A PROPOSED SINGLE EWMA CHART COMBINING THE $\bar{X}$ AND *R* CHARTS FOR A JOINT MONITORING OF THE PROCESS MEAN AND VARIANCE

YEONG KAH WAI

UNIVERSITI SAINS MALAYSIA

2008

A PROPOSED SINGLE EWMA CHART COMBINING THE $\overline{X}$ AND R CHARTS FOR A JOINT MONITORING OF THE PROCESS MEAN AND VARIANCE

by

YEONG KAH WAI

Thesis submitted in fulfillment of the requirements for the Degree of Master of Science

MAY 2008

ACKNOWLEDGEMENTS

I would like to express my gratitude to all those who gave me the possibility to complete this thesis. First of all, I want to thank my thesis supervisor, Assoc. Prof. Michael Khoo Boon Chong from the School of Mathematical Sciences, Universiti Sains Malaysia, whose advice, stimulating suggestions and encouragement helped me in my research and writing of this thesis. His expertise in Statistical Quality Control has improved my research skills and prepared me for future challenges.

I am grateful to the Institute of Postgraduate Studies and the School of Mathematical Sciences, Universiti Sains Malaysia, for allowing me to undertake this research. I also wish to thank the Dean, Assoc. Porf. Ahmad Izani Md. Ismail and his deputies, Assoc. Prof. Abd. Rahni Mt. Piah and Assoc. Prof. Norhashidah Mohd. Ali, all the lecturers and staffs of the department for their assistance and support throughout this research.

At the same time, I am indebted to the library of Universiti Sains Malaysia for providing me with related reference books and journals for the completion of my thesis.

I am also thankful to my company, Penang Seagate Technology (M) Sdn Bhd, my manager and my colleagues. This work would not have been possible without their invaluable support and patience.

Last but not least, I would like to thank all those who have helped and supported me in one way or another throughout my research.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	**ii**
TABLE OF CONTENTS	**iii**
LIST OF TABLES	**vii**
LIST OF FIGURES	**viii**
LIST OF APPENDICES	**ix**
ABSTRAK	**x**
ABSTRACT	**xi**

CHAPTER 1

INTRODUCTION

1.1	Overview of Quality Control and Quality Improvement	1
1.2	Objective of the Study	3
1.3	Organization of the Thesis	3

CHAPTER 2

VARIABLES CONTROL CHARTS

2.1 Introduction 5

2.2 Types of Variables Control Charts 6

2.2.1 Conventional Control Charts 6

2.2.2 Memory Control Charts 7

2.3 Importance of Control Charts 9

CHAPTER 3

SINGLE VARIABLES CONTROL CHARTS FOR SIMULTANEOUS MONITORING OF PROCESS MEAN AND VARIANCE

3.1 Introduction 11

3.2 A Review of Single Variables Control Charts 12

3.2.1 The Shewhart-Type Control Charts 12

3.2.1.1 The Box Plot Simultaneous Control Chart 13

3.2.1.2 The Single Variables T Control Chart 18

3.2.1.3 The Alternate Variables Control Chart 19

3.2.1.4 The Semicircle Control Chart 21

3.2.1.5 The Max Chart 23

3.2.1.6 The L Chart 26

3.2.1.7 The Non-Central Chi-Square Chart 27

3.2.1.8 The Single Weighted Loss Function Chart 29

3.2.1.9 Other Single Shewhart-Type Charts 30

3.2.2 The CUSUM-Type Control Charts 31

3.2.2.1 The Unified CUSUM Chart 31

3.2.2.2 The Weighted Loss Function CUSUM Chart 34

3.2.2.3 The Self-Starting CUSUM Chart 35

3.2.2.4 Other Single CUSUM-Type Charts 37

3.2.3 The EWMA-Type Control Charts 38

3.2.3.1 The Omnibus EWMA Schemes 38

3.2.3.2 The MaxEWMA Chart 40

3.2.3.3 The EWMA-SC Chart 42

3.2.3.4 The Single EWMA Chart 44

3.2.3.5 Other Single EWMA-Type Charts 46

3.2.4 The MA-Type Control Charts 46

CHAPTER 4

USING A SINGLE EWMA $\bar{X}-R$ CONTROL CHART TO JOINTLY MONITOR THE PROCESS MEAN AND THE PROCESS VARIANCE

4.1 Introduction 49

4.2 The Proposed Single EWMA $\bar{X}-R$ Chart 50

4.3 Implementation of the Single EWMA $\bar{X}-R$ Chart 55

4.4 Performance Comparison with the Other Control Charts 58

4.4.1 Comparison with the MaxEWMA Chart 58

4.4.2 Comparison with the Combined $\bar{X}-R$ Charts 61

4.4.3 Comparison with the Combined EWMASM and EWMASR Charts 62

4.5 An Example of Application 68

4.6 Discussion 73

CHAPTER 5

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

5.1 Introduction 75

5.2 Contributions of the Thesis 75

5.3 Areas for Further Research 76

5.4 Concluding Remarks 78

REFERENCES 80

APPENDICES

Appendix A 84

Appendix B 87

Appendix C 91

Appendix D 102

LIST OF TABLES

		Page
Table 3.1	Q-Spreads and Simplified Q-Spreads	15
Table 3.2	Factors for the Three-Sigma Limits of the Box-Plot Simultaneous Chart	17
Table 3.3	Radius Constants, q	22
Table 3.4	CL and UCL s of the Max Chart for Various Values of Type-I Error Probability, α	25
Table 4.1	The (λ, L) Combinations for Selected In-Control ARLs of 185, 250 and 370 with any Sample Size, $n \geq 2$	54
Table 4.2	ARL Profiles of the Single EWMA $\overline{X} - R$, the MaxEWMA, the Combined $\overline{X} - R$ and the Combined $\overline{X} - S$ Charts based on $\text{ARL}_0 = 185.4$ and $n = 5$	60
Table 4.3	ARL Profiles of the Single EWMA $\overline{X} - R$ Chart and the Combined EWMASM and EWMASR Charts based on $\text{ARL}_0 = 250$ and $n = 5$	64
Table 4.4	A Comparison of the Diagnostic Abilities of the Single EWMA $\overline{X} - R$ Chart and the Combined EWMASM and EWMASR Charts based on $\lambda = \lambda_1 = \lambda_2 = 0.1$	67
Table 4.5	The DSC Apparatus Data Set of Van Nuland (1992)	68
Table 4.6	The Single EWMA $\overline{X} - R$ Control Chart Statistics for $\lambda = 0.3$ and $\lambda = 1$	70
Table A1	Standard Deviations of the Sample Median from a Normal Population	85
Table A2	Means and Standard Deviations of Q-Spread from a Normal Population	86
Table D1	The Notations, Checking Conditions and Descriptions Used for the Construction of the Single EWMA $\overline{X} - R$ Chart	102

LIST OF FIGURES

		Page
Figure 4.1	The Single EWMA $\overline{X} - R$ Chart ($\lambda = 1$, $L = 2.695$) for the DSC Apparatus Data	71
Figure 4.2	The Single EWMA $\overline{X} - R$ Chart ($\lambda = 0.3$, $L = 3.017$) for the DSC Apparatus Data	72
Figure B1	The Region, $(s,t) \in \mathrm{R}_2$, is a One-to-One Mapping of the Region, $(y_{i(1)}, y_{i(n_i)}) \in \mathrm{R}_1$, by the Transformation $S = Y_{i(n_i)} - Y_{i(1)}$ and $T = Y_{i(1)}$	89

LIST OF APPENDICES

		Page
Appendix A	Means and Standard Deviations of the Sample Median and Q-Spread	84
Appendix B	A Proof to Show that $B_i \sim N(0,1)$ in Equation (4.2)	87
Appendix C	Computer Programs	91
Appendix C1	SAS Program for the Estimation of the (λ, L) Combinations in Table 4.1 and the Computation of the ARLs in Table 4.2 for the Single EWMA $\bar{X}-R$ Chart	91
Appendix C2	SAS Program for the Computation of ARLs for the MaxEWMA Chart	93
Appendix C3	SAS Program for the Computation of ARLs for the Combined $\bar{X}$ and R Charts	94
Appendix C4	SAS Program for the Computation of ARLs for the Combined EWMASM and EWMASR Charts	95
Appendix C5	SAS Program for the Computation of the Diagnostic Abilities of the Single EWMA $\bar{X}-R$ Chart	96
Appendix C6	SAS Program for the Computation of the Diagnostic Abilities of the Combined EWMASM and EWMASR Charts	98
Appendix C7	SAS Program for the Computation of B_i	100
Appendix C8	SAS Program for the Computation of W_i, Z_i and M_i	101
Appendix D	List of Notations for The Single EWMA $\bar{X}-R$ Chart	102

CADANGAN SUATU CARTA EWMA TUNGGAL YANG MENGGABUNGKAN CARTA-CARTA $\overline{X}$ DAN R UNTUK KAWALAN SERENTAK MIN DAN VARIANS PROSES

ABSTRAK

Dua carta kawalan biasanya digunakan untuk kawalan min proses dan varians proses secara berasingan di industri pengeluaran. Dalam 15 tahun yang lepas, banyak carta tunggal telah dicadangkan untuk membolehkan penggunaan hanya satu carta bagi kawalan serentak min dan varians proses untuk tujuan penambahbaikan keberkesanan dan pengoptimuman sumber. Kebanyakan kaedah adalah berdasarkan transformasi statistik-statistik min sampel dan varians sampel kepada dua statistik, setiap satu mempunyai skala piawai, diikuti dengan sama ada memplotkan kedua-dua statistik itu pada carta yang sama atau menggabungkan kedua-dua statistik itu untuk memberikan suatu statistik tunggal, yang berfungsi sebagai statistik memplot untuk mewakili min proses dan varians proses. Penggunaan julat sampel berbanding dengan varians sampel dalam kawalan varians proses untuk saiz sampel yang kecil, katakan $n \leq 10$, memberikan kelebihan, seperti dalam mengurangkan kerja, masa, kos dan kesilapan pengiraan serta membolehkan pengesanan mudah cerapan terpencil dalam sampel data permulaan. Tesis ini mempertimbangkan transformasi statistik-statistik min sampel dan julat sampel kepada dua pembolehubah rawak berlainan yang bertaburan normal piawai secaman. Daripada dua pembolehubah rawak normal piawai ini, dua statistik purata bergerak berpemberatkan eksponen (EWMA) yang sepadan akan dihitung dan kemudiannya digabungkan untuk menjadi suatu statistik tunggal bagi membentuk statistik memplot carta yang dicadangkan, yang akan dinamakan sebagai carta EWMA $\overline{X} - R$ tunggal. Daripada simulasi yang dijalankan, didapati bahawa carta EWMA $\overline{X} - R$ tunggal adalah berkesan dalam pengesanan kedua-dua peningkatan dan penyusutan min dan/atau varians.

A PROPOSED SINGLE EWMA CHART COMBINING THE $\overline{X}$ AND R CHARTS FOR A JOINT MONITORING OF THE PROCESS MEAN AND VARIANCE

ABSTRACT

In manufacturing industries, two control charts are usually used to monitor the process mean and the process variance separately. In the last 15 years, numerous single charts are proposed to enable the use of only one single chart to jointly monitor both the process mean and variance for the purpose of efficiency improvement and resource optimization. Most approaches are based on transforming the sample mean and the sample variance statistics into two statistics, each having a standard scale, followed by either plotting them on the same chart or combining them into a single statistic, which serves as the plotting statistic to represent the process mean and the process variance. Using the sample range instead of the sample variance in the monitoring of the process variance, for small sample sizes, say $n \leq 10$, provides advantages, such as in reducing computation work, time, cost and error as well as enabling easier detection of outliers in the initial data sample. This thesis considers the transformation of the sample mean and the sample range statistics into two different random variables which follow the same standard normal distribution. From these two standard normal random variables, two corresponding exponentially weighted moving average (EWMA) statistics are computed which are then combined into a single statistic to form the plotting statistic of the proposed chart, that will be named as the single EWMA $\overline{X}-R$ chart. From the simulation conducted, it is found that the single EWMA $\overline{X}-R$ chart is effective in detecting both increases and decreases in the mean and/or variance.

CHAPTER 1

INTRODUCTION

1.1 Overview of Quality Control and Quality Improvement

Quality is one of the most important consideration and decision factor for the success of a business, both in the field of manufacturing and services. There are different views from different people about quality. This is because a business is always customer orientated and therefore quality is commonly being defined as meeting the customers' needs and expectations or simply suitable for intended use.

Quality is inversely proportional to variability (Montgomery, 2005), would be a more specific definition for manufacturing industries. This definition implies that in order to improve the quality of a product, the variability in the critical characteristics of the product must be reduced. These critical characteristics would be measured and the measured values are usually being evaluated relative to the specifications. The variability of the characteristics is considered acceptable if the measured values are within the specification limits defined. Therefore, Deming (1986) defines quality as "Uniformity about target".

There are different sources of variability and these include differences in material, equipment, operation and measurement method. In order to make sure that the manufactured product conforms to specifications, the variability of each of these sources has to be reduced. At the same time, the detection of the sources of the variability and the measure of their degree of impact on the quality of the product are as important.

Quality control is not just to ensure the quality of the manufactured product but also to enable the manufacturer to take the necessary improvement actions on the manufacturing process.

Therefore, variability must be studied and analyzed in a systematic approach, i.e., using statistical methods to detect the assignable causes of process shift. Furthermore, the information obtained when performing analysis on the sources of variability can be an indication of the cause of low quality.

Quality improvement is a continuous effort. Through the use of statistical process control methods, the quality of the manufactured product will not only be controlled but the variability of the process will also be reduced through improvement activities and thus, achieving process stability.

In order to manage and implement the quality improvement activities, total quality management (TQM) was started in the early 1980s, with the philosophies of Deming and Juran as the focal point (Montgomery, 2005). Basically, TQM is an approach that needs the involvement of every level of management in the organization. This might be one of the reasons why TQM only obtained moderate success.

Six Sigma is another disciplined, data-driven approach and methodology that drives continuous quality improvement and variation reduction through project-based activities. This approach uses the DMAIC (define, measure, analyze, improve and control) improvement system. Many manufacturing industries have successfully applied Six Sigma and benefited from it.

All of these approaches promote the use of the seven statistical process control (SPC) tools for variability detection, analysis of possible causes and drilling down for variability reduction. The use of a control chart is one of the most useful and common approaches to start with.

1.2 Objective of the Study

It is important to monitor both the process mean (process location) and the process variance (process spread) in order to control and improve the quality of a manufactured product. The most common approach is to perform the monitoring separately by using two control charts, one for plotting the sample mean and the other the sample variance.

However, in the last decade, much effort is being put in to use a single chart to simultaneously monitor both the process mean and the process variance. The objective of this study is to propose the use of a single variables EWMA $\overline{X} - R$ control chart to jointly monitor both the process mean and variance. The effectiveness of this proposed chart in detecting both increases and decreases in the mean and/or variance while at the same time indicating the source and direction of the shift is also shown.

1.3 Organization of the Thesis

This section will discuss the organization of the thesis.

Chapter 1 highlights the importance of quality control and quality improvement, being the foundation and motivation to use control charts for process monitoring. The objective of this thesis is also presented in this chapter.

Chapter 2 gives an overview of the different types of variables control charts and their characteristics. This chapter also introduces some commonly used memory control charts, such as the moving average (MA), cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts. The importance of these charts as compared to the conventional charts is also discussed.

Chapter 3 mainly focuses on the review of the single variables control charts for a simultaneous monitoring of both the process mean and variance. The discussion includes the Shewhart-type, CUSUM-type, EWMA-type and MA-type single variables control charts. The respective advantages and disadvantages are also being discussed in this chapter.

Chapter 4 presents the proposed Single EWMA $\overline{X} - R$ chart. The detailed procedures and calculations of the chart's statistics and control limits for its construction and implementation are demonstrated. The outcomes of the performance comparisons with the other single variables control charts such as the MaxEWMA chart, the combined $\overline{X} - R$ charts, the combined $\overline{X} - S$ charts and the combined EWMASM and EWMASR charts are also discussed here. In addition to that, an example of application will also be included in this chapter.

Chapter 5 is the conclusion of the study in this thesis. A brief discussion on the contributions of this study to industries is also given. Besides that, topics for further research are also being suggested in this chapter.

CHAPTER 2

VARIABLES CONTROL CHARTS

2.1 Introduction

A variables control chart is one of the most important and widely used SPC tools in the manufacturing field for online monitoring of the process mean and variability when dealing with quality characteristics of a continuous type. It will help to detect the occurrence of a process shift due to assignable causes as early as possible so that appropriate verifications and corrective actions can be performed. With these actions in place and the monitoring being carried out continuously and systematically, the assignable causes can be identified and eliminated from the process at the initial stage and thus the process variability can then be reduced.

Generally, variables control charts can be classified into two types, namely the conventional control charts and the memory control charts. Most of the conventional control charts are based on the principles of control charts developed by Shewhart and they are sometimes called the Shewhart control charts (Montgomery, 2005). These charts are very useful in detecting large process shifts due to assignable causes. However, these charts are not very sensitive to small process shifts as they ignore the information about all the points within the entire time sequence but use only the one contained in the last sample point.

Memory control charts can be very useful and effective when small process shifts are of a major interest. They outperform the Shewhart control charts in detecting small process shifts as they have all the information given by the entire sequence of points.

2.2 Types of Variables Control Charts

2.2.1 Conventional Control Charts

There are various types of conventional control charts being used in industries depending on the quality characteristics being monitored or the purpose of the implementation.

The $\overline{X}-R$ charts are the most commonly used technique when working with variables data. These charts can be easily implemented with minimum sampling or inspection cost as only a small sample is required ($n<10$). Besides giving the information on the process mean of a quality characteristic, the information on the spread or variability of the process can also be obtained easily from the $\overline{X}-R$ charts. The $\overline{X}$ chart, which plots the subgroup average is used for process mean monitoring while the R chart which plots the subgroup range monitors the spread or process variability.

If a subgroup size is large enough (say, $n \geq 10$), and more sensitive charts are desired, the $\overline{X}-S$ charts can be selected instead. In addition to that, if the subgroup sizes are unequal, the $\overline{X}-S$ charts should also be used. The $\overline{X}-S$ charts have the same function as the $\overline{X}-R$ charts. The only difference is that the process variability is being monitored with the subgroup standard deviation, S, instead of the subgroup range, R. With the development of many computational programs, the calculation of the subgroup standard deviation should not be a problem in the implementation of the $\overline{X}-S$ charts.

The S^2 chart plots the subgroup variance, S^2, directly for process variability monitoring. This chart is defined with probability limits. Sometimes, the use of this chart is also recommended as the calculation procedures become easier with the availability of computer programs.

There are also situations where the subgroup size used for process monitoring consists of only an individual unit. Examples are when the production rate is very slow and the subgroups are not possible to be accumulated before the analysis or when the difference in measurements across a continuous production is very little resulting in very small standard deviation and it is almost impossible to detect a shift or change in the standard deviation (Montgomery, 2005). Therefore, control charts for individual measurements, such as the $I - MR$ charts are useful in these situations. The control procedure uses the individual chart for process mean monitoring and the moving range chart of two successive measurements to estimate the process variability.

2.2.2 Memory Control Charts

Memory control charts are effective alternatives to the conventional control charts when small process shifts are of interest. These charts contain the information for the entire sequence of sample points and this feature enables the charts to signal when there is a small process shift, even on the order of 1.5σ or less. Among the examples of memory control charts are the cumulative sum (CUSUM) chart, the exponentially weighted moving average (EWMA) chart and the moving average (MA) chart.

The CUSUM control chart is preferred compared to the conventional Shewhart control chart when the magnitude of a change in the process is relatively small (Lowry et al., 1995, Lucas, 1976 & 1982 and Hawkins & Olwell, 1997). This is because a CUSUM control chart gives a smaller out-of-control ARL in the detection of small process shifts when the in-control ARL of both the Shewhart and CUSUM charts are maintained at the same level (Yeh et al., 2004). In addition to the process mean, CUSUM control charts can also be used for the monitoring of the process standard deviation. A CUSUM control chart is also effective to be used in industries when the subgroup size is one.

The CUSUM chart directly combines all the information from the sequence of the sampled points when the cumulative sums of the deviations of the sample values from a target value are tabulated or plotted. If a process is in-control at the target value, the CUSUM values will fluctuate around zero and an upward or downward trend will be clearly observed even when there is a small shift in the process.

The EWMA control chart is also a good alternative in detecting small shifts for both the process mean and the process variability. The EWMA control chart generally has similar performance to the CUSUM control chart. However, in some ways, the EWMA control chart is easier to be set up and operate (Montgomery, 2005).

The EWMA control chart is used extensively in time series modeling and forecasting (Montgomery, 2005) as it considers the weighted average of all past and current sample statistics. The EWMA control chart is also preferred for individual observations.

Unlike the Shewhart control chart, the EWMA control chart does not perform well against large process shifts. However, it is often more effective than the CUSUM control chart to detect large process shifts (Montgomery, 2005). There are also some efforts being put in to improve the sensitivity of the EWMA control chart in detecting large process shifts without sacrificing the effectiveness of detecting smaller shifts. One example is to use the sample means and squared deviations from target for the monitoring of the process mean and the process variance (Reynolds and Stoumbos, 2005).

The moving average (MA) chart is another memory control chart based on simple and unweighted moving averages as the chart’s statistics. There will be a fixed time span for the statistics computed. All the observations within this time span will be averaged and the process variables of interest are plotted on the chart. The ‘oldest’ observation will be excluded as a new observation is obtained. Generally, if a more effective chart is preferred for small process shifts, the time span should be set longer.

The moving average chart is of course more effective in detecting small process shifts as compared to the conventional Shewhart control chart. However, it is not as effective as the CUSUM and EWMA charts. The MA chart is sometimes preferred for simple and direct calculation or when the subgroup size is one.

2.3 Importance of Control Charts

Control charts are very important tools in SPC. With the implementation of control charts, the process variations can be reduced so that the manufactured products can meet customers' requirements.

The main objectives of a control chart are to detect the assignable causes and reduce the process variability. These can be achieved through continuous investigations and improvement of a process after the out-of-control situation is detected and more process information is obtained. Furthermore, the respective improvement strategies can also be evaluated with the employment of a control chart. If there is a need to modify or change the action plans, it can always be carried out as early as possible.

On the other hand, a control chart can also be incorporated into some cost control activities to help in decision making and performance evaluations. Therefore, a control chart is not only applicable to the manufacturers' processes but also to the suppliers' processes.

A control chart is also used as an estimating device (Montgomery & Runger, 1999). The process mean and standard deviation can be estimated from a control chart under an in-control situation. The estimation can then be used for process capability study purposes.

CHAPTER 3

SINGLE VARIABLES CONTROL CHARTS FOR SIMULTANEOUS MONITORING OF PROCESS MEAN AND VARIANCE

3.1 Introduction

When dealing with SPC control charts in manufacturing industries, the major interest is to monitor the mean and variance of a quality characteristic. A process deterioration is detected if either the mean shifts away from the target value or the variance increases. This monitoring used to be performed by plotting two control charts separately, one for monitoring the mean and the other for monitoring the variance, The $\overline{X}$ chart which plots the sample mean, is the traditional and most widely used chart for monitoring the process mean. Efforts are then being put in to develop more powerful charts to meet different objectives and efficiency requirements. The EWMA and CUSUM control charts are among the popular ones.

Process variability is as important as the process mean and it should be well monitored and controlled. A common practice is to plot either the R chart for the sample range or the S chart for the sample standard deviation in order to make sure that the process variability is within an acceptable limit. Lately, there are also modified EWMA and CUSUM control charts for monitoring the process variability. Useful information related to the status of a process could be obtained from a variance shift, either increasing or decreasing. This will enhance the effectiveness of a control chart in detecting process deterioration or process improvement.

For many processes, there is a possibility of the process mean and variance shifting at the same time due to special causes and it is therefore more appropriate and meaningful to monitor the behavior of the combined mean and variance information on the same chart to have a reasonably good control of the whole process (Duncan, 1986).

In other cases, if the problems or causes resulting in either or both the process mean and variability shifts are related, it is more advantageous to simultaneously monitor and control them in a single chart. In addition to that, running two charts, one for the mean and the other for the variance, may not always be reliable in identifying the nature of the change as shown by Reynolds and Stoumbos (2004) and mentioned again by Costa and Rahim (2006). Cheng and Thaga (2006) also commented that this practice requires more resources such as quality control practitioners, time and other resources needed for process monitoring.

3.2 A Review of Single Variables Control Charts

In the last 15 years, numerous charts are proposed to enable the use of a single chart to simultaneously monitor both the process mean and variability. Most approaches are based on transforming the sample mean and the sample variance statistics either into two standard scaled statistics or a single statistic. These statistics will then be plotted on a single chart. The single variables charts can be generally classified as the Shewhart-type charts, the CUSUM-type charts, the EWMA-type charts and the MA-type charts. This chapter will review some of these single variables charts found in the literature.

3.2.1 The Shewhart-Type Control Charts

Numerous charting methods have been proposed to improve the effectiveness of the Shewhart-type single variables control chart in the process monitoring activities. Besides that, it is also important to consider the implementation procedures to ensure that the calculations or plotting methods involved are not too complicated to the shop-floor or line workers.

3.2.1.1 The Box Plot Simultaneous Control Chart

White and Schroeder (1987) introduce a simultaneous control chart to monitor the process mean and variance on the same chart. This chart is different from the conventional control charts because it is designed using resistant measures and a modified box plot display. Besides controlling the process level and variability simultaneously on a single chart, box plots also provide additional process control information, such as its distribution, specifications and the location of the individual units and outliers. These could help to enhance the effectiveness of the control chart to be a decision aid. Iglewicz and Hoaglin (1987) and Karuthan et al. (2000) also support the use of box plot for this purpose.

The box plot simultaneous control chart uses resistant measures, the median and the Q-spread as the statistics for monitoring the process location and variability respectively. The median, m, is the middle order statistic if the sample size, n, is odd and the average of the two center order statistics if the sample size, n, is even (White and Schroeder, 1987).

Hence,

$$m = X_{((n+1)/2)}\text{ , if } n \text{ is odd} \tag{3.1a}$$

and

$$m = 0.5\left(X_{(n/2)} + X_{(n/2+1)}\right)\text{, if } n \text{ is even,} \tag{3.1b}$$

where $X_{(i)}$ denotes the i th order statistic of measurements $X_1, X_2, \ldots, X_n$.

The Q-spread is defined as the distance between the 'hinges'. It is originally based on the idea of folding ordered data into four sections, having the median and the 'hinges' of the data set located as shown in the example below.

For the data set 1, 2, 3, 4, 5, 6, 7, 8, 9, the median is 5 and the hinges are 3 and 7 (White and Schroeder, 1987).

1 5 9

2 4 6 8

3 7

Tukey (1977) recommends the interpolation of half values for the Q-spread, resulting in the median of 4.5 and the hinges of 2.5 and 6.5 for the data set 1, 2, 3, 4, 5, 6, 7, 8 (White and Schroeder, 1987).

4.5

1 4 5 8

2 3 6 7

2.5 6.5

Table 3.1 shows the order statistics comprising of the Q-spread for samples of size, $n = 2$ to $n = 20$. White and Schroeder (1987) suggest the use of the simplified version of Q-spread developed by Tukey (1977), in this box plot simultaneous control chart. It uses an approximation of the Q-spread, i.e.,

$$Q = X_U - X_L, \tag{3.2}$$

where X_U and X_L depend on the sample size, n, as shown in Table 3.1.

Table 3.1 Q-Spreads and Simplified Q-Spreads

Sample Size, n	Q-spread	Approximation
2	$x_2 - x_1$	——
3	$x_3 - x_1$	——
4	$0.5(x_3 + x_4 - x_1 - x_2)$	$x_3 - x_2$
5	$x_4 - x_2$	——
6	$x_5 - x_2$	——
7	$0.5(x_5 + x_6 - x_2 - x_3)$	$x_5 - x_3$
8	$0.5(x_6 + x_7 - x_2 - x_3)$	$x_6 - x_3$
9	$x_7 - x_3$	——
10	$x_8 - x_3$	——
11	$0.5(x_8 + x_9 - x_3 - x_4)$	$x_8 - x_4$
12	$0.5(x_9 + x_{10} - x_3 - x_4)$	$x_9 - x_4$
13	$x_{10} - x_4$	——
14	$x_{11} - x_4$	——
15	$0.5(x_{11} + x_{12} - x_4 - x_5)$	$x_{11} - x_5$
16	$0.5(x_{12} + x_{13} - x_4 - x_5)$	$x_{12} - x_5$
17	$x_{13} - x_5$	——
18	$x_{14} - x_5$	——
19	$0.5(x_{14} + x_{15} - x_5 - x_6)$	$x_{14} - x_6$
20	$0.5(x_{15} + x_{16} - x_5 - x_6)$	$x_{15} - x_6$

Source: White and Schroeder (1987)

Both the median and Q-spread are more efficient and robust against non-normality and outliers as we cannot always expect the normality assumption from the processes in a real manufacturing environment.

The box plot simultaneous control chart requires two sets of control limits, one for the median and the other for the Q-spread. Generally, the K sigma control limits can be defined as

$$\mu_m \pm K\sigma_m \text{ for the median} \qquad (3.3a)$$

and

$$\mu_Q \pm K\sigma_Q \text{ for the } Q\text{-spread} \quad (3.3b)$$

where μ_m, σ_m and μ_Q, σ_Q are the means and variances of the distribution of the sample median and the Q-spread statistics respectively. The formulae for calculating these means and variances and their values for the various sample sizes, n, are given in Appendix A.

Table 3.2 lists the simplified factors for calculating the three-sigma control limits. The M_2 factor for the median chart is analogous to the standard A_2 factor for the $\overline{X}$ chart and the factors Q_3 and Q_4 for the Q-spread charts are analogous to the D_3 and D_4 factors for the R chart (White and Schroeder ,1987). Therefore,

$$UCL = \overline{m} + M_2\left(\overline{Q}\right) \text{ and } LCL = \overline{m} - M_2\left(\overline{Q}\right) \text{ for the median chart} \quad (3.4a)$$

while

$$UCL = Q_4\left(\overline{Q}\right) \text{ and } LCL = Q_3\left(\overline{Q}\right) \text{ for the } Q\text{-spread chart.} \quad (3.4b)$$

Here, $\overline{Q}$ is computed as the average of the Q-spreads, based on a preliminary dataset with sample size, n.

Table 3.2. Factors for the Three-Sigma Limits of the Box-Plot Simultaneous Chart

Sample Size n	Median M_2	Q-Spread Q_3	Q_4
2	1.88	0	3.27
3	1.19	0	2.57
4	2.76	0	3.27
5	1.62	0	3.72
6	1.08	0	2.38
7	1.95	0	2.81
8	1.30	0	2.43
9	1.07	0	2.22
10	0.85	0	2.08
11	1.20	0	2.26
12	0.96	0	2.11
13	1.05	0	2.18
14	0.72	0.08	1.92
15	0.93	0	2.02
16	0.79	0.06	1.93
17	0.73	0.13	1.87
18	0.64	0.81	1.81
19	0.78	0.12	1.88
20	0.69	0.21	1.79

Source: White and Schroeder (1987)

The box plot display is able to provide information on both the control limits and specification limits graphically. Unlike the conventional $\overline{X}$ and R charts, the probability limits for the box plot simultaneous control chart are relatively consistent against the sample size, n, and the process distribution (White and Schroeder, 1987).

The major disadvantage of using resistant measures, such as the median and the Q-spread statistics, is the lower sensitivity of the control chart to outliers and process changes as compared to the non-resistant measures. Other drawbacks which limit the use of a box plot simultaneous control chart is its ineffectiveness in dealing with small sample sizes, control limits for any measure of dispersion and multi-characteristic processes (Spiring and Cheng, 1998). Furthermore, monitoring a process with a box plot simultaneous control chart can only be performed with the availability of computer and automatic data acquisition techniques. The shop-floor workers might also be confused by its complexity and the overwhelming information contained in this single plot. Cheng and Thaga (2006) also comment that the box plot simultaneous control chart is difficult to interpret, especially the Q-spread charts.

3.2.1.2 The Single Variables *T* Control Chart

The single variables T control chart is proposed by Cheng and Li (1993). This chart measures the proximity of an observation to its target value as well as the process variability with a single standard variable plotted on the chart for both parameters.

The plotting statistic, W is defined as follows (Cheng and Li, 1993):

$$W = \left| Y_{(1)} - T \right| + \left| Y_{(n)} - T \right|, \qquad (3.5)$$

where $Y_i = \dfrac{X_i - \mu}{\sigma}$, for $i = 1, 2, \ldots, n$, $T = \dfrac{T^* - \mu}{\sigma}$ and $Y_{(1)}$ and $Y_{(n)}$ are the first and nth order statistics corresponding to $Y_1, Y_2, \cdots, Y_n$. Also, X_i and T^* denote the observation of a process at time i and its target value, respectively.

A major disadvantage of this chart is that it could not tell which parameter, i.e., the process mean or the variability, has shifted when there is an out-of-control situation.

3.2.1.3 The Alternate Variables Control Chart

Spiring and Cheng (1998) develop the alternate variables control chart that monitors both the process mean and standard deviation on a single chart. Besides being more superior to the single variables T chart, this chart also provides an alternative to the box plot style of simultaneous control charts as it has the advantage of performing equally well for both large and small subgroup sizes. The implementation of this chart is also simple and straightforward.

The suggested measures being monitored for samples of size n on this alternate variables control chart can be the squared difference from the target,

$$\left(\overline{X} - T\right)^2 , \tag{3.6}$$

the mean square error around the target value,

$$MSE = (n-1)^{-1} \sum_{i=1}^{n} \left(X_i - T\right)^2 , \tag{3.7}$$

or the sample variance,

$$S^2 = (n-1)^{-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2 , \tag{3.8}$$

where T is the target value of the process (Spiring and Cheng, 1998).

As small values of these measures are desired, only the upper control limits are being implemented on this chart.

If the process measurements are normally distributed, i.e., $X \sim N\left(\mu, \sigma^2\right)$, then $\left(\overline{X} - T\right)^2 \sim n^{-1}\sigma^2\chi^2_{1,\lambda}$, where $\chi^2_{1,\lambda}$ denotes the non-central chi-square distribution with one degree of freedom (df) and non-centrality parameter $\lambda = n\left[(\mu - T)/\sigma\right]^2$ (Spiring and Cheng, 1998).

The UCL is defined to be the $(1-\alpha)100$th percentile of the distribution function associated with $(\overline{X}-T)^2$. Therefore,

$$UCL_{(\overline{X}-T)^2} = n^{-1}\sigma^2\chi^2_{1,\lambda,(1-\alpha)}. \tag{3.9}$$

The MSE is distributed as $(n-1)^{-1}\sigma^2\chi^2_{n,\lambda}$ with n df and $\lambda = n[(\mu-T)/\sigma]^2$. The UCL to be applied is

$$UCL_{MSE} = (n-1)^{-1}\sigma^2\chi^2_{n,\lambda,(1-\alpha)}. \tag{3.10}$$

S^2 has a distribution of $\sigma^2\chi^2_{n-1,0}$, where $\chi^2_{n-1,0}$ denotes the central chi-square $(\lambda = 0)$ distribution. Therefore, its UCL is defined as

$$UCL_{S^2} = (n-1)^{-1}\sigma^2\chi^2_{n-1,0,(1-\alpha)}. \tag{3.11}$$

For an effective process monitoring, Spiring and Cheng (1998) suggest plotting on the same chart $(\overline{X}-T)^2$ and MSE with their respective upper control limits rather than S^2. However, the control limits for S^2 should also be included in the chart. This would be an additional "warning" for large or increasing S^2 even if MSE does not exceed its limit.

The alternate variables control chart has a comparable ARL to the conventional Shewhart control chart for process changes or shifts detection. However, the ability of this alternate variables control chart to detect changes in the process depends greatly on various situations. For example, the detection of a shift in the process is poorer if the process is currently running closer to the target. Therefore, in the literature, Spiring and Cheng (1998) also include some comments and suggestions on how the common detection rules could be applied to this alternate variables control chart.

Not only bringing the process within its control limits, this control chart is also able to make sure that the process is being controlled to its target all the time through the control of $\left(\bar{X}-T\right)^2$ without losing much of the other information needed about the process. This is because the information for both the process variability and the proximity to the target can be provided as this chart incorporates the target into its charting procedures.

3.2.1.4 The Semicircle Control Chart

Chao and Cheng (1996) propose a semicircle control chart, where a semicircle is used to plot a single plotting statistic to represent the position of the mean and the standard deviation, by plotting the standard deviation on the y-axis and the mean on the x-axis. The detection of the out-of-control signals and the interpretation of the semicircle control chart are quite simple. When a point plots outside of the semicircle indicating an out-of-control signal, the chart can easily tell whether the mean, the variance or both the parameters have shifted. With its straightforward calculations, this can be considered a new alternative to the combination of $\bar{X}$ and S charts. In addition to the ease of interpretation, the implementation of this chart is also quick and simple for the shop-floor workers.

The plotting statistic proposed by Chao and Cheng (1996),

$$T=\left(\bar{X}-\mu\right)^2+S^{*2}, \qquad (3.12)$$

is based on the sample mean, $\bar{X}$, and the root mean square, $S^*=\left(\frac{n-1}{n}\right)^{1/2}S$, where $S=\sqrt{\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}{n-1}}$ is the sample standard deviation. Under the normality assumption, $\left(\frac{n}{\sigma^2}\right)T$ is distributed exactly as a χ_n^2 for any sample size, n.

The value S^* for each sample will be plotted on the y-axis while the value $\overline{X}$ will be plotted on the x-axis. As $S^* > 0$, the control limit will only need to form a semicircle centered at $\left(\overline{\overline{X}}, 0\right)$, with the radius, r, calculated as follows (Chao and Cheng, 1996):

$$P\left(T < r^2\right) = P\left(\frac{n}{\sigma^2} T < \frac{n}{\sigma^2} r^2\right) = P\left(\frac{n}{\sigma^2} T < \chi^2_{n,(1-\alpha)}\right) = 1 - \alpha . \tag{3.13}$$

Then,

$$\frac{n}{\sigma^2} r^2 = \chi^2_{n,(1-\alpha)} \Rightarrow r = \left(\frac{\chi^2_{n,(1-\alpha)}}{n}\right)^{1/2} \sigma , \tag{3.14}$$

where $\chi^2_{n,(1-\alpha)}$ is the $100(1-\alpha)$th percentile of the χ^2_n distribution, and $\overline{\overline{X}}$ and $\overline{S^*}$ will be substituted for μ and σ.

Therefore,

$$\hat{r} = q\overline{S^*} , \tag{3.15}$$

where $q = \left(\frac{\chi^2_{n,(1-\alpha)}}{n}\right)^{1/2}$.

The values of the radius constant, q, for various sample sizes, n, are shown in Table 3.3 below (Chao and Cheng, 1996).

Table 3.3. Radius Constants, q

n	q	n	q	n	q
2	2.2850	5	1.8202	8	1.6489
3	2.0553	6	1.7491	9	1.6117
4	1.9161	7	1.6937	10	1.5802

Source: Chao and Cheng (1996)

The data points, $\left(\overline{X}_i, S_i^*\right)$ are plotted on the semicircle chart. Any point being plotted out of the semicircle control limit, $\hat{r}$, will be considered out-of-control and investigation needs to be carried out.

A disadvantage of this approach is that one cannot keep track of the time sequence of the plotted points. To take care of that, all the points can be identified with a number as the time sequence indication.

3.2.1.5 The Max Chart

Chen and Cheng (1998) develop the Max chart, which combines the $\overline{X}$ chart and the *S* chart into a single chart. It jointly monitors the mean and the variance using a single plotting statistic and it has the ability to indicate the source and direction of a shift when an out-of-control signal is detected.

Let X_{ij}, $i = 1, 2, \ldots,$ and $j = 1, 2, \ldots, n_i$ be measurements of X arranged in samples of size n_i, with i representing the sample number. Assume that for each i, $X_{i1}, X_{i2}, \ldots, X_{in_i}$ is a random sample from the normal distribution with mean, $\mu + a\sigma$, and standard deviation, $b\sigma$, where $a = 0$ and $b = 1$ indicate that the process is in-control, otherwise, the process has shifted.

The plotting statistic $M(n_i)$ is defined as the maximum absolute values of the standardized mean, U_i, and the standardized variance, V_i, given as follows (Chen and Cheng, 1998):

$$M(n_i) = \max\{|U_i|, |V_i|\}, \tag{3.16}$$

where

$$U_i = \frac{\left(\overline{X}_i - \mu\right)}{\sigma/\sqrt{n_i}} \tag{3.17}$$

and

$$V_i = \Phi^{-1}\left\{H\left(\frac{(n_i - 1)S_i^2}{\sigma^2}; n_i - 1\right)\right\}. \tag{3.18}$$

Here, $\Phi(z) = P(Z \le z)$ for $Z \sim N(0,1)$, the standard normal distribution. Then $\Phi^{-1}(\,\cdot\,)$ is the inverse of $\Phi(\,\cdot\,)$, and $H(w;v) = P(W \le w|v)$ for $W \sim \chi_v^2$, the chi-squared distribution with v degrees of freedom.

To find the distribution of $M(n_i)$, let $\chi_{\gamma,v}^2$ satisfy $P\left(S \le \chi_{\gamma,v}^2\right) = \gamma$, where $\gamma \in (0,1)$ and $S \sim \chi_v^2$. If $F(\,\cdot\,)$ represents the distribution function of $M(n_i)$, then for any $y > 0$ (Chen and Cheng, 1998),

$$\begin{aligned} F(y; n_i, a, b) &= P(M(n_i) \le y) \\ &= P(|U_i| \le y, |V_i| \le y) \\ &= \left\{\Phi\left(\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\right) - \Phi\left(-\frac{y}{b} - \frac{a}{b}\sqrt{n_i}\right)\right\} \\ &\times \left\{H\left(\frac{\chi^2_{\Phi(y),n_i-1}}{b^2}; n_i - 1\right) - H\left(\frac{\chi^2_{\Phi(-y),n_i-1}}{b^2}; n_i - 1\right)\right\}. \end{aligned} \tag{3.19}$$

Letting $a = 0$ and $b = 1$ in Equation (3.19), we obtain (Chen and Cheng, 1998)

$$F(y; n_i, 0, 1) = \{\Phi(y) - \Phi(-y)\}^2 = P\left(S \le y^2\right)^2, \tag{3.20}$$

where $S \sim \chi_1^2$.