Range Restricted Interpolation Using Cubic Bézier Triangles

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ABSTRACT

A range restricted C^1 interpolation local scheme to scattered data is derived. Each macro triangle of the triangulated domain is split into three mini triangles and the interpolating surface on each mini triangle is a cubic Bézier triangle. Sufficient conditions derived for the non-negativity of these cubic Bézier triangles are expressed as lower bounds to the Bézier ordinates. The non-negativity preserving interpolation scheme extends to the construction of a range restricted interpolating surface with lower or upper constraints which are polynomial surfaces of degree up to three. The scheme is illustrated with graphical examples.

Keywords

Scattered data, interpolation, range-restricted, positivity preserving, cubic Bézier triangle.

1. INTRODUCTION

The construction of a surface in computer aided geometric design usually involves generating a set of surface patches which are smoothly connected together with certain degree of continuity. Besides, one is often interested in preserving some properties inherent in the data such as positivity, monotonity and convexity as displayed by its piecewise linear interpolant. For example, in scientific visualization when physical quantities like densities and rainfall are reconstructed graphically the non-negativity of their values should be preserved for otherwise negative values are not physically meaningful.

The preservation of non-negativity refers to that the generated interpolating surface will be non-negative if the given data are non-negative. Non-negativity preserving interpolation or more generally range restricted interpolation has been considered, for example in [Goo91a, Ong92a, Opf88a, Sch88a, Wev88] for the univariate cases and in [Bro95a, Cha01a, Mul94a, Mul94b] for the bivariate cases.

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[Cha01a] describes a local scheme for scattered data range restricted C^1 interpolation. The interpolating surface is piecewise a convex combination of three cubic Bézier patches. As the coefficients of the convex combination involve rational functions, thus the interpolant is piecewise a rational patch. Sufficient conditions for the non-negativity of a cubic Bézier triangle are derived and these conditions prescribe lower bounds to the Bézier ordinates. Non-negativity is achieved by modifying if necessary the first order partial derivatives at the data sites and some Bézier ordinates.

Given scattered data points (x_i, y_i, z_i) with $z_i > 0$, $1 = 1, 2, \cdots, N$, $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$. In this paper we have constructed a C^1 non-negativity preserving piecewise cubic polynomial surface $z = F(x, y)$ with $F(x_i, y_i) = z_i$, $i = 1, 2, \cdots, N$, and then extended the scheme to construct a range restricted C^1 interpolant subject to polynomial constraint surface of degree up to three. An approach similar to [Cha01a] is adopted, but the interpolant has a simpler and different structure, being piecewise a cubic polynomial Bézier triangle instead of a rational function of degree seven. This is achieved by subdividing each triangle, referred as a macro triangle, in the triangulated domain into three mini triangles as in the Clough-Tocher split [Clo65a] and constructing a Bézier triangle on each mini triangle. This subdivision of a macro triangle into three mini triangles is driven by the fact that it is generally not possible to solve the scattered data C^1 interpolation problem with cubic polynomials defined over the triangulated data. We describe in Section 2 the conditions for C^1 continuity between two adjacent Bézier triangles. In Section 3, we derive sufficient conditions on the Bézier ordinates to ensure the non-negativity for a composite C^1 triangular patch consisting of three adjacent cubic Bézier triangles on the three mini triangles of a macro triangle. In Section 4 a local C^1 non-negativity preserving scattered data interpolation scheme applying these sufficient non-negativity conditions is derived. Section 5 extends the results to range restricted interpolation which considers polynomial surfaces up to degree three as lower bound or upper bound. Lastly, two numerical examples are presented graphically.

2. C^1 CONTINUITY BETWEEN ADJACENT CUBIC BÉZIER TRIANGLES

Let T be the triangle on the x-y plane with vertices V_1, V_2, V_3 and barycentric coordinates u, v and w such that any point V on the triangle can be expressed as

$$V = uV_1 + vV_2 + wV_3,$$

$$u+v+w=1, \quad u, v, w \geq 0.$$

A cubic Bézier triangle S on T is defined as

$$S(u,v,w) = \sum_{\substack{i+j+k=3 \\ i,j,k\geq 0}} b_{i,j,k} \frac{3!}{i!\,j!\,k!} u^i v^j w^k$$

$$u+v+w=1, \quad u, v, w \geq 0,$$

with $b_{i,j,k}$ denoting Bézier ordinates of S. Note that S interpolates the Bézier ordinates $b_{3,0,0}$, $b_{0,3,0}$, $b_{0,0,3}$ at the vertices V_1, V_2, V_3 of T respectively since the barycentric coordinates of these vertices are (1,0,0), (0,1,0) and (0,0,1). Ordinate $b_{i,j,k}$ (except $b_{1,1,1}$) is referred as a boundary Bézier ordinate and $b_{1,1,1}$ is referred as the inner Bézier ordinate of the cubic Bézier triangle S. The boundary Bézier ordinates are determined by the first order partial derivatives at the vertices along the corresponding boundary. For example,

$$b_{2,1,0} = S(V_1) + \frac{1}{3}\frac{\partial S}{\partial e_{12}}(V_1),$$

$$b_{2,0,1} = S(V_1) - \frac{1}{3}\frac{\partial S}{\partial e_{31}}(V_1),$$

where $\partial S / \partial e_{ij}$ are the directional derivatives along the respective edges of the triangle (see Figures 1 and 2). Consider two adjacent cubic Bézier triangles with the same boundary curve along the common edge of the domain triangles. We shall recall two sets of conditions for C^1 continuity along the common boundary of the two adjacent patches for the later reference.

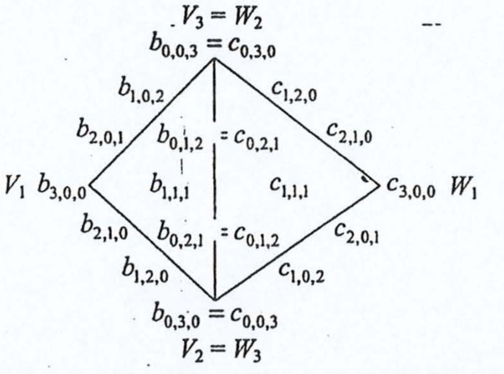

Figure 1. Two adjacent Bézier triangular patches

The first to be noted are the necessary and sufficient conditions described in [Far96a]. Let $\Delta V_1V_2V_3$, $\Delta W_1W_2W_3$ be two adjacent triangles on the x-y plane with $V_2 = W_3$ and $V_3 = W_2$. Suppose that the cubic Bézier triangles on these two triangles have Bézier ordinates $b_{i,j,k}$ dan $c_{i,j,k}$ respectively (see Figure 1). These two cubic Bézier triangles have the same boundary curve along the common boundary V_2V_3, thus $b_{0,0,3} = c_{0,3,0}$, $b_{0,1,2} = c_{0,2,1}$, $b_{0,2,1} = c_{0,1,2}$ and $b_{0,3,0} = c_{0,0,3}$. Then the neccesary and sufficient conditions for C^1 continuity between the two patches are

$$c_{1,0,2} = \alpha b_{1,2,0} + \beta b_{0,3,0} + \gamma b_{0,2,1} \tag{2.1}$$

$$c_{1,1,1} = \alpha b_{1,1,1} + \beta b_{0,2,1} + \gamma b_{0,1,2} \tag{2.2}$$

$$c_{1,2,0} = \alpha b_{1,0,2} + \beta b_{0,1,2} + \gamma b_{0,0,3} \tag{2.3}$$

where $W_1 = \alpha V_1 + \beta V_2 + \gamma V_3$, α, β and γ are constants which sum to 1. Conditions (2.1) and (2.3) will be automatically fulfilled when the Bézier triangles have common first order partial derivatives at V_2 and V_3. If we associate the Bézier ordinates $b_{i,j,k}$ with $(x_{i,j,k}, y_{i,j,k})$ for $0 \leq i, j, k \leq 3$ where

$$(x_{i,j,k}, y_{i,j,k}) = \tfrac{1}{3}(iV_i + jV_j + kV_k)$$

so as to obtain $B_{i,j,k}$ with $B_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, b_{i,j,k})$ and the points $C_{i,j,k}$ are similarly defined, then the three conditions above have a geometrical interpretation, *i.e.* the four points in each set, $\{C_{1,0,2}, B_{1,2,0}, B_{0,3,0}, B_{0,2,1}\}$, $\{C_{1,1,1}, B_{1,1,1}, B_{0,2,1}, B_{0,1,2}\}$, $\{C_{1,2,0}, B_{1,0,2}, B_{0,1,2}, B_{0,0,3}\}$, are coplanar.

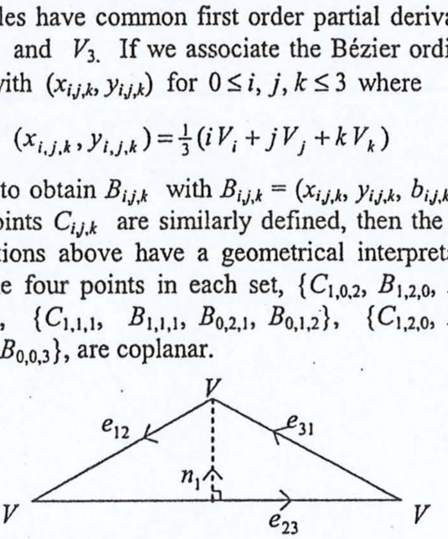

Figure 2. Notation on the triangle

The second set of sufficient conditions is quoted from [Goo91b] to determine the inner Bézier ordinates $b_{1,1,1}$ and $c_{1,1,1}$ so as to attain C^1 continuity along the common boundary. This is achieved by fixing the normal derivative to vary linearly along the common boundary. Consider the edge e_{ij} of the triangle from vertex V_i to V_j. Let n_1 be the inward normal to the edge e_{23} (see Figure 2). The normal derivative $\partial P/\partial n_1$ on e_{23} being linear yields

$$b_{1,1,1} = [\, b_{1,2,0} + b_{1,0,2} + h_1 (\, 2\, b_{0,1,2} - b_{0,2,1} - b_{0,0,3}\,) + (1-h_1)(\, 2\, b_{0,2,1} - b_{0,3,0} - b_{0,1,2}\,)\,] \,/\, 2 \qquad (2.4)$$

where $h_1 = -(e_{12}\cdot e_{23})\,/\,|e_{23}|^2$. Ordinate $c_{1,1,1}$ of the adjacent Bézier triangle T is determined similarly. In our scheme, each macro triangle in the triangulated domain is subdivided into three mini triangles at an interior point G of T (see Figure 3). G is chosen as the centroid of T as this yields a more even subdivision of T into three triangles but otherwise G can be chosen arbitrarily. C^1 continuity along the common boundary of two adjacent macro triangles is obtained by using the second set of conditions while C^1 continuity along the common boundary between two adjacent mini triangles which are in the same macro triangle is achieved by using the first set of conditions.

Suppose that each cubic Bézier triangle on the three mini triangles which are in the macro triangle T have Bézier ordinates $\{a_{i,j,k}\}$, $\{b_{i,j,k}\}$ and $\{c_{i,j,k}\}$, $0 \le i, j, k \le 3$, $i+j+k=3$, respectively as shown in the Figure 3. The three cubic Bézier patches are required to meet with C^1 continuity and their normal derivatives vary linearly along the three edges of the macro triangle T. By C^0 continuity between these three cubic Bézier triangles along GV_i, $i=1, 2, 3$, we have

$$a_{0,3,0}=c_{0,0,3},\quad a_{1,2,0}=c_{1,0,2},\quad a_{2,1,0}=c_{2,0,1},$$
$$a_{0,0,3}=b_{0,3,0},\quad b_{1,2,0}=a_{1,0,2},\quad b_{2,1,0}=a_{2,0,1},$$
$$b_{0,0,3}=c_{0,3,0},\quad c_{1,2,0}=b_{1,0,2},\quad c_{2,1,0}=b_{2,0,1},$$
$$a_{3,0,0}=b_{3,0,0}=c_{3,0,0}.$$

Denote the sets

$$M_1=\{a_{2,1,0}=c_{2,0,1},\, b_{2,1,0}=a_{2,0,1},\, c_{2,1,0}=b_{2,0,1}\},$$
$$M_2=\{a_{1,1,1},\, b_{1,1,1},\, c_{1,1,1}\},$$
$$M_3=\{c_{1,0,2}=a_{1,2,0},\, a_{1,0,2}=b_{1,2,0},\, b_{1,0,2}=c_{1,2,0}\},$$
$$M_4=\{a_{0,2,1}, a_{0,1,2}, b_{0,2,1}, b_{0,1,2}, c_{0,2,1}, c_{0,1,2}\},$$
$$M_5=\{a_{0,3,0}=c_{0,0,3},\, a_{0,0,3}=b_{0,3,0},\, b_{0,0,3}=c_{0,3,0}\}.$$

As noted earlier, the Bézier ordinates and the first order partial derivatives at the vertices V_i will determine the boundary Bézier ordinates in M_4. From this, the elements in M_3 are determined by C^1 continuity (with the geometrical interpretation) as

$$a_{1,0,2}=(b_{0,2,1}+a_{0,1,2}+b_{0,3,0})\,/\,3$$
$$b_{1,0,2}=(c_{0,2,1}+b_{0,1,2}+c_{0,3,0})\,/\,3 \qquad (2.5)$$
$$c_{1,0,2}=(a_{0,2,1}+c_{0,1,2}+a_{0,3,0})\,/\,3.$$

If we now fix the choice of the three inner Bézier ordinates in M_2, then by C^1 continuity the remaining four Bézier ordinates will be determined, namely

$$a_{2,0,1}=(a_{1,0,2}+b_{1,1,1}+a_{1,1,1})\,/\,3$$
$$b_{2,0,1}=(b_{1,0,2}+c_{1,1,1}+b_{1,1,1})\,/\,3 \qquad (2.6)$$
$$c_{2,0,1}=(c_{1,0,2}+a_{1,1,1}+c_{1,1,1})\,/\,3$$

$$a_{3,0,0}=(a_{2,0,1}+b_{2,0,1}+c_{2,0,1})\,/\,3. \qquad (2.7)$$

3. SUFFICIENT CONDITIONS FOR THE NON-NEGATIVITY OF THE CUBIC BÉZIER PATCHES

Suppose that the Bézier ordinates at the three vertices of a triangle T are positive, *i.e.* $m \ge \ell$, $\forall\, m \in M_5$, for some $\ell > 0$. We shall derive sufficient conditions to ensure that the three cubic Bézier patches defined on the mini triangles of T interpolating the given positive data values are non-negative while C^1 continuity is maintained along the common edges (see Figure 3). These sufficient non-negativity conditions prescribe lower bounds for Bézier ordinates. Let us first observe conditions for the non-negativity of a cubic Bézier curve described in following theorem quoted from [Goo91a].

Theorem 1

Let $r(x) = A(1-x)^3 + 3B\,(1-x)^2 x + 3\,C\,(1-x)\,x^2 + D\,x^3,\ \ 0 \le x \le 1$,

where A and D are positive, and at least one of B or C, is negative. Then $r(x) < 0$ for some $x \in (0, 1)$ [resp. $r(x) = 0$ for only one point in $(0, 1)$] if and only if $3B^2C^2 + 6\,ABCD - 4(AC^3 + B^3D) - A^2D^2 > 0$ [resp. $= 0$]. (3.1)

With $r(x)$, $0 \le x \le 1$, as in Theorem 1 where $A, D > 0$, denote

$$\Phi = 3B^2C^2 + 6\,ABCD - 4(AC^3 + B^3D) - A^2D^2.$$

If $B = -A/3$ and $C = -D/3$, then

$$\Phi = 4AD(A-D)^2/27 \ge 0.$$

Thus in this case, $\Phi = 0$ if and only if $A = D$; and $\Phi > 0$ if and only if $A \ne D$. (3.2)

Also note that if $A = D = \ell > 0$ and $B = C = -\ell/3a$ where $a > 1$, then we have

$$r(x) \ge \frac{\ell(a-1)}{4a} > 0, \qquad \forall\, x \in [0, 1]. \qquad (3.3)$$

In view of Theorem 1 and (3.2), we fix the lower bounds of the boundary Bézier ordinates of T as

$$m_4 \geq -\ell/3a, \text{ with } a > 1, \forall m_4 \in M_4. \quad (3.4)$$

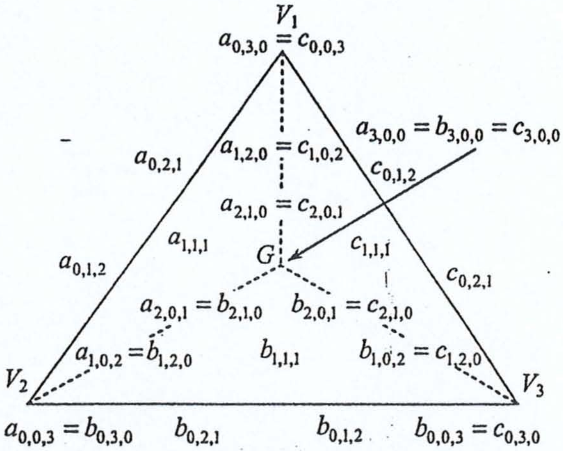

Figure 3. Bézier ordinates of the mini triangles

By (2.5), we obtain

$$m_3 \geq \frac{1}{3}\left(\ell + \frac{-\ell}{3a} + \frac{-\ell}{3a}\right) = \frac{\ell}{3}(1 - \frac{2}{3a}), \quad \forall m_3 \in M_3.$$

If in addition, the lower bound of the inner Bézier ordinates is also fixed as $-\ell/3a$, *i.e.*,

$$m_2 \geq -\ell/3a, \quad \forall m_2 \in M_2, \qquad (3.5)$$

then from (2.6), $\forall m_1 \in M_1$,

$$m_1 \geq \frac{1}{3}\left(\frac{\ell}{3}(1 - \frac{2}{3a}) + \frac{-\ell}{3a} + \frac{-\ell}{3a}\right) = \frac{\ell}{9}(1 - \frac{8}{3a}).$$

The Bézier ordinates $a_{3,0,0} = b_{3,0,0} = c_{3,0,0}$ at the centroid G is the value of the interpolating surface at G, so $b_{3,0,0} \geq 0$ is necessary to ensure non-negativity of the triangluar Bézier patches. Moreover, it is also necessary that $m_1 \geq 0$, $\forall m_1 \in M_1$, otherwise negative values of m_1 would lead to negative partial derivatives at G along the corresponding edges and the corresponding Bézier patches will not be non-negative. By (2.7),

$$b_{3,0,0} = \frac{1}{3}\ (a_{2,0,1} + b_{2,0,1} + c_{2,0,1}) \geq \frac{\ell}{9}(1 - \frac{8}{3a}).$$

For $b_{3,0,0} \geq 0$, it is required to have

$$a \geq 8/3.$$

So if $\forall m \in M_2 \cup M_4$, $m \geq -\ell/3a \geq -\ell/8$ with $a \geq 8/3$, we obtain as described above

$$m_3 \geq \ell/4, \quad \forall m_3 \in M_3,$$

$$g \geq 0, \quad \forall g \in M_1 \cup \{ b_{3,0,0} \}.$$

Now consider the triangular Bézier patch

$$S(u,v,w) = \sum_{i+j+k=3} b_{i,j,k} \frac{3!}{i!\,j!\,k!} u^i v^j w^k$$

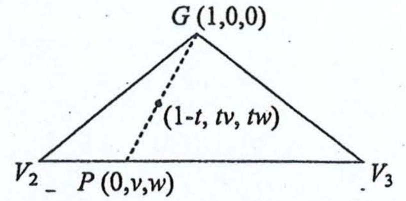

Figure 4. Notation on the mini triangle

on the mini triangle GV_2V_3. Let $P(0, v, w)$, where $v + w = 1$, denote a point along the edge V_2V_3 opposite the vertex G. With t as the parameter which varies between 0 to 1 from vertex G to the point P, the barycentric coordinates for a point on the line segment GP can be written as $(1-t, tv, tw)$, $0 \leq t \leq 1$ (see Figure 4). Then the curve on the cubic Bézier triangular patch S along the line segment GP is given by

$$S(1-t, t(1-w), tw) = A(w)B_0^3(t) + B(w)B_1^3(t)$$
$$+ C(w)B_2^3(t) + D(w)B_3^3(t) \quad (3.6)$$

where $A(w) = b_{3,0,0}$, $B(w) = (1-w)\ b_{2,1,0} + wb_{2,0,1}$, $C(w) = (1-w)^2 b_{1,2,0} + 2(1-w)wb_{1,1,1} + w^2 b_{1,0,2}$ and $D(w) = (1-w)^3 b_{0,3,0} + 3(1-w)^2 wb_{0,2,1} + 3(1-w)w^2 b_{0,1,2} + w^3 b_{0,0,3}$, and the Bernstein polynomial $B_i^3(t) = 3!\ (1-t)^{3-i}\, t^i\ /\ (3-i)!\,i!$. As $\forall t \in [0,1]$, $B_i^3(t) \geq 0$, so if $A(w)$, $B(w)$, $C(w)$ and $D(w)$ are non-negative, then $S(1-t, t(1-w), tw) \geq 0$.

Taking $a \geq 8/3$, then from the above discussion $b_{3,0,0} \geq 0$, $b_{2,1,0} \geq 0$, $b_{2,0,1} \geq 0$, thus $A(w) \geq 0$ and $B(w) \geq 0$. Since $b_{1,1,1} \geq -\frac{\ell}{8}$ and $b_{1,2,0}$, $b_{1,0,2} \geq \frac{\ell}{4}$, so $C(w) \geq 0$. Note that by (3.3), $D(w) > 0$. Thus the curve $S(1-t, t(1-w), tw) \geq 0$, $t \in [0,1]$. The triangular patch $S(u, v, w)$ is made up of these univariate cubic Bézier curves along GP where $P \in V_2V_3$, so the patch $S(u, v, w)$ will be non-negative.

With the same argument, the Bézier triangles with Bézier ordinates $\{a_{i,j,k}\}$ and $\{c_{i,j,k}\}$ on the other two mini triangles are also non-negative. The result of the above discussion is summarized as below.

Proposition 1

Let T be a triangle on the plane which is split into three mini triangles at its centroid. Let the triangular cubic Bézier patches defined on each of these mini triangles respectively have Bézier ordinates $\{a_{i,j,k}\}$, $\{b_{i,j,k}\}$ and $\{c_{i,j,k}\}$, $0 \leq i, j, k \leq 3$, $i + j + k = 3$.

Suppose the three Bézier triangles form a C^1 triangular patch Q on T. If $\forall m \in M_5$, $m \geq \ell$, where $\ell > 0$ and $\forall m \in M_4 \cup M_2$, $m \geq -\ell/3a$, with $a \geq 8/3$, then $Q(x,y) \geq 0$, $\forall (x, y) \in T$.

4. CONSTRUCTION OF THE C^1 NON-NEGATIVITY PRESERVING INTERPOLATING SURFACE

Given data points (x_i, y_i, z_i) with $z_i > 0$, $1 = 1, 2, \cdots, N$, $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$. We describe the construction of a C^1 non-negativity preserving function $F(x, y)$ with $F(x_i, y_i) = z_i$, $i = 1, 2, \cdots, N$. The construction process consists of the three usual steps for scattered data interpolation.

(i) The domain Ω of the function F is the convex hull of $\{V_i = (x_i, y_i): i = 1, \cdots, N\}$. Points V_i, $i = 1, \cdots, N$ are used as the vertices of the triangulation of the domain Ω. The Delaunay triangulation the method [Fan92a] is used to triangulate the domain Ω.

(ii) Estimation of first order partial derivatives, *i.e.*, F_x and F_y at each $V_i(x_i, y_i)$ for surface F, is obtained by using the method in [Goo94a].

(iii) For every macro triangle in the domain, a triangular patch will be generated.

Here we will concentrate on the third step. We shall discuss how to construct on each macro triangle a non-negative C^1 triangular patch. Each macro triangle is subdivided into three mini triangles at its centroid and a cubic Bézier triangle is constructed on a mini triangle. The determination of the Bézier ordinates of these three Bézier triangles, $\{a_{i,j,k}\}$, $\{b_{i,j,k}\}$ and $\{c_{i,j,k}\}$, is described as follows.

The first order partial derivatives F_x and F_y at each vertex $V_i(x_i, y_i)$ in the triangulated domain Ω are estimated by the method in [Goo94a]. For each patch Q on a macro triangle, the partial derivatives at its vertex V_i in the direction along the edge e_{ij} from V_i to V_j is given by

$$\frac{\partial Q}{\partial e_{ij}}(V_i) = (x_j - x_i)\frac{\partial F}{\partial x}(V_i) + (y_j - y_i)\frac{\partial F}{\partial y}(V_i).$$

From the given data, the ordinates at the vertices of each macro triangle are determined. For example, on the macro triangle T (see Figure 3),

$$a_{0,3,0} = c_{0,0,3} = F(V_1) = z_1 .$$

From the estimated derivatives at each vertex, the boundary Bézier ordinates in M_4 are determined. However these ordinates determined need not ensure that the resulting patch is non-negative. To ensure this, we need to impose conditions on these boundary Bézier ordinates, *i.e.*, we require

$$m_4 \geq -\ell / 8, \quad \forall m_4 \in M_4,$$

with $\ell = \min\{F(V_1), F(V_2), F(V_3)\}$ as in Proposition 1. To achieve it, the first order partial derivatives at V_i is modified if necessary. The modification of the derivatives F_x and F_y at a vertex V_i is performed by scaling each of them with a positive factor $\alpha < 1$ by taking into consideration all the triangular patches on the macro triangles sharing that vertex. We proceed as follows:

Let O be a vertex in our triangulated domain and let π_i, $i = 1, \cdots, k$, be the macro triangles in the triangulation which have O as a vertex. Consider the triangle π_1 (see Figure 5) and lower bound $-\ell_{\pi_1}/8$ where $\ell_{\pi_1} = \min\{F(O), F(A), F(B)\}$.

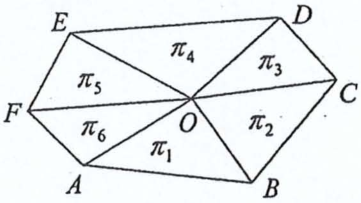

Figure 5. Triangles in the triangulation with common vertex O.

Denote the partial derivatives at O along OA and OB by $\partial F / \partial e_{OA}$ and $\partial F / \partial e_{OB}$ respectively. The scaling factors α_{OA} and α_{OB} are defined as follows.

If $F(O) + \frac{1}{3}\frac{\partial F}{\partial e_{OA}} \geq -\ell_{\pi_1}/8$, then $\alpha_{OA} = 1$, otherwise α_{OA} is determined by the equation $F(O) + \alpha_{OA}\frac{1}{3}\frac{\partial F}{\partial e_{OA}} = -\ell_{\pi_1}/8$. Similarly if $F(O) + \frac{1}{3}\frac{\partial F}{\partial e_{OB}} \geq -\ell_{\pi_1}/8$, then $\alpha_{OB} = 1$, otherwise α_{OB} is defined by the equation $F(O) + \alpha_{OB}\frac{1}{3}\frac{\partial F}{\partial e_{OB}} = -\ell_{\pi_1}/8$. Then we define $\alpha_{\pi_1} = \min\{\alpha_{OA}, \alpha_{OB}\}$. By using the same method, α_{π_i}, $i = 2, 3, \cdots, k$ are found. For all the Bézier ordinates adjacent to O to fulfill the non-negativity conditions, we choose $\alpha_O = \min\{\alpha_{\pi_1}, \alpha_{\pi_2}, \cdots, \alpha_{\pi_k}\}$. If $\alpha_O < 1$, then the first partial derivatives at O are scaled by the factor α_O and the Bézier ordinates adjacent to O are determined accordingly. The above process is repeated at all the vertices V_i in the domain Ω. Thus all Bézier ordinates along the edges of the macro triangles can be determined as described above. By the C^1 continuity property, ordinates in M_3

will be determined by using the relations in (2.5) (see Figure 3).

Next the inner Bézier ordinates in M_2 are determined by using (2.4). Proposition 1 imposes a lower bound on these inner Bézier ordinates. Here since the boundary Bézier ordinates of the macro triangle are already fixed, we could use their actual values to relax the bound on the inner Bézier ordinates suggested in Proposition 1. Observe that to ensure that $C(w)$ in (3.6) is non-negative, we require that

$$b_{1,1,1} \geq -\min\{b_{1,2,0}, b_{1,0,2}\}. \quad (4.1)$$

Similarly, we require

$$a_{1,1,1} \geq -\min\{a_{1,2,0}, a_{1,0,2}\}$$

$$c_{1,1,1} \geq -\min\{c_{1,2,0}, c_{1,0,2}\}. \quad (4.2)$$

Moreover, since $a_{2,0,1}$, $b_{2,0,1}$ and $c_{2,0,1}$ have to non-negative, we require by (2.6) that

$$a_{1,0,2} + b_{1,1,1} + a_{1,1,1} \geq 0$$

$$b_{1,0,2} + c_{1,1,1} + b_{1,1,1} \geq 0 \quad (4.3)$$

$$c_{1,0,2} + a_{1,1,1} + c_{1,1,1} \geq 0$$

To fulfill (4.1), (4.2) and (4.3), it suffices to have

$$a_{1,1,1} \geq -\tfrac{1}{2}\min\{a_{1,2,0}, a_{1,0,2}\},$$

$$b_{1,1,1} \geq -\tfrac{1}{2}\min\{b_{1,2,0}, b_{1,0,2}\}, \quad (4.4)$$

$$c_{1,1,1} \geq -\tfrac{1}{2}\min\{c_{1,2,0}, c_{1,0,2}\}.$$

Note that these lower bounds in (4.4) are less than or equal to $-\ell/8$ where $\ell = \min\{a_{0,0,3}, b_{0,0,3}, c_{0,0,3}\}$. If the initial values of $a_{1,1,1}$, $b_{1,1,1}$ and $c_{1,1,1}$ do not satisfy (4.4), then they are increased to the corresponding bound. This will suffice to ensure the control ordinates $A(w), B(w), C(w), D(w)$ of the cubic Bézier curve in (3.6) are non-negative and thus the Bézier triangle concerned is non-negative. Finally $a_{2,0,1}$, $b_{2,0,1}$, $c_{2,0,1}$ and $a_{3,0,0}$ are obtained via (2.6) and (2.7). When an inner Bézier ordinate has been modified, the C^1 continuity along the boundary of the macro triangle is maintained by modifying the corresponding inner Bézier ordinate of the adjacent macro triangle according to (2.2) and recomputing the Bézier ordinates which are dependent on the modified inner Bézier ordinate. This adjustment to maintain C^1 continuity will not upset the non-negativity property because of (2.2) and the lower bound in (3.4).

The triangular patch on a macro triangle, consisting of three Bézier triangles with its Bézier ordinates $\{a_{i,j,k}\}$, $\{b_{i,j,k}\}$, $\{c_{i,j,k}\}$ respectively, thus generated is non-negative and is C^1 along the common boundary curve with the adjacent patch.

Denote the triangular patch formed by this way on the macro triangle T as Q_T. Then the interpolating surface F which is C^1 and preserves non-negativity can be defined on the triangulated domain as $F|_T = Q_T$ for every macro triangle T in the domain.

5. RANGE RESTRICTED INTERPOLATION

So far we have only discussed the construction of the C^1 interpolating surface which is constrained to lie above the plane $z = 0$. Now we would like to extend our scheme to include a larger set of constraint surfaces besides the plane $z = 0$. The constraint surfaces to be considered are of the form $z = D(x, y)$ where $D(x, y)$ is a constant, linear, quadratic or cubic polynomial, *i.e.*,

$$D(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

where $a, b, c, d, e, f, g, h, i$ and j are real numbers. These surfaces are considered because they can be expressed as a cubic Bézier triangle on each mini triangle of the domain.

Given the data points (x_i, y_i, z_i), $i = 1, \cdots, N$, $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$, which lie on one side of the given constraint surface $z = D(x, y)$ we would like to generate a C^1 interpolating surface $z = F(x, y)$ that lies on the same side of the constraint surface as the data points. This problem can be reduced to the problem of non-negativity preserving interpolation which we have considered in Section 4. Suppose that the data points lie above the constraint surface. As before, the partial derivatives F_x and F_y at (x_i, y_i) are estimated by using the method in [Goo94a].

Let $H(x, y) = F(x, y) - D(x, y)$. A new set of data points (x_i, y_i, z_i^*), $i = 1, 2, \cdots, N$, is obtained from the original data set and the constraint function $D(x, y)$ by defining $z_i^* = z_i - D(x_i, y_i)$. With this, the problem of constructing a C^1 interpolating surface $z = F(x, y)$ subject to the constraint surface $z = D(x, y)$ is transformed to the problem of constructing a non-negative C^1 interpolating surface $z = H(x, y)$ with $H(x_i, y_i) = z_i^*$, and initial derivatives $H_x(x_i, y_i) = F_x(x_i, y_i) - D_x(x_i, y_i)$ and $H_y(x_i, y_i) = F_y(x_i, y_i) - D_y(x_i, y_i)$. By using the scheme in Section 4, the function $H(x, y)$ which is made up of non-negative cubic Bézier triangles with each of its domain on a mini triangle can be generated. Then F can be obtained as

$F(x,y)=H(x,y)+D(x,y)$. As a result, the C^1 interpolating surface F is piecewise a cubic Bézier triangle and it lies on one side of the constraint $z=D(x,y)$.

Suppose the data points lie below the constraint surface $z=D(x,y)$. By using the same construction as above with $H(x,y)=D(x,y)-F(x,y)$, we can generate a C^1 interpolating surface $z=F(x,y)$ which also lies below the constraint surface.

6. GRAPHICAL EXAMPLES

To illustrate our scheme, we use the following two test functions:-

$$f(x,y)=\sin x\cos y,\ \ (x,y)\in[-3,3]\times[-3,3],$$

$$g(x,y)=\begin{cases} 2(y-x), & 0\le y-x\le 0.5 \\ 1, & y-x\ge 0.5 \\ 0.5\cos\left(4\pi\sqrt{(x-1.5)^2+(y-0.5)^2}\right)+0.5, & (x-1.5)^2+(y-0.5)^2\le 1/16 \\ 0, & \text{elsewhere on } [0,2]\times[0,1] \end{cases}$$

The first example consists of 25 data points obtained from the function f. These data are bounded below by the constraint surface $z=-0.55x^2-1.35x-0.2xy-0.2y-1.35$. The triangulation of the domain is given in Figure 6(a). As a comparison, we show in Figure 6(b) the C^1 interpolating surface generated without applying the non-negativity conditions. Indeed it crosses the constraint surface. After the non-negativity conditions are imposed, the interpolating surface does not cross the constraint surface anymore as shown in Figure 6(c).

The second example consists of 36 data points obtained from the function g (quoted from [Lan86a]) which are bounded above by the plane $z=1.001$ and bounded below by the plane $z=-1.001$. The triangulation of the domain is given in Figure 7(a) and the unconstrained interpolating surface is shown in Figure 7(b). It oscillates at a number of places and crosses the upper and lower bounding planes. When the upper and lower constraints are imposed, the range restricted interpolating surface in Figure 7(c) does not oscillate unnecessarily and it stays between the two bounding planes.

7. ACKNOWLEDGMENTS

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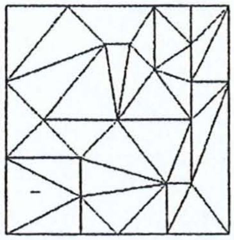

Figure 6(a). Triangulated domain of f

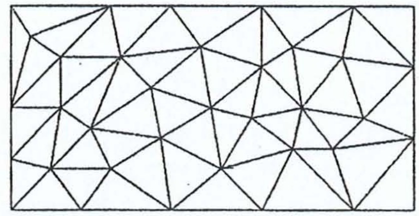

Figure 7(a). Triangulated domain of g

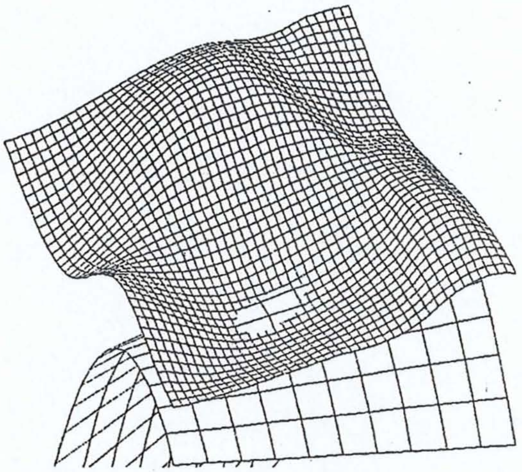

Figure 6(b). The unconstrained interpolating surface to data from f (with the constraint surface)

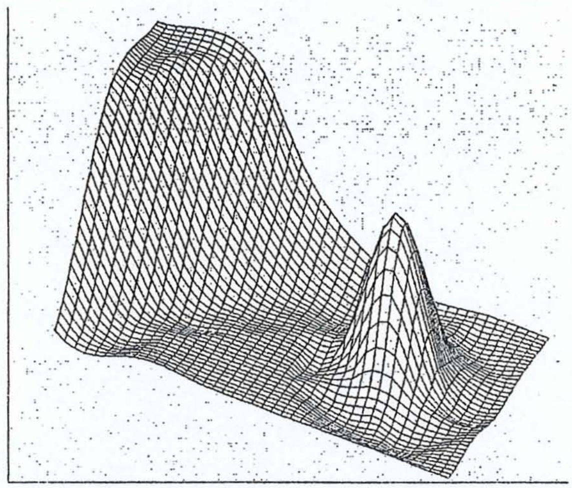

Figure 7(b). The unconstrained interpolating surface to data from g

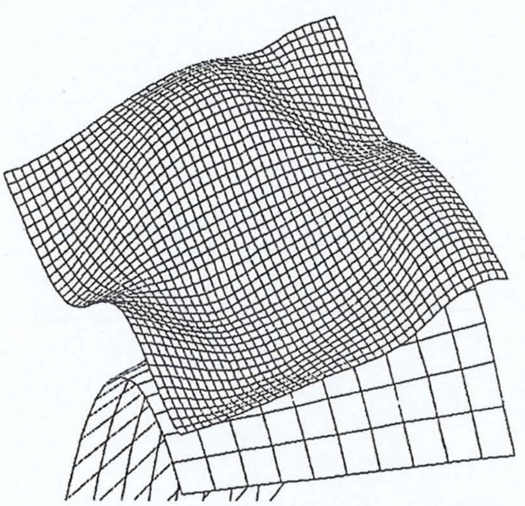

Figure 6(c). The constrained interpolating surface to data from f (with the constraint surface)

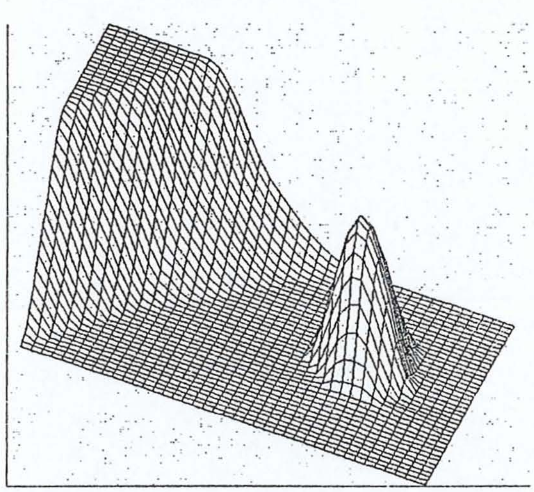

Figure 7(c). The constrained interpolating surface to data from g (without displaying both constraint planes)