gamma^{(1)}\right) = \left(-\frac{\kappa^2 E_c^2}{128\pi^2} - \frac{3\kappa^2\Lambda}{256\pi^2}\ln E_c^2 + \frac{e^2}{48\pi^2}\ln E_c^2\right)\int d^4x\bar{F}^2 . \quad (18)$$

$$\mathrm{divp}\left(\Gamma^{(1)}\right) = \frac{1}{16}\left(-\frac{\kappa^2 E_c^2}{8\pi^2} - \frac{3\kappa^2\Lambda}{16\pi^2}\ln E_c^2 + \frac{e^2}{3\pi^2}\ln E_c^2\right)\int d^4x\bar{F}^2 . \quad (18b)$$

From this the renormalization group function in (1) that governs the running electric charge to be calculated to be

$$\beta(E,e) = \frac{e^3}{12\pi^2} - \frac{\kappa^2}{32\pi^2}\left(E^2 + \frac{3}{2}\Lambda\right)e . \quad (19)$$

The first term on the right hand side of (19) is that present in the absence of gravity and results in the electric charge increasing with the energy. The second term on the right hand side of (19) represents the correction due to quantum gravity. For pure gravity with no cosmological constant, or for small cosmological constant Λ, the quantum gravity contribution to the renormalization group β-function is negative and therefore tends to result in asymptotic freedom.

2. On some equations concerning the gravitational contributions to the running of gauge couplings. [2]

The action of Einstein-Yang-Mills theory is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{\kappa^2}R - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\mathfrak{F}^a_{\mu\nu}\mathfrak{F}^a_{\alpha\beta}\right] \quad (20)$$

where R is Ricci scalar and $\mathfrak{F}^a_{\mu\nu}$ is the Yang-Mills fields strength $\mathfrak{F}_{\mu\nu} = \nabla_\mu \mathrm{A}_\nu - \nabla_\nu \mathrm{A}_\mu - ig\left[\mathrm{A}_\mu, \mathrm{A}_\nu\right]$. It is hard to quantize this lagrangian because of gravity-part's non-linearity and minus-dimension coupling constant $\kappa = \sqrt{16\pi G}$. Usually, one expands the metric tensor around a background metric

$\bar{g}_{\mu\nu}$ and treats graviton field as quantum fluctuation $h_{\mu\nu}$ propagating on the background space-time determined by $\bar{g}_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} . \quad (21)$$

Furthermore, we can rewrite the lagrangian (20) also as follows:

$$S = \int d^4 x \sqrt{-g} \left[\frac{1}{16\pi G} R - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} \mathcal{F}^a_{\mu\nu} \mathcal{F}^a_{\alpha\beta} \right] . \quad (20b)$$

Let us set $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. $h_{\mu\nu}$ is interpreted as graviton field, fluctuating in flat space-time. The lagrangian can be arranged to different orders of $h_{\mu\nu}$ or κ. The free part of gravitation is of order unit and gives the graviton propagator

$$P_G^{\mu\nu\rho\sigma}(k) = \frac{i}{k^2} \left[g^{\nu\rho} g^{\mu\sigma} + g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma} \right] \quad (21b)$$

in the harmonic gauge $C^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h^\nu_\nu = 0$. For simplicity, the metric $g^{\mu\nu}$ is understood as $\eta^{\mu\nu}$. The interactions of gauge field and gravity field are determined by expanding the second term of the lagrangian (20). We can compute the β function by evaluating two and three point functions of gauge fields. These Green functions are general divergent, so counter-terms are needed to cancel these divergences. The relevant counter-terms to the β function are

$$T^{\mu\nu} = i \delta_{ab} Q^{\mu\nu} \delta_2 , \quad T^{\mu\nu\rho} = g f^{abc} V^{\mu\nu\rho}_{qkp} \delta_1 , \quad Q^{\mu\nu} \equiv q^\mu q^\nu - q^2 g^{\mu\nu} ,$$
$$V^{\mu\nu\rho}_{qkp} \equiv g^{\nu\rho} (q-k)^\mu + g^{\rho\mu} (k-p)^\nu + g^{\mu\nu} (p-q)^\rho . \quad (22)$$

The β function is defined as $\beta(g) = g\mu \frac{\partial}{\partial\mu} \left(\frac{3}{2} \delta_2 - \delta_1 \right)$. With the consistency condition $I^R_{2\mu\nu} = \frac{1}{2} g_{\mu\nu} I^R_2$, we obtain for two and three point functions

$$T^{(a+b)\mu\nu}_{LR} = 2\kappa^2 Q^{\mu\nu} \int dx \left[-\frac{3}{2} I^R_2(0) + 2 I^R_2\left(\mathcal{M}^2_q\right) + q^2 \left(3x^2 - x\right) I^R_0\left(\mathcal{M}^2_q\right) \right] \quad (23)$$

$$T^{(d+e)\mu\nu\rho}_{LR} = 2ig\kappa^2 \int dx \left[\frac{1}{2} V^{\mu\nu\rho}_{qkp} I^R_2(0) + \left(g^{\mu\nu} q^\rho - q^\mu g^{\nu\rho} \right) I^R_2\left(\mathcal{M}^2_q\right) + \left(g^{\nu\rho} k^\mu - k^\nu g^{\rho\mu} \right) I^R_2\left(\mathcal{M}^2_k\right) + \right.$$
$$\left. + \left(g^{\rho\mu} p^\nu - p^\rho g^{\mu\nu} \right) I^R_2\left(\mathcal{M}^2_p\right) \right] \quad (24)$$

from which we can directly read off the two-point and three-point counter-terms δ^κ_2 and δ^κ_1 respectively

$$\delta^\kappa_2 = \kappa^2 \frac{1}{16\pi^2} \left[M^2_c - \mu^2_s \left(\ln \frac{M^2_c}{\mu^2_s} - \gamma_w + 1 + y_2 \left(\frac{\mu^2_s}{M^2_c} \right) \right) \right] ;$$

$$\delta_1^{\kappa} = \kappa^2 \frac{1}{16\pi^2}\left[M_c^2 - \mu_s^2 \left(\ln\frac{M_c^2}{\mu_s^2} - \gamma_w + 1 + y_2\left(\frac{\mu_s^2}{M_c^2}\right)\right)\right]. \quad (25)$$

Putting δ_1^{κ} and δ_2^{κ} in the following equation

$$\Delta\beta^{\kappa} = g\mu\frac{\partial}{\partial\mu}\left(\frac{3}{2}\delta_2^{\kappa} - \delta_1^{\kappa}\right), \quad (26)$$

we obtain the gravitational corrections to the gauge β function

$$\Delta\beta^{\kappa} = g\kappa^2\frac{-\mu_s^2}{16\pi^2}\left[\ln\frac{M_c^2}{\mu_s^2} + 1 - \gamma_w + y_0\left(\frac{\mu_s^2}{M_c^2}\right)\right] \quad (26b)$$

In general, the total β function of gauge field theories including the gravitational effects may be written as follows

$$\beta^{\kappa} = -\frac{1}{16\pi^2}b_0 g^3 + g\kappa^2\frac{-\mu_s^2}{16\pi^2}\left[\ln\frac{M_c^2}{\mu_s^2} + 1 - \gamma_w + y_0\left(\frac{\mu_s^2}{M_c^2}\right)\right]. \quad (27)$$

The interesting feature of gauge theory interactions is the possible gauge couplings unification at ultra-high energy scale when the gravitational effects are absent. Where the running of gauge coupling in the Model Standard Super Symmetric (MSSM) without gravitational contributions is known to be

$$\alpha_e^{-1}(\mu) = \alpha_e^{-1}(M) + \frac{33}{10\pi}\ln\frac{M}{\mu}; \quad \alpha_w^{-1}(\mu) = \alpha_w^{-1}(M) + \frac{1}{2\pi}\ln\frac{M}{\mu}; \quad \alpha_s^{-1}(\mu) = \alpha_s^{-1}(M) - \frac{3}{2\pi}\ln\frac{M}{\mu} \quad (28)$$

with experimental input at M_Z

$$\alpha_e^{-1}(M_Z) = 58.97 \pm 0.05; \qquad \alpha_w^{-1}(M_Z) = 29.61 \pm 0.05; \qquad \alpha_s^{-1}(M_Z) = 8.47 \pm 0.22 \quad (29)$$

We note that these values are equal to the following values connected with the aurea ratio:

38,12461180 + 20,56230590 = 58,68691770;   29,12461180;   8,49844719.

Indeed, we have:

$$\left[(\Phi)^{35/7} + (\Phi)^{7/7}\right]\times 3 = (11{,}09016994 + 1{,}61803399)\times 3 = 12{,}70820393\times 3 \cong 38{,}12461180;$$

$$\left[(\Phi)^{28/7}\right]\times 3 = 6{,}85410197\times 3 \cong 20{,}56230590;$$

$$\left[(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{-21/7}\right]\times 3 = (6{,}85410197 + 2{,}61803399 + 0{,}23606798)\times 3 =$$
$$= 9{,}70820394\times 3 \cong 29{,}12461180;$$

$$\left[(\Phi)^{14/7} + (\Phi)^{-28/7} + (\Phi)^{-42/7} + (\Phi)^{-63/7}\right]\times 3 = (2{,}61803399 + 0{,}145898033 + 0{,}055728089 +$$

$+0{,}013155617)\times 3 = 2{,}832815729\times 3 \cong 8{,}49844719$.

As usual, $\Phi = \frac{\sqrt{5}+1}{2} \cong 1{,}61803399$, i.e. the value of the aurea ratio.

3. On various equations concerning some quantum effects in the theory of gravitation [3]

The general spherical static metric is given by

$$ds^2 = -f(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega^2, \quad (30)$$

where $f(r)$ and $h(r)$ are arbitrary functions of the coordinate r. The angular part of the metric is diagonal and given by

$$d\Omega^2 = \sum_{i=1}^{d-2}\left((d\theta^i)^2 \prod_{j=i+1}^{d-2} \sin^2\theta^j \right). \quad (31)$$

Consider an interesting classical scalar field $\phi(x)$ living in a spacetime with the metric (30). This field has an action given by:

$$S = \int d^d x\sqrt{-g}\left(-\phi\nabla^2\phi - \sum_{n=2}^{\infty}\lambda_n\phi^n \right). \quad (32)$$

The λ_n are a set of arbitrary coupling constants. In particular, $\lambda_2 \equiv m^2$ gives the mass of a weakly coupled excitation of this field. We will expand ϕ in the eigenmodes of the free, classical wave equation such that

$$\phi(x) = \int d\mu_p a_p \phi_p(x), \quad (33)$$

$$\nabla^2\phi_p = m^2\phi_p. \quad (34)$$

A sufficiently large set of quantum numbers p label the eigenbasis. The abstract formal expression $d\mu_p$ simply represents an appropriate measure over the modes under which

$$\int d\mu_p \phi_p(x)\phi_p(y) = \delta^d(x-y),$$

and

$$\int d^d x\sqrt{-g}\,\phi_p(x)\phi_q(x) = \delta_{pq}.$$

Now we want to consider the energy density, ρ, of a massless scalar field in a infinitely large hypercubic blackbody cavity at temperature T. Consider a real scalar field $\phi^I(x,t)$, where I is some kind of p-dimensional polarization index representing p internal degrees of freedom (for example, in a well-chosen gauge, the transverse polarization of an Abelian vector field behaves essentially like an internal index on a scalar field with $p = d-2$). Further assume that the field is sufficiently weakly coupled that each polarization component can be treated as an independent field obeying an action similar to eq. (32) with all interaction coefficients higher than $\lambda_2 \equiv M^2$ set equal to zero. Then each field component obeys the classical equation of motion

$$\nabla^2 \phi^I(x) = M^2 \phi^I(x) . \quad (35)$$

The solutions to equation (35) may be expressed as a sum over modes labelled by a wave vector k_a obeying $k^2 = -M^2$:

$$\phi_k^I = A_k^I \sin(k_a x^a) + B_k^I \cos(k_a x^a), \quad (36)$$

for arbitrary real coefficients A_k^I and B_k^I. We take the state to be labelled by the $d-1$ spatial components of k_a and fix the frequency of each mode by $\omega_k^2 \equiv k_0^2 = M^2 + k_i k^i$. We now confine the field to live in a cubic box of side length L by demanding Dirichlet boundary conditions at $x^i = 0, L$ for $i = 1 ... d-1$. This demands $B_k^I = 0$ and

$$k_i = \frac{\pi}{L} m_i , \quad (37)$$

where m_i is a spatial vector whose components are non-negative integers. The total energy in the hypercube is given by

$$U = \sum_{I=1}^{p} \sum_{m_1=0}^{\infty} ... \sum_{m_{d-1}=0}^{\infty} \omega_k n_{mI} , \quad (38)$$

where we understand that ω_k is given by

$$\omega_k = \sqrt{M^2 + \frac{\pi^2}{L^2} \sum_{i=1}^{d-1} m_i^2} . \quad (39)$$

The set of integers n_{mI} defining the quantum state must obey the appropriate statistics for the field ϕ^I. We have described a pure quantum state of the theory. At a finite temperature T and zero chemical potential, the system will be in a mixed state governed by the partition function

$$Q = \sum_{\{n_{mI}\}} e^{-\beta \omega_k} , \quad (40)$$

where $\beta \equiv 1/T$. This can be evaluated to give

$$\ln Q = -p \xi \sum_{m_1=0}^{\infty} ... \sum_{m_{d-1}=0}^{\infty} \ln\left(1 - \xi e^{-\beta \omega_k}\right), \quad (41)$$

where $\xi = 1$ for bosons and $\xi = -1$ for fermions. The overall factor of p occurs because the energy is independent of p, so each polarization mode contributes equally. The average occupation number of a given momentum mode in the thermal state is then given by

$$\langle n_m \rangle = \sum_{I=1}^{p} \langle n_{mI} \rangle = -\frac{1}{\beta}\frac{\partial}{\partial \omega_k}\ln Q = \frac{p}{e^{\beta\omega_k} - \xi}. \quad (42)$$

The total energy in the hypercube can now be found by combining the expressions (38) and (42), or by

$$\langle U \rangle = -\frac{\partial}{\partial \beta}\ln Q = \sum_{m_1=0}^{\infty} \ldots \sum_{m_{d-1}=0}^{\infty} \omega_k \langle n_m \rangle = \sum_{m_1=0}^{\infty} \ldots \sum_{m_{d-1}=0}^{\infty} \frac{p\omega_k}{e^{\beta\omega_k} - \xi}. \quad (43)$$

We should pass from a state labelling in terms of quantum numbers m_i to a labelling in terms of physical momenta k_i, with a mode density determined by the differential limit of equation (37). The sums over m_i then become integrals over k_i as

$$\langle U \rangle \overset{L\to\infty}{\rightarrow} \int d^{d-1}k \frac{L^{d-1}}{(2\pi)^{d-1}} \frac{p\omega_k}{e^{\beta\omega_k} - \xi}, \quad (44)$$

where ω_k is understood as $\sqrt{M^2 + k_i k^i}$. The factors of 2 in the denominator of the measure arise because the integrals over the k_i run over both positive and negative values, whereas the m_i were only summed over non-negative values. Equation (44) scales properly with the volume, so that even in the infinite volume limit we can define the spectral energy density over k_i modes. Using the spherical symmetry of the infinite volume limit and defining $k = \sqrt{k_i k^i} = |k_i|$, eq. (44) becomes

$$\frac{\langle U \rangle}{V} = \rho = p\frac{Vol(S^{d-2})}{(2\pi)^{d-1}} \int_0^\infty dk \frac{k^{d-2}\sqrt{M^2 + k^2}}{e^{\beta\sqrt{M^2+k^2}} - \xi}. \quad (45)$$

This defines the spectral energy density over the magnitude of the spatial momentum, via $\rho \equiv \int dk u_k(k)$, as

$$u_k(k) = p\frac{Vol(S^{d-2})}{(2\pi)^{d-1}} \frac{k^{d-2}\sqrt{M^2 + k^2}}{e^{\beta\sqrt{M^2+k^2}} - \xi}. \quad (46)$$

Similarly, we can define the spectral energy density over the frequency as

$$u_\omega(\omega) = p\frac{Vol(S^{d-2})}{(2\pi)^{d-1}} \frac{\omega^2(\omega^2 - M^2)^{(d-3)/2}}{e^{\beta\omega} - \xi}, \quad (47)$$

where ω runs over $[M, \infty]$.

The total energy density can now be evaluated using either eq. (46) or eq. (47) match. Simple analytic results can be found for the case $M = 0$, which will also apply when $T >> M$. In this case, $\omega = k$ and equations (46) and (47) match. They give

$$\rho = pT^d \frac{Vol(S^{d-2})}{(2\pi)^{d-1}} \int_0^\infty dx \frac{x^{d-1}}{e^x - \xi}. \quad (48)$$

The integral can be evaluated by pulling the exponential into the numerator, performing a Taylor series in e^{-x}, doing the integral, and resumming the Taylor series. The result for the general definite integral is given by

$$\int_a^b \frac{x^{d-1}dx}{e^x-\xi} = \left(1-\frac{2}{2^d}\right)^{(1-\xi)/2} \zeta(d)(d-1)! \sum_{n=0}^{d-1} \frac{a^n e^{-a} - b^n e^{-b}}{n!}, \quad (49)$$

where $\zeta(s)$ is the Riemann zeta function, which can be defined for real $s>1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (50)$$

This series arises in the evaluation of the integral with the $\xi=1$, for bosons. For fermions, $\xi=-1$, the corresponding series is

$$\mathcal{R}(s) \equiv \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}. \quad (51)$$

This series can be evaluated using

$$\zeta(s) - \mathcal{R}(s) = \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \ldots = \frac{2}{2^s}\zeta(s), \quad (52)$$

so that

$$\mathcal{R}(s) = \left(1-\frac{2}{2^s}\right)\zeta(s), \quad (53)$$

which is the origin of this factor in eq. (49). So, eq. (48) becomes

$$\rho = p\left(1-\frac{2}{2^d}\right)^{(1-\xi)/2} \frac{\zeta(d)(d-1)!}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)\pi^{(d-1)/2}} T^d. \quad (54)$$

Furthermore, we can rewrite the eq. (48) also as follows:

$$\rho = pT^d \frac{Vol(S^{d-2})}{(2\pi)^{d-1}} \int_0^\infty dx \frac{x^{d-1}}{e^x-\xi} = p\left(1-\frac{2}{2^d}\right)^{(1-\xi)/2} \frac{\zeta(d)(d-1)!}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)\pi^{(d-1)/2}} T^d. \quad (54b)$$

Before, we have considered the energy density, ρ, of a massless scalar field in a infinitely large hypercubic cavity at temperature T. We now want to calculate the energy flux, Φ, emitted from a blackbody with this same temperature. For the flux calculation, instead of $Vol(S^{d-2})$ we will encounter

$$\int d^{d-2}\Omega \cos(\theta^{d-2}) \Theta\left(\frac{\pi}{2} - \theta^{d-2}\right) =$$

$$= \int_0^{2\pi} d\theta^1 \int_0^{\pi} d\theta^2 \ldots \int_0^{\pi} d\theta^{d-3} \int_0^{\pi/2} d\theta^{d-2} \left\{ \cos\left(\theta^{d-2}\right) \prod_{i=2}^{d-3} \left(\sin\left(\theta^i\right)\right)^{i-1} \right\} = \frac{2\pi^{(d-2)/2}}{(d-2)\Gamma\left(\frac{d-2}{2}\right)} = \frac{1}{d-2} Vol\left(S^{d-3}\right) =$$

$$= Vol\left(B^{d-2}\right) = \frac{1}{\sqrt{\pi}(d-2)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-2}{2}\right)} Vol\left(S^{d-2}\right), \quad (55)$$

where $\Theta(x)$ is the step function and B^n is the n-dimensional unit ball: the compact subspace of R^n bounded by S^{n-1}(for example, B^3 is the unit 3-ball of volume $4\pi/3$ bounded by the sphere S^2 of area 4π). The fact that the expression for the energy density becomes that for the flux when $Vol\left(S^{d-2}\right)$ is replaced by $Vol\left(B^{d-2}\right)$ makes physical sense, since B^n is the projection of S^n onto R^n. We are left with the relationship of flux to energy density as

$$\Phi = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}(d-2)\Gamma\left(\frac{d-2}{2}\right)} \rho = \frac{Vol\left(B^{d-2}\right)}{Vol\left(S^{d-2}\right)} \rho . \quad (56)$$

Thus, the d-dimensional Stefan-Boltzmann law is given by

$$\Phi = p\left(1 - \frac{2}{2^d}\right)^{(1-\xi)/2} \frac{\zeta(d)(d-1)!}{2^{d-2}(d-2)\Gamma\left(\frac{d-2}{2}\right)\pi^{d/2}} T^d . \quad (57)$$

4. On some equations concerning the supersymmetric Yang-Mills theory applied in string theory and some lemmas and equations concerning various gauge fields in any non-trivial quantum field theory for the pure Yang-Mills Lagrangian. [4] [5]

The fields of the minimal $N = 2$ supersymmetric Yang-Mills theory are the following: a gauge field A_m, fermions λ_α^i and $\overline{\lambda}_{\dot{\alpha}i}$ transforming as $(1/2,0,1/2)$ and $(0,1/2,1/2)$ under $SU(2)_L \times SU(2)_R \times SU(2)_I$, and a complex scalar B - all in the adjoint representation of the gauge group. Covariant derivatives are defined by

$$D_m \Phi = \left(\partial_m + iA_m\right)\Phi , \quad (58)$$

and the Yang-Mills field strength is

$$F_{mn} = \partial_m A_n - \partial_n A_m + \left[A_m, A_n\right]. \quad (59)$$

The supersymmetry generators transform as $(1/2,0,1/2) \oplus (0,1/2,1/2)$; introducing infinitesimal parameters η_α^i and $\overline{\eta}_{\dot{\alpha}i}$, furthermore $D = \left[B, \overline{B}\right]$. The minimal Lagrangian is

$$L=\frac{1}{e^2}\int_M d^4xTr\left(\frac{-1}{4}F_{mn}F^{mn}-i\overline{\lambda}_i^{\dot{\alpha}}\sigma^m_{\alpha\dot{\alpha}}D_m\lambda^{\alpha i}-D_m\overline{B}D^mB-\frac{1}{2}\left[\overline{B},B\right]^2-\frac{i}{\sqrt{2}}\overline{B}\varepsilon_{ij}\left[\lambda^{\alpha i},\lambda^j_\alpha\right]+\frac{i}{\sqrt{2}}B\varepsilon^{ij}\left[\overline{\lambda}_{\dot{\alpha}i},\overline{\lambda}^{\dot{\alpha}}_i\right]\right) \quad (60)$$

Here Tr is an invariant quadratic form on the Lie algebra which for $G=SU(N)$ we can conveniently take to be the trace in the N dimensional representation.
It is possible to realize a mass term for the $N=1$ matter multiplet (which consists of B and $\psi_\alpha=\lambda^{\dot{2}}_\alpha$) by adding to the Lagrangian a term of the form $I(\omega)+\{Q_1,...\}$, where Q_1 is the charge corresponding to the ρ_1 transformation. Furthermore, Q_1 is the only essential symmetry. We obtain:

$$\hat{L}=L+I(\omega)+\{Q_1,...\}=L-\frac{1}{2}\int_M d^4xTr\left(m\lambda^{\dot{2}}_\alpha\lambda^{\alpha\dot{2}}+\overline{m}\overline{\lambda}^{\dot{\sigma}}_{\dot{2}}\overline{\lambda}_{\dot{\sigma}\dot{2}}\right)-e^2\int_M d^4x\sqrt{2}\overline{m}mTr\overline{B}B, \quad (61)$$

with

$$m=\sigma_{mn\dot{2}\dot{2}}\omega_{kl}\varepsilon^{mnkl}. \quad (62)$$

The mass is proportional to the holomorphic two-form ω. The $N=1$ gauge multiplet, consisting of the gauge field A_m and the gluino $\lambda_\alpha=\lambda^i_\alpha$, remains massless.

With regard the Λ^{16}-amplitude in the type IIB description, the classical action for the operator $(\Lambda)^{16}$ in the $AdS_5\times S^5$ supergravity action is

$$S_\Lambda[J]=e^{-2\pi\left(\frac{1}{g_s}i\chi_0\right)}g_s^{-12}V_{S^5}\int d^4x_0d\rho_0\frac{1}{\rho_0^5}t_{16}\prod_{p=1}^{16}\left[K_4(\rho_0,x_0;x_p)\frac{1}{\sqrt{\rho_0}}\left(\rho_0\gamma^{\hat{5}}+(x_0-x_p)^\mu\gamma_{\hat{\mu}}\right)J_\Lambda(x_p)\right]. \quad (63)$$

This result is in agrees with the following expression obtained in the Yang-Mills calculation:

$$G_{16}(x_p)=g^8_{YM}e^{-\frac{8\pi^2}{g^2_{YM}}+i\theta_{YM}}\int d^4x_0d\rho_0\frac{1}{\rho_0^5}\int d^8\eta d^8\overline{\xi}\prod_{p=1}^{16}\left[\frac{\rho_0^4}{\left[\rho_0^2+(x_p-x_0)^2\right]^4}\frac{1}{\sqrt{\rho_0}}\left(\rho_0\eta^{A_p}_{\alpha_p}+(x_p-x_0)_\mu\sigma^\mu_{\alpha_p\dot{\alpha}_p}\overline{\xi}^{\dot{\alpha}_pA_p}\right)\right] \quad (64)$$

i.e. the correlation function in the super Yang-Mills description.

The low energy effective action for type IIB superstring theory in ten dimensions includes the interaction (in the string frame)

$$S\approx l_s^4\int d^{10}x\sqrt{-g}\left(\zeta(3)^2e^{-2\phi}+2\zeta(3)\zeta(2)+6\zeta(4)e^{2\phi}+\frac{2}{9}\zeta(6)e^{4\phi}+...\right)D^6\mathcal{R}^4, \quad (65)$$

where the … involve contributions form D-instantons.
We consider the perturbative contributions to the $D^6\mathcal{R}^4$ interaction. The sum of the contributions to the four graviton amplitude at tree level and at one loop in type II string theory compactified on T^2 is proportional to

$$\left[-V_2e^{-2\phi}\frac{\Gamma(-l_s^2s/4)\Gamma(-l_s^2t/4)\Gamma(-l_s^2u/4)}{\Gamma(1+l_s^2s/4)\Gamma(1+l_s^2t/4)\Gamma(1+l_s^2u/4)}+2\pi I\right]\mathcal{R}^4, \quad (66)$$

where V_2 is the volume of T^2 in the string frame, s,t,u are the Mandelstam variables, and I is obtained from the one loop amplitude. The amplitude is the same for type IIA and type IIB string theories. Now I is given by

$$I = \int_{\mathcal{F}} \frac{d^2\Omega}{\Omega_2^2} Z_{lat} F(\Omega, \overline{\Omega}) \quad (67)$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,Z)$, and $d^2\Omega = d\Omega d\overline{\Omega}/2$. In the above expression, the lattice factor Z_{lat} which depends on the moduli is given by

$$Z_{lat} = V_2 \sum_{A \in Mat(2\times 2, Z)} \exp\left[-2\pi i T(\det A) - \frac{\pi T_2}{\Omega_2 U_2} \left| \begin{pmatrix} 1 & U \end{pmatrix} A \begin{pmatrix} \Omega \\ 1 \end{pmatrix} \right|^2 \right]. \quad (68)$$

Expanding eq. (67) to sixth order in the momenta, we get that

$$I = \frac{l_s^6}{3}\left(s^3 + t^3 + u^3\right)\left[\hat{I}_1 + \hat{I}_2\right], \quad (69)$$

where

$$\hat{I}_1 = 4\int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{lat} \int_{\mathcal{F}} \prod_{i=1}^{3} \frac{d^2 v_i}{\Omega_2} \ln \hat{\chi}(v_1 - v_2;\Omega) \ln \hat{\chi}(v_1 - v_3;\Omega) \hat{\chi}(v_2 - v_3;\Omega), \quad (70)$$

and

$$\hat{I}_2 = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{lat} \int_{\mathcal{F}} \prod_{i=1}^{3} \frac{d^2 v_i}{\Omega_2} [\ln \hat{\chi}(v_1 - v_2;\Omega)]^3. \quad (71)$$

We have also that

$$\frac{(4\pi)^3}{4} \hat{I}_1 = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{lat} E_3(\Omega, \overline{\Omega})^{SL(2,Z)} = \hat{I}_1^1 + \hat{I}_1^2 + \hat{I}_3^3, \quad (72)$$

where $\hat{I}_1^1, \hat{I}_1^2$, and $\hat{I}_1^3$ are the contributions from the zero orbit, the non-degenerate orbits and the degenerate orbits of $SL(2,Z)$ respectively. In the eq. (72), we have used the expression

$$E_3(\Omega, \overline{\Omega})^{SL(2,Z)} = 2\zeta(6)\Omega_2^3 + \frac{3\pi}{4\Omega_2^2}\zeta(5) + \pi^3\sqrt{\Omega_2} \sum_{m_1 \neq 0, m_2 \neq 0} \left|\frac{m_1}{m_2}\right|^{5/2} K_{5/2}\left(2\pi\Omega_2 |m_1 m_2|\right) e^{2\pi i m_1 m_2 \Omega_1}. \quad (73)$$

Thence, we can rewrite the eq. (72) also as follows:

$$\frac{(4\pi)^3}{4} \hat{I}_1 = \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} Z_{lat} \times 2\zeta(6)\Omega_2^3 + \frac{3\pi}{4\Omega_2^2}\zeta(5) + \pi^3\sqrt{\Omega_2} \sum_{m_1 \neq 0, m_2 \neq 0} \left|\frac{m_1}{m_2}\right|^{5/2} K_{5/2}\left(2\pi\Omega_2 |m_1 m_2|\right) e^{2\pi i m_1 m_2 \Omega_1}. \quad (74)$$

The contribution from the zero orbit gives

$$\hat{I}_1^1 = V_2 \int_{\mathcal{F}_L} \frac{d^2\Omega}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} = 0 . \quad (75)$$

The contribution from the non-degenerate orbits gives

$$\hat{I}_1^2 = 2V_2 \int_{-\infty}^{\infty} d\Omega_1 \int_0^{\infty} \frac{d\Omega_2}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} \sum_{k>j\ge 0,\, p\neq 0} e^{-2\pi i T k p - \frac{\pi T_2}{\Omega_2 U_2}|k\Omega + j + pU|^2} =$$

$$= 2\sqrt{T_2} E_3(U,\overline{U})^{SL(2,Z)} \sum_{p\neq 0, k\neq 0} \left|\frac{p}{k}\right|^{5/2} K_{5/2}(2\pi T_2|pk|) e^{2\pi i p k T_1} . \quad (76)$$

Furthermore, the contribution from the degenerate orbits gives

$$\hat{I}_1^3 = V_2 \int_{-1/2}^{1/2} d\Omega_1 \int_0^{\infty} \frac{d\Omega_2}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\Omega_2 U_2}|j+pU|^2} = \frac{2}{\pi^3}\left(2\zeta(6) T_2^3 + \frac{3\pi\zeta(5)}{4T_2^2}\right) E_3(U,\overline{U})^{SL(2,Z)} . \quad (77)$$

<u>Now we want to show two lemmas and equations concerning the gauge fields as described in the **Jormakka's paper** *"Solutions to Yang-Mills equations* *[5]*</u>". Thence, in the next Section, we describe some possible mathematical connections between some equations concerning this interesting argument and some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, and some Ramanujan's identities concerning π.

Lemma 1

Let the gauge field satisfy

$$A_\mu^{a,R}(0,x_1,x_2,x_3) = s_a c_\mu e^{-\beta^2 \sum_{j=1}^3 \rho_j^2} \quad (78)$$

where ρ_j *and* c_μ *are as in the following expressions*:

$$r_j = \rho_j + i\sigma_j , \quad \rho_j,\sigma_j \in R; \quad \rho_1 = x_1 - x_2 + \sqrt{2}x_3; \quad \rho_2 = x_1 - x_2 - \sqrt{2}x_3; \quad \rho_3 = -x_1 + \frac{1}{\sqrt{2}}x_3;$$

$$\sigma_1 = \sigma_2 = 0; \quad \sigma_3 = \frac{1}{\sqrt{2}}x_0; \quad h(r_j) = u(\rho_j,\sigma_j) + iv(\rho_j,\sigma_j) \quad (79)$$

$$c_0 = \sqrt{2}; \quad c_1 = c_2 = 1; \quad c_3 = 0; \quad e_0 = -\sqrt{2}; \quad e_1 = e_2 = 1; \quad e_3 = 0, \quad (80)$$

and $\beta, s_a \in R$. *Then*

$$\int d^3x \left(A_k^a(0,x_1,x_2,x_3)\right)^2 = s_a^2 c_k^2 \left(\frac{\pi}{2}\right)^{3/2} \frac{1}{\beta^3} . \quad (81)$$

We change the variables to y_1, y_2, y_3

$$y_1 = \sqrt{3}x_1 - \frac{2}{\sqrt{3}}x_2 - \frac{1}{\sqrt{6}}x_3; \quad y_2 = \sqrt{\frac{2}{3}}x_2 - \frac{1}{\sqrt{3}}x_3; \quad y_3 = 2x_3 \quad (82)$$

Then

$$\sum_{j=1}^{3}\rho_j^2 = y_1^2 + y_2^2 + y_3^2 \quad (83)$$

As y_2 and y_3 are not functions of x_1 we can change the order of integration

$$\int d^3xe^{-\beta^2\sum_{j=1}^{3}\rho_j^2} = \int d^3xe^{-\beta^2\sum_{j=1}^{3}y_j^2} = \int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\int dx_1e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} =$$
$$= \int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} \quad (84)$$

As y_3 is not a function of x_2 we can change the order of integration

$$\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$
$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3e^{-\beta^2y_3^2}\int dx_2e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3e^{-\beta^2y_3^2}\int dy_2e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2} =$$
$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3e^{-\beta^2y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3e^{-\beta^2y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}. \quad (85)$$

Thus

$$\int d^3xe^{-2\beta^2\sum\rho_j^2} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{1}{\beta^3} \Rightarrow \int d^3xe^{-2\beta^2\sum\rho_j^2} = \sqrt{\left(\frac{\pi}{2}\right)^3}\frac{1}{\beta^3} \Rightarrow$$
$$\Rightarrow \int d^3x\left(A_k^a(0,x_1,x_2,x_3)\right)^2 = s_a^2c_k^2\left(\frac{\pi}{2}\right)^{3/2}\frac{1}{\beta^3}. \quad (86)$$

Thence, we can conclude that the eq. (81) is true.
We can rewrite the eq. (85) also as follows:

$$\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2} = \int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3} =$$
$$= \left[\int d^2xe^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6}. \quad (86b)$$

Lemma 2

Let the gauge field satisfy

$$A_\mu^{a,R}(0,x_1,x_2,x_3) = s_a c_\mu e^{-\beta^2 \sum_{j=1}^{3}\rho_j^2} \quad (87)$$

where ρ_j *and* c_μ *are as in* (79), (80) *and* $\beta, s_a \in R$. *Then*

$$\int d^3 \mathcal{L}_{\mathfrak{R}} = -\frac{1}{\beta}\frac{\sqrt{\pi^3}}{16}\sum_a s_a^2 B \quad (88)$$

where in Minkowski's metric at $x_0 = 0$, *we have* $B = 0$. *In the negative definite metric*

$$(g_{\mu\nu})_{\mu,\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (88b)$$

we have that $B = \frac{13}{3} + \frac{2}{3} + 4$.

From (79) and (82) follows that

$$\rho_1 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3; \quad \rho_2 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{2}}y_3; \quad \rho_3 = -\frac{1}{\sqrt{3}}y_1 - \sqrt{\frac{2}{3}}y_2. \quad (89)$$

For Minkowski's metric

$$P(\rho) = 0 = B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3 \quad (90)$$

where $B_k = 0$ for all k. For the metric (88b)

$$P(\rho) = 4\rho_1^2 + 4\rho_2^2 + \rho_3^2 - 4\rho_2\rho_3 = B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3 \quad (91)$$

where

$$B_1 = \frac{13}{3}; \quad B_2 = \frac{2}{3}; \quad B_3 = 4; \quad B_4 = -\frac{4}{3}\sqrt{2}; \quad B_5 = -\frac{4}{\sqrt{6}}; \quad B_6 = -\frac{4}{\sqrt{3}}. \quad (92)$$

We do the integration with generic parameters B_j. Then

$$\int d^3 x e^{-\frac{1}{2}(2\beta)^2\left(\rho_1^2+\rho_2^2+\rho_3^2\right)} P(\rho) = \int d^3 x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)} \left(B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3\right). \quad (93)$$

As y_2 and y_3 are not functions of x_1 we can change the order of integration and change the integration parameter x_1 to y_1

$$\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(\rho_1^2+\rho_2^2+\rho_3^2\right)} P(\rho) = \int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)} \ (B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3) =$$

$$= \int d^2x e^{-\frac{1}{2}(2\beta)^2\left(y_2^2+y_3^2\right)} \left(B_1 \int dx_1 y_1^2 e^{-\frac{1}{2}(2\beta)^2 y_1^2} + \left(B_2 y_2^2 + B_3 y_3^2 + B_6 y_2 y_3\right) \int dx_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} + \right.$$

$$\left. + \left(B_4 y_2 + B_5 y_3\right) \int dx_1 y_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} \right) =$$

$$= \frac{1}{\sqrt{3}} \int d^2x e^{-\frac{1}{2}(2\beta)^2\left(y_2^2+y_3^2\right)} \left(B_1 \int dy_1 y_1^2 e^{-\frac{1}{2}(2\beta)^2 y_1^2} + \left(B_2 y_2^2 + B_3 y_3^2 + B_6 y_2 y_3\right) \int dy_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} + \right.$$

$$\left. + \left(B_4 y_2 + B_5 y_3\right) \int dy_1 y_1 e^{-\frac{1}{2}(2\beta)^2 y_1^2} \right) =$$

$$= \frac{1}{\sqrt{3}} \int d^2x e^{-\frac{1}{2}(2\beta)^2\left(y_2^2+y_3^2\right)} \left(B_1 \sqrt{2\pi} \frac{1}{(2\beta)^3} + \left(B_2 y_2^2 + B_3 y_3^2 + B_6 y_2 y_3\right) \sqrt{2\pi} \frac{1}{2\beta} \right). \quad (94)$$

As y_3 is not a function of x_2 we can change the order of integration and change the integration parameter x_2 to y_2:

$$\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(\rho_1^2+\rho_2^2+\rho_3^2\right)} P(\rho) = \int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)} \ (B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3) =$$

$$= \frac{1}{\sqrt{3}} \int d^2x e^{-\frac{1}{2}(2\beta)^2\left(y_2^2+y_3^2\right)} \left(B_1 \sqrt{2\pi} \frac{1}{(2\beta)^3} + \left(B_2 y_2^2 + B_3 y_3^2 + B_6 y_2 y_3\right) \sqrt{2\pi} \frac{1}{2\beta} \right) =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \int dx_3 e^{-\frac{1}{2}(2\beta)^2 y_3^2} \left(B_1 \frac{1}{(2\beta)^3} \int dy_2 e^{-\frac{1}{2}(2\beta)^2 y_2^2} + B_2 \frac{1}{2\beta} \int dy_2 y_2^2 e^{-\frac{1}{2}(2\beta)^2 y_2^2} + \right.$$

$$\left. + B_3 y_3^2 \frac{1}{2\beta} \int dy_2 e^{-\frac{1}{2}(2\beta)^2 y_2^2} + B_6 y_3 \frac{1}{2\beta} \int dy_2 y_2 e^{-\frac{1}{2}(2\beta)^2 y_2^2} \right) =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \sqrt{2\pi} \int dx_3 e^{-\frac{1}{2}(2\beta)^2 y_3^2} \left(B_1 \frac{1}{(2\beta)^3} \sqrt{2\pi} \frac{1}{2\beta} + B_2 \frac{1}{2\beta} \sqrt{2\pi} \frac{1}{(2\beta)^3} + B_3 y_3^2 \frac{1}{2\beta} \sqrt{2\pi} \frac{1}{2\beta} \right) =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} 2\pi \int dy_3 e^{-\frac{1}{2}(2\beta)^2 y_3^2} \left(B_1 \frac{1}{(2\beta)^4} + B_2 \frac{1}{(2\beta)^4} + B_3 y_3^2 \frac{1}{(2\beta)^2} \right) =$$

$$= \frac{1}{\sqrt{3}} \sqrt{\frac{3}{2}} \frac{1}{2} (2\pi)^{\frac{3}{2}} (B_1 + B_2 + B_3) \frac{1}{(2\beta)^5} = \pi^{\frac{3}{2}} \frac{1}{(2\beta)^5} (B_1 + B_2 + B_3). \quad (95)$$

We can rewrite this equation also as follows:

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)} \left(B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3\right) \right]^2 = \frac{\pi^3}{4} \times \frac{81}{\left(16\beta^5\right)^2}. \quad (95b)$$

Thence, we can conclude that the eq. (88) is true. Indeed, we have that:

$$\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(\rho_1^2+\rho_2^2+\rho_3^2\right)} P(\rho) = \int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)} \ (B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_1 y_2 + B_5 y_1 y_3 + B_6 y_2 y_3) =$$

$$= \pi^{\frac{3}{2}} \frac{1}{(2\beta)^5}(B_1 + B_2 + B_3) = \int d^3 \mathcal{L}_{\mathcal{R}} = -\frac{1}{\beta}\frac{\sqrt{\pi^3}}{16}\sum_a s_a^2 B. \quad (96)$$

5. Ramanujan's equations, zeta strings and mathematical connections

Now we describe some mathematical connections with some sectors of String Theory and Number Theory, principally with some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, the Ramanujan's identities concerning π and the zeta strings.

3.1 Ramanujan's equations **[6] [7]**

With regard the Ramanujan's modular functions, we note that the number 8, and thence the numbers $64 = 8^2$ and $32 = 2^2 \times 8$, are connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)} \right]} . \quad (97)$$

Furthermore, with regard the number 24 (12 = 24 / 2 and 32 = 24 + 8) this is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)} \right]} . \quad (98)$$

It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad (99)$$

and

$$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad (100)$$

where

$$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identities (Ramanujan's paper: "Modular equations and approximations to π" Quarterly Journal of Mathematics, 45 (1914), 350-372.):

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \qquad (100a)$$

and

$$\pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \qquad (100b)$$

From (100b), we have that

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \qquad (100c)$$

Let $u(q)$ denote the Rogers-Ramanujan continued fraction, defined by the following equation

$$u := u(q) := \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots}, \quad |q|<1 \qquad (101)$$

and set $v = u(q^2)$. Recall that $\psi(q)$ is defined by the following equation

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \qquad (102)$$

Then

$$\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \log\left(u^2v^3\right) + \sqrt{5}\log\left(\frac{1+\left(\sqrt{5}-2\right)uv^2}{1-\left(\sqrt{5}+2\right)uv^2}\right). \qquad (103)$$

We note that $1+\left(\sqrt{5}-2\right)=2\cdot 0{,}61803398$ and that $1-\left(\sqrt{5}+2\right)=2\cdot 1{,}61803398$, where $\phi=0{,}61803398$ and $\Phi=1{,}61803398$ are the aurea section and the aurea ratio respectively. Let $k:=k(q):=uv^2$. Then from page 326 of Ramanujan's second notebook, we have

$$u^5=k\left(\frac{1-k}{1+k}\right)^2 \text{ and } v^5=k^2\left(\frac{1+k}{1-k}\right). \quad (104)$$

It follows that

$$\log\left(u^2v^3\right)=\frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right). \quad (105)$$

If we set $\varepsilon=\left(\sqrt{5}+1\right)/2=1{,}61803398$, i.e. the aurea ratio, we readily find that $\varepsilon^3=\sqrt{5}+2$ and $\varepsilon^{-3}=\sqrt{5}-2$. Then, with the use of (105), we see that (103) is equivalent to the equality

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=\frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right)+\sqrt{5}\log\left(\frac{1+\varepsilon^{-3}k}{1-\varepsilon^3k}\right). \quad (106)$$

Now from Entry 9 (vi) in Chapter 19 of Ramanujan's second notebook,

$$\frac{\psi^5(q)}{\psi(q^5)}=25q^2\psi(q)\psi^3(q^5)+1-5q\frac{d}{dq}\log\frac{f(q^2,q^3)}{f(q,q^4)}. \quad (107)$$

By the Jacobi triple product identity

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad (108)$$

we have

$$\frac{f(q^2,q^3)}{f(q,q^4)}=\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}=q^{1/5}\frac{u(q)}{v(q)}, \quad (109)$$

by the following expression

$$u(q)=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \quad (110)$$

Using (109) in (107), we find that

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=40\int q\psi(q)\psi^3(q^5)dq+\int\frac{8}{5q}dq-8\int\frac{d}{dq}\log\left(q^{1/5}u/v\right)dq=$$

$$=40\int q\psi(q)\psi^3(q^5)dq-8\log(u/v)=40\int q\psi(q)\psi^3(q^5)dq+\frac{8}{5}\log k-\frac{24}{5}\log\frac{1-k}{1+k}, \quad (111)$$

where (104) has been employed. We note that we can rewrite the eq. (111) also as follows:

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=40\int q\psi(q)\psi^3(q^5)dq+\frac{8}{5}\log k-\frac{24}{5}\log\frac{1-k}{1+k}. \quad (112)$$

Multiplying both sides for $\frac{5}{64}$, we obtain the following identical expression:

$$\frac{1}{8}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = 25\cdot\frac{1}{8}\int q\,\psi(q)\psi^3(q^5)dq + \frac{1}{8}\log k - 3\cdot\frac{1}{8}\log\frac{1-k}{1+k}, \quad (112b)$$

or

$$\frac{1}{8}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{8}\left(25\int q\,\psi(q)\psi^3(q^5)dq + \log k - 3\log\frac{1-k}{1+k}\right). \quad (112c)$$

In the Ramanujan's notebook part IV in the Section "Integrals" are examined various results on integrals appearing in the 100 pages at the end of the second notebook, and in the 33 pages of the third notebook. Here, we have showed some integrals that can be related with some arguments above described.

$$\int_0^\infty e^{-2a^2 n}\psi(n)dn = \frac{1}{8\pi a^2} + 4a^2\sum_{k=1}^{\infty}\frac{k}{\left(e^{2\pi k}-1\right)\left(4a^4+k^4\right)} - 4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)}; \quad (113)$$

$$4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{1}{4a} - \frac{\pi}{4} + a\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2}. \quad (114)$$

Multiplying both sides for π^2, we obtain the equivalent expression:

$$\pi^2\cdot 4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{\pi^2}{4a} - \frac{\pi^3}{4} + a\pi^2\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2}. \quad (114b)$$

Let $n \geq 0$. Then

$$\int_0^\infty \frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi}{4} - 2\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}. \quad (115)$$

Also here, multiplying both sides for π^2, we obtain the equivalent expression:

$$\pi^2\int_0^\infty \frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi^3}{4} - 2\pi^2\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}. \quad (115b)$$

Now we analyze the following integral:

$$I := \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx = \frac{\pi^2}{15}; \quad (116)$$

Let $u=\left(1+\sqrt{1+4x}\right)/2$, so that $x=u^2-u$. Then integrating by parts, setting $u=1/v$ and using the following expression $Li_2(z)=-\int_0^z\frac{\log(1-w)}{w}dw$, $z\in C$, and employing the value $Li_2\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{10}-\log^2\left(\frac{\sqrt{5}-1}{2}\right)$, we find that

$$I=\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du=-\int_1^{(\sqrt{5}+1)/2}\frac{\log\left(u^2-u\right)}{u}du=-\int_1^{(\sqrt{5}+1)/2}\left(\frac{\log u}{u}+\frac{\log(u-1)}{u}\right)du=$$

$$=-\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right)+\int_1^{(\sqrt{5}+1)/2}\frac{\log(1-v)-\log v}{v}dv=$$

$$=-\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right)-Li_2\left(\frac{\sqrt{5}-1}{2}\right)+Li_2(1)-\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right)=-\frac{\pi^2}{10}+\frac{\pi^2}{6}=\frac{\pi^2}{15}. \quad (117)$$

Thence, we obtain the following equation:

$$I=\int_0^1\frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx=\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du=\frac{\pi^2}{15}. \quad (118)$$

Multiplying both sides for $\frac{15}{4}\pi$, we obtain the equivalent expression:

$$I=\frac{15}{4}\pi\int_0^1\frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx=\frac{15}{4}\pi\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du=\frac{15}{4}\pi\cdot\frac{\pi^2}{15}=\frac{\pi^3}{4};$$

$$I=\frac{15}{4}\pi\int_0^1\frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx=\frac{15}{4}\pi\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du=\frac{\pi^3}{4}. \quad (118b)$$

In the Ramanujan's paper: "Some definite integrals connected with Gauss's sums" (Ramanujan, 1915 – *Messanger of Mathematics, XLIV, 75-85*), are described the following integrals:

$$\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx=\frac{1}{16}\left(\frac{1}{4}-\frac{3}{\pi^2}\right), \quad (119) \qquad \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx=\frac{1}{16\pi}, \quad (120)$$

$$\int_0^\infty\frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx=\frac{1}{64}\left(\frac{1}{2}-\frac{3}{\pi}+\frac{5}{\pi^2}\right), \quad (121) \qquad \int_0^\infty\frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx=\frac{1}{256}\left(1-\frac{5}{\pi}+\frac{5}{\pi^2}\right). \quad (122)$$

Now, we sum (119) and (120) obtaining:

$$\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx+\int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx=\frac{1}{16}\left(\frac{1}{4}-\frac{3}{\pi^2}\right)+\frac{1}{16\pi}=\frac{1}{64}-\frac{3}{16\pi^2}+\frac{1}{16\pi}.$$

Now we multiply both the sides for π^3 and obtain:

$$\pi^3(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} dx) = \frac{\pi^3}{64} - \frac{3\pi^3}{16\pi^2} + \frac{\pi^3}{16\pi} = \frac{\pi^3}{64} - \frac{3\pi}{16} + \frac{\pi^2}{16} = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right),$$

thence

$$\pi^3\left(\int_0^\infty x^3 \frac{\cos \pi x^2}{\sinh \pi x} dx + \int_0^\infty x^3 \frac{\sin \pi x^2}{\sinh \pi x} dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right). \quad (123)$$

Now, we sum (121) and (122) obtaining:

$$\int_0^\infty \frac{x \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx + \int_0^\infty \frac{x^2 \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx = \left(\frac{1}{128} - \frac{3}{64\pi} + \frac{5}{64\pi^2} + \frac{1}{256} - \frac{5}{256\pi} + \frac{5}{256\pi^2}\right).$$

Now we multiply both the sides for π^3 and obtain the following equivalent equation:

$$\pi^3\left(\int_0^\infty \frac{x \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx + \int_0^\infty \frac{x^2 \cos 2\pi x}{e^{2\pi\sqrt{x}} - 1} dx\right) = \frac{1}{16} \cdot \frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right). \quad (124)$$

In the work of Ramanujan, [i.e. the modular functions,] the number 24 (8 x 3) appears repeatedly. This is an example of what mathematicians call magic numbers, which continually appear where we least expect them, for reasons that no one understands. Ramanujan‘s function also appears in string theory. Modular functions are used in the mathematical analysis of Riemann surfaces. Riemann surface theory is relevant to describing the behavior of strings as they move through space-time. When strings move they maintain a kind of symmetry called "conformal invariance". Conformal invariance (including "scale invariance") is related to the fact that points on the surface of a string's world sheet need not be evaluated in a particular order. As long as all points on the surface are taken into account in any consistent way, the physics should not change. Equations of how strings must behave when moving involve the Ramanujan function. When a string moves in space-time by splitting and recombining a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function. The KSV loop diagrams of interacting strings can be described using modular functions. The "Ramanujan function" (an elliptic modular function that satisfies the need for "conformal symmetry") has 24 "modes" that correspond to the physical vibrations of a bosonic string. When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10) for fermionic strings.

With regard the **Section 1**, we have the following mathematical connections between the eqs. (16b) and (18b) and the eqs. (112c), (123) and (124):

$$\Gamma_{quad}^{(1)} = -\frac{1}{8}\frac{\kappa^2 E_c^2}{4\pi^2}\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{5}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\int d^4x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \frac{1}{8}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{8}\left(25\int q\psi(q)\psi^3(q^5)dq + \log k - 3\log\frac{1-k}{1+k}\right); \quad (125)$$

$$\Gamma_{quad}^{(1)} = -\frac{1}{16}\frac{\kappa^2 E_c^2}{2\pi^2}\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{5}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\int d^4 x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3 \frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (126)$$

$$\Gamma_{quad}^{(1)} = -\frac{1}{16}\frac{\kappa^2 E_c^2}{2\pi^2}\left(\frac{3}{8} - \frac{3}{4}\omega + \frac{5}{8}\omega^2 + \frac{3}{8}\xi - \frac{1}{2}\omega\xi + \frac{3}{8}\omega\zeta - \frac{1}{32}\omega\xi\zeta + \frac{1}{32}\omega^2\zeta\right)\int d^4 x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx + \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx\right) = \frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right); \quad (127)$$

$$\text{divp}\left(\Gamma^{(1)}\right) = \frac{1}{8}\cdot\frac{1}{2}\left(-\frac{\kappa^2 E_c^2}{8\pi^2} - \frac{3\kappa^2\Lambda}{16\pi^2}\ln E_c^2 + \frac{e^2}{3\pi^2}\ln E_c^2\right)\int d^4 x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \frac{1}{8}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{8}\left(25\int q\psi(q)\psi^3(q^5)dq + \log k - 3\log\frac{1-k}{1+k}\right); \quad (128)$$

$$\text{divp}\left(\Gamma^{(1)}\right) = \frac{1}{16}\left(-\frac{\kappa^2 E_c^2}{8\pi^2} - \frac{3\kappa^2\Lambda}{16\pi^2}\ln E_c^2 + \frac{e^2}{3\pi^2}\ln E_c^2\right)\int d^4 x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3 \frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (129)$$

$$\text{divp}\left(\Gamma^{(1)}\right) = \frac{1}{16}\left(-\frac{\kappa^2 E_c^2}{8\pi^2} - \frac{3\kappa^2\Lambda}{16\pi^2}\ln E_c^2 + \frac{e^2}{3\pi^2}\ln E_c^2\right)\int d^4 x\overline{F}^2 \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx + \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx\right) = \frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right). \quad (130)$$

With regard the **Section 2**, we have the following mathematical connections between the eqs. (24) and (27) and the eqs. (123) and (124):

$$T_{LR}^{(d+e)\mu\nu\rho} = 2ig(16\pi G)\int dx\left[\frac{1}{2}V_{qkp}^{\mu\nu\rho}I_2^R(0) + \left(g^{\mu\nu}q^\rho - q^\mu g^{\nu\rho}\right)I_2^R\left(\mathcal{M}_q^2\right) + \left(g^{\nu\rho}k^\mu - k^\nu g^{\rho\mu}\right)I_2^R\left(\mathcal{M}_k^2\right) + \right.$$
$$\left. + \left(g^{\rho\mu}p^\nu - p^\rho g^{\mu\nu}\right)I_2^R\left(\mathcal{M}_p^2\right)\right] \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3 \frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (131)$$

$$T_{LR}^{(d+e)\mu\nu\rho} = 2ig(16\pi G)\int dx\left[\frac{1}{2}V_{qkp}^{\mu\nu\rho}I_2^R(0) + \left(g^{\mu\nu}q^\rho - q^\mu g^{\nu\rho}\right)I_2^R\left(\mathcal{M}_q^2\right) + \left(g^{\nu\rho}k^\mu - k^\nu g^{\rho\mu}\right)I_2^R\left(\mathcal{M}_k^2\right) + \right.$$
$$\left. + \left(g^{\rho\mu}p^\nu - p^\rho g^{\mu\nu}\right)I_2^R\left(\mathcal{M}_p^2\right)\right] \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx + \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}} - 1}dx\right) = \frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right); \quad (132)$$

$$\beta^{\kappa} = -\frac{1}{16\pi^2} b_0 g^3 + g\kappa^2 \frac{-\mu_s^2}{16\pi^2}\left[\ln\frac{M_c^2}{\mu_s^2} + 1 - \gamma_w + y_0\left(\frac{\mu_s^2}{M_c^2}\right)\right] \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3 \frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3 \frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (133)$$

$$\beta^{\kappa} = -\frac{1}{16\pi^2} b_0 g^3 + g\kappa^2 \frac{-\mu_s^2}{16\pi^2}\left[\ln\frac{M_c^2}{\mu_s^2} + 1 - \gamma_w + y_0\left(\frac{\mu_s^2}{M_c^2}\right)\right] \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx + \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx\right) = \frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right). \quad (134)$$

Thence, mathematical connections with the Ramanujan's modular equations that are related to the physical vibrations of the superstrings and with the Ramanujan's identities concerning π.

With regard the **Section 4**, we have the following mathematical connections between the eqs. (85) and (95b), that derive from the (95), and the eqs. (114b), (115b), (118b), (123) and (124):

$$\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^2\cdot 4a^2\int_0^\infty \frac{x\,dx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{\pi^2}{4a} - \frac{\pi^3}{4} + a\pi^2\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2}; \quad (135)$$

$$\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6} \Rightarrow$$

$$\Rightarrow \pi^2\int_0^\infty \frac{\sin(2nx)\,dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi^3}{4} - 2\pi^2\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}; \quad (136)$$

$$\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1} =$$

$$=\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$=\left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2=8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6}\Rightarrow$$

$$\Rightarrow I=\frac{15}{4}\pi\int_0^1\frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx=\frac{15}{4}\pi\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du=\frac{\pi^3}{4};\quad(137)$$

$$\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2}=\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$=\left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2=8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6}\Rightarrow$$

$$\Rightarrow\pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx+\int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right)=\frac{1}{16}\left(\frac{\pi^3}{4}-3\pi+\pi^2\right);\quad(138)$$

$$\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_1^2}=\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}\left(\sqrt{2}\beta\right)^2 y_2^2}=$$

$$=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dx_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)\left(\sqrt{2}\beta\right)^{-2}\int dy_3 e^{-\beta^2 y_3^2}=\frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}\left(\sqrt{2}\beta\right)^{-3}$$

$$=\left[\int d^2x e^{-\beta^2\left(y_2^2+y_3^2\right)}\frac{1}{\sqrt{3}}\sqrt{2\pi}\left(\sqrt{2}\beta\right)^{-1}\right]^2=8\frac{\pi^3}{4}\times\frac{1}{\left(\sqrt{2}\beta\right)^6}\Rightarrow$$

$$\Rightarrow\pi^3\left(\int_0^\infty\frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx+\int_0^\infty\frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx\right)=\frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4}-\frac{3}{2}\pi^2+\frac{5}{2}\pi+\frac{1}{8}\pi^3-\frac{5}{8}\pi^2+\frac{5}{8}\pi\right).\quad(139)$$

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)}\left(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_1y_2+B_5y_1y_3+B_6y_2y_3\right)\right]^2=\frac{\pi^3}{4}\times\frac{81}{\left(16\beta^5\right)^2}\Rightarrow$$

$$\Rightarrow\pi^2\cdot 4a^2\int_0^\infty\frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)}=\frac{\pi^2}{4a}-\frac{\pi^3}{4}+a\pi^2\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2};\quad(140)$$

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)}\left(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_1y_2+B_5y_1y_3+B_6y_2y_3\right)\right]^2=\frac{\pi^3}{4}\times\frac{81}{\left(16\beta^5\right)^2}\Rightarrow$$

$$\Rightarrow \pi^2 \int_0^\infty \frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi^3}{4} - 2\pi^2 \sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)n} \cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}; \quad (141)$$

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)}\left(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_1y_2+B_5y_1y_3+B_6y_2y_3\right)\right]^2 = \frac{\pi^3}{4}\times\frac{81}{\left(16\beta^5\right)^2} \Rightarrow$$

$$\Rightarrow I = \frac{15}{4}\pi\int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx = \frac{15}{4}\pi\int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du = \frac{\pi^3}{4}; \quad (142)$$

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)}\left(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_1y_2+B_5y_1y_3+B_6y_2y_3\right)\right]^2 = \frac{\pi^3}{4}\times\frac{81}{\left(16\beta^5\right)^2} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right); \quad (143)$$

$$\left[\int d^3x e^{-\frac{1}{2}(2\beta)^2\left(y_1^2+y_2^2+y_3^2\right)}\left(B_1y_1^2+B_2y_2^2+B_3y_3^2+B_4y_1y_2+B_5y_1y_3+B_6y_2y_3\right)\right]^2 = \frac{\pi^3}{4}\times\frac{81}{\left(16\beta^5\right)^2} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx + \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}dx\right) = \frac{1}{16}\cdot\frac{1}{2}\left(\frac{\pi^3}{4} - \frac{3}{2}\pi^2 + \frac{5}{2}\pi + \frac{1}{8}\pi^3 - \frac{5}{8}\pi^2 + \frac{5}{8}\pi\right). \quad (144)$$

Thence, mathematical connections with the Ramanujan's identities concerning π, precisely $\frac{\pi^3}{4}$.

3.2 Zeta Strings **[8]**

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (145)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq 1} C_n \mathcal{L}_n = \sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (146)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (147)$$

Employing usual expansion for the logarithmic function and definition (147) we can rewrite (146) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (148)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon, \quad (149)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string". Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (150)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (151)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (152)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (153)$$

and one can easily see trivial solution $\phi = \theta = 0$.

With regard the **Section 3**, we have the following mathematical connection between the eq. (54b) and the eq. (150):

$$\zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \rho = pT^d \frac{Vol(S^{d-2})}{(2\pi)^{d-1}}\int_0^\infty dx \frac{x^{d-1}}{e^x-\xi} = p\left(1-\frac{2}{2^d}\right)^{(1-\xi)/2} \frac{\zeta(d)(d-1)!}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)\pi^{(d-1)/2}} T^d . \quad (154)$$

With regard the **Section 4**, we have the following mathematical connections between the eqs. (74) and (77) and the eqs. (150):

$$\frac{(4\pi)^3}{4}\hat{I}_1 = \int_{\mathcal{F}_L}\frac{d^2\Omega}{\Omega_2^2}Z_{lat} \times 2\zeta(6)\Omega_2^3 + \frac{3\pi}{4\Omega_2^2}\zeta(5) + \pi^3\sqrt{\Omega_2}\sum_{m_1\neq 0, m_2\neq 0}\left|\frac{m_1}{m_2}\right|^{5/2} K_{5/2}(2\pi\Omega_2|m_1 m_2|)e^{2\pi i m_1 m_2 \Omega_1} \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} ; \quad (155)$$

$$\hat{I}_1^3 = V_2\int_{-1/2}^{1/2} d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\Omega_2 U_2}|j+pU|^2} = \frac{2}{\pi^3}\left(2\zeta(6)T_2^3 + \frac{3\pi\zeta(5)}{4T_2^2}\right)E_3(U,\overline{U})^{SL(2,Z)} \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} . \quad (156)$$

In conclusion, multiplying for $\frac{1}{8}\pi^6$ the eq. (77), we obtain the following equivalent equation:

$$\hat{I}_1^3 = \frac{1}{8}\pi^6 \times V_2\int_{-1/2}^{1/2} d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\Omega_2 U_2}|j+pU|^2} = \frac{\pi^3}{4}\left(2\zeta(6)T_2^3 + \frac{3\pi\zeta(5)}{4T_2^2}\right)E_3(U,\overline{U})^{SL(2,Z)}$$

(157)

This equation can be related with the expression (138) and with the eq. (150), giving the following interesting complete mathematical connections:

$$\hat{I}_1^3 = \frac{1}{8}\pi^6 \times V_2\int_{-1/2}^{1/2} d\Omega_1 \int_0^\infty \frac{d\Omega_2}{\Omega_2^2} E_3(\Omega,\overline{\Omega})^{SL(2,Z)} \sum_{(j,p)\neq(0,0)} e^{-\frac{\pi T_2}{\Omega_2 U_2}|j+pU|^2} = \frac{\pi^3}{4}\left(2\zeta(6)T_2^3 + \frac{3\pi\zeta(5)}{4T_2^2}\right)E_3(U,\overline{U})^{SL(2,Z)} \Rightarrow$$

$$\Rightarrow \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\int dy_1 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_1^2} = \int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}(\sqrt{2}\beta)^{-1} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{2\pi}(\sqrt{2}\beta)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dx_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\sqrt{2\pi}(\sqrt{2}\beta)^{-1}\int dx_3 e^{-\beta^2 y_3^2}\int dy_2 e^{-\frac{1}{2}(\sqrt{2}\beta)^2 y_2^2} =$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}(2\pi)(\sqrt{2}\beta)^{-2}\int dx_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)(\sqrt{2}\beta)^{-2}\int dy_3 e^{-\beta^2 y_3^2} = \frac{1}{\sqrt{3}}\sqrt{\frac{3}{2}}\frac{1}{2}(2\pi)^{\frac{3}{2}}(\sqrt{2}\beta)^{-3}$$

$$= \left[\int d^2x e^{-\beta^2(y_2^2+y_3^2)}\frac{1}{\sqrt{3}}\sqrt{2\pi}(\sqrt{2}\beta)^{-1}\right]^2 = 8\frac{\pi^3}{4}\times\frac{1}{(\sqrt{2}\beta)^6} \Rightarrow$$

$$\Rightarrow \pi^3\left(\int_0^\infty x^3\frac{\cos\pi x^2}{\sinh\pi x}dx + \int_0^\infty x^3\frac{\sin\pi x^2}{\sinh\pi x}dx\right) = \frac{1}{16}\left(\frac{\pi^3}{4} - 3\pi + \pi^2\right) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (158)$$

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