PARAMETRIC SPIRAL AND ITS APPLICATION AS TRANSITION CURVE

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TABLE OF CONTENTS

LIST OF FIGURES

LINGKARAN PARAMETRIK DAN PENGGUNAANNYA SEBAGAI LENGKUNG PERALIHAN

ABSTRAK

Lengkung Bezier merupakan suatu perwakilan lengkungan yang paling popular digunakan di dalam applikasi Rekabentuk Berbantukan Komputer (RBK) dan Rekabentuk Geometrik Berbantukan Komputer (RGBK). Lengkungan berkenaan ditakrifkan secara geometri sebagai lokus titik-titik di dalam ruang tiga dimensi yang terjana oleh nilai-nilai parameternya. Oleh kerana lengkung Bezier juga merupakan suatu fungsi polinomial maka ia amat bersesuaian untuk diimplimentasi di dalam persekitaran interaktif komputer grafik. Walaubagaimanapun terdapat kekangan semulajadi fungsi polynomial yang menjadi penghalang bagi memperolehi sesuatu rupabentuk yang diingini. Bagi lengkung yang berdarjah rendah, umpamanya kubik dan kuartik, kita mungkin memperolehi bentuk-bentuk seperti juring, gelung dan titik lengkokbalas. Suatu lengkungan itu dikatakan 'baik' apabila ia memiliki bilangan ektrema kelengkungan sebagaimana kehendak perekanya. Pada amnya, ini tidak berlaku apabila kita menggunakan perwakilan Bezier. Oleh itu pembinaan lengkung yang baik mampu dicapai apabila kita menggunakan lingkaran kubik dan juga kuartik yang terkawal.

Tesis ini mengkaji tentang lingkaran kubik Alternative dan kuartik Bezier, serta penggunaannya sebagai suatu alternatif kepada perwakilan-perwakilan fungsi lingkaran yang sedia ada. Bagi lingkaran-lingkaran ini, ia telah diperolehi dengan menggunakan kaedah manipulasi aljabar ke atas variasi kelengkungan monoton bagi setiap perwakilan tersebut. Hasil kajian berjaya menunjukkan bahawa peningkatan

darjah kebebasan mampu memberi kebebasan kawalan terhadap panjang lengkung serta berkemampuan untuk melaras fungsi kelengkungan yang berkaitan. Beberapa kaedah dan algoritma telah ditunjukkan bagi pembinaan lingkaran peralihan G^1, G^2, dan juga G^3. Selanjutnya, penggunaan lingkaran untuk masalah-masalah yang lazim dan baru juga telah dibincangkan, yang mana ia boleh digunapakai di dalam bidang RBK/RGBK.

PARAMETRIC SPIRAL AND ITS APPLICATION AS TRANSITION CURVE

ABSTRACT

The Bezier curve representation is frequently utilized in computer-aided design (CAD) and computer-aided geometric design (CAGD) applications. The curve is defined geometrically, which means that the parameters have geometric meaning; they are just points in three-dimensional space. Since they are also polynomial, resulting algorithms are convenient for implementation in an interactive computer graphics environment. However, their polynomial nature causes problems in obtaining desirable shapes. Low degree (cubic and quartic curve) segments may have cusps, loops, and inflection points. Since a fair curve should only have curvature extrema wherever explicitly desired by the designer. But, generally curves do not allow this kind of behavior. Therefore, it would be required to constrain the proposed cubics and quartics, so that the spirals are designed in a favorable way.

This thesis investigated the use of cubic spirals and quartic Bezier spirals as the alternative parametric representations to other spiral functions in the literature. These new parametric spirals were obtained by algebraic manipulation methods on the monotone curvature variation of each curve. Results are reported showing that the additional degree of freedom offers the designer a precise control of total-length and the ability to fine-tune their curvature distributions. The methods and algorithms to construct the G^1, G^2, and G^3 transition spirals have also been presented. We explore some common and new cases that may arise in the use of such spiral segments for practical application of CAD/CAGD.

CHAPTER 1

INTRODUCTION

In this thesis the discussion is centered on the construction of the parametric curve with monotone curvature in computer aided geometric design (CAGD), a discipline in its own right after the 1974 conference at the University of Utah (Barnhill and Riesenfield, 1974). A curve with monotone curvature of constant sign is referred as spiral. Such curves are useful in various types of transition curves. Two representations of parametric polynomial planar curves are considered in this research; cubic Alternative curves and quartic Bezier curves. For these representations, the manipulation method on monotone curvature condition is used to achieve the sufficient condition of the spirals. As the result of this research, the theory and algorithms developed here may give important contributions to the scientific and engineering community.

Computer-Aided Geometric Design (CAGD) deals with the mathematical representation and approximation of three-dimensional physical objects. A major task of CAGD at early ages is to automate the design process of such objects as ship hull, car bodies, airplane wings, and propeller blades; usually represented by smooth meshes of curves and surfaces (Farin, 1997). Although the origin of CAGD was the use of geometry in engineering, CAGD is now extended in a wide range of applications such as Computer-Aided Design (CAD)/ Computer-Aided Manufacture (CAM).

Curves are considered as important graphical primitives to define geometric objects in computer graphics applications. It arises in many applications such as art, industrial design, mathematics, and numerous computer drawing packages and computer aided design packages have been developed to facilitate the creation of curves. A particularly illustrative application is that of computer fonts which are defined by curves that specify the outline of each character in the font. Special font effects can be obtained by applying various transformations such as shears, rotations, and scaling. Other tasks that are also needed in achieving the desirable curves are modifying, fairing, analyzing, and visualizing the curves. In order to execute such operations a mathematical representation for curves is required. In literatures, research in curve designs has been largely dominated by the theory of parametric polynomial curves, or just parametric curves for brevity, due to their highly desirable properties for controlled curve design and trimmed surface design.

Two of the most important mathematical representation of curves and surfaces used in computer graphics and computer aided design are the Bezier and B-spline forms. The original development of Bezier curves took place in the automobile industry during the period 1958-60 by two Frenchman, Pierre Bezier at Renault and Paul de Casteljau at Citroen. The development of B-spline followed the publication in 1946 of a landmark paper on splines (Farin, 2006). Their popularity is due to the fact that they possess a number of mathematical properties which enable their manipulation and analysis, yet no deep mathematical knowledge is required in order to use the curves. According to Farouki and Rajan (1987), the Bezier curves are numerically more stable that other curve forms. Among the Bezier representation, the low degree curves are widely used in the CAGD application, such as quadratic, cubic, quartic, and quintic Bezier curves. A convex shape definitely exists by the use of quadratic Bezier curve. Cubic Bezier curves provide a greater range of shapes than quadratic Bezier curves, since they can exhibit loops, cusps, and inflections. Since quartic Bezier curve is polynomial of degree four so it is more flexible than cubic Bezier curve.

For purposes of this research, we are interested in simple curves. In mathematical terms, simple curves are curves made of a single polynomial span. These curves have the special properties because they are the fairest curves possible and make very simple surfaces that are easy to edit.

1.1 Motivation

For CAD systems that are used for designing curves and surfaces, it is necessary to generate smooth curves or surfaces that satisfy the designer's task purpose. The overall smoothness of the curves or surfaces always referred as fairness; which is a very general property of the curves. It is not only used to illustrate object in terms of aesthetic values (for example, car bodies) but also important on functional values (for example, highway design).

Fairness is a somewhat slippery concept; there is no commonly agreed upon criterion for quantifying it. It is still an open question (Klass, 1980; Farin and Sapidis, 1989). No one can exactly define it, but they know when they see it. While the concept of fairness is subtle, there certainly is agreement on a coarse level. All measures of fairness agree that a circle is the fairest curve to traverse 2π of angle. At the other extreme, a polygon, with its sharp corners at the control points, is definitely not fair. There are many definitions of the fairness of curves. For example, there are the curves with minimum strain energy, the curves that can be drawn with a small number of French curve segments and the curves whose curvature plots consist of a few monotone pieces (Farin and Sapidis, 1989, Yoshida and Saito, 2006). The interested reader is referred to (Moreton, 1992) for a collection of definitions.

Most measures of fairness are based on curvature. Intuitively, curvature is the position of the steering wheel when driving a car along the curve. When driving along a fair curve, thus steering wheel can be described as in “sweet” or smooth motion, and this steering wheel motions is economical. A fair curve would not have sharp or wild variations in curvature. Preserving the sign is useful, especially for curves with inflection points; turning the steering wheel from one side to the other, passing through the central position in which the wheels are straight (zero curvature). Absolute curvature is also equal to the reciprocal of the radius of the osculating circle; the circle that locally “kisses” the curve. A circular arc, of course, has constant curvature.

This steering wheel analogy is an interesting way to visualize the underlying geometry of curvature, and has directly relevant in the application of highway and railway track design. For high-speed trains, even higher derivatives have direct physical meaning. When curvature varies, the forces experienced at the front of the train are different from those experienced at the back, causing stresses and noise. The discontinuous speed of curvature corresponds directly to forces experienced by passengers.

We can plot curvature versus arc length and thus obtain the curvature plot of the curve. The curvature plot can be used for the definition of fair curves. According to Farin, (1997), “A curve is fair if its curvature plot consists of relatively few monotone pieces”. The curvature plot of a parametric curve is a display of the function, $\kappa(s)$, where κ is the curvature at the arc-length s. From the shape of curvature plot and its derivatives, $\kappa^{(n)}(s)$, $n \geq 1$, we may gain more precise information about the behavior of the curves. For simple curves, we might have convex curve, inflection points, or singularities (a loop and a cusp) (Sakai, 1999). And for spline curves, we might obtain more than for a single curve, plus the roughness of curvature plot, which refers it to discontinuity of curvature's derivative. In addition, for monotonic curvature plot of a parametric curve; either increasing or decreasing, we will obtain a spiral.

Spirals are visually pleasing curves of monotone curvature; and they have the advantage of not containing curvature maxima, curvature minima, inflection points and singularities. Many authors have advocated their use in the design of fair curves (Farin, 1997). These spirals are desirable for applications and the benefit of using such curves in the design of surfaces, in particular surfaces of revolution and swept surfaces, is the control of unwanted flat spots and undulations (Walton and Meek, 1998b). The spiral curve has also been used widely in practice because of its functional values, such as in highway design, or the design of robot trajectories. For example, it is desirable that a transition curve between two circular arcs in the horizontal layout of highway design be free of curvature extrema (except at its endpoint) (Baass, 1984). In the discussion about geometric design standards in AASHO, Hickerson in (Hickerson, 1964) states that "sudden changes between curve of widely different radii or between long tangents and sharp curves should be avoided by the use of curves of gradually increasing or decreasing radii without at the same time introducing an appearance of forced alignment". The transition curve which based on combination of clothoid spirals is popular mainly because its curvature is a linear function of its arc length (Baass, 1984). Use of this form of transition spiral has been a part of standard practice in North American railroad track design for many years and continues to be the standard practice today (Klauder, 2001). And the importance of this design feature is highlighted in (Gibreel et al., 1999) that links vehicle accidents to inconsistency in highway geometric design.

Curvature is central in many other application domains as well. There is compelling evidence that the human visual system perceives curves in terms of curvature features, motivating a substantial body of literature on shape completion, or inferring missing segments of curves when shapes are occluded in the visual field (Moreton, 1992).

1.2 Problem Statement

The lack of ease of curvature control of a parametric cubic segment in geometric design has been discussed in

- Fairing of curves (Farin, 1992; Sapidis, 1994).
- Identifying which segments of polynomial curve have monotone curvature, and using such segments in the design of curves and surface (Walton and Meek, 1996a,b; 1998a,b)

Methods for curve fairing typically depend on an examination of curvature plot. Techniques such as knot removal, adjustment of data point, degree elevation and reduction, are then applied to improve the curvature plot, such as reduce the number of monotone pieces. This process is usually iterative and may change the original curve.

An alternative technique of curve fairing is to design with fair curves. In this approach, the designer works with spiral segments (i.e., curve segments of monotone curvature) and fit them together to form a curve whose curvature plot has relatively few monotone pieces. The advantage of this method is that the designer knows a prior that the segments of the curve have monotone curvature and can thus adjust them interactively to obtain a desired curve; a posterior examination of curvature plots, or fairing, is not necessary and the curve need not be changed later to make it fair (Walton and Meek, 1990; 1998a). The disadvantage of this approach is that such spiral segments are not as flexible as the usual NURBS curves, and thus not always suitable for practical applications.

The curvature of some spirals, for example the clothoid spiral (Walton and Meek, 1989; 1992), and the logarithmic spiral (Baumgarten and Farin, 1997), are simple functions of their parameters and are thus more easily controlled than the curvature of a parametric cubic curve. Unfortunately, such curves which usually mean more overhead on implementation. They are also not as flexible as cubic curve segments. Furthermore, many existing CAD software packages are based on NURBS, hence addition of non-polynomial based curve drawing may not be feasible. As the alternative, Walton and Meek (1996a,b) introduce a planar cubic Bezier and Pythagorean hodograph (PH) spiral and show the suitability in the applications such as highway design, in which the clothoid has been traditionally used. Those spirals tended the G^2 orders of continuity, and contain zero curvature at one end point of its segment.

Despite the growing interest in cubic Bezier and PH quintic spiral representation, not in literature has yet recorded about the use of other polynomials representation; i.e. quartic Bezier curves, and curves with shape parameters such as cubic Alternative curves (Jamaludin,1994), which is a cubic Bezier-like curves. Furthermore, G^2 continuity at joints when composing a curve from the segments using cubic Bezier and PH quintic spiral seem as a smooth transition, but it is not adequate for application such as highway design if road-vehicle dynamics is taken into account.

This research desires to construct the parametric spiral as alternative to Walton and Meek (1996a,b), by considering two parametric polynomial curves; quartic Bezier and cubic Alternative curves. Moreover, we analyze the application of these representations as transition curves in CAD/CAGD.

1.3 Objectives

The objectives of this research are;

- To characterize the shape of a cubic Alternative curve.
- To derive the spiral condition of a cubic Alternative curve for transition curve between two circles.
- To construct a family of quartic Bezier spiral that suitable for approximating G^2 Hermite data of clothoid.
- To analyze the application of quartic Bezier spiral in the various G^2 transition curves.
- To construct a family of G^3 quartic Bezier spiral for horizontal geometry of route designs.

1.4 Outline of the thesis

The following is a brief outline of the thesis. Chapter 1 presents the central motivation and problem as the essential of the thesis. Chapter 2 presents the outline of previous works and some basic concepts that contain in CAGD, which is needed to understand the sequel. Much of this material is available in standard texts, to which the interested reader is referred for more details.

In the Chapter 3, we focus on finding the necessary and sufficient condition of shape parameter of cubic Alternative curve on convexity of curve, inflection point and singularities. Some additional remarks on constructing curves that satisfy the monotone curvature condition is presented. This chapter is preliminary step toward understanding the capability of shape parameters of the cubic Alternative curve.

Chapter 4 presents a necessary and sufficient condition of cubic Alternative curve for transition curve between two circles whereby one circle is inside the other. The degree of smoothness of contact tended are G^2 and G^3. The key problem is to analyze the relationship between the first derivation of curvature of parametric curve as a monotone curvature test and its sign, as well as its shape parameter.

In Chapter 5, we discuss a method for constructing a family of planar quartic Bezier spiral. The method is based on analyzing the monotone curvature condition. We also defined theorems and corollaries that related to this spiral. Furthermore, the coordinate free function of quartic Bezier spiral and the algorithm for fulfill the given G^1 Hermite data is described. This is a theoretical result, which is proven by a rather long proof. Chapter 6 discusses the comparison between quartic Bezier spiral obtained from Chapter 5 and the standard clothoid. It is start with detail discussion on finding the allowable region of end point of the quartic spiral, which allow the approximation of G^2 Hermite data of clothoid.

Chapter 7 presents a family of quartic Bezier spiral that have G^3 contact at one end. Various G^2 transition curves which involves combination of point-circle, straight line-circle, two straight lines, and two circles; S-shape, C-shape, and oval shape, are showed. Chapter 8 presents the family of quartic Bezier spiral that hold G^3 contact at two endpoints. Curvature profile of this spiral and the application on the transition between two straight lines are discussed. A comparison of the new transition curve that based on quartic spiral over classical transition curve is exclusively discussed, which relate to the vehicle-road dynamics. Finally, the conclusion on the accomplishment of the thesis and suggestion for further work is presented in Chapter 9.

CHAPTER 2

LITERATURE REVIEW AND BACKGROUND THEORY

This chapter reviews the literature and background theory used in characterization of parametric curves, the development of parametric spiral of G^2 continuity, and G^3 transition spiral. The theorems presented in this chapter have been used frequently in the following chapters

2.1 Previous works

2.1.1 Characterization of parametric curves

Characterization of a curve is carried out to identify whether a curve has any inflection points, cusps, or loops. The characterization of the cubic curve has wide-ranging applications, for instance, in numerically controlled milling operations. In the design of highways, many of the algorithms rely on the fact that the trace of the curve or route is fair; an assumption that is violated if a cusp is present. Inflection points often indicate unwanted oscillations in applications such as the automobile body design and aerodynamics, and a surface that has a cross section curve possessing a loop cannot be manufactured.

Previous work in this area has been done by Wang (1981), who produced algorithms based on algebraic properties of the coefficients of the parametric polynomial and included some geometric tests using the B-spline control polygon. Su and Liu (1983) have presented a specific geometric solution for the Bezier representation, and Forrest (1980) has studied rational cubic curves. DeRose and

Stone (1989) describes a geometric method for determining whether a parametric cubic curve such as a Bezier curve, or a segment of a B-spline, has any loops, cusps, or inflection points. Since the characteristic of the curve does not change under affine transformations (the transformations including rotation, scaling, translation, and skewing), the curve can be mapped onto a canonical form so that the coordinates of three of the control points are fixed. Sakai and Usmani (1996) has extended the case of parametric cubic segments earlier resolved by Su and Liu (1983) to the case of the cubic/quadratic model for computing a visually pleasing vector-valued curve to planar data. A general purpose method to detect cusps in polynomial or rational space curve of arbitrary degree is presented in Manocha and Canny (1992). Using homogeneous coordinates, a rational curve can be represented in a nonrational form. Based on such a nonrational representation of a curve, Li and Cripps (1997) proposed a method to identify inflection points and cusps on 2-D and 3-D rational curves.

Walton and Meek (2001), and Habib and Sakai (2003a) have presented results on the number and location of curvature extrema for whole cubic parametric cubic curves. Thus, the number and location is determined without its practical computation. In Habib and Sakai (2003a), a characteristic diagram or shape diagram of nonrational cubic Bezier is shown.

Yang and Wang (2004) studies the characterization of a hybrid polynomial, C-curve. C-curve is affine images of trochoids or sine curves and uses this relation to investigate the occurrence of inflection points, cusps, and loop. The results are summarized in a shape diagram of C-Bezier curves; this shape diagram is like the one in Su and Liu (1983). For a Bezier-like curve, Azhar and Jamaludin (2006) have characterized rational cubic Alternative representation by using the shoulder point methods and it is only restricted for trimmed shape parameters. From our observation, many of stated authors use discriminant method for the characterization process. We used the similar method to study the characteristic of cubic Alternative curve for untrimmed shape parameters, as discussed in Chapter 3.

In recent studies, Juhasz (2006) derived more general condition by examining parametric curves that can be described by combination of control points and basis functions. The curve can either be in space or in plane. All of the related control points are fixed but one. The locus of the varying control point that yields a zero curvature point on the curve is a developable surface.

2.1.2 Parametric spiral

Curvature continuous curves with curvature extrema only at specified locations are desirable for applications such as the design of highway or railway routes, or the trajectories of mobile robots. Such curves are referred to as being fair (Farin, 1997). Fair curves and surfaces are also desirable in the design of consumer products and several other computer-aided design (CAD) and computer-aided geometric design (CAGD) applications (Sapidis, 1994). Curve segments, which have no interior curvature extrema, are known as spiral, and thus are suitable for the design of fair curves. Besides conic sections, spirals are the curves that have been most frequently used (Baass, 1984; Svensen, 1941).

Work on designing G^2 curve from polynomials curve, which has no interior curvature extrema, was denominated by Walton and Meek. The cubic Bezier and Pythagorean hodograph (PH) quintic spirals were developed by Walton and Meek (1996a,1996b). These are polynomial and are thus usable in NURBS based CAD packages. In both the cubic and PH quintic Bezier spirals, the spiral condition was determined by imposing $\kappa'(1)=0$ where the curvature of the corresponding curve is $\kappa(t)$. The Pythagorean hodograph curve, introduced by Farouki and Sakkalis (1990), has the attractive properties that its arc-length is a polynomial of its parameter, and the formula for its offset is a rational algebraic expression. Both of these curves; cubic Bezier and Pythagorean hodograph curve quintic has eight degrees of freedom. It was forced to be a spiral by placing restrictions on it to ensure monotone curvature (Walton and Meek, 1996b). In doing so the number of degrees of freedom was reduced to five. Although thus spirals can be used to obtain smooth transition curve, but not free to approximating clothoid which also has five degrees of freedom. The main contribution of these researches is the determination of fair curve composed of two cubic and PH spiral segments, which its used for the various transitions encountered in general curve and highway design, as identified by Baass(1984). Thus transition curve cases in highway design, namely, straight line to circle, straight line to straight line, circle to circle with a C-shape, circle to circle with a S-shape, and circle to circle where one circle lies inside the other with an oval transition.

In Walton and Meek (1998b), expressions for regular quadratic and PH cubic spiral segments are derived for the cases of starting at a non-inflection point with a given radius of curvature with decreasing or increasing curvature magnitude, up to a given non-inflection point. This limitation allows them to be used only in the transition curve between circle to circle with an oval-shape. The advantage in studying this work is the determination of a fair curve by a single quadratic and PH cubic spiral segments.

In further works, the number of degrees of freedom in the cubic spiral has been increased to six (Walton and Meek, 1998a). This additional freedom is then exploited to draw guided G^2 curves composed of spiral segments. Similarly, PH quintic spiral has been increased to six in two different ways; (i) by moving the second control point along the line segment joining the first and third control points (Farouki, 1997), and (ii) by relaxing the requirement of a curvature extremum at $t = 1$ (Walton and Meek, 1998b). In Habib and Sakai (2003b) a tension parameter is introduced for the cubic Bezier and PH quintic transition curves developed in Walton and Meek (1996a; 1996b). This parameter is used as an additional degree of freedom to fix one endpoint of the transition curve. With relaxing one of the constraints in Walton and Meek (1996b), Habib and Sakai (2003b) constructed a PH quintic spiral similar to that developed in (Farouki, 1997), i.e., maintaining the constraint $\kappa'(1) = 0$. Thus additional freedom is used for shape control of transition curves, composed of a pair of PH quintic spirals, between two fixed circles.

A further generalization, increasing the number of degrees of freedom in the cubic Bezier spiral to seven, was recently developed (Walton et al., 2003). With this generalization, two spirals joined at their point of zero curvature such that their tangent directions are parallel (hence a G^2 join) can be used for G^2 Hermite interpolation. Walton and Meek (2004) has increased the number of degrees of freedom in PH quintic spiral to seven. This additional degree of freedom allows both endpoints of a PH quintic spiral to be specified, followed by ranges from which an ending tangent angle and an ending curvature can be selected. Not only does the additional freedom provide the PH quintic spiral with more flexibility than the clothoid, but it also allows the construction of a PH quintic spiral that matches the G^2 Hermite data of a clothoid. Recently, Habib and Sakai (2007) introduced a tension parameter, similar to that in Habib and Sakai (2003b) to construct G^2 transition curves between two circles with shape control. And a method of examining conditions under which such G^2 Hermite interpolation can be done was presented by Walton and Meek (2007). The problem considered in this paper is G^2 transition curve with two fixed points. Most recently, Cai and Wang (2009) presented a new method for drawing a transition curves joining circular arcs by using a single C-Bezier curve with shape parameter.

2.1.3 Transition spiral

Although in many applications G^1 continuity is adequate, for applications that depend on the fairness of a curve or surface, especially those that depend on a smooth transition of reflected light, e.g., automobile bodies, or those that require smooth transition of high-speed motion, e.g., horizontal geometry of highways, G^1 or even G^2 continuity is not adequate. For these applications, at least C^3 continuity is required to achieve the desired results (Farin, 1997).

G^3 continuity in geometrical design of highway and railroad track has exclusively related to the vehicle-road dynamics. Although curve shape in which the curvature changes linearly with distance along the curve is visually pleasing but it is not the good form of spiral from the point of view of vehicle-road dynamics. The problem is that there is an abrupt change in acceleration, it is perceived as a "jerk" experience by moving body on the combination curve in which the spiral is used. In the literature, there is a number of studies in designing a new curve has been done. Curves such as Blosss, Sinusoidal, Cosine (Klauder, 2001), and POLUSA (Lipicnik, 1998) were discovered. The recent transition curve began with Baykal et al. (1997) and continued with Tari (2003), and Tari and Baykal (2005). Alternative forms of curvature of new curves presented by those authors provide the continuity of its first derivatives at the start and end of the spiral; this transition curve is definite as G^3

fashion. Deriving the curves from those curvatures always ends up with numerical technique and it is neither polynomial nor rational representation. This major drawback motivates us to consider parametric curves because NURB representation is widely used in geometrical design packages.

2.2 Reviewing of Parametric Curve

We begin by recalling some definitions and properties of parametric polynomial curves represented in Bezier and Alternative form.

2.2.1 The General Bezier Curve

Given $n+1$ control points $P_0, P_1, ..., P_n$, the Bezier curve of degree n is defined to be (Hoschek and Lasser, 1993)

$$R(t) = \sum_{i=0}^{n} P_i \mathrm{B}_{i,n}(t) \tag{2.0}$$

where

$$\mathrm{B}_{i,n}(t) = \begin{cases} \dfrac{n!}{(n-i)!i!}(1-t)^{n-i} t^i & if \quad 0 \le i \le n \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

are called the Bernstein polynomials or Bernstein basis functions of degree n. They are often referred to as integral Bezier curves to distinguish from rational Bezier curves. The polygon formed by joining the control points $P_0, P_1, ..., P_n$ in the specified order is called the Bezier control polygon. The quantities $\dfrac{n!}{(n-i)!i!}$ are called binomial coefficients and are denoted by $\begin{pmatrix} n \\ i \end{pmatrix}$ or ${}^{n}C_i$.

Properties of the Bernstein Polynomials

The Bernstein polynomials have a number of important properties which give rise to properties of Bezier curve.

i. Positivity

The Bernstein polynomials are non-negative on the interval [0,1],

$$\mathrm{B}_{i,n}(t) \geq 0 \qquad\qquad t \in [0,1].$$

ii. Partition of Unity

The Bernstein polynomials of degree n sum to one on the interval [0,1],

$$\sum_{i=0}^{n} \mathrm{B}_{i,n}(t) = 1 \qquad\qquad t \in [0,1].$$

iii. Symmetry

$$\mathrm{B}_{n-i,n}(t) = \mathrm{B}_{i,n}(1-t), \text{ for } i = 0,\ldots,n.$$

iv. Recursion

The Bernstein polynomials of degree n can be expressed in terms of the polynomials of degree $n-1$.

$$\mathrm{B}_{i,n}(t) = (1-t)\mathrm{B}_{i,n-1}(t) + t\mathrm{B}_{i-1,n-1}(t), \quad \text{for } i = 0,\ldots,n,$$

where $\mathrm{B}_{-1,n-1}(t) = 0$ and $\mathrm{B}_{n,n-1}(t) = 0$.

The positivity and partition of unity properties lead to two important properties of Bezier curves, namely the convex hull property and the invariance under affine transformation which we shall describe in the next section.

Properties of the Bezier Curves

A Bezier curve $R(t)$ of degree n with control point $P_0, P_1, \ldots, P_n$ satisfies the following properties.

i. Endpoint Interpolation Property

$$R(0) = P_0 \text{ and } R(1) = P_n .$$

ii. Endpoint Tangent Property

$$R'(0) = n(P_1 - P_0) \text{ and } R'(1) = n(P_n - P_{n-1}).$$

iii. Convex Hull Property (CHP)

Thus every point of a Bezier curve lies inside the convex hull of its defining control points. The convex hull of the control points is often referred to as the convex hull of the Bezier curve.

iv. Invariance under Affine Transformations

Let Φ be an affine transformation (for example, a rotation, translation, or scaling). Then

$$\Phi\left(\sum_{i=0}^{n} P_i \mathrm{B}_{i,n}(t)\right) \equiv \sum_{i=0}^{n} \Phi(P_i)\mathrm{B}_{i,n}(t).$$

v. Variation Diminishing Property (VDP)

For a planar Bezier curve $R(t)$, the VDP states that the number of intersections of given line with $R(t)$ is less than or equal to the number of intersections of the line with the control polygon.

Finally, the quartic Bezier curve is defined in global parameter $0 \le t \le 1$ from (2.0)-(2.1) when $n = 4$.

2.2.2 Cubic Alternative Curve

Alternative basis function is a relatively new set of basis function in the field of computer aided geometric design (Jamaludin, 1994). As compared to the cubic Bezier basis function, these basis functions have two parameters to change the shape of their curve. It is convenient to control the curve by adjusting the parameters rather than to change the control points as what happens to the cubic Bezier polynomial curve. The discussion on shape control of this parametric curve can be found in (Jamaludin et al., 1996; Azhar and Jamaludin, 2003; 2004; 2006). The cubic Alternative basis functions are defined for $t \in [0,1]$ as below;

$$\mathrm{M}_0(t) = (1-t)^2(1+(2-\alpha)t)$$

$$\mathrm{M}_1(t) = \alpha(1-t)^2 t$$

$$\mathrm{M}_2(t) = \beta t^2(1-t)$$

$$\mathrm{M}_3(t) = t^2(1+(2-\beta)(1-t)). \qquad (2.2)$$

where α and β are shape parameters. For $\alpha = \beta = 3$, the cubic Alternative functions reduce to the cubic Bernstein Bezier basis functions. We will obtain cubic Ball basis functions when $\alpha = \beta = 2$, and if $\alpha = \beta = 4$ then (2.2) are known as basis functions for cubic Timmer.

The cubic Alternative basis functions satisfy the following properties;

i. Positivity

If $0 \le \alpha, \beta \le 3$, the cubic Alternative basis function in (2.2) are nonnegative on the interval $t \in [0,1]$, i.e.

$$\mathrm{M}_i(t) \ge 0 \qquad i = 0,1,2,3$$

ii. Partition of Unity

The sum of the cubic Alternative basis function is one on the interval $t \in [0,1]$, i.e.,

$$\sum_{i=0}^{3} \mathrm{M}_i(t) = 1$$

Referring to Jamaludin (1994), cubic Alternative curve which has been used is defined by

$$R(t) = P_0\mathrm{M}_0(t) + P_1\mathrm{M}_1(t) + P_2\mathrm{M}_2(t) + P_3\mathrm{M}_3(t); \quad 0 \le \mathrm{t} \le 1 \qquad (2.3)$$

where P_0, P_1, P_2, P_3 are control points of the curve. In general, the cubic Alternative curve possesses some interesting properties:

- Endpoint Interpolation Property - $R(0) = P_0$ and $R(1) = P_3$.
- Endpoint Tangent Property - $R'(0) = \alpha(P_1 - P_0)$ and $R'(1) = \beta(P_3 - P_2)$.
- Invariance under Affine Transformations.

  Let Φ be an affine transformation. Then

$$\Phi\left(\sum_{i=0}^{3} P_i\mathrm{M}_i(t)\right) \equiv \sum_{i=0}^{3} (\Phi P_i)\mathrm{M}_i(t)$$

- Convex Hull Property

In general, the cubic Alternative curve violated the Convex Hull property for $\alpha, \beta \in \mathbb{R}$. But if $0 \le \alpha, \beta \le 3$, it satisfies the positivity and the partition of unity properties this lead to the convex hull property of the curve.

The above properties of the cubic Alternative curve assist us to understand the behavior of the curve. The values of α and β will affect the cubic Alternative curve geometrically. The parameter α has a stronger influence on the first half of the curve while parameter β has a stronger influence on the second half. This phenomenon is illustrated in Figure 2.1. This figure showed an example of curves segments for given control points on the edge of a rectangle and (α, β) is given by constant $\alpha = 3$, and $\beta = -2, -1, 0, 1, 2, 3, 4, 5$.

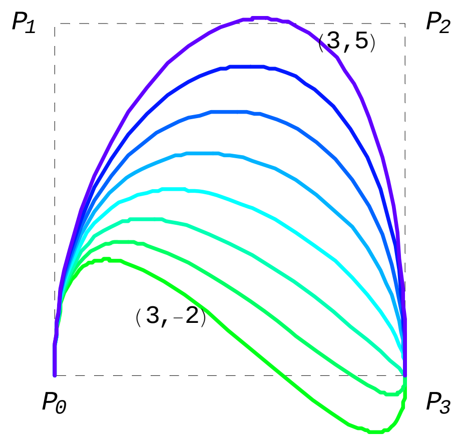

Figure 2.1 An example of cubic Alternative curves with one of the parameter is fixed.

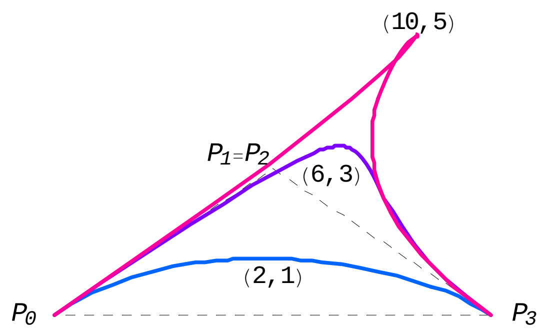

Figure 2.2 An example of cubic Alternative curves with $(\alpha,\beta)=(2,1),(6,3),(10,5)$

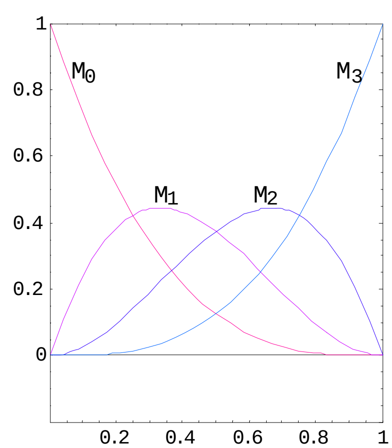

Figure 2.3 Basis function with $(\alpha,\beta)=(3,3)$

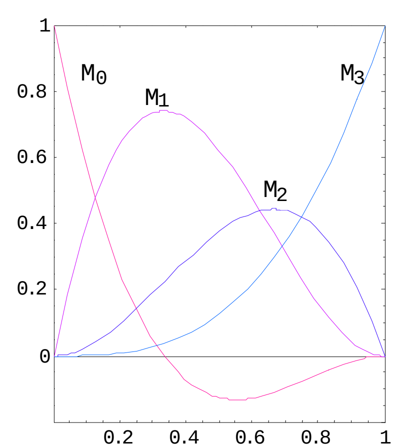

Figure 2.4 Basis function with $(\alpha,\beta)=(5,3)$

Figure 2.2 shows three curves drawn with $(\alpha,\beta)=(2,1),(6,3),(10,5)$ for $P_0P_1(=P_2)P_3$ is control polygon. Figure 2.3 and 2.4, show basis functions with $(\alpha,\beta)=(3,3)$ and $(\alpha,\beta)=(5,3)$, respectively. It is clear that parameters α, β give a great impact on the shape control of a cubic Alternative curve.

2.3 Reviewing of Differential Geometry

The following treatment of a planar curve can be found in books of differential geometry such as Boehm and Prautzsh (1994), Marsh (1999), Rogers (2001), Guggenheimer (1963) and many related papers.

2.3.1 Curvature

In this thesis, a planar curve is defined as follows.

Definition: An oriented planar curve is an ordered set in $\mathbb{R}^2$, given by

$$R(t)=\big(x(t),y(t)\big) \qquad t\in[0,1], \tag{2.4}$$

with direction from $t=0$ to $t=1$. If $R(u)=R(v)$ with $0<u\neq v<1$, then $R(t)$ is a self-intersection point at u (or at v). If $R(t)$ is differentiable at $R(u)$ and $R'(u)=0$, then $R(u)$ is an irregular point. Generally, the position of $R(t)$ at t is the point given by itself: $R(t)$; for regular point, the tangent at t or $R'(t)$ is the oriented line which passes through point $R(t)$, with direction given by the vector $R'(t)$. The signed curvature of a plane curve $R(t)$ is (Hoschek and Lasser, 1993)

$$\kappa(t)=\frac{R'(t)\times R''(t)}{\|R'(t)\|^3} \tag{2.5}$$

As we often use only the sign of the curvature, some times the denominator $\|R'(t)\|^3$ is omitted. The signed radius of curvature $r(t)$ is the reciprocal of (2.5), which for $\kappa(t) \neq 0$, it is given by;

$$r(t) = \frac{1}{\kappa(t)} \tag{2.6}$$

The magnitude of $r(t)$ can be geometrically interpreted as the radius of the osculating circles, i.e. the circle constructed in the limit passing through three consecutive points on the curve (Faux and Pratt, 1988). If $\kappa(t) = 0$, then the radius of curvature is infinite. $\kappa(t)$ is a positive sign when the curve segment bends to left and it is a negative sign if it bends to right at t. Let $R'(t)$ and $R''(t)$ be the first and second derivations of $R(t)$, respectively. From the first derivative of $\kappa(t)$ gives

Lemma 2.0

$$\kappa'(t) = \frac{\Lambda(t)}{\|R'(t)\|^5}. \tag{2.7}$$

where

$$\Lambda(t) = \{R'(t) \bullet R'(t)\}\frac{d}{dt}\{R'(t) \times R''(t)\} - 3\{R'(t) \times R''(t)\}\{R'(t) \bullet R''(t)\}. \tag{2.8}$$

Straight forward from (2.7), the second derivative of $\kappa(t)$ gives

Lemma 2.1

$$\kappa''(t) = \frac{\Upsilon(t)}{\|R'(t)\|^7}, \tag{2.9}$$

where

$$\Upsilon(t) = \{R'(t) \bullet R'(t)\}\frac{d}{dt}\Lambda(t) - 5\Lambda(t)\{R'(t) \bullet R''(t)\}. \tag{2.10}$$