BOOTSTRAP TECHNIQUE FOR EXAMINING THE PROPERTIES OF THE MAXIMUM LIKELIHOOD ESTIMATOR OF THE SPATIAL UNILATERAL AUTOREGRESSIVE MODEL

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ABSTRACT

Many methods and procedures have been developed and proposed to overcome the estimation problem in spatial modelling, for example, by defining classes of models called separable models and the unilateral models. A special type of spatial models that received much attention is the spatial unilateral autoregressive models denoted as AR(p_1,1). Several procedures have been proposed to estimate the parameters of this model. Awang and Shitan (2008) used the maximum likelihood method with some modifications at the border in their proposed procedure but the properties and sampling distribution of the estimates were not discussed. Hence, in this paper, we examine the properties and the sampling distribution of the estimate by the method based on resampling, i.e. bootstrapping.

Keywords: Bootstrap, maximum likelihood, sampling distribution, spatial modelling, spatial unilateral autoregressive model.

1. INTRODUCTION

Many methods and procedures have been developed and proposed to overcome the estimation problem in spatial modelling, however, some of the estimators are biased or inconsistent and computational difficulties exist. Some effort had been made to remedy these problems, for example, Martin (1979, 1990 and 1996) studied extensively a class of models called separable models, while Tjøstheim (1978 and 1983) and Basu and Reinsel (1992, 1993 and 1994) considered the unilateral models. A special feature of separable models is that it has a product correlation structure, which in turn simplifies the estimation, whereas, the unilateral models can be analyzed using extension of time series theory in some special cases.

A special type of spatial models that received much attention is the spatial unilateral autoregressive models denoted as AR(p_1,1) and defined by

$$Y_{ij} = \alpha_{10} Y_{i-1,j} + \alpha_{01} Y_{i,j-1} + \cdots + \alpha_{p_1,1} Y_{i-p_1,j-1} + \varepsilon_{ij} \qquad (1.1)$$

where Y_{ij} is an associated random variable and ε_{ij} is an independent and identically random variable with $E(\varepsilon_{ij}) = 0$ and $Var(\varepsilon_{ij}) = \sigma^2$ at site labelled (*i*, *j*) and α_{ij}s are parameters to be estimated for processes on rectangular grid of size $m \times n$. Tjøstheim (1978) discussed the Yule-Walker method for estimating the parameters of the spatial unilateral autoregressive model of any orders. The method can be used to estimate the parameters of the AR(p_1,1) model as defined in (1) above. Shitan and Brockwell (1996) also proposed a procedure to estimate the parameters of the model, where the approach is to transform the 2-dimensional spatial series to a multiple time-series, treating one of the coordinates as a time index and the other coordinates as a multivariate index and then carried out the multivariate least squares estimation procedures. Awang and Shitan (2006) looked at the same problem of estimation from a different perspective where the approach is to use the maximum likelihood method with some modifications at the border. They focused on the second order model, i.e. the AR(2,1) model. The method is then extended for the AR(p_1,1) model defined in equation (1) above (refer to Awang and Shitan, 2008).

In this paper, we examine the properties of the estimator proposed by Awang and Shitan (2008) by using the resampling method, i.e. *bootstrapping*. It is a computer-based method which helps to simplify the theoretical analysis of the estimator. Bootstrap method is considered since it is proven useful in the analysis of time series (see Efron and Tibshirani, 1986). In the next section, the maximum likelihood method as proposed by Awang and Shitan (2008) for estimating the parameters of the AR(p_1,1) model is reviewed. Section 3 discusses about the bootstrap method in general, whereas in section 4 details about the bootstrap procedure applied in this research and the results of the analysis are presented. Finally, in section 5, some conclusions and directions for further research are given.

2. MAXIMUM LIKELIHOOD METHOD FOR THE SPATIAL AR(p_1,1) MODEL

We consider a non-separable spatial unilateral autoregressive, AR(p_1,1) model defined as,

$$Y_{ij} = \alpha_{10}Y_{i-1,j} + \alpha_{01}Y_{i,j-1} + \alpha_{11}Y_{i-1,j-1} + ... + \alpha_{p_1 0}Y_{i-p_1,j} + \alpha_{p_1 1}Y_{i-p_1,j-1} + \varepsilon_{ij}, \quad i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n, \tag{2.1}$$

where $\{Y_{ij}\}$ is a sequence of two-dimensional random variable with zero mean and the errors ε_{ij} are assumed to be normally distributed with mean 0 and common variance σ^2.

By assuming that the unobserved values to be zeroes, and letting the observation vector, $\mathbf{Y} = (Y_{11}, Y_{12}, ..., Y_{1n}, Y_{21}, Y_{22}, ..., Y_{2n}, ..., Y_{m1}, Y_{m2}, ..., Y_{mn})' = (\mathbf{Y_1}, \mathbf{Y}_2, ..., \mathbf{Y_m})'$, where $\mathbf{Y}_i = (Y_{i1}, Y_{i2}, ..., Y_{in})'$, $i = 1, 2, ..., m$ and the error vector, $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{12}, ..., \varepsilon_{1n}, \varepsilon_{21}, \varepsilon_{22}, ..., \varepsilon_{2n}, ..., \varepsilon_{m1}, \varepsilon_{m2}, ..., \varepsilon_{mn})' = (\boldsymbol{\varepsilon_1}, \boldsymbol{\varepsilon_2}, ..., \boldsymbol{\varepsilon_m})'$, where $\boldsymbol{\varepsilon_i} = (\varepsilon_{i1}, \varepsilon_{i2}, ..., \varepsilon_{in})'$, $i = 1, 2, ..., m$ we can rewrite equation (2.1) in the matrix form as,

$$\begin{pmatrix} \mathbf{Y_1} \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \vdots \\ \mathbf{Y}_m \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} \boldsymbol{\Phi}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Phi}_{p_1} & \boldsymbol{\Phi}_{p_1-1} & \cdots & \boldsymbol{\Phi}_2 & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_0 \end{array}\right) \begin{pmatrix} \mathbf{Y_1} \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \vdots \\ \mathbf{Y}_m \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon_1} \\ \boldsymbol{\varepsilon}_2 \\ \boldsymbol{\varepsilon}_3 \\ \vdots \\ \boldsymbol{\varepsilon}_m \end{pmatrix}, \quad (2.2)$$

where $\boldsymbol{\Phi}_j$ s are $n \times n$ matrices defined as,

$$\boldsymbol{\Phi}_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{01} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{01} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{01} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{01} & 0 \end{pmatrix}, \text{ and } \quad \boldsymbol{\Phi}_j = \begin{pmatrix} \alpha_{j0} & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{j1} & \alpha_{j0} & 0 & \cdots & 0 & 0 \\ 0 & \alpha_{j1} & \alpha_{j0} & \cdots & 0 & 0 \\ 0 & 0 & \alpha_{j1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{j1} & \alpha_{j0} \end{pmatrix}.$$

for $j = 1, 2, \ldots, p_1$.

Equation (2.2) can be written more compactly as,

$$\mathbf{Y} = \boldsymbol{\Phi}\mathbf{Y} + \boldsymbol{\varepsilon}, \quad (2.3)$$

where $\boldsymbol{\Phi}$ is $N \times N$ matrix, $N = m \times n$. It is clear that $\boldsymbol{\Phi}$ is a lower triangular matrix with zeroes on the main diagonal. Then, if we decompose $\boldsymbol{\Phi}$ into $2p_1 + 1$ matrices such that it isolates different parameters, we obtain

$$\mathbf{Y} = (\alpha_{10}\mathbf{W}_{10} + \alpha_{01}\mathbf{W}_{01} + \alpha_{11}\mathbf{W}_{11} + \ldots + \alpha_{p_1 0}\mathbf{W}_{p_1 0} + \alpha_{p_1 1}\mathbf{W}_{p_1 1})\mathbf{Y} + \boldsymbol{\varepsilon}, \quad (2.4)$$

where, $\boldsymbol{\Phi} = \alpha_{10}\mathbf{W}_{10} + \alpha_{01}\mathbf{W}_{01} + \alpha_{11}\mathbf{W}_{11} + \ldots + \alpha_{p_1 0}\mathbf{W}_{p_1 0} + \alpha_{p_1 1}\mathbf{W}_{p_1 1}$ and $\mathbf{W}_{jk}$, $j = 1, 2, \ldots, p_1$; $k = 0, 1$ are the $N \times N$ lower triangular weight matrices with elements ones and zeros.

Equation (2.4) can then be written as,

$$\mathbf{Y} = \left(\mathbf{I} - (\alpha_{10}\mathbf{W}_{10} + \alpha_{01}\mathbf{W}_{01} + \alpha_{11}\mathbf{W}_{11} + \ldots + \alpha_{p_1 0}\mathbf{W}_{p_1 0} + \alpha_{p_1 1}\mathbf{W}_{p_1 1})\right)^{-1}\boldsymbol{\varepsilon} \quad (2.5)$$

or

$$\mathbf{Y} = (\mathbf{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\varepsilon} \quad (2.6)$$

where $\mathbf{I}$ is an $N \times N$ identity matrix. Therefore, the covariance matrix of $\mathbf{Y}$, $\mathbf{V}$ is given as,

$$\mathbf{V} = \sigma^2 (\mathbf{I} - \boldsymbol{\Phi})^{-1}\left[(\mathbf{I} - \boldsymbol{\Phi})^{-1}\right]'. \quad (2.7)$$

The square root of the determinant of $\mathbf{V}$ is given as,

$$|\mathbf{V}|^{1/2} = \left(\sigma^2\right)^{N/2}\left|(\mathbf{I} - \boldsymbol{\Phi})^{-1}\right|. \quad (2.8)$$

Since $(\mathbf{I}-\mathbf{\Phi})$ is the lower triangular matrix with diagonal elements 1, $\left|(\mathbf{I}-\mathbf{\Phi})^{-1}\right| = 1$. This leads to

$$|\mathbf{V}|^{1/2} = \left(\sigma^2\right)^{N/2}. \tag{2.9}$$

Therefore, the likelihood function l is given as,

$$\begin{aligned} l &= \frac{1}{(2\pi)^{N/2}|\mathbf{V}|^{1/2}}\exp\left\{-\frac{1}{2}\mathbf{Y}'\mathbf{V}^{-1}\mathbf{Y}\right\} \\ &= (2\pi)^{-N/2}(\sigma^2)^{-N/2}\exp\left\{-\frac{1}{2\sigma^2}\mathbf{Y}'\left[(\mathbf{I}-\mathbf{\Phi})^{-1}(\mathbf{I}-\mathbf{\Phi})^{-1}\right]^{-1}\mathbf{Y}\right\} \\ &= (2\pi)^{-N/2}(\sigma^2)^{-N/2}\exp\left\{-\frac{1}{2\sigma^2}\mathbf{Y}'(\mathbf{I}-\mathbf{\Phi})(\mathbf{I}-\mathbf{\Phi})\mathbf{Y}\right\}. \end{aligned}$$

Thus we obtain the log likelihood, L as

$$L = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\mathbf{Y}'(\mathbf{I}-\mathbf{\Phi}')(\mathbf{I}-\mathbf{\Phi})\mathbf{Y}. \tag{2.10}$$

The partial derivative of L with respect to α_{jk}, $j = 1, 2, \ldots, p_1$; $k = 0, 1$ is given by

$$\frac{\partial L}{\partial \alpha_{jk}} = -\frac{1}{\sigma^2}\left[-\mathbf{Y}'\mathbf{W}'_{jk}\mathbf{Y} + \alpha_{jk}\mathbf{Y}'\mathbf{W}'_{jk}\mathbf{W}_{jk}\mathbf{Y} + \sum_{\forall r\neq j}\sum_{\forall s\neq k}\alpha_{rs}\mathbf{Y}'\mathbf{W}'_{rs}\mathbf{W}_{jk}\mathbf{Y}\right] \tag{2.11}$$

for $j = 1, 2, \ldots, 5$.

Equating (2.11) to zero leads to

$$\left[\alpha_{jk}\mathbf{Y}'\mathbf{W}'_{jk}\mathbf{W}_{jk}\mathbf{Y} + \sum_{\forall r\neq j}\sum_{\forall s\neq k}\alpha_{rs}\mathbf{Y}'\mathbf{W}'_{rs}\mathbf{W}_{jk}\mathbf{Y}\right] = \mathbf{Y}'\mathbf{W}'_{jk}\mathbf{Y}. \tag{2.12}$$

Therefore, denoting $\mathbf{Z}_i = \mathbf{W}_i\mathbf{Y}$, the maximum likelihood for α_{jk} s can be obtained by solving the equation

$$\begin{pmatrix} \mathbf{Z}'_{10}\mathbf{Z}_{10} & \mathbf{Z}'_{01}\mathbf{Z}_{10} & \cdots & \mathbf{Z}'_{p_10}\mathbf{Z}_{10} & \mathbf{Z}'_{p_11}\mathbf{Z}_{10} \\ \mathbf{Z}'_{10}\mathbf{Z}_{01} & \mathbf{Z}'_{01}\mathbf{Z}_{01} & \cdots & \mathbf{Z}'_{p_10}\mathbf{Z}_{01} & \mathbf{Z}'_{p_11}\mathbf{Z}_{01} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{Z}'_{10}\mathbf{Z}_{p_10} & \mathbf{Z}'_{01}\mathbf{Z}_{p_10} & \cdots & \mathbf{Z}'_{p_10}\mathbf{Z}_{p_10} & \mathbf{Z}'_{p_11}\mathbf{Z}_{p_10} \\ \mathbf{Z}'_{10}\mathbf{Z}_{p_11} & \mathbf{Z}'_{01}\mathbf{Z}_{p_11} & \cdots & \mathbf{Z}'_{p_10}\mathbf{Z}_{p_11} & \mathbf{Z}'_{p_11}\mathbf{Z}_{p_11} \end{pmatrix} \begin{pmatrix} \alpha_{10} \\ \alpha_{01} \\ \vdots \\ \alpha_{p_10} \\ \alpha_{p_11} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}'\mathbf{Z}_{10} \\ \mathbf{Y}'\mathbf{Z}_{01} \\ \vdots \\ \mathbf{Y}'\mathbf{Z}_{p_10} \\ \mathbf{Y}'\mathbf{Z}_{p_11} \end{pmatrix},$$

$$\text{or} \quad \begin{pmatrix} \alpha_{10} \\ \alpha_{01} \\ \vdots \\ \alpha_{p_1 0} \\ \alpha_{p_1 1} \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{10}'\mathbf{Z}_{10} & \mathbf{Z}_{01}'\mathbf{Z}_{10} & \cdots & \mathbf{Z}_{p_1 0}'\mathbf{Z}_{10} & \mathbf{Z}_{p_1 1}'\mathbf{Z}_{10} \\ \mathbf{Z}_{10}'\mathbf{Z}_{01} & \mathbf{Z}_{01}'\mathbf{Z}_{01} & \cdots & \mathbf{Z}_{p_1 0}'\mathbf{Z}_{01} & \mathbf{Z}_{p_1 1}'\mathbf{Z}_{01} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{Z}_{10}'\mathbf{Z}_{p_1 0} & \mathbf{Z}_{01}'\mathbf{Z}_{p_1 0} & \cdots & \mathbf{Z}_{p_1 0}'\mathbf{Z}_{p_1 0} & \mathbf{Z}_{p_1 1}'\mathbf{Z}_{p_1 0} \\ \mathbf{Z}_{10}'\mathbf{Z}_{p_1 1} & \mathbf{Z}_{01}'\mathbf{Z}_{p_1 1} & \cdots & \mathbf{Z}_{p_1 0}'\mathbf{Z}_{p_1 1} & \mathbf{Z}_{p_1 1}'\mathbf{Z}_{p_1 1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{Y}'\mathbf{Z}_{10} \\ \mathbf{Y}'\mathbf{Z}_{01} \\ \vdots \\ \mathbf{Y}'\mathbf{Z}_{p_1 0} \\ \mathbf{Y}'\mathbf{Z}_{p_1 1} \end{pmatrix}. \tag{2.13}$$

3. BOOTSTRAP

The bootstrap is a computer-intensive method and it is based on resampling the data. The primary objective of bootstrapping is to obtain the properties of the sampling distribution of a a random variable without assuming any distributional function of the random variable. For independent data, the classical bootstrap is done by resampling with replacement the data of size n, $x_1, x_2, ..., x_n$. In another words, a random sample of size n is drawn and here, we assign a probability mass of $1/n$ on $x_1, x_2, ..., x_n$. From this random sample, the estimator of interest is computed. These steps are repeted B times and from these B estimates, the standard deviation or other estimator's accuracy are calculated.

For dependent data or data with more complicated structure, the procedure above cannot be done in a similar manner. For spatial data, two bootstrap methods may be applied. The first method is the block bootstrap where the data is divided into congruent subregions and the resampling with replacement is done for these subregions. The second method is based on resampling with replacement the residuals, instead of the data. This method is adopted in our study and details about the procedure are presented in the next section.

4. METHODOLOGY AND RESULTS

Having obtained the estimator given in equation (2.13), the question is, how accurate it is as an estimator for $\boldsymbol{\alpha}$? In this research, we choose the standard error of the estimator as the measure of the estimator's accuracy and we choose the bootstrapping the residual method as a tool to determine the value of standard deviation. We focus our analysis on the first order model, AR(1,1) defined as

$$Y_{ij} = \alpha_{10} Y_{i-1,j} + \alpha_{01} Y_{i,j-1} + \alpha_{11} Y_{i-1,j-1}. \tag{4.1}$$

The grid size, $m \times n$ and the value of parameters are the first to be determined. Following the previous research, we choose the grid sizes of 10×6, 8×8, 10×8, 15×6, 10×10, 20×5, 20×10, 15×15, 20×15, 20×16, 30×12, 25×15 and 20×20. Three set of $\boldsymbol{\alpha}$'s values are considered, $\boldsymbol{\alpha} = (\alpha_{10} = 0.3, \alpha_{01} = 0.2, \alpha_{11} = 0.2)$, $\boldsymbol{\alpha} = (\alpha_{10} = 0.5, \alpha_{01} = 0.3, \alpha_{11} = 0.1)$ and $\boldsymbol{\alpha} = (\alpha_{10} = 0.8, \alpha_{01} = 0.7, \alpha_{11} = -0.6)$. The series $\{\varepsilon_{ij}\}$, $i = 1, 2, ..., m$ and $j = 1, 2,$

..., n is then generated from independent standard normal with mean 0 and variance 1. The border values of $\{Y_{ij}\}$ is then determined by assuming that the cells bordering the lattice have fixed values of zeros. Then the remaining values of $\{Y_{ij}\}$ are obtained recursively from equation (4.1). The estimates of the parameters are then obtained by equation (2.13). From these estimates, the estimate of residuals $\{\hat{\varepsilon}_{ij}\}$ is computed by $\hat{\varepsilon}_{ij} = Y_{ij} - \hat{Y}_{ij}$, where $\hat{Y}_{ij} = \hat{\alpha}_{10}Y_{i-1,j} + \hat{\alpha}_{01}Y_{i,j-1} + \hat{\alpha}_{11}Y_{i-1,j-1}$. A bootstrap sample $\left\{\overset{*}{Y}_{ij}\right\}$ is created from $\left\{\overset{*}{\varepsilon}_{ij}\right\}$, where $\left\{\overset{*}{\varepsilon}_{ij}\right\}$ is obtained by sampling with replacement from the residual $\{\hat{\varepsilon}_{ij}\}$. Then, the parameters are reestimated from $\left\{\overset{*}{Y}_{ij}\right\}$ by the same equation (equation 2.13). The estimate is denoted as $\overset{*}{\alpha} = \left(\overset{*}{\alpha}_{10}, \overset{*}{\alpha}_{01}, \overset{*}{\alpha}_{11}\right)$. The steps above are repeated B times (B = 100, 200 and 500). Finally, the standard deviation of the estimate is calculated by

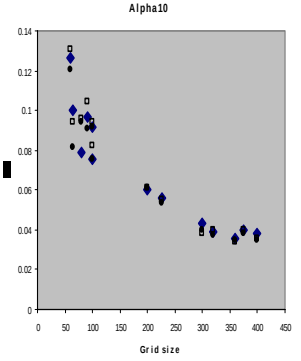
$$s_{\hat{\alpha}} = \sqrt{\sum (\overset{*}{\alpha} - \overset{\bar{}}{\overset{*}{\alpha}})^2 / B - 1}.$$

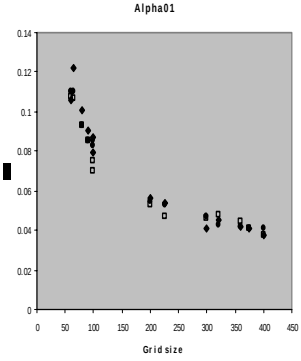
The results of the analysis are presented in Table 1. It displays the value of estimates together with its standard deviation (in bracket, B =100, 200 and 500). We notice that as the grid size increases, the standard deviation of the estimates decreases. For example, for $\boldsymbol{\alpha} = (\alpha_{10} = 0.3, \alpha_{01} = 0.2, \alpha_{11} = 0.2)$ and B = 100, the standard error for $\hat{\alpha}_{10}$ is 0.1440 for the grid size 10×6, but it reduces to 0.0518 when the grid size is 20×20. From the table we may observe that the number of replication (B) does not affecting the value of the standard deviations. As the number of replication increases, the improvement is very small and in many cases, we may conclude that it does not improve at all. Also, we may notice that the standard deviations (as well as the absolute biases) are smaller for $\boldsymbol{\alpha} = (\alpha_{10} = 0.8, \alpha_{01} = 0.7, \alpha_{11} = -0.6)$ compared to the other two set of $\boldsymbol{\alpha}$ values. Another aspect that we may observe is the shape of the grid. Generally, the square grid gives smaller standard deviation (and the absolute bias) than the rectangle grid for the same grid size. For instance, for $\boldsymbol{\alpha} = (\alpha_{10} = 0.3, \alpha_{01} = 0.2, \alpha_{11} = 0.2)$ and B = 200, the standard deviation is (0.1041, 0.0952, 0.0.1121) for 10×10 grid but for 20×5, the standard deviation is (0.1060, 0.0957, 0.1235). To see the results clearly, we plot the graph (see Figure 1) of grid size versus the value of standard deviation for $\boldsymbol{\alpha} = (\alpha_{10} = 0.8, \alpha_{01} = 0.7, \alpha_{11} = -0.6)$.

Table 1: The maximum likelihood estimates and its standard deviation (in bracket: B = 100, 200 and 500) for three set of $\boldsymbol{\alpha}$ values.

a) $\alpha_{10} = 0.3$, $\alpha_{01} = 0.2$, $\alpha_{11} = 0.2$			
Grid size	$\hat{\alpha}_{10}$	$\hat{\alpha}_{01}$	$\hat{\alpha}_{11}$
10 × 6	0.1517 (0.1440, 0.1455, 0.1342)	0.1143 (0.1216, 0.1310, 0.1325)	0.2781 (0.1477, 0.1477, 0.1477)
8 × 8	0.2946 (0.1242, 0.1314, 0.1224)	0.0259 (0.1352, 0.1315, 0.1270)	– 0.0115 (0.1462, 0.1337, 0.1458)
10 × 8	0.2528 (0.1013, 0.1151, 0.1135)	0.1354 (0.1178,0.1130, 0.1143)	0.2198 (0.1213, 0.1217, 0.1294)
15 × 6	0.1599 (0.1139, 0.1202, 0.1105)	0.1343 (0.1182, 0.1078, 0.1048)	0.2341 (0.1202, 0.1126, 0.1230)
10 × 10	0.2076 (0.0973, 0.1041, 0.0989)	0.1412 (0.1028, 0.0952, 0.1009)	0.1869 (0.1070, 0.1121, 0.1087)
20 × 5	0.1057 (0.1172, 0.1060, 0.1061)	0.1043 (0.0995, 0.0957, 0.1017)	0.0652 (0.1223, 0.1235, 0.1163)
20 × 10	0.1340 (0.0817, 0.0800, 0.0747)	0.1362 (0.0710, 0.0682, 0.0652)	0.1164 (0.0736, 0.0770, 0.0764)
15 × 15	0.2336 (0.0701, 0.0734, 0.0678)	0.1754 (0.0693, 0.0591, 0.0631)	0.2851 (0.0698, 0.0665, 0.0704)
20 × 15	0.2127 (0.0600, 0.0558, 0.0583)	0.1264 (0.0529, 0.0586, 0.0601)	0.1373 (0.0679, 0.0655, 0.0631)
20 × 16	0.2114 (0.0562, 0.0530, 0.0528)	0.1133 (0.0490, 0.0521, 0.0561)	0.1483 (0.0585, 0.0594, 0.0579)
30 × 12	0.2942 (0.0468, 0.0500, 0.0498)	0.1278 (0.0552, 0.0507, 0.0525)	0.2873 (0.0569, 0.0510, 0.0532)
25 × 15	0.2769 (0.0497, 0.0457, 0.0499)	0.1355 (0.0520, 0.0542, 0.0510)	0.2310 (0.0573, 0.0564, 0.0513)
20 × 20	0.2807 (0.0518, 0.0493, 0.0481)	0.1355 (0.0489, 0.0480, 0.0504)	0.1660 (0.0543, 0.0554, 0.0520)
b) $\alpha_{10} = 0.5$, $\alpha_{01} = 0.3$, $\alpha_{11} = 0.1$			
Grid size	$\hat{\alpha}_{10}$	$\hat{\alpha}_{01}$	$\hat{\alpha}_{11}$
10 × 6	0.3456 (0.1410, 0.1428, 0.1297)	0.2099 (0.1202, 0.1274, 0.1315)	0.2043 (0.1432, 0.1646, 0.1459)
8 × 8	0.5073 (0.1192, 0.1245, 0.1137)	0.1380 (0.1380, 0.1312, 0.1273)	– 0.0763 (0.1574, 0.1316, 0.1453)
10 × 8	0.4142 (0.0940, 0.1092, 0.1076)	0.2482 (0.1179, 0.1114, 0.1131)	0.1345 (0.1186, 0.1222, 0.1292)
15 × 6	0.3478 (0.1107, 0.1162, 0.1060)	0.2280 (0.1162, 0.1068, 0.1040)	0.1540 (0.1214, 0.1123, 0.1257)
10 × 10	0.3938 (0.0917, 0.0991, 0.0932)	0.2413 (0.1011, 0.0927, 0.0998)	0.0897 (0.1073, 0.1136, 0.1092)
20 × 5	0.3098 (0.1106, 0.1033, 0.1013)	0.2201 (0.0967, 0.0919, 0.1012)	0.0280 (0.1167, 0.1200, 0.1144)
20 × 10	0.3443 (0.0768, 0.0749, 0.0728)	0.2437 (0.0685, 0.0656, 0.0631)	0.0648 (0.0751, 0.0760, 0.0759)
15 × 15	0.4223 (0.0661, 0.0679, 0.0629)	0.2683 (0.0677, 0.0557, 0.0612)	0.2137 (0.0686, 0.0696, 0.0708)
20 × 15	0.4179 (0.0530, 0.0515, 0.0527)	0.2242 (0.0512, 0.0572, 0.0587)	0.0780 (0.0677, 0.0676, 0.0638)

20 × 16	0.4177 (0.0525, 0.0490, 0.0487)	0.2112 (0.0494, 0.0514, 0.0547)	0.0885 (0.0629, 0.0611, 0.0582)
30 × 12	0.5101 (0.0429, 0.0443, 0.0443)	0.2155 (0.0546, 0.0504, 0.0505)	0.1907 (0.0574, 0.0508, 0.0552)
25 × 15	0.4658 (0.0486, 0.0417, 0.0455)	0.2361 (0.0519, 0.0525, 0.0497)	0.1518 (0.0591, 0.0570, 0.0512)
20 × 20	0.4647 (0.0484, 0.0417, 0.0446)	0.2361 (0.0487, 0.0465, 0.0494)	0.0920 (0.0562, 0.0584, 0.538)
c) $\alpha_{10} = 0.8$, $\alpha_{01} = 0.7$, $\alpha_{11} = -0.6$			
Grid size	$\hat{\alpha}_{10}$	$\hat{\alpha}_{01}$	$\hat{\alpha}_{11}$
10 × 6	0.6510 (0.1265, 0.1306, 0.1208)	0.6781 (0.1062, 0.1075, 0.1105)	– 0.5207 (0.1329, 0.1438, 0.1441)
8 × 8	0.8730 (0.1000, 0.0942, 0.0818)	0.6464 (0.1224, 0.1067, 0.1105)	– 0.7538 (0.1580, 0.1230, 0.1315)
10 × 8	0.7179 (0.0787, 0.0956, 0.0946)	0.6949 (0.1008, 0.0933, 0.0933)	– 0.5737 (0.1040, 0.1070, 0.1165)
15 × 6	0.6853 (0.0964, 0.1043, 0.0908)	0.6997 (0.0906, 0.0856, 0.0856)	– 0.5635 (0.1121, 0.1045, 0.1130)
10 × 10	0.7078 (0.0753, 0.0820, 0.0757)	0.6762 (0.0873, 0.0749, 0.0857)	– 0.5908 (0.0980, 0.1075, 0.0974)
20 × 5	0.6138 (0.0920, 0.0942, 0.0915)	0.6617 (0.0798, 0.0703, 0.0831)	– 0.5576 (0.1055, 0.1049, 0.1048)
20 × 10	0.6896 (0.0599, 0.0607, 0.0611)	0.6241 (0.0567, 0.0531, 0.0544)	– 0.5611 (0.0668, 0.0646, 0.0661)
15 × 15	0.7311 (0.0559, 0.0554, 0.0532)	0.6774 (0.0541, 0.0472, 0.0530)	– 0.5136 (0.0586, 0.0613, 0.0652)
20 × 15	0.7532 (0.0430, 0.0385, 0.0399)	0.6123 (0.0413, 0.0461, 0.0472)	– 0.5618 (0.0557, 0.0560, 0.0538)
20 × 16	0.7453 (0.0394, 0.0396, 0.0370)	0.6169 (0.0449, 0.0477, 0.0429)	– 0.5500 (0.0564, 0.0576, 0.0506)
30 × 12	0.8121 (0.0353, 0.0340, 0.0344)	0.6638 (0.0419, 0.0443, 0.0419)	– 0.5561 (0.0490, 0.0467, 0.0492)
25 × 15	0.7676 (0.0402, 0.0402, 0.0381)	0.6492 (0.0407, 0.0406, 0.0413)	– 0.5380 (0.0510, 0.0466, 0.0439)
20 × 20	0.7681 (0.0383, 0.0354, 0.0350)	0.6477 (0.0378, 0.0373, 0.0407)	– 0.5612 (0.0430, 0.0480, 0.0469)





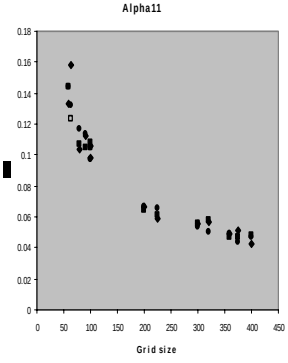


Figure1 : Grid size vs. Standard deviation for $\boldsymbol{\alpha} = (\alpha_{10} = 0.8, \alpha_{01} = 0.7, \alpha_{11} = -0.6)$ – (♦: B = 100, □: B = 200, •: B = 500)

5. COMMENTS AND CONCLUSION

In this analysis, we applied the bootstrap method to examining the properties of the maximum likehood estimator as proposed by Awang and Shitan (2008). The method was based on resampling with replacement the residual of the AR(1,1) model. The standard deviation of the estimator was choosed as the measure for estimator's accuracy. Three set of $\boldsymbol{\alpha}$ values and nine grid sizes were considered. The numbers of the replication were 100, 200 and 500. The results showed that as the grid size increased, the standard deviation of the estimates decreased and the size of replication did not give huge differences for the value of the standard deviations.

From this analysis, we may conclude that bootstrap can be considered as a tool to obtain the properties of sampling distribution of a random variable when the traditional statistical analysis are complicated. Further research may take into cosideration the block bootstrap and comparison can be done as which method of bootstraping is better for examining the properties of the estimator. The research may also be extended to higher order model.

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REFERENCES

1. Awang, N. and Shitan, S. (2006). Estimating the Parameters of the Second Order Spatial Unilateral Autoregressive Model. *International Journal of Statistical Sciences*, 5, 37-58.
2. Awang, N. and Shitan, M. (2008). Maximum likelihood estimation for the non-separable spatial unilateral autoregressive model. Proceedings of the 3rd International Conference on Mathematics and Statistics (ICoMS-3), Institut Pertanian Bogor, Indonesia.
3. Basu, S. and Reinsel, G. C. (1992). A note on properties of spatial Yule-Walker estimators. *Journal of Statistical Computing and Simulation*, **41**, pp. 243-255.
4. Basu, S. and Reinsel, G. C. (1993). Properties of the spatial unilateral first order ARMA model. *Advance in Applied Probability,* 25, 631-648.
5. Basu, S. and Reinsel, G. C. (1994). Regression models with spatially correlated errors. *Journal of the American Statistical Association: Theory and Method*, 89(425), 88-99.
6. Efron, B. and Tibshirani, R. J. (1986). Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statistical Sciences*, 1(1), 54-77.
7. Efron, B. and Tibshirani, R. J. (1993). *An Introduction to the Bootstrap*. New York : Chapman & Hall.

8. Martin, R. J. (1979). A subclass of lattice processes applied to a problem of planar sampling, *Biometrika*, 66, 209-217.
9. Martin, R. J. (1990). The use of time series models and methods in the analysis of agricultural field trials. *Communication in Statistics - Theory and Method*, 19(1), 55-81.
10. Martin, R. J. (1996). Some results on unilateral ARMA lattice processes, *Journal of Statistical Planning and Inference*, 50, 395-411.
11. Shitan, M. and Brockwell, P. J. (1996). An alternative estimation procedure of the spatial AR(p_1,1) model. Research Report No.2, Dept of Statistics and Operations Research, RMIT, Australia.
12. Tjøstheim, D. (1978). Statistical spatial series modelling. *Advances in Applied Probability,* 10, 130-154.
13. Tjøstheim, D. (1983). Statistical spatial series modelling II: Some further results on unilateral lattice processes. *Advances in Applied Probability*, 15, 562-584.