STOCHASTIC OPTIMIZATION FOR FINANCIAL DECISION MAKING: PORTFOLIO SELECTION PROBLEM

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by

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PENGOPTIMUMAN BERSTOKASTIK UNTUK PEMBUATAN KEPUTUSAN KEWANGAN: MASAALAH PEMILIHAN PORTFOLIO

ABSTRAK

Tesis ini mengaplikasikan pengoptimuman berstokastik sebagai penyelesaian kepada masaalah pemilihan portfolio. Pemilihan portfolio merupakan satu bidang penting dalam pembuatan keputusan kewangan. Ciri penting bagi masaalah dalam pasaran kewangan umumnya terpisah dan tertakrif dengan jelas.

Ketidakpastian merupakan ciri yang tidak dapat dipisahkan dalam pembuatan keputusan kewangan. Untuk menangani masaalah ini, kaedah berkebarangkalian perlu di gunakan bersama dengan teknik pengoptimuman. Pengaturcaraan stokastik satu tahap dan pengaturcaraan stokastik dua tahap dibangunkan dengan pembetulan risiko. Model-model dibangunkan untuk pelabur yang mengelak risiko dengan objektif untuk meminimumkan risiko pelaburan.

Pengaturcaraan stokastik menyediakan rangka kerja untuk memodelkan ketidakpastian dan membolehkan keputusan yang dibuat adalah secara umumnya baik. Taburan kebarangkalian diberikan kepada data tak pasti dan pengaturcaraan matematik dibentuk. Pendekatan yang digunakan dikenali sebagai “di sini dan sekarang” yang mana pembuat keputusan membuat keputusan “sekarang” sebelum nilai sebenar bagi parameter berstokastik berlaku.

Pembuatan keputusan kewangan berkenaan pembentukan portfolio yang terdiri daripada saham-saham pelaburan pertimbangkan. Model-model diaplikasikan kepada pengoptimuman portfolio dalam satu masa, dengan andaian pelabur mempunyai bajet

permulaan dan pelabur mahu membentuk portfolio daripada saham-saham berisiko yang mana ciri terpenting di dalam masalah pengoptimuman ini adalah ketidakpastian mengenai pulangan pelaburan pada masa depan.

Pertama, pembentukan model dimulakan dengan membentuk model pengaturcaraan stokastik tahap satu di mana pelabur membuat keputusan dengan mempertimbangkan pergerakan harga-harga saham pada masa depan. Kemudian, masalah ini di bentuk pula kepada model pengaturcaraan stokastik tahap dua yang mana di samping pergerakan harga saham, pelabur juga mengambilkira peluang untuk mengimbangkan semula komposisi portfolio apabila pulangan sebenar berlaku. Dalam kes ini, dengan menggunakan model program stokastik dengan pembetulan, pelabur menggunakan polisi dengan mempertimbangkan pergerakan harga saham pada masa depan dan juga menggunakan maklumat yang sedia ada berkenaan dengan pergerakan harga tersebut.

Kami membandingkan pencapaian antara model pengaturcaraan stokastik satu tahap dengan model pengaturcaraan stokastik dua tahap. Perbandingan antara model–model juga dilakukan dengan menggunakan dua ukuran risiko pelaburan. Pencapaian model pengaturcaraan stokastik dua tahap mengatasi pencapaian model stokastik satu tahap apabila diaplikasikan dalam jangkamasa tempoh pelaburan selama satu bulan.

STOCHASTIC OPTIMIZATION FOR FINANCIAL DECISION MAKING: PORTFOLIO SELECTION PROBLEM

ABSTRACT

In this thesis stochastic optimization was applied to solve portfolio selection problem. Portfolio selection problem is one of the important areas in financial decision making. An important distinguishing feature of problems in financial markets is that they are generally separable and well defined.

Uncertainty is an inseparable property in financial decision making. To handle such problems, one needs to utilize probabilistic methods alongside with optimization techniques. Single stage and two stage stochastic programming with risk recourse were developed. The models were developed for risk-averse investors and the objective of the stochastic programming models was to minimize risk.

Stochastic Programming provides a generic framework to model uncertainties and enables decisions that will perform well in the general case. A probability distribution to the uncertain data (return) was assigned and a mathematical programming model was formed. The so-called "**Here-and-Now**" approach where the decision-maker makes decision "now" before observing the actual value for the stochastic parameter was used.

The financial decision making that was considered was that of portfolio selection of common stocks. The models were applied to a single period portfolio optimization problem assuming that an investor has an initial budget and seeks to form a portfolio from risky assets where the most important character within this

optimization problem is the uncertainty of the future returns.

First, a single stage stochastic programming model where the investor makes decision by considering the future movements of stock prices was formulated. Then the problem is formulated as two-stage programming model where the investor may consider both future movements of stock prices as well as rebalancing the portfolio positions as prices change. In this case, using stochastic programming with recourse model, the investor seeks a policy that not only considers future observations of stock prices but also takes into account temporarily available information to make recourse decisions.

The performance of single stage and two stage models was compared and similar comparison was also done between two different deviation risk measures. Two stage models outperformed the single stage models for both risk measures when they were applied to investment period of one month.

CHAPTER 1

INTRODUCTION

1.1 Background

Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and the knowledge of all aspects of the field has grown even more profound (Floudas and Pardalos, 2002; Pardalos and Resende, 2002). At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the fields. Optimization today is a basic research tool in all areas of engineering, medicine and sciences. The decision making tools based on optimization procedures are successfully applied in a wide range of practical problems including finance.

The problems of finding the "best" and the "worst" have always been of great interest. Mathematical Programming (MP) is the generic name for optimization models which are used in planning. MP is characterized by the use of an objective function which must be optimized and a set of linear or non linear equations or inequalities called constraints which must be satisfied. The objective function is introduced to obtain a desirable or in some sense the best solution. This is because in general there are many (often infinitely many) different ways the constraints can be satisfied. However, the MP models turn this into a problem of making the best decision in contrast to any feasible decision. Through the use of objectives and goals therefore MP models are formulated which lead to best decisions.

Optimization has been applied to problems in finance for at least the last half century. Markowitz (1952, 1987) developed the application of optimization to finance when he specified portfolio theory as a quadratic programming problem. He formalized the diversification principles in portfolio selection and earned him the 1990 Nobel prize in economics. Since then, MP techniques have become essential tools in financial decision making, and thus are being increasingly applied in practice.

An important distinguishing feature of problems in financial markets is that they are generally separable and well defined. The objective is usually to maximize profit or minimize risk, and the relevant variables are amenable to quantification, almost always in monetary term. In finance problems, the relationships between the variables are usually well defined, so that, for example, the way in which an increase in the proportion of a portfolio invested in a particular asset affects the mean and variance of the portfolio is clear.

Uncertainty is an inseparable property of any financial decision making. The essence of financial decision making is the study of allocation and deployment of economic resources, both spatially and across time, in an uncertain environment. In many optimization problems the input data such as asset prices, returns, and interest rates, are stochastic. Uncertainties are examined indirectly by sensitivity analyses, or by performing scenario tests on key parameters to see how the optimal solutions vary. While these approaches have their value, the analysis is incomplete because each model of optimization assumes there is only one scenario of the future, and moreover that it will occur with certainty.

To handle such problems, one needs to utilize probabilistic methods alongside with optimization techniques. This leads to the development of a new era called stochastic programming (Precopa, 1995), whose objective is to provide tools helping to design and control stochastic systems with the goal of optimizing their performances. Stochastic Programming provides a generic framework to model uncertainties and enables to make decisions that will perform well in the general case. In Stochastic Programming uncertain data is assigned a probability distribution and a mathematical programming model is formed.

The stochastic properties of the optimization problem models are characterized by analyzing three alternative problems:

i. The Expected–Value approach where the stochastic parameters are replaced by their expected values.

ii. The Wait-and-See approach that relies on perfect information where it is assumed that we can somehow wait until the uncertainty is resolved at the end of the planning horizon before the optimal decision is made.

iii. A third approach so-called "**Here-and-Now**" where the decision-maker makes decision "now" before observing the actual outcome for the stochastic parameter.

The solution of the decision problem for "Here-and Now" approach for the stochastic optimization can be divided into four stages:

1) Model formulation, which requires the selection of essential decision variables, constraints and stochastic components for the problem, and choosing an appropriate objective function;

2) Description of stochastic components, which requires the definition of an appropriate stochastic process and estimation of the model parameters;

3) Discretization of the problem for numerical solution;

4) Solution of the discretized problem with appropriate optimization algorithms.

Portfolio optimization has been one of the important research fields in financial decision making. In this thesis, the financial decision we consider is that of portfolio selections of common stocks. We assume that an investor has an initial budget and seeks to form a portfolio from risky assets. The most important character within this optimization problem is the uncertainty of the future returns.

In this thesis, the mathematical programming models for portfolio selection were extended to explicitly treat uncertainties and risk. Stochastic programming models that allow an investor to consider simultaneously multiple scenarios of an uncertain future were formulated. The models calculate optimal contingency plans for each scenario which in turn are explicitly considered in the calculation of an optimal here-and-now purchasing strategy for the portfolio. In effect, the here-and-now strategy is an optimal hedge against future uncertainties, taking into account the contingency plans that have been predetermined for each scenario.

Stochastic programming allows a direct and intuitive modeling of uncertainties. All that is required is an objective or subjective forecast or assessment of the scenarios to be considered and their associated probabilities. First, single stage stochastic programming models where the investor makes decision by considering the future movements of stock prices were formulated. Then the problems were formulated as two stage programming models where the investor may consider both future movements of stock prices as well as rebalancing the portfolio positions as prices change. In this case, using stochastic programming with recourse model, the investor seeks a policy that not only considers future observations of stock prices but also takes into account temporarily available information to make recourse decisions.

The stochastic components are the returns and the technique of time series forecasting is employed to define the stochastic process of the returns and to estimate the model parameters. The continuous probability distributions of the returns were discretized for numerical solution of the model and the discretized distributions were described in a form of scenario path with finite number of realizations. The discretization of probability distribution results in finite dimensional, usually very large optimization problem, which can be solved with numerical optimization techniques.

1.2 Objective of the study

This study explored the effectiveness of the stochastic optimization techniques to solve portfolio selection problem. Optimization techniques were applied to the selection of common stocks listed on the Main Board of Bursa Malaysia. The objectives of applying these techniques are:

1) To construct portfolio optimization decision models which capture return and risk (due to uncertainty).
2) To combine models of optimum resource allocation and models of randomness in the formation of a portfolio.
3) To compare empirically two downside risk measures on the performance of portfolio returns.

4) To compare empirically the performance of two different types of stochastic programming models when applied to portfolio selection problem.

The research in this thesis concerns the following topics:

1) The adoption of appropriate methods for scenario generation that properly capture the uncertainty of the random variables, without assuming explicitly a particular functional form for the distribution of these variables.
2) The adoption of appropriate measures to control for risk of portfolios when the distribution of asset returns is not normal. Asymmetric return of the underlying assets leads us to adopt downside risk measures, appropriate for asymmetric distributions.
3) The adoption of stochastic optimization as the optimization method to solve the problem of the best selection of stocks in the formation of a portfolio.
4) The extension of optimization models to two-stage stochastic programming with recourse by considering the rebalancing of portfolio composition once the uncertainty of returns is realized.
5) The modification of the risk measures by considering the fluctuation of returns in any year of consideration.

1.3 Methodology

Uncertainty about future economic conditions plays a key role in financial decision making particularly in portfolio optimization problems. The models developed in this thesis support optimal decision making in the face of uncertainty. Stochastic programming models possess several attractive features that make them applicable in portfolio optimization problems. Uncertainty in input parameters of these models is represented by means of discrete distributions (scenarios) that show the joint co-variation of the random variables. The scenarios are not restricted to follow any specific distribution or stochastic process. Thus, any joint distribution of the random variables can be flexibly accommodated. Asymmetric distributions in random variables (which are often the case in financial problems) can be readily incorporated in the portfolio optimization model. The choice of stochastic programming is made for several reasons:

1) They can accommodate general distributions by means of scenarios. We do not have to explicitly assume a specific stochastic process for the return of the assets; instead; we rely on the empirical distribution of these returns.
2) They can address practical issues such as transaction costs, turnover constraints, limits on assets, no short sales, etc. Regulatory, institutional, market specific or other constraints can be included.
3) They can flexibly use different risk measures. We have the choice to optimize the appropriate (for the specific problem) risk measure or a utility function.

In this study, single period portfolio optimization problems were modeled as single-stage stochastic programming models for the mean-risk portfolio selection problem. Downside semi-absolute deviation of portfolio returns from the mean and the maximum downside deviation of portfolio returns from the mean were used as the measure of risk. The objective is to minimize the downside risk and achieving the target wealth at the end of the planning horizon. The optimal portfolio returns between risk measures were compared.

The models were then extended to two stage stochastic linear programming with recourse. This was done by assuming that the investor may revise his portfolio composition once in the future when the uncertain return is revealed. The comparisons of the optimal portfolio returns were made between the two risk measures and also between single stage and two stage models.

The models were then modified by extending the planning horizon to a year and the objective of the model is to minimize the downside risk during the planning year while achieving the target wealth at the end of the year.

The scenarios of asset returns and their associated probabilities constituted the necessary inputs to the stochastic optimization models that determine portfolio compositions. The parametric optimization models traded off expected portfolio return against the relevant risk measure. The efficient risk-return frontiers were thus traced for the respective risk measures. Comparisons of these frontiers enabled a relative assessment of alternative models. These static evaluations compared potential performance profiles at a single point in time.

Back testing experiments were also carried out, whereby the models were repeatedly applied in several successive time periods and the attained returns of their selected portfolios were determined on the basis of observed data. The results of back testing provided a more reliable basis for comparative assessment of the models as they reflected realized performance over longer time periods.

The stochastic programming models developed were evaluated using the historical data obtained for stocks listed on the Main Board of Bursa Malaysia. All models were solved by direct method using revised simplex method and LINGO software was used to analyze the results.

1.4 Research Contributions

The contributions of this research are the following:

1) The development of integrated risk management and optimization framework for portfolio selection problems.
2) The implementation of portfolio optimization models that jointly select the appropriate investments among risky assets and minimizing the risk of investment.
3) The development of single, as well as multistage stochastic programming models for optimal selection of portfolio. The models apply a deviation risk measure that does not rely on the assumption of the normal distribution for portfolio return.
4) The empirical validation of the models through numerical tests using real market data.

1.5 Report Outline

The thesis is organized as follow: Chapter 2 includes literature review and provides background material related to the notation and terminology of later chapters. Mean-risk portfolio optimization models and various risk measures were reviewed. Theoretical properties and different class of risk measures found in literature were included. In particular, two risk measures that can transform the objective function of the mathematical programming into linear functions and the respective mean-risk portfolio optimization models were reviewed. The downside risk of portfolio returns and the single period downside semi absolute deviation model for an optimal portfolio were discussed. Financial applications of stochastic programming and scenario generation approach were also discussed. In particular, the formulation of stochastic programming models for the problem was described in detail in the introductory section of chapter three.

Chapter 3 discusses risk measures formulations considered in this study. They were formulated in terms of scenarios and stochastic programming models for the portfolio optimization problems developed. First, single-stage stochastic programming models and two stage stochastic linear programming models with recourse models were formulated. In these models, investment period of one month was considered. Then, the planning horizon was extended to one year and the risk measures were modified by considering the annual return and the monthly returns during the year differently to quantify risk of investment. The single stage and two stage stochastic programming models were developed using the modified return and risk. This chapter also discussed the modeling of stochastic components of the model and the set of scenarios were used as input to the stochastic programming models. Two sets of scenarios were generated. The first set was generated from the empirical distribution, where each historical data were used as scenarios. The second set of scenarios was generated using multivariate time series model, i.e., the Vector Autoregressive model.

In Chapter 4, the statistical characteristics of the historical data set were examined, the computational tests were described, and the empirical results were discussed. The numerical analysis of the different models discussed in the previous chapter was presented and comparison of the results between two different risks was made. The models were solved by direct solvers using LINGO software and the solutions from different models explained in the previous chapter were discussed. In the first set of models where the one month planning horizon was considered, the first set of scenarios was used as input. The second set of scenarios was used as the input to the modified models where the planning horizon was extended to one year.

Finally, Chapter 5 presents our conclusions from the application of the model, and discusses some shortcomings of the methods proposed in this thesis. The current and future directions that are directly related or have been motivated by some of the issues raised in this thesis were summarized. Figure 1.1 below gives an overview of this study.

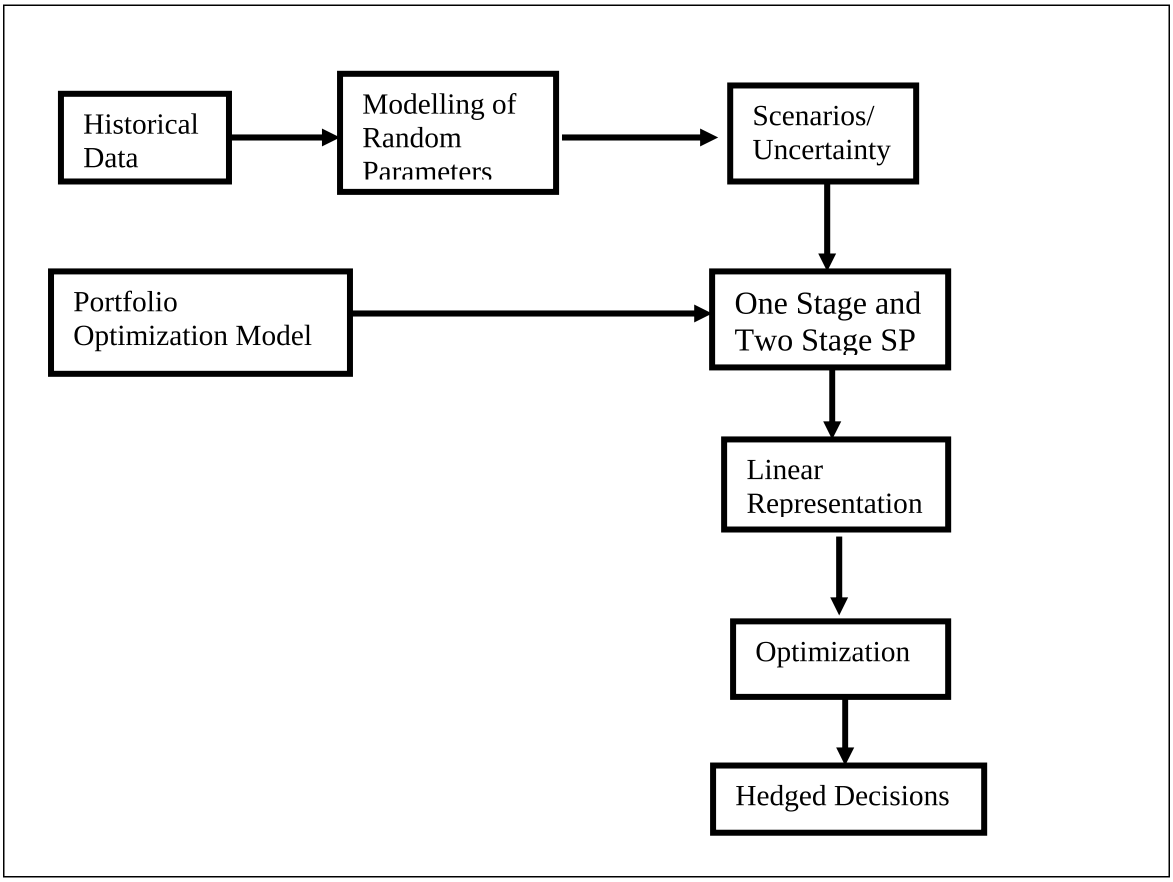

Figure 1.1: Overview of the study

CHAPTER 2

LITERATURE REVIEW

2.0 Introduction

Optimization has been applied to problems in finance for at least the last half century. An important distinguishing feature of problems in financial markets is that they are generally separable and well defined. The objective is usually to maximize profit or minimize risk, and the relevant variables are amenable to quantification, most of the times in monetary term.

Mathematical Programming (MP) is characterized by the use of an objective function which must be optimized and a set of linear or non linear equations or inequalities called constraints which must be satisfied. The objective function is introduced to obtain a desirable or in some sense the best solution. This is because in general there are many (often infinitely many) different ways the constraints can be satisfied. However, the MP models turn this into a problem of making the best decision in contrast to any feasible decision.

A classical problem in finance is as follow. An investor has some initial capital, and wants to make the best use of his/her capital. So the investor turns to a financial market, where certain investment instruments such as bonds, stocks, and derivative securities are provided.

Portfolio selection can be viewed as selecting securities to include in a portfolio and then determining the appropriate weighting: proportional representations of the securities in the portfolio. Now the investor is facing his portfolio optimization problem.

The literature on models for portfolio optimization is vast and dates back to the seminal contribution of Markowitz (1952) where he established a quantitative framework for asset allocation into a portfolio that is well known. The Markowitz model indicates that the proper goal of portfolio construction should be to generate a portfolio that provides the highest return at a given level of risk. A portfolio having this characteristic is known as an efficient portfolio and has generally been accepted as the paradigm of optimal portfolio selection.

Investors aim to select investment portfolios that yield the maximum possible return, while at the same time ensuring an acceptable level of risk exposure. Risk derives from potential losses in portfolio value due to possible reduction in the market values of financial assets resulting from changes in equity prices. Diversification into multiple securities can practically eliminate idiosyncratic risk, that is, potential severe losses from any individual security.

Risk management is the discipline that provides tools to measure the risks and techniques to help us shape and make rational decisions about them. The first task is to measure the risk. Risk measurement identifies the risk factors that affect the performance of portfolios. The definition of risk is a particularly difficult task. It can be defined as the uncertainty of the future outcome of a decision today. To measure the risk, a number of measures have been proposed in the literature. The standard deviation, also known as volatility, has been the most widely used measure. However, this measure relies on the assumption that the portfolio return distribution is symmetric and implies that the sensitivity of the investor is the same on the upside as on the downside. In order to take the asymmetry of the portfolio return distribution into account, the use of downside measures has been advocated. In this chapter we also review studies on risk management as risk measures, and we justify our choice of a measure to control the excess shortfall risk of portfolio optimization problem.

In many real world problems, the uncertainty relating to one or more parameters can be modeled by means of probability distributions. In essence, every uncertain parameter is represented by a random variable over some canonical probability space; this in turn quantifies the uncertainty. Stochastic programming (SP) enables modeler to incorporate this quantifiable uncertainty into an underlying optimization model. Stochastic Programming model combine the paradigm of mathematical programming with modeling of random parameters, providing decisions which hedge against future uncertainties.

Stochastic programming is our approach to deal with uncertainty. This approach can deal the management of portfolio risk and the identification of optimal portfolio simultaneously. The stochastic programming method overcomes the limitations of other techniques such as dynamic programming or optimal control. These methods provide significant insights about fundamental issues in investments and risk managements, and are good approaches for theoretical contributions, but their practical use is limited by the many simplifying assumptions that are needed to derive the solutions.

In stochastic programming the uncertainty is treated using discrete random variables, that is, random variables that take on a countable set of values. With each of these values we associate a probability. Particularly, we assume that as time evolves, the random variables take one from a finite set of values, which is a discrete scenario. The set of these scenarios with the associated probabilities is a discrete distribution. The scenarios are then used as inputs to the stochastic programming models for managing portfolios. Thus, the scenario generation is an important step for the portfolio optimization problem. We review the major applications of stochastic programming models to financial problems, as well as scenario generation procedures. The review justifies our choice to use stochastic programming models, and the choice of the particular scenario generation procedure that is adopted in this thesis.

In the following section we review the portfolio selection models, different risk measures used in the models, stochastic programming and the generation of random parameters which are related to our models of study.

2.1 Portfolio Optimization Models

Portfolio Optimization models often take the following form: an appropriate risk measure is optimized subject to operating constraints and a parametric constraint that a desirable performance measure (such as expected portfolio return) meets a pre specified target level.

Markowitz (1952) established a quantitative framework for asset allocation into a portfolio assuming asset returns follow a multivariate normal distribution or that investors have a quadratic utility function. He proposed to express the risk of an asset's return by means of the deviation from the expected return (i.e., by the variance). This approach shows that characteristics of a portfolio of assets can be completely described by the mean and the variance (risk) and so is described as a mean-variance (MV) portfolio model. So, for a portfolio of correlated assets the risk must be gauged via the covariances between all pairs of investments. The main innovation introduced by Markowitz is to measure the risk of a portfolio by taking into account the joint (multivariate) distribution of returns of all assets in the portfolio. For a particular universe of assets, the set of portfolios of assets that offers the minimum risk for a given level of returns forms the efficient frontier.

2.1.1 Mean Variance Model

Markowitz' portfolio selection problem (Markowitz, 1952, 1959), also called the mean-variance optimization problem, can be formulated in three different ways. The Markowitz model, put forward in 1952, is a multi (two) objective optimization model which is used to balance the expected return and variance of a portfolio. Markowitz (1952) showed how rational investors can construct optimal portfolios under conditions of uncertainty. For an investor, the returns (for a given portfolio) and the stability or its absence (volatility) of the returns are the crucial aspects in the choice of portfolio. Markowitz used the statistical measurements of expectation and variance of return to describe, respectively, the benefit and risk associated with an investment. The objective is either to minimize the risk of the portfolio for a given level of return, or to maximize the expected level of return for a given level of risk.

Consider an investor who has a certain amount of money to be invested in a number of different securities with random returns. Let R_i denote the return in the next time period for each security i, $i = 1,2,...,n,$ and estimates of its expected return,

μ_i, and variance, σ_i^2, are given. Furthermore, for any two securities *i* and *j*, their correlation coefficient ρ_i^2 is also assumed to be known. If we represent the proportion of the total funds invested in security *i* by x_i, one can compute the expected return and the variance of the resulting portfolio $\boldsymbol{x}=(x_1,\ldots,x_n)$ as follows:

$$E[\boldsymbol{x}]=x_1\mu_1+\ldots+x_n\mu_n=\sum_{i=1}^{n} x_i\mu_i \tag{2.1}$$

and

$$Var[\boldsymbol{x}]=\sum_{i,j}\rho_{ij}\sigma_i\sigma_j x_i x_j=\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\,\sigma_{ij} \tag{2.2}$$

where

σ_{ij} = the coefficients of the (*nxn*) variance-covariance matrix *V* defined for stock *i* and stock *j*

($\sigma_{ii} = \sigma_i$ is the diagonal coefficients for the stock *i*)

The classical Mean-Variance (MV) model for minimizing variance and constraining the expected portfolio return yields at least a target value at the end of holding period is set out below (Markowitz, 1952, 1959).

$$\text{Minimize } Z=\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\,\sigma_{ij} \tag{2.3}$$

Subject to

$\sum_{i=1}^{n} x_i\mu_i \geq \beta$, β is the minimal rate of return required by investor

$\sum_{i=1}^{n} x_i = 1$; budget constraint, weight sum must be equal to 1

$0 \le x_i \le U_{x_i}$ $i = 1,...,n$; U_{x_i} is the maximum allowable amount for investment in stock i.

Mathematically, this formulation produces a convex quadratic programming problem. There are non negativity constraints on the weights since short selling is expensive for individual investors and are not generally permissible for most institutional investors. Further, to ensure that the portfolio is diversified, the weight of a stock cannot exceed the upper limit.

A feasible portfolio x is called efficient if it has the maximal expected return among all portfolios with the same variance, or alternatively, if it has a minimum variance among all portfolios that have at least a certain expected return. The collection of efficient portfolios forms the efficient frontier of the portfolio universe. Varying the desired level of return, β, in (2.3) and repeatedly solving the quadratic program identifies the minimum variance portfolio for each value of β. These are the efficient portfolios that compose the efficient set.

The MV model form a quadratic programming problem and it requires the use of complex non-linear numerical algorithms to solve the portfolio problem. The practical application of such models was severely limited until computers were powerful enough to handle even the smallest problems. Sharpe (1971) commented that if the portfolio problem could be formulated as a linear programming problem, the prospect for practical application would be greatly enhanced. Since then, many attempts have been made to linearize the portfolio optimization procedure (see Speranza, 1993). Several alternative risk measures that can be transformed to linear programming have been proposed.

The mean absolute deviation (MAD) was very early considered in the portfolio analysis by Sharpe (1971) who suggested the Mean Absolute Deviation (MAD) measure to quantify risk. The absolute deviation of a random variable is the expected absolute value of the difference between the random variable and its mean. MAD is a dispersion-type risk linear programming (LP) computable measures that may be viewed as some approximations to the variance itself generated by the use of the absolute values replacing the squares. Konno and Yamazaki (1991) presented and analyzed the complete portfolio optimization model based on this risk measure instead of variance. Since the use of MAD by Konno and Yamazaki (1991) as an alternative to the classical Mean-Variance measure of a portfolio's volatility, MAD models have been applied to various portfolio optimization problems such as Kang and Zenios (1993) and Beltratti *et al*. (2004).

The MAD model is a special case of the piecewise linear risk model. Konno (1990) and Konno and Yamazaki (1991) showed that the MAD approach is equivalent to the MV model if the returns are multivariate normally distributed. That is, under this assumption, the minimization of the sum of absolute deviations of portfolio returns about the mean is equivalent to the minimization of the variance. Since MAD is a piecewise linear convex function of portfolio positions, it allows for fast efficient portfolio optimization procedures by means of linear programming, in contrast to Mean-Variance approach, which leads to quadratic programming. Konno and Shirakawa (1994) showed that MAD-optimal portfolios exhibit properties, similar to those of

Markowitz Mean-Variance-optimal portfolios.

The primary aim of the mean absolute deviation (MAD) portfolio optimization model, proposed by Konno (1990) and Konno and Yamazaki (1991), was to overcome the limits of Capital Asset Pricing Model (CAPM), a model used in the pricing of risky securities and describes the relationship between risk and expected return. The proposed MAD model improves and accelerates the computation of optimal portfolios. Consequently, because of its computational simplicity in solving portfolio selection problem, the absolute deviation was intensively used in the last decade.

Yitzhaki (1982) introduced the mean-risk model using Gini's mean (absolute) difference as a risk measure and showed that minimizing the Gini's mean difference for a discrete random variable is linear programming (LP) computable.

If the rates of return are normally distributed, then most of the LP computable risk measures become proportional to the standard deviation (Kruskal and Tanur, 1978). Hence, the corresponding LP solvable mean-risk models are then equivalent to the Markowitz mean-variance model. However, the LP solvable mean-risk models do not require any specific type of the return distributions.

In a mean-risk world the uncertainty of future wealth can be described with two summary statistics: risk and the mean representing reward. Thus, several portfolio selection approaches differ in the risk measure used.

Risk is an inevitable consequence of productive activity. This is especially true for financial activities where the results of investment decisions are observed or realized in the future, in unpredictable circumstances. Risk is the degree of uncertainty in attaining a certain level of portfolio return. It reflects the chance that the actual return of the portfolio may be different than the expected return. Risk in portfolio selection problem arises due to the uncertainties in the return of the assets. Investors can neither ignore nor insure themselves completely against these risks. They must be aware of their exposures to these risk factors and take them into consideration in their decision process so as to properly manage their total risk.

Although the MV model is the most popular approach, it relies on the assumptions that returns are either normally distributed or that the investor's expected utility is quadratic. If either of these conditions holds, it can be shown that choosing among risky investments is compatible with maximization of an investor's expected utility (Tobin, 1958). However, many authors have pointed out that both of the assumptions underlying the MV model generally do not hold –either theoretically or in practice. Apart from the criticisms of the assumptions underpinning the MV model, it has been argued also that the use of variance as a measure of risk implies that investors are indifferent between returns above and below the mean. Variance, as risk measure considers the positive and the negative deviations from the means as a potential risk. It is a dispersion measure which is a measure of uncertainty: however uncertainty is not necessarily risk. In order to overcome this anomaly, Markowitz (1959) proposes the semi variance as risk measure.

Alternative approaches attempt to conform the model's assumptions to reality by, for example, dismissing the normality hypothesis in order to account for fat-tailedness and asymmetry of asset return that cannot always be reduced to standard linear or quadratic programs. As a consequence, other measures of risk, such as value-at-risk, expected shortfall, mean-semi-absolute deviation, and semi variance have been proposed.

It is often argued that the variability of the rate of return above the mean should not be penalized since the investors are concerned of an underperformance rather than the over performance of a portfolio. This led Markowitz (1959) to propose downside risk measures such as (downside) semi variance to replace variance as the risk measure.

Markowitz describes the dependence structure of the random returns via the linear (Pearson) correlation coefficient between each pair of random returns. Variance is appropriate when asset returns follow normal distributions. Using variance as a risk measure is an initial step towards more realistic risk measures that are better suited when investment returns exhibit skewed, leptokurtic and/or heavy tailed distributions as it is the case in practice (e.g., Fama, 1965). It is also important to make a distinction between negative and positive returns if the distribution of the portfolio returns is asymmetric and most stock return distributions are skewed (Fama, 1965). In order to deal with these problems, downside risk measures have been introduced (e.g., Bawa and Lindenberg, 1977, Fishburn, 1977, Harlow, 1991 and Sortino and van der Meer, 1991). Downside-risk measures only penalize returns below a given threshold level, specified by the investor. Consequently, one observes growing popularity of downside risk models for portfolio selection (Sortino and Forsey, 1996). Fienstein and Thapa (1993) and Speranza (1993) pointed out that the MAD models opens up opportunities for more specific modeling of the downside risk. Moreover, the models may be extended with some piece-wise linear penalty (risk) functions to provide opportunities for more specific modeling of the downside risk (Konno, 1990 and Carino *et al*., 1998). Michalowski and Ogryczak (2001) extended the MAD portfolio optimization model to incorporate downside risk aversion.

Bawa (1975) and Bawa and Lindenberger (1977) suggested models based on Lower Partial Moments (LPM) over *n* orders. LPMs of order 2 are measures of portfolio risk that focus on returns below some target level, so that for example, the semi-variance is just a special case LPM when expected return is used as the target return.

The Markowitz mean-variance model provides the main theoretical background for the modern portfolio theory. Nevertheless, the optimization model itself is frequently criticized as not consistent with the stochastic dominance rules. Stochastic dominance is a form of stochastic ordering. The term is used in decision theory to refer to situations where a probability distribution over outcomes can be ranked as superior to another. It is based on preferences regarding outcomes but requires only limited knowledge of preferences with regard to distributions of outcomes, which depend e.g. on risk aversion. The relation of stochastic dominance is one of the fundamental concepts of decision theory (Levy, 1992 and Whitmore and Findlay, 1978).

While theoretically attractive, stochastic dominance order is computationally difficult, as it is a multi-objective model with a continuum of objectives. If the rates of return are multivariate normally distributed, then the MAD and most of the LP solvable mean-risk models are equivalent to the Markowitz mean-variance model which in this specific case is consistent with the stochastic dominance rules. However, the LP solvable mean-risk models do not require any specific type of return distributions. Ogryczak and Ruszczynski (2001) showed that MAD model is consistent with the second degree stochastic dominance (SSD) rules for general random variables.

Mansini, *et al*., (2003) saw the need to classify and compare the different LP solvable portfolio optimization models presented in the literature. They provided a systematic overview of the models with a wide discussion of their theoretical properties. We see that the variety of LP solvable portfolio optimization models provide a good background for the development of stochastic linear programming for portfolio selection problem. This is the major goal of this thesis.

In the following sections we review in detail the Minimax model by Young (1998) and the MAD model by Konno and Yamazaki (1991).

2.1.2 The Minimax Model (MM)

The minimax rule has a long tradition in models of uncertainty because, in situations with conflicting alternatives, the most rational choice is that which seeks to minimize the maximum loss (negative gain). Given an historic time series of returns, the optimum portfolio under the minimax rule is defined as that which would minimize the maximum loss over all past periods, subject to restriction that some minimum average return is achieved across the observed time periods.