SUBCLASSES OF MULTIVALENT HARMONIC MAPPINGS DEFINED BY CONVOLUTION

ROSIHAN M. ALI, B. ADOLF STEPHEN, AND K. G. SUBRAMANIAN

1. Introduction

Harmonic mappings have been recently investigated from the perspective of geometric function theory. These mappings are important in the study of minimal surfaces. Although harmonic mappings need not be analytic, they have been studied as generalizations of conformal mappings. The seminal works of Clunie and Sheil-Small [4] and Sheil-Small [8] showed that while certain classical results for conformal mappings have analogues for harmonic mappings, many other basic questions remain unsolved. In this work, we introduce two new subclasses of multivalent harmonic univalent functions defined by convolution. The subclasses generate a number of known subclasses of multivalent harmonic mappings, and thus provide a unified treatment in the study of these subclasses. Sufficient coefficient conditions are obtained that are also shown to be necessary when the functions have negative coefficients. Growth estimates and extreme points are also determined.

Definition 1.1. Let σ be a real constant and $\phi(z) = z^m + \sum_{n=2}^{\infty} \phi_{n+m-1} z^{n+m-1}$ be a given analytic function in U. A harmonic function $f = h + \overline{g} \in S_H^0$ where

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \quad g(z) = \sum_{n=2}^{\infty} b_{n+m-1} z^{n+m-1}, \tag{1.1}$$

belongs to the class $SH^0(\phi, \sigma, m, \alpha)$ if

$$\Re\left\{ \frac{z(h*\phi)'(z) - \sigma \overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma \overline{(g*\phi)(z)}} \right\} > m\alpha, \tag{1.2}$$

where $0 \le \alpha < 1; m \ge 1; z \in U$. Here $*$ is the convolution operator.

Equivalently, with $F(z) = (\phi + \sigma\overline{\phi}) * (h(z) + \overline{g(z)})$, the function $f \in S_H^0(\phi, \sigma, m, \alpha)$ provided $\frac{\partial}{\partial\theta} arg(F(re^{i\theta}) \ge m\alpha$ on $|z| = r$. The subclasses $S_H^0(m, \alpha)$ and $K_H^0(m, \alpha)$ of multivalent harmonic functions are special cases of the new class $S_H^0(\phi, \sigma, m, \alpha)$ for suitable choices of ϕ and σ. In fact $S_H^0(\frac{z^m}{1-z}, 1, m, \alpha)$ and $K_H^0(\frac{z^m}{(1-z)^2} - \frac{(1-m)z^m}{1-z}, -1, m, \alpha)$ are respectively the classes $S_H^0(m, \alpha)$ of multivalent harmonic starlike functions and $K_H^0(m, \alpha)$ of multivalent harmonic convex functions investigated by Ahuja and Jahangiri [2]. In Section 2 of this note, a necessary and sufficient convolution condition is obtained for $S_H^0(\phi, \sigma, m, \alpha)$ and for another class $SP_H^0(\phi, \sigma, m, \alpha)$. Sufficient coefficient conditions are obtained for these two classes, which in Section 3 are also shown to be necessary when f

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has negative coefficients. Section 3 is also devoted to determining growth estimates and extreme points for the class $S_H^0(\phi,\sigma,m,\alpha)$.

2. Main Results

We now derive a convolution characterization for functions in the class $S_H^0(\phi,\sigma,m,\alpha)$.

Theorem 2.1. *Let $f=h+\overline{g}\in S_H^0$. Then $f\in S_H^0(\phi,\sigma,m,\alpha)$ if and only if*

$$(h*\phi)*\left[\frac{z^m+\frac{x+1-2m+2m\alpha}{2m-2m\alpha}z^{m+1}}{(1-z)^2}\right]-\sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}^m-\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\overline{z}^{m+1}}{(1-\overline{z})^2}\right]\neq 0,$$

where $|x|=1, 0<|z|<1$.

Proof. A necessary and sufficient condition for $f=h+\overline{g}$ to be in the class $S_H^0(\phi,\sigma,m,\alpha)$, with h and g of the form (1.1), is given by (1.2). Since

$$\frac{z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}}=1$$

at $z=0$, the condition (1.2) is equivalent to

$$\frac{1}{(m-m\alpha)}\left\{\frac{z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}}-m\alpha\right\}\neq\frac{x-1}{x+1};\quad |x|=1, x\neq-1, 0<|z|<1. \tag{2.1}$$

By a simple algebraic manipulation, (2.1) yields

$$\begin{aligned}0&\neq(x+1)[z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}]-m\alpha(x+1)[(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}]\\&\qquad-(x-1)(m-m\alpha)[(h*\phi)(z)+\sigma\overline{(g*\phi)(z)}]\\&=(h*\phi)*\left[(x+1)\left(\frac{z^m}{(1-z)^2}-\frac{(1-m)z^m}{1-z}\right)-\frac{(xm+2m\alpha-m)z^m}{1-z}\right]\\&\quad-\sigma\overline{(g*\phi)}*\left[\overline{(\overline{x}+1)\left(\frac{z^m}{(1-z)^2}-\frac{(1-m)z^m}{1-z}\right)}+\overline{\frac{(\overline{x}m+2m\alpha-m)z^m}{1-z}}\right].\end{aligned}$$

The latter condition together with (1.2) establishes the result for all $|x|=1$. □

Necessary coefficient conditions for the multivalent harmonic starlike functions and multivalent harmonic convex functions were obtained in [1]. We now derive a sufficient coefficient condition for multivalent harmonic functions to belong to the class $S_H^0(\phi,\sigma,m,\alpha)$.

Theorem 2.2. *Let $f=h+\overline{g}\in S_H^0$. Then $f\in S_H^0(\phi,\sigma,m,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}|a_{n+m-1}||\phi_{n+m-1}|+|\sigma|\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}|b_{n+m-1}||\phi_{n+m-1}|\leq 1.$$

Proof. For h and g given by (1.1), (2.1) gives

$$\left|(h*\phi)*\left[\frac{z^m+\frac{x+1-2m+2m\alpha}{2m-2m\alpha}z^{m+1}}{(1-z)^2}\right]-\sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}^m-\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\overline{z}^{m+1}}{(1-\overline{z})^2}\right]\right|$$

$$= \left| z^m + \sum_{n=2}^{\infty} \left[n + (n-1)\frac{x+1-2m+2m\alpha}{2m-2m\alpha} \right] a_{n+m-1}\phi_{n+m-1} z^{n+m-1} \right.$$
$$\left. -|\sigma| \sum_{n=2}^{\infty} \left[n\frac{x+\alpha}{1-\alpha} - (n-1)\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha} \right] \overline{b_{n+m-1}\phi_{n+m-1} z^{n+m-1}} \right|$$
$$> |z^m| \Big[1 - \sum_{n=2}^{\infty} \frac{n+m(1-\alpha)-1}{m(1-\alpha)} |a_{n+m-1}||\phi_{n+m-1}|$$
$$-|\sigma| \sum_{n=2}^{\infty} \frac{n+m(1+\alpha)-1}{m(1-\alpha)} |b_{n+m-1}||\phi_{n+m-1}| \Big].$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f \in S_H^0(\phi, \sigma, m, \alpha)$. □

The sufficient coefficient condition for the classes $S_H^0(m, \alpha)$, $K_H^0(m, \alpha)$ are special cases of Theorem 2.2.

Related to the analytic univalent classes of uniformly convex functions and parabolic starlike functions, a survey of which can be found in [5], classes of harmonic functions were introduced by several authors. Such subclasses of harmonic functions include the classes $G_H(\alpha)$ and $GK_H(\alpha)$ of Goodman-Rønning-type harmonic functions studied in [6, 7]. The multivalent cases of these classes can be given a unified treatment by considering the following class of functions.

Definition 2.1. Let σ be a real constant and $\phi(z) = z^m + \sum_{n=2}^{\infty} \phi_{n+m-1} z^{n+m-1}$ be a given analytic function in U. A harmonic function $f = h + \overline{g} \in S_H^0$ where h and g are given by (refeq1.1) belongs to the class $SP_H^0(\phi, \sigma, m, \alpha)$ if

$$\Re\left\{ (1+e^{i\gamma}) \frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}} - me^{i\gamma} \right\} > m\alpha, \quad (\gamma \text{ real}, \quad 0 \le \alpha < 1, m \ge 1, z \in U). \tag{2.2}$$

Theorem 2.3. *Let $f = h + \overline{g} \in S_H^0$. Then $f \in SP_H^0(\phi, \sigma, m, \alpha)$ if and only if*

$$(h*\phi) * \left[\frac{z^m + \frac{(x+1)e^{i\gamma}+x+(1-2m+2m\alpha)}{2m-2m\alpha} z^{m+1}}{(1-z)^2} \right]$$
$$- \sigma\overline{(g*\phi)} * \left[\frac{\frac{(x+1)e^{i\gamma}+x+\alpha}{1-\alpha} \overline{z}^m - \frac{(2m(x+1)-x-1)e^{i\gamma}+2xm-x-1+2m\alpha}{2m-2m\alpha} \overline{z}^{m+1}}{(1-\overline{z})^2} \right] \neq 0,$$

where $|x| = 1, 0 < |z| < 1$.

Proof. A necessary and sufficient condition for f in $SP_H^0(\phi, \sigma, m, \alpha)$, with h and g of the form (1.1), is given by (2.2). Since

$$(1+e^{i\gamma}) \frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}} - me^{i\gamma} = 1$$

at $z = 0$, condition (2.2) is equivalent to

$$\frac{1}{m(1-\alpha)} \left\{ (1+e^{i\gamma}) \frac{z(h*\phi)' - \sigma\overline{z(g*\phi)'}}{(h*\phi) + \sigma\overline{(g*\phi)}} - me^{i\gamma} - m\alpha \right\} \neq \frac{x-1}{x+1}; \quad |x| = 1, x \neq -1, z \neq 0.$$

This now yields the desired result. □

Proceeding similarly as in Theorem 2.2, the following sufficient coefficient condition for the class $SP_H^0(\phi,\sigma,m,\alpha)$ is easily derived.

Theorem 2.4. *Let $f=h+\overline{g}\in S_H^0$. Then $f\in SP_H^0(\phi,\sigma,m,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{2n+m(1-\alpha)-2}{m(1-\alpha)}|a_{n+m-1}||\phi_{n+m-1}|+|\sigma|\sum_{n=2}^{\infty}\frac{2n+m(3+\alpha)-2}{m(1-\alpha)}|b_{n+m-1}||\phi_{n+m-1}|\leq 1.$$

3. The Class $TS_H^0(\phi,\sigma,m,\alpha)$

Several subclasses of analytic functions with negative coefficients have been introduced and studied following the work of Silverman [9]. A unified class of analytic p-valent functions with negative coefficients defined by convolution was introduced in [3] that included many well-known subclasses of analytic functions with negative coefficients as special cases. In this section, we shall devote attention to the subclass $TS_H^0(\phi,\sigma,m,\alpha)$ of $S_H^0(\phi,\sigma,m,\alpha)$ consisting of multivalent harmonic functions $f=h+\overline{g}$ of the form

$$h(z)=z^m-\sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1},\quad g(z)=\sigma\sum_{n=2}^{\infty}b_{n+m-1}z^{n+m-1},\quad a_{n+m-1}\geq 0, b_{n+m-1}\geq 0. \tag{3.1}$$

Theorem 3.1. *For f of the form (3.1), $f\in TS_H^0(\phi,\sigma,m,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}a_{n+m-1}\phi_{n+m-1}+\sigma^2\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}b_{n+m-1}\phi_{n+m-1}\leq 1. \tag{3.2}$$

Proof. If f belongs to $TS_H^0(\phi,\sigma,m,\alpha)$, then (1.2) is equivalent to

$$\operatorname{Re}\left\{\frac{m(1-\alpha)z^m-\sum_{n=2}^{\infty}(n+m(1-\alpha)-1)a_n\phi_{n+m-1}z^{n+m-1}-\sigma^2\sum_{n=2}^{\infty}(n+m(1+\alpha)-1)b_{n+m-1}\phi_{n+m-1}\overline{z}^{n+m-1}}{z^m-\sum_{n=2}^{\infty}a_{n+m-1}\phi_{n+m-1}z^{n+m-1}+\sigma^2\sum_{n=2}^{\infty}b_{n+m-1}\phi_{n+m-1}\overline{z}^{n+m-1}}\right\}>0$$

for $z\in U$. Letting $z\to 1^-$ through real values yields condition (3.2). Conversely, for h and g given by (3.1),

$$\left|(h*\phi)*\left[\frac{z^m+\frac{x+1-2m+2m\alpha}{2m-2m\alpha}z^{m+1}}{(1-z)^2}\right]-\sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha}{1-\alpha}\overline{z}^m-\frac{(2m-1)x-1+2m\alpha}{2m-2m\alpha}\overline{z}^{m+1}}{(1-\overline{z})^2}\right]\right|$$
$$>|z|\left[1-\sum_{n=2}^{\infty}\frac{n+m(1-\alpha)-1}{m(1-\alpha)}|a_{n+m-1}||\phi_{n+m-1}|-\sigma^2\sum_{n=2}^{\infty}\frac{n+m(1+\alpha)-1}{m(1-\alpha)}||\phi_{n+m-1}|\right]$$

which is non-negative by hypothesis, thus proving sufficiency of condition (3.2). □

Corollary 3.1. *Let* $\phi(z) = z^m + \sum_{n=2}^{\infty} \phi_{n+m-1} z^{n+m-1}$ *with* $\phi_{n+m-1} \geq \phi_{m+1}$ $(n \geq 2)$, *and* $|\sigma| \geq \frac{1+m(1-\alpha)}{1+m(1+\alpha)}$. *If* $f \in TS_H^0(\phi, \sigma, m, \alpha)$, *then for* $|z| = r < 1$,

$$r^m - \frac{m(1-\alpha)}{(1+m(1-\alpha)\phi_{m+1}} r^{m+1} \leq |f(z)| \leq r^m + \frac{m(1-\alpha)}{(1+m(1-\alpha)\phi_{m+1}} r^{m+1}.$$

We now determine its extreme points.

Theorem 3.2. *Let*

$$h_m(z) := z^m, h_{n+m-1}(z) := z^m - \frac{m(1-\alpha)}{(n+m(1-\alpha)\phi_{n+m-1}} z^{n+m-1},$$

and

$$g_{n+m-1}(z) := z^m + \frac{m(1-\alpha)}{\sigma(n+m(1+\alpha)-1)\phi_n} \overline{z}^{n+m-1}, \quad (n = 2, 3, \cdots).$$

A function $f \in TS_H^0(\phi, \sigma, m, \alpha)$ *if and only if* f *can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1} + \gamma_{n+m-1} g_{n+m-1}),$$

where $\lambda_{n+m-1} \geq 0, \gamma_{n+m-1} \geq 0, \lambda_m = 1 - \sum_{n=2}^{\infty}(\lambda_{n+m-1} + \gamma_{n+m-1})$, *and* $\gamma_m = 0$. *In particular, the extreme points of* $TS_H^0(\phi, \sigma, m, \alpha)$ *are* $\{h_{n+m-1}\}$ *and* $\{g_{n+m-1}\}$.

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School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
E-mail address: rosihan@cs.usm.my,adolf@usm.my,kgs@usm.my

Current Address: School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia, Permanent Address: Department of Mathematics, Madras Christian College, Tambaram, Chennai 600059 India
E-mail address: adolf@usm.my

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
E-mail address: kgs@usm.my