DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR ANALYTIC AND MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATORS

N. SEENIVASAGAN

UNIVERSITI SAINS MALAYSIA
2007

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by

N. SEENIVASAGAN

Thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

May 2007

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ACKNOWLEDGEMENT

I would like to express my gratitude to my Supervisor, Professor Dato' Rosihan M. Ali, Deputy Vice-Chancellor (Academic and International Affairs), Universiti Sains Malaysia, and my Co-Supervisor, Dr. V. Ravichandran, School of Mathematical Sciences, Universiti Sains Malaysia, for their guidance, encouragement and support throughout my research. Because of their untiring guidance, care and affection, I could work continuously without any slackness. In fact, I am fortunate enough to work with such great exponents.

I am thankful to Associate Professor Dr Ahmad Izani Md. Ismail, Dean, School of Mathematical Sciences, Universiti Sains Malaysia, and to Dr Adli Mustaffa, Deputy Dean (Graduate Studies and Research), School of Mathematical Sciences, Universiti Sains Malaysia. I am also thankful to the authorities of Universiti Sains Malaysia for providing me an opportunity to pursue my research at Universiti Sains Malaysia.

My research is supported by a Research Assistantship from an *Intensification of Research in Priority Areas (IRPA)* grant (09-02-05-00020 EAR) to Professor Dato' Rosihan M. Ali.

My appreciation also goes to all my friends, especially S. Sivaprasad Kumar, for their encouragement and motivation.

Finally, my heartfelt gratitude goes to my wife S. Santha, and my children.

SUBORDINASI DAN SUPERORDINASI PEMBEZA UNTUK FUNGSI ANALISIS DAN FUNGSI MEROMORFI YANG TERTAKRIF OLEH PENGOPERASI LINEAR

ABSTRAK

Suatu fungsi f yang tertakrif pada cakera unit terbuka U dalam satah kompleks $\mathcal{C}$ disebut univalen jika fungsi tersebut memetakan titik berlainan dalam U ke titik berlainan dalam C. Suatu fungsi f disebut subordinat terhadap suatu fungsi univalen g jika $f(0) = g(0)$ dan $f(U) \subset g(U)$. Fungsi ternormalkan f disebut bak-bintang Janowski jika $zf'(z)/f(z)$ adalah subordinat terhadap $(1 + Az)/(1 + Bz), (-1 \leq B \leq A \leq 1)$. Dengan menggunakan teori subordinasi pembeza peringkat pertama, beberapa syarat cukup untuk fungsi f menjadi bak-bintang Janowski diperoleh. Syarat-syarat cukup ini diperoleh dengan mengkaji implikasi

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) \prec \frac{1 + Az}{1 + Bz}$$

dimana simbol $\prec$ menandai subordinasi antara fungsi analisis dan implikasi-implikasi lain yang serupa yang melibatkan $1 + \beta zp'(z)$, $1 + \beta\frac{zp'(z)}{p(z)}$ dan $1 + \beta\frac{zp'(z)}{p^2(z)}$. Keputusan-keputusan ini digunakan kemudian untuk memperoleh syarat-syarat cukup bagi fungsi analisis menjadi bak-bintang Janowski.

Andaikan Ω sebagai set dalam $\mathcal{C}$, fungsi q_1 adalah univalen dan q_2 adalah analisis dalam $\mathcal{U}$. Juga andaikan $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$. Miller dan Mocanu [*Differential Subordinations*, Dekker, New York, 2000.] telah mengkaji teori subordinasi pembeza peringkat pertama dan kedua. Baru-baru ini, Miller dan Mocanu (Subordinants of differential superordinations, Complex Var. Theory Appl. **48**(10) (2003), 815–826.) mengkaji konsep kedualan superordinasi pembeza dan berjaya mendapat beberapa keputusan 'tersepit'. Dengan menggunakan teori subordinasi pembeza, kami menentukan kelas fungsi sedemikian

$$\psi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}\right) \prec h$$

mengimplikasikan $zf'(z)/f(z) \prec q(z)$, dimana simbol $\{f, z\}$ menandai terbitan Schwarz fungsi f. Beberapa keputusan serupa yang melibatkan nisbah antara fungsi yang tertakrif melalui operator linear Dziok-Srivastava dan transformasi pendarab bagi fungsi analisis

juga telah diperoleh. Kami juga memperoleh bagi superordinasi yang sepadan beberapa keputusan tersepit. Tambahan pula, kajian telah dijalankan bagi masalah yang serupa bagi fungsi meromorfi yang tertakrif melalui pengoperasi linear Liu-Srivastava dan transformasi pendarab.

Bagi fungsi analisis $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ yang tetap dan tertakrif pada cakera unit terbuka dan $\gamma < 1$, andaikan $T_g(\gamma)$ sebagai kelas semua fungsi analisis $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ yang memenuhi syarat $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. Bagi fungsi $f \in T_g(\gamma)$ dan fungsi cembung h, kami menunjukkan bahawa

$$\frac{g_2}{2g_2 + 1 - \gamma}(f * h) \prec h,$$

dan menggunakan keputusan ini untuk memperoleh batas bawah bagi $\Re f(z)$. Keputusan ini merangkumi beberapa keputusan awal sebagai kes khas.

Andaikan $\varphi(z)$ fungsi analisis dengan bahagian nyata positif pada cakera unit U, dengan $\varphi(0) = 1$ and $\varphi'(0) > 0$, yang memetakan U secara keseluruh kesuatu rantau bak-bintang terhadap 1 dan simetri terhadap paksi nyata. Ma dan Minda (A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Int. Press, Cambridge, MA.) memperkenalkan kelas $\mathcal{S}^*(\varphi)$ yang terdiri daripada semua fungsi analisis ternormalkan $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ sedemikian $zf'(z)/f(z) \prec \varphi(z)$. Andaikan $\mathcal{A}_p$ menandakan kelas semua fungsi analisis berbentuk $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\})$. Andaikan $S_p^*(\varphi)$ sebagai subkelas $\mathcal{A}_p$ yang ditakrifkan sebagai

$$S_p^*(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \varphi(z) \right\}.$$

Bagi fungsi $f \in S_p^*(\varphi)$, batas atas tepat bagi fungsian pekali $|a_{p+2} - \mu a_{p+1}^2|$ dan $|a_{p+3}|$ telah diperoleh; batas-batas ini menghasilkan batas atas tepat bagi pekali kedua, ketiga dan keempat. Seterusnya dikaji masalah pekali yang serupa bagi fungsi-fungsi dalam subkelas yang tertakrif dengan ungkapan $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)$, $1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)$, $\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)}$, $\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ dan $\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$. Keputusan-keputusan ini digunakan kemudian untuk memperoleh ketaksamaan bak Fekete-Szegö bagi beberapa kelas fungsi yang tertakrif melalui konvolusi.

DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR ANALYTIC AND MEROMORPHIC FUNCTIONS DEFINED BY LINEAR OPERATORS

ABSTRACT

A function f defined on the open unit disk U of the complex plane $\mathcal{C}$ is univalent if it maps different points of U to different points of $\mathcal{C}$. The function f is subordinate to an univalent function g if $f(0) = g(0)$ and $f(U) \subset g(U)$. A normalized function f is Janowski starlike if $zf'(z)/f(z)$ is subordinated to $(1+Az)/(1+Bz)$, $(-1 \leq B \leq A \leq 1)$. By making use of the theory of first order differential subordination, we obtain several sufficient conditions for a function f to be Janowski starlike. These sufficient conditions are obtained by investigating the implication

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}$$

where $\prec$ denotes subordination between analytic functions and other similar implications involving $1+\beta zp'(z)$, $1+\beta\frac{zp'(z)}{p(z)}$ and $1+\beta\frac{zp'(z)}{p^2(z)}$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

Let Ω be any set in $\mathcal{C}$ and the functions q_1 be univalent and q_2 be analytic in U. Let $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$. Miller and Mocanu [*Differential Subordinations*, Dekker, New York, 2000.] have investigated the theory of first and second order differential subordination. Recently Miller and Mocanu (Subordinants of differential superordinations, Complex Var. Theory Appl. **48**(10) (2003), 815–826.) investigated the dual concept of differential superordination to obtain several sandwich results. By using the theory of differential subordination, we determine the class of functions so that

$$\psi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}\right) \prec h$$

implies $zf'(z)/f(z) \prec q(z)$, where $\{f,z\}$ denotes the Schwarzian derivative of the function f. We also obtain similar results involving the ratios of functions defined through the Dziok-Srivastava linear operator and the multiplier transformation of analytic functions. We also obtain the corresponding superordination and sandwich type results. Further, we

investigate similar problems for meromorphic functions defined through the Liu-Srivastava linear operator and the multiplier transformation.

For a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ defined on the open unit disk and $\gamma < 1$, let $T_g(\gamma)$ denote the class of all analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. For a function $f \in T_g(\gamma)$ and a convex function h, we show that

$$\frac{g_2}{2g_2 + 1 - \gamma}(f * h) \prec h$$

and use this to obtain the lower bound for $\Re f(z)$. These results includes several earlier results as special cases.

Let $\varphi(z)$ be an analytic function with positive real part in the unit disk U with $\varphi(0) = 1$ and $\varphi'(0) > 0$, maps U onto a region starlike with respect to 1 and symmetric with respect to real axis. Ma and Minda (A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Int. Press, Cambridge, MA.) introduced the class $\mathcal{S}^*(\varphi)$ consisting of all normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $zf'(z)/f(z) \prec \varphi(z)$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\})$. Let $S_p^*(\varphi)$ be a subclass of $\mathcal{A}_p$ defined by

$$S_p^*(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \varphi(z) \right\}.$$

For the function $f \in S_p^*(\varphi)$, the sharp upper bounds for the coefficient functionals $|a_{p+2} - \mu a_{p+1}^2|$ and $|a_{p+3}|$ are obtained; these bounds yield the sharp upper bounds for the second, third and fourth coefficients. Further we investigate a similar coefficient problem for functions in the subclasses defined by the expressions $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)$, $1 + \frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}} - 1\right)$, $\frac{1+\alpha(1-p)}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\frac{z^2 f''(z)}{f(z)}$, $\frac{1-\alpha}{p}\frac{zf'(z)}{f(z)} + \frac{\alpha}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ and $\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}$. These are then applied to obtain Fekete-Szegö-like inequalities for several classes of functions defined by convolution.

SYMBOLS

Symbol	Description
$\mathcal{A}_p$	Class of all p-valent analytic functions of the form
	$f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \quad (z \in U)$
$\mathcal{A} := \mathcal{A}_1$	Class of analytic functions of the form
	$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U)$
$(a)_n$	Pochhammer symbol or shifted factorial
$\arg$	Argument
$\mathcal{C}$	Complex plane
C	Class of normalized convex functions in U
$C(\alpha)$	Class of normalized convex functions of order α in U
$f * g$	Convolution or Hadamard product of functions f and g
$\{f, z\}$	Schwarzian derivative of f
${}_1F_1(a, b, c; z)$	Confluent hypergeometric functions
${}_2F_1(a, b, c; z)$	Gaussian hypergeometric functions
${}_lF_m\left(\begin{array}{ll}\alpha_1, \ldots, \alpha_l; & \\ \beta_1, \ldots, \beta_m; & z\end{array}\right)$	Generalized hypergeometric functions
$\mathcal{H}(U)$	Class of analytic functions in U
$\mathcal{H}[a, n]$	Class of analytic functions in U of the form
	$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in U)$
$\mathcal{H}_0 := \mathcal{H}[0, 1]$	Class of analytic functions in U of the form
	$f(z) = a_1 z + a_2 z^2 + \cdots \quad (z \in U)$
$\mathcal{H} := \mathcal{H}[1, 1]$	Class of analytic functions in U of the form
	$f(z) = 1 + a_1 z + a_2 z^2 + \cdots \quad (z \in U)$
$H_p^{l,m}$	Dziok-Srivastava / Liu-Srivastava linear operator
$\prec$	Subordinate to
$\Im$	Imaginary part of a complex number
$I_p(n, \lambda),\ (\lambda + p > 0, n \in \mathcal{N})$	Multiplier transformation from $\mathcal{A}_p \to \mathcal{A}_p$
K	Class of close-to-convex functions in $\mathcal{A}$

$k(z)$	Koebe function
$\mathcal{N}$	Set of all positive integers
$\mathcal{R}$	Set of all real numbers
$R[A,B]$	$\{f \in \mathcal{A} : f'(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1)\}$
$R[\alpha]$	$\{f \in \mathcal{A} : \lvert f'(z)-1\rvert < 1-\alpha \quad (z \in U,\ 0 \leq \alpha < 1)\}$
$\Re$	Real part of a complex number
$\mathcal{S}$	Class of all normalized univalent functions of the form
	$f(z) = z + a_2 z^2 + \cdots \quad z \in U$
S^*	Class of normalized starlike functions in U
S^*_α	Class of normalized starlike functions of order α in U
$\mathcal{S}^*[A,B]$	$\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1)\}$
$T_g(\gamma)$	$\{f(z) \in \mathcal{A} : \sum_{n=2}^{\infty} \lvert a_n g_n\rvert \leq 1-\gamma,$
	$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, g_n \geq g_2 > 0, n \geq 2, \gamma < 1\}$
Σ_p	Class of all p-valent functions of the form
	$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in U^*)$
$\Sigma := \Sigma_1$	Class of all functions of the form
	$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (z \in U^*)$
U	Open unit disk $\{z \in \mathcal{C} : \lvert z\rvert < 1\}$
U^*	Punctured unit disk $U \setminus \{0\}$
U_r	Open disk of radius r, $\{z \in \mathcal{C} : \lvert z\rvert < r\}$
∂U	Boundary of unit disk U, $\{z \in \mathcal{C} : \lvert z\rvert = 1\}$
$\Psi_n[\Omega,q],\ \Phi_H[\Omega,q],\ \Phi_I[\Omega,q]$	
$\Theta_H[\Omega,q],\ \Theta_I[\Omega,q],\ \Theta_I[\Omega,M]$	Classes of admissible functions
$\mathcal{Z}$	Set of all integers

CHAPTER 1

INTRODUCTION

1.1. UNIVALENT FUNCTIONS

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$. Let $\mathcal{H}(U)$ be the class of functions analytic in U. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ and let $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the class of all analytic functions defined in U and normalized by $f(0) = 0, f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series of the form

(1.1.1) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U).$$

More generally, let $\mathcal{A}_p$ denote the class of all analytic functions of the form

(1.1.2) $$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in U,\ p \in \mathcal{N} := \{1, 2, 3 \ldots\}).$$

A function $f \in \mathcal{H}(U)$ is *univalent* if it is one-to-one in U. The function $f \in \mathcal{H}(U)$ is *locally univalent* at $z_0 \in U$ if it is univalent in some neighborhood of z_0. The function $f(z)$ is p-*valent* (or *multivalent* of order p) if for each w_0 with infinity included, the equation $f(z) = w_0$ has at most p roots in U, where the roots are counted with their multiplicities, and for some w_1 the equation $f(z) = w_1$ has exactly p roots in U [**31**]. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Thus $\mathcal{S}$ is the class of all normalized univalent functions in U.

In 1916, Bieberbach [**12**] studied the second coefficient a_2 of a function $f \in \mathcal{S}$ of the form (1.1.1). He has shown that $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function $k(z) = z/(1-z)^2$ and he mentioned "$|a_n| \leq n$ is generally valid". This statement is known as the Bieberbach conjecture. The Koebe function

(1.1.3) $$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} n z^n \quad (z \in U)$$

which maps U onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$ is the "largest" function in $\mathcal{S}$. The function $e^{-i\beta}k(e^{i\beta}z)$, $(\beta \in \mathcal{R})$ also belongs to $\mathcal{S}$ and is referred to as a rotation of the Koebe function. These functions play a very important role in the study of the class $\mathcal{S}$ and are the only extremal functions for various extremal problems in $\mathcal{S}$.

In 1923, Löwner [**53**] proved the Bieberbach conjecture for n $=$ 3. Schaeffer and Spencer [**96**], Jenkins [**37**], Garabedian and Schiffer [**29**], Charzyński and Schiffer [**16, 17**], Pederson [**76, 77**], Ozawa [**66**] and Pederson and Schiffer [**78**] have investigated the Bieberbach conjecture for certain values of n. Finally, in 1985, de Branges proved the Bieberbach conjecture for all coefficients with the help of the hypergeometric functions.

Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions. A set $\mathcal{D}$ in the complex plane is called *convex* if for every pair of points w_1 and w_2 lies in the interior of $\mathcal{D}$, the line segment joining w_1 and w_2 lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps U onto a convex domain, then f is called a convex function. Let C denotes the class of all convex functions in $\mathcal{A}$. An analytic description of the class C is given by $C := \{f \in \mathcal{A} : \Re\left(1 + zf''(z)/f'(z)\right) > 0\}$ [**24, 30, 31, 32, 80**]. Let w_0 be an interior point of $\mathcal{D}$. A set $\mathcal{D}$ in the complex plane is called *starlike* with respect to w_0 if the line segment joining w_0 to every other point $w \in \mathcal{D}$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps U onto a domain starlike, then f is called a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically, $\mathcal{S}^* := \{f \in \mathcal{A} : \Re\left(zf'(z)/f(z)\right) > 0\}$ [**24, 30, 31, 32, 80**].

A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there is a convex function $g(z)$ such that $\Re\left(f'(z)/g'(z)\right) > 0$ for all $z \in U$. The class of all close-to-convex functions in $\mathcal{A}$ is denoted by K.

A function in any one of these classes is characterized by either of the quantities $1 + zf''(z)/f'(z)$, $zf'(z)/f(z)$ or $f'(z)/g'(z)$ lying in a given region in the right half plane; the region is often convex and symmetric with respect to the real axis [**54**]. Let

f and F be members of $\mathcal{H}(U)$. The function $f(z)$ is said to be *subordinate* to $F(z)$, or $F(z)$ is *superordinate* to $f(z)$, if there exists a function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = F(w(z))$. In such a case, we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let φ be an analytic function with positive real part in the unit disk U, $\varphi(0) = 1$ and $\varphi'(0) > 0$, and map U onto a region starlike with respect to 1 and symmetric with respect to real axis. Ma and Minda [**54**] introduced the classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ by

(1.1.4) $$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

(1.1.5) $$C(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

The classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ include the subclasses of starlike and convex functions as special cases. When

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \le B \le A \le 1),$$

the classes $\mathcal{S}^*(\varphi)$ and $C(\varphi)$ reduce to the class $\mathcal{S}^*[A, B]$ of *Janowski starlike functions* and the class $C[A, B]$ of *Janowski convex functions* respectively [**35, 79**]. Thus

$$\mathcal{S}^*[A, B] =: \mathcal{S}^*\left(\frac{1 + Az}{1 + Bz}\right) \quad \text{and} \quad C[A, B] =: C\left(\frac{1 + Az}{1 + Bz}\right).$$

Also

$$\mathcal{S}^* = \mathcal{S}^*[1, -1] = \mathcal{S}^*\left(\frac{1 + z}{1 - z}\right) \quad \text{and} \quad C = C[1, -1] = C\left(\frac{1 + z}{1 - z}\right)$$

are the familiar classes of starlike and convex functions respectively.

For $0 \le \alpha < 1$, the class $\mathcal{S}^*[1 - 2\alpha, -1]$ is the class $\mathcal{S}^*_\alpha$ of starlike functions of order α. An equivalent analytic description of $\mathcal{S}^*_\alpha$ is given by

$$\mathcal{S}^*_\alpha := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (0 \le \alpha < 1) \right\}.$$

For $0 \le \alpha < 1$,

$$\mathcal{S}^*(\alpha) := \mathcal{S}^*[1 - \alpha, 0] = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in U,\ 0 \le \alpha < 1) \right\}.$$

For $0 < \alpha \leq 1$, Parvatham [**74**] introduced and studied the class $S^*[\alpha]$ where

$$\mathcal{S}^*[\alpha] := \mathcal{S}^*[\alpha, -\alpha] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\alpha z}{1-\alpha z} \right\}$$

(1.1.6) $$= \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \left| \frac{zf'(z)}{f(z)} + 1 \right| \quad (z \in U,\ 0 < \alpha \leq 1) \right\}.$$

For $0 \leq \alpha < 1$, $C(\alpha) := C[1-2\alpha, -1]$ is the class of convex functions of order α. Equivalently

$$C(\alpha) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \right\}$$
$$= \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (0 \leq \alpha < 1) \right\}.$$

The transform

$$\int_0^z \frac{f(t)}{t} dt$$

is called the Alexander transform of $f(z)$. It is clear that $f \in C(\alpha)$ if and only if $zf' \in \mathcal{S}^*_\alpha$ or equivalently $f \in \mathcal{S}^*_\alpha$ if and only if the Alexander transform of $f(z)$ is in $C(\alpha)$.

For real α, let

$$\mathcal{M}(\alpha, f; z) \equiv (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right).$$

The class of α-convex functions is defined by

$$\mathcal{M}_\alpha = \{ f \in \mathcal{A} : \Re \mathcal{M}(\alpha, f; z) > 0 \}.$$

This class $\mathcal{M}_\alpha$ is a subclass of $\mathcal{S}$, and was introduced and studied by Miller et al. [**56**]. It has the additional properties that $\mathcal{M}_\alpha \subset \mathcal{M}_\beta \subset \mathcal{M}_0 = \mathcal{S}^*$ for $0 \leq \alpha/\beta \leq 1$, and $\mathcal{M}_\alpha \subset \mathcal{M}_1 \subset C$ for $\alpha \geq 1$.

More information on univalent functions can be found in the text books [**24, 30, 31, 32, 33, 58, 80**].

1.2. HYPERGEOMETRIC FUNCTIONS

The use of the hypergeometric functions in the celebrated de Branges proof of the Bieberbach conjecture prompted renewed interest in the investigation of special functions.

Prior to this proof, there had been only a few articles in the literature dealing with the relationships between these special functions and univalent function theory.

Let a and c be any complex numbers with $c \neq 0, -1, \cdots$, and consider the function defined by

$$(1.2.1)\quad \Phi(a,c;z) = {}_1F_1(a,c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2\,!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3\,!} + \cdots .$$

This function, called a *confluent (or Kummer) hypergeometric function* is analytic in $\mathcal{C}$ and satisfies Kummer's differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0.$$

The Pochhammer symbol $(a)_n$ is defined by

(1.2.2)

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \text{ and } a \in \mathcal{C} \setminus \{0\}; \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n \in \mathcal{N} \text{ and } a \in \mathcal{C} \end{cases}$$

where $\Gamma(a)$, $(a \in \mathcal{C})$ denotes the Gamma function. Then (1.2.1) can be written in the form

$$(1.2.3)\qquad \Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k\,!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}.$$

Let a, b and c be any complex numbers with $c \neq 0, -1, \cdots$, and consider the function defined by

$$(1.2.4)\qquad F(a,b,c;z) = {}_2F_1(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2\,!} + \cdots .$$

This function, called a *(Gaussian) hypergeometric function* is analytic in U and satisfies the hypergeometric differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0.$$

Using the notation (1.2.2) in (1.2.4), F can be written as

$$(1.2.5)\qquad F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k\,!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}.$$

More generally, for $\alpha_j \in \mathcal{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathcal{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function*

$$_lF_m(z) := \; _lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$$

is defined by the infinite series

$$_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathcal{N}_0 := \mathcal{N} \cup \{0\}, z \in U)$$

where $(a)_n$ is the Pochhammer symbol defined by (1.2.2). The absence of parameters is emphasized by a dash. For example,

$$_0F_1(-; b; z) = \sum_{k=0}^{\infty} \frac{z^k}{(b)_k k\,!},$$

is the Bessel's function. Also

$$_0F_0(-; -; z) = \sum_{k=0}^{\infty} \frac{z^k}{k\,!} = \exp(z)$$

and

$$_1F_0(a; -; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k\,!} = \frac{1}{(1-z)^a}.$$

Similarly,

$$_2F_1(a, b; b; z) = \frac{1}{(1-z)^a}, \qquad _2F_1(1, 1; 1; z) = \frac{1}{1-z},$$

$$_2F_1(1, 1; 2; z) = \frac{-\ln(1-z)}{z}, \text{ and } \; _2F_1(1, 2; 1; z) = \frac{1}{(1-z)^2}.$$

For two functions $f(z)$ given by (1.1.2) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the *Hadamard product* (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2.6}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^p \; _lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the *Dziok-Srivastava operator* [**25**] (see also [**107**])

$$H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}_p \to \mathcal{A}_p$$

is defined by the Hadamard product

$$H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) := h_p(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)*f(z)$$

$$= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p}\ldots(\alpha_l)_{n-p}}{(\beta_1)_{n-p}\ldots(\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!}. \tag{1.2.7}$$

For brevity,

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z).$$

The linear (convolution) operator $H_p^{l,m}[\alpha_1]f(z)$ includes, as its special cases, many earlier linear (convolution) operators investigated in geometric function theory. Some of these special cases are described below.

The linear operator $\mathcal{F}(\alpha,\beta,\gamma)$ defined by

$$\mathcal{F}(\alpha,\beta,\gamma) = H_1^{2,1}(\alpha,\beta;\gamma)f(z)$$

is Hohlov linear operator [**34**]. The linear operator $\mathcal{L}(\alpha,\gamma)$ defined by

$$\mathcal{L}(\alpha,\gamma) = H_1^{2,1}(\alpha,1;\gamma)f(z) = \mathcal{F}(\alpha,1,\gamma)$$

is the Carlson and Shaffer linear operator [**15**]. The differential operator $\mathcal{D}^\lambda : \mathcal{A} \to \mathcal{A}$ defined by the Hadamard product:

$$\mathcal{D}^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) = H_1^{2,1}(\lambda+1,1;1)f(z), \quad (\lambda \geq 1, f \in \mathcal{A})$$

is the Ruscheweyh derivative operator [**92**]. This operator can also be defined by,

$$\mathcal{D}^n f(z) := \frac{z(z^{n-1}f(z))^{(n)}}{n\,!}, \quad (n \in \mathcal{N}_0, f(z) \in \mathcal{A}).$$

In 1969, Bernardi [**10**] considered the linear integral operator $F : \mathcal{A} \to \mathcal{A}$ defined by

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt. \tag{1.2.8}$$

When $c = 1$, this operator was investigated by Libera [**47**] and Livingston [**48**]. Therefore the operator in (1.2.8) is called the generalized Bernardi-Libera-Livingston linear operator. Clearly

$$F(z) = H_1^{2,1}(c+1,1;c+2)f(z), \quad (c > -1, f \in \mathcal{A}).$$

It is well-known [**10**] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi-Libera-Livingston integral operator.

Definition 1.2.1. [**67, 71**] The *fractional integral of order* λ is defined by

$$D_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0)$$

where $f(z)$ is an analytic function in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\zeta)^{1-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 1.2.2. [**67, 71**] The *fractional derivative of order* λ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1)$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.2.1 above.

Definition 1.2.3. [**67, 71**] Under the hypothesis of Definition 1.2.2, the *fractional derivative of order* $n + \lambda$ is defined, by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1, \ n \in \mathcal{N}_0).$$

In 1987, Srivastava and Owa [**105**] studied a fractional derivative operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\Omega^{\lambda} f(z) := \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z).$$

The fractional derivative operator is a special case of the Dziok-Srivastava linear operator since

$$\begin{aligned} \Omega^{\lambda} f(z) &= H_1^{2,1}(2, 1; 2-\lambda) f(z) \\ &= \mathcal{L}(2, 2-\lambda) f(z), \quad (\lambda \notin \mathcal{N} \setminus \{1\}, f \in \mathcal{A}). \end{aligned}$$

1.3. MULTIPLIER TRANSFORMATIONS

The Sălăgean [**95**] derivative operator $\mathcal{D}^m f(z)$ of order m $(m \in \mathcal{N})$ is defined by

$$\mathcal{D}^m f(z) := f(z) * \left(z + \sum_{n=2}^{\infty} n^m z^n \right) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad (m \in \mathcal{N}, f \in \mathcal{A}).$$

Clearly $\mathcal{D}^0 f(z) = f(z)$, $\mathcal{D}^1 f(z) = z f'(z)$ and in general

$$\mathcal{D}^m f(z) = z(\mathcal{D}^{m-1} f(z))' \quad (m \in \mathcal{N}, f \in \mathcal{A}).$$

In 1990, Komatu [**42**] introduced a certain integral operator $\mathcal{I}_a^\lambda$ $(a > 0, \lambda \geq 0)$ defined by

$$\begin{aligned}\mathcal{I}_a^\lambda f(z) &:= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda-1} f(zt)dt, \\ &= z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\lambda a_n z^n \quad (z \in U,\ a > 0,\ \lambda \geq 0,\ f \in \mathcal{A}).\end{aligned} \tag{1.3.1}$$

When $a = 2$, the integral operator $\mathcal{I}_a^\lambda f(z)$ is essentially the multiplier transformation studied by Flett [**27**]. Subsequently Jung *et al.* [**38**] studied the following one-parameter families of integral operators:

$$P^\alpha f(z) := \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^1 \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \quad (\alpha > 0),$$

$$Q_\alpha^\beta f(z) := \binom{\alpha+\beta}{\alpha} \frac{\beta}{z^\alpha} \int_0^1 t^{\alpha-1} \left(1 - \frac{t}{z}\right)^{\beta-1} f(t)dt \quad (\beta > 0, \alpha > -1),$$

and

$$F(z) := \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)dt \quad (\alpha > -1),$$

where $\Gamma(\alpha)$ is Gamma function, $\alpha \in \mathcal{N}$. The operator P^α, Q_α^β and $F(z)$ were considered by Bernardi [**10, 11**]. Further, for a real number $\alpha > -1$, the operator $F(z)$ was used by several authors [**70, 74, 103, 104**].

For $f \in \mathcal{A}$ given by (1.1.1), Jung *et al.* [**38**] obtained

$$P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \quad (\alpha > 0), \tag{1.3.2}$$

$$Q_\alpha^\beta f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \quad (\beta > 0, \alpha > -1), \tag{1.3.3}$$

and

$$F(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \quad (\alpha > -1). \tag{1.3.4}$$

By virtue of (1.3.1), (1.3.2), (1.3.3) and (1.3.4), we see that

$$\mathcal{I}_2^\lambda f(z) = P^\lambda f(z) \quad (\lambda > 0)$$

and

$$F(z) = Q_\alpha^1 f(z) \quad (\alpha > -1).$$

Ali and Singh [**4**] and Fournier and Ruscheweyh [**28**] have studied integral operators V_λ of functions $f \in \mathcal{A}$ by

(1.3.5)
$$V_\lambda f(z) := \int_0^1 \lambda(t) \frac{f(zt)}{t} dt \quad (\lambda(t) \in \Phi),$$

where

$$\Phi := \left\{ \lambda(t) : \lambda(t) \geq 0 \quad (0 \leq t \leq 1) \quad \text{and} \quad \int_0^1 \lambda(t) dt = 1 \right\}.$$

Recently, Li and Srivastava [**46**] have studied an integral operator V_λ^α of functions $f \in \mathcal{A}$ defined by

(1.3.6)
$$V_\lambda^\alpha f(z) := \int_0^1 \lambda_\alpha(t) \frac{f(zt)}{t} dt,$$

where the real valued functions λ_α and $\lambda_{\alpha-1}$ satisfy the following conditions:

(i) For a suitable parameter α,

$$\lambda_{\alpha-1} \in \Phi, \lambda_\alpha \in \Phi \quad \text{and} \quad \lambda_\alpha(1) = 0;$$

(ii) There exists a constant c $(-1 < c \leq 2)$ such that

$$c\lambda_\alpha(t) - t\lambda'_{\alpha-1}(t) = (c+1)\lambda_{\alpha-1} \quad (0 < t < 1; -1 < c \leq 2).$$

Further Li and Srivastava [**46**] found a relation between $V_\lambda^\alpha f(z)$, $P^\alpha f(z)$ and $Q_\alpha^\beta f(z)$ by setting particular values of $\lambda_\alpha(t)$. By setting

$$\lambda_\alpha(t) = \binom{\alpha+\beta}{\alpha} \alpha(1-t)^{\alpha-1} t^\beta \quad (\alpha > 0, \beta > -1)$$

in (1.3.6), they obtained $V_\lambda^\alpha f(z) = Q_\beta^\alpha f(z)$; similarly by setting

$$\lambda_\alpha(t) = \frac{2^\alpha}{\Gamma(\alpha)} t \left(\log \frac{1}{t} \right)^{\alpha-1} \quad (\alpha > 0),$$

in (1.3.6), they obtained $V_\lambda^\alpha f(z) = P^\alpha f(z)$.

Motivated by these operators , Cho and Kim [**20**, Definition, p. 400] introduced a more general linear operator called the multiplier transformation. For any integer n, the multiplier transformation $I_\lambda^n : \mathcal{A} \to \mathcal{A}$ is defined by

(1.3.7)
$$I_\lambda^n f(z) := z + \sum_{k=2}^\infty \left(\frac{k+\lambda}{1+\lambda} \right)^n a_k z^k \quad (\lambda \geq 0, \ n \in \mathcal{Z}).$$

For $\lambda = 1$, the operator I_λ^n was studied by Uralegaddi and Somanatha [**111**]. The operator I_λ^n is closely related to the Komatu integral operators [**42**] and the differential and integral operators defined by Sălăgean [**95**].

Motivated by the multiplier transformation on $\mathcal{A}$, we define the operator $I_p(n,\lambda)$ on $\mathcal{A}_p$ by the following infinite series

$$I_p(n,\lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k \quad (\lambda \geq -p,\ n \in \mathcal{Z}). \tag{1.3.8}$$

The operator $I_1(m,0)$ is the Sălăgean derivative operator $\mathcal{D}^m$ [**95**]. The operator $I_1(n,\lambda)$ was studied recently by Cho and Srivastava [**19**] and Cho and Kim [**20**]. The operator $I_1(n,1)$ was studied by Uralegaddi and Somanatha [**111**]. The operator $I_1(-1,c)$ is the generalized Bernardi-Libera-Livingston linear operator [**10, 11**].

1.4. SUBORDINATION AND SUPERORDINATION

Let $\psi(r,s,t;z) : \mathcal{C}^3 \times U \to \mathcal{C}$ and let $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.4.1}$$

then $p(z)$ is called a *solution* of the differential subordination. The univalent function $q(z)$ is called a *dominant of the solution* of the differential subordination or more simply *dominant*, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.4.1). A dominant $q_1(z)$ satisfying $q_1(z) \prec q(z)$ for all dominants $q(z)$ of (1.4.1) is said to be the *best dominant* of (1.4.1). The best dominant is unique up to a rotation of U. If $p(z) \in \mathcal{H}[a,n]$, then $p(z)$ will be called an (a,n)-*solution*, $q(z)$ an (a,n)-*dominant*, and $q_1(z)$ the *best* (a,n)-*dominant*. Let $\Omega \subset \mathcal{C}$ and let (1.4.1) be replaced by

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \text{ for all } z \in U. \tag{1.4.2}$$

Even though this is a "differential inclusion" and $\psi(p(z), zp'(z), z^2p''(z); z)$ may not be analytic in U, the condition in (1.4.2) will also be referred as a *second order differential subordination*, and the same definition of solution, dominant and best dominant as given above can be extended to this generalization. See [**58**] for more information on differential subordination.

Let $\psi(r,s,t;z) : \mathcal{C}^3 \times U \to \mathcal{C}$ and let $h(z)$ be analytic in U. If $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z); z)$$

are univalent in U and satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.4.3}$$

then $p(z)$ is called a *solution* of the differential superordination. An analytic function $q(z)$ is called a *subordinant of the solution* of the differential superordination or more simply *subordinant*, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.4.3). A univalent subordinant $q_1(z)$ satisfying $q(z) \prec q_1(z)$ for all subordinants $q(z)$ of (1.4.3) is said to be the *best subordinant* of (1.4.3). The best subordinant is unique up to a rotation of U. Let $\Omega \subset \mathcal{C}$ and let (1.4.3) be replaced by

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) | z \in U\}. \tag{1.4.4}$$

Even though this more general situation is a "*differential containment*", the condition in (1.4.4) will also be referred as a *second order differential superordination* and the definition of solution, subordinant and best subordinant can be extended to this generalization. See [**59**] for more information on the differential superordination.

Denote by $\mathcal{Q}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{U} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.4.1. [**58**, Definition 2.3a, p. 27] Let Ω be a set in $\mathcal{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathcal{C}^3 \times U \to \mathcal{C}$ that satisfy the admissibility condition

$$\psi(r,s,t;z) \notin \Omega \tag{1.4.5}$$

whenever $r = q(\zeta),\ s = k\zeta q'(\zeta)$, and

$$\Re\left\{\frac{t}{s}+1\right\} \geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

If $\psi : \mathcal{C}^2 \times U \to \mathcal{C}$, then the admissible condition (1.4.5) reduces to

(1.4.6) $$\psi(q(\zeta), k\zeta q'(\zeta); z) \notin \Omega,$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}$. In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.4.2. [**59**, Definition 3, p. 817] Let Ω be a set in $\mathcal{C}$, $q(z) \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathcal{C}^3 \times \overline{U} \to \mathcal{C}$ that satisfy the admissibility condition

(1.4.7) $$\psi(r, s, t; \zeta) \in \Omega$$

whenever $r = q(z), s = \frac{zq'(z)}{m}$, and

$$\Re\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. When $n = 1$, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

If $\psi : \mathcal{C}^2 \times \overline{U} \to \mathcal{C}$ If $\psi : \mathcal{C}^2 \times \overline{U} \to \mathcal{C}$, then the admissible condition (1.4.7) reduces to

(1.4.8) $$\psi(q(z), zq'(z)/m; \zeta) \notin \Omega,$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n$.

1.5. SCOPE AND MOTIVATION OF THIS WORK

In the present work, certain properties of analytic functions and meromorphic functions are investigated. In particular, certain sufficient conditions for Janowski starlikeness are obtained for various classes of analytic functions. Certain general differential subordination and superordination results are also obtained. These are then used to obtain differential sandwich results. Also non-linear coefficient problems involving the first three coefficients of a p-valent function are discussed.

In 1969, Bernardi [**10**] introduced and studied a linear integral operator

$$F(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1),$$

now called the *Bernardi integral operator*. He proved that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi integral operator. In the year 2000, Parvatham [**74**] extended Bernardi's results for functions in $\mathcal{S}^*[\alpha]$ defined in (1.1.6). We have extended the results of Parvatham [**74**] by considering a more general subordinate function. In the year 1999, Silverman [**99**] introduced and studied classes of functions obtained by the quotient of expressions defining the convex and starlike functions. Later Obradovič and Tuneski [**64**] and Tuneski [**109**] improved the results of Silverman [**99**]. Further many researchers [**61, 62, 63, 85, 88**] have studied these classes. These results are extended by considering a more general subordinate function. Sufficient conditions for *Janowski starlikeness* for functions in several subclasses of analytic functions are also obtained. These results are presented in Chapter 2.

By using differential subordination, Miller and Mocanu [**58**] found some sufficient conditions relating the Schwarzian derivative to the starlikeness or convexity of $f \in \mathcal{A}$. Aouf et al. [**7**] and Kim and Srivastava [**41**] derived several inequalities associated with some families of integral and convolution operators that are defined for the class of normalized analytic functions in the open unit disk U. Recently Aghalary et al. [**2**] obtained some inequalities for analytic functions in the open unit disk that are associated with the Dziok-Srivastava linear operator and the multiplier transformation. Similar results for meromorphic functions defined through a linear operator are considered by Liu and

Owa [**49**]. These results motivate our main results in Chapters 3 and 4. Chapter 3 deals with the applications of differential subordination and superordination to obtain sufficient conditions on the *Schwarzian derivative* of normalized analytic functions. Subordination and superordination results for analytic functions associated with the *Dziok-Srivastava linear operator* and *multiplier transformation* are obtained. Additionally sandwich results are obtained. In Chapter 4, we consider the Liu-Srivastava linear operator for the case of meromorphic functions [**51, 52**] and also the multiplier transformation for meromorphic functions. Differential subordination and superordination results are obtained for meromorphic functions in the punctured unit disk that are associated with the *Liu-Srivastava linear operator* and the *multiplier transformation*. These results are obtained by investigating appropriate class of admissible functions. Certain related sandwich-type results are also obtained.

In 1975, Silverman [**98**] studied the class of analytic functions whose Taylor coefficients are negative. Several other authors (see for example, Al-Amiri [**3**], Attiya [**9**], Srivastava and Attiya [**102**], Owa and Srivastava [**72**], as well as Owa and Nishiwaki [**68**]) have studied several classes of analytic functions with negative coefficients. These results are shown to be special cases of our main results in Chapter 5. For a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ defined on the open unit disk and $\gamma < 1$, let $T_g(\gamma)$ denote the class of all analytic functions $f \in \mathcal{A}$ of the form (1.1.1) satisfying $\sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma$. For functions in $T_g(\gamma)$, a subordination result is derived involving the convolution with a normalized convex function.

In 1992, Ma and Minda [**54**] obtained sharp distortion, growth, rotation and covering theorems for the classes $C(\varphi)$ and $\mathcal{S}^*(\varphi)$. In addition, they obtained some sharp results for coefficient problems, particularly, the sharp bound on the coefficient functional $|a_3 - \mu a_2^2|$, $-\infty < \mu < \infty$, which implies sharp upper bounds for the second and third coefficients. They also studied some convolution properties. Also, several authors [**1, 23, 39, 40, 43, 82, 89**] have studied the coefficient problems for various classes of univalent functions. In Chapter 6, sharp upper bounds for the coefficient functionals $|a_{p+2} - \mu a_{p+1}^2|$ and $|a_{p+3}|$ are derived for certain p-valent analytic functions. These are then applied to obtain Fekete-Szegö like inequalities for several classes of functions defined by convolution.

CHAPTER 2

SUFFICIENT CONDITIONS FOR JANOWSKI STARLIKENESS

2.1. INTRODUCTION

For the class $\mathcal{S}^*[\alpha] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1+\alpha z)/(1-\alpha z) \quad (0 < \alpha \le 1)\}$, Parvatham proved the following:

THEOREM 2.1.1. [**74**, Theorem 1, p. 438] *Let $c \ge 0$, $0 < \alpha \le 1$ and δ be given by*

$$\delta := \alpha \left[\frac{2 + \alpha + c(1-\alpha)}{1 + 2\alpha + c(1-\alpha)} \right].$$

If $f \in \mathcal{S}^*[\delta]$, then the function $F(z)$ given by the Bernardi's integral as defined in (1.2.8) belong to $\mathcal{S}^*[\alpha]$.

It is well-known [**10**] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi's integral operator. Since $\delta \ge \alpha$, Theorem 2.1.1 extends the result of Bernardi [**10**].

Parvatham also considered a similar problem for the class $R[\alpha]$ of functions $f \in \mathcal{A}$ satisfying

$$|f'(z) - 1| < \alpha \, |f'(z) + 1| \quad (z \in U, \ 0 < \alpha \le 1),$$

or equivalently

$$f'(z) \prec \frac{1 + \alpha z}{1 - \alpha z} \quad (z \in U, \ 0 < \alpha \le 1),$$

and proved the following:

THEOREM 2.1.2. [**74**, Theorem 2, p. 440] *Let $c \ge 0$, $0 < \alpha \le 1$ and δ be given by*

$$\delta := \alpha \left[\frac{2 - \alpha + c(1-\alpha)}{1 + c(1-\alpha)} \right].$$

If $f \in R[\delta]$, then the function $F(z)$ given by the Bernardi's integral (1.2.8) is in $R[\alpha]$.

The class $R[\alpha]$ can be extended to the general class $R[A,B]$ consisting of all analytic functions $f(z) \in \mathcal{A}$ satisfying

$$f'(z) \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1),$$

or the equivalent inequality,

$$|f'(z)-1| < |A - Bf'(z)| \quad (z \in U, \ -1 \leq B < A \leq 1).$$

For $0 \leq \alpha < 1$, the class $R[1-2\alpha, -1]$ consists of functions $f \in \mathcal{A}$ for which

$$\Re f'(z) > \alpha \quad (z \in U, \ 0 < \alpha \leq 1),$$

and $R[1-\alpha, 0] =: R_\alpha$ is the class of functions $f \in \mathcal{A}$ satisfying the condition

$$|f'(z) - 1| < 1 - \alpha \quad (z \in U, \ 0 \leq \alpha < 1).$$

When $0 < \alpha \leq 1$, the class $R[\alpha, -\alpha]$ is the class $R[\alpha]$ considered by Parvatham [**74**].

Silverman [**99**], Obradovič and Tuneski [**64**] and many others (see [**61, 62, 63, 85, 88**]) have studied properties of functions defined in terms of the quotient

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}.$$

In fact, Silverman [**99**] have obtained the order of starlikeness for functions in the class G_b defined by

$$G_b := \left\{ f \in \mathcal{A} : \left| \frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b, \ 0 < b \leq 1, \ z \in U \right\}.$$

Obradovič and Tuneski [**64**] improved the result of Silverman [**99**] by showing

$$G_b \subset \mathcal{S}^*[0, -b] \subset \mathcal{S}^*(2/(1+\sqrt{1+8b})).$$

Later Tuneski [**109**] obtained conditions for the inclusion $G_b \subset \mathcal{S}^*[A,B]$ to hold. If we let $zf'(z)/f(z) =: p(z)$, then $G_b \subset \mathcal{S}^*[A,B]$ becomes

(2.1.1) $$1 + \frac{zp'(z)}{p(z)^2} \prec 1 + bz \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}.$$

Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. Frasin and Darus [**26**] have shown that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha.$$

By writing $z^2f'(z)/(f(z))^2$ as $p(z)$, we see that the above implication is a special case of

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}.$$

Another special case of the above implication was considered by Ponnusamy and Rajasekaran [**81**].

Obradovič *et. al.* [**60**] have shown that if $p(z)$ is analytic in U, $p(0)=1$ and

$$1+zp'(z) \prec 1+z, \quad \text{then } p(z) \prec 1+z.$$

Using this, they have obtained a criterion for a normalized analytic function to be univalent.

In this chapter, we extend Theorems 2.1.1 and 2.1.2 to hold true for the more general classes $\mathcal{S}^*[A,B]$ and $R[A,B]$ respectively. In fact a more general result for functions $p(z)$ with $p(0)=1$ satisfying

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec \frac{1+Dz}{1+Ez} \text{ implies } p(z) \prec \frac{1+Az}{1+Bz}$$

is obtained and by applying this result, we investigate the Bernardi's integral operator on the classes $S^*[A,B]$ and $R[A,B]$. Similar results are obtained by considering the expressions $1+\beta zp'(z)$, $1+\beta\frac{zp'(z)}{p^2(z)}$ and $1+\beta\frac{zp'(z)}{p(z)}$. These results are then applied to obtain sufficient conditions for analytic functions to be Janowski starlike.

2.2. A BRIOT-BOUQUET DIFFERENTIAL SUBORDINATION

THEOREM 2.2.1. *Let* $-1 \le B < A \le 1$ *and* $-1 \le E \le 0 < D \le 1$. *For* $\beta \ge 0$ *and* $\beta+\gamma>0$, *let* $G := A\beta+B\gamma$, $H := (\beta+\gamma)(D-E)$, $I := (A\beta+B\gamma)(D-E)+(BD-AE)(\beta+\gamma)-kE(A-B)$, $J := (A\beta+B\gamma)(BD-AE)$, *and* $L := \beta+\gamma+k$. *In addition, for all* $k \ge 1$, *let*

$$(L^2+G^2)[(H+J)I-4H|J|]+4LGHJ \ge LG[(H-J)^2+I^2]. \tag{2.2.1}$$

Further assume that

$$\frac{[\beta(1+A)+\gamma(1+B)+1](A-B)}{[\beta(1+A)+\gamma(1+B)][D(1+B)-E(1+A)]-E(A-B)} \ge 1. \tag{2.2.2}$$

Let $p(z)$ be analytic in U with $p(0) = 1$. If

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Dz}{1 + Ez},$$

then

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

PROOF. Define $P(z)$ by

$$P(z) := p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \tag{2.2.3}$$

and $w(z)$ by

$$w(z) := \frac{p(z) - 1}{A - Bp(z)},$$

or equivalently by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.2.4}$$

Then $w(z)$ is meromorphic in U and $w(0) = 0$. We need to show that $|w(z)| < 1$ in U. By a computation from (2.2.4), it follows that

$$p'(z) = \frac{(A - B)w'(z)}{(1 + Bw(z))^2}$$

and using this in (2.2.3),

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)} + \frac{(A - B)zw'(z)}{(1 + Bw(z))[\beta(1 + Aw(z)) + \gamma(1 + Bw(z))]}.$$

Therefore

$$\frac{P(z) - 1}{D - EP(z)} = \frac{(A - B)[(\beta + \gamma)w(z) + (A\beta + B\gamma)w^2(z) + zw'(z)]}{\begin{array}{c}[(D - E) + (BD - AE)w(z)][\beta + \gamma \\ +(A\beta + B\gamma)w(z)] - E(A - B)zw'(z)\end{array}}.$$

Assume that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then by [**93**, Lemma 1.3, p. 28], there exists $k \geq 1$ such that $z_0 w'(z_0) = kw(z_0)$. Let $w(z_0) = e^{i\theta}$. For this z_0, we have

$$\left|\frac{P(z_0) - 1}{D - EP(z_0)}\right| = \left|\frac{(A - B)[L + Gw(z_0)]}{H + Iw(z_0) + Jw(z_0)^2}\right| = (A - B)[\varphi(\cos\theta)]^{\frac{1}{2}}$$

where

$$\begin{aligned}
\varphi(\cos\theta) &:= \frac{|L+Ge^{i\theta}|^2}{|He^{-i\theta}+Je^{i\theta}+I|^2} \\
&= \frac{L^2+G^2+2LG\cos\theta}{[I+(H+J)\cos\theta]^2+[(J-H)\sin\theta]^2} \\
&= \frac{L^2+G^2+2LG\cos\theta}{I^2+(H+J)^2\cos^2\theta+2I(H+J)\cos\theta+(J-H)^2\sin^2\theta} \\
&= \frac{L^2+G^2+2LG\cos\theta}{H^2+J^2+I^2+2HJ\cos 2\theta+2I(H+J)\cos\theta}.
\end{aligned}$$

Define the function

$$\varphi(t) := \frac{L^2+G^2+2LGt}{4HJt^2+2I(H+J)t+(H-J)^2+I^2}. \tag{2.2.5}$$

We shall show that $\varphi(t)$ is a decreasing function. A simple computation using (2.2.5) yields

$$\varphi'(t) = -\frac{4LGHJt^2+4HJ(L^2+G^2)t+I(L^2+G^2)(H+J)-LG[I^2+(H-J)^2]}{[4HJt^2+2I(H+J)t+(H-J)^2+I^2]^2}.$$

The function $\varphi(t)$ is a decreasing function if $\varphi'(t)<0$ or equivalently if

$$4LGHJt^2+4HJ(L^2+G^2)t+I(L^2+G^2)(H+J)-LG[I^2+(H-J)^2]\geq 0.$$

In view of the fact that

$$\min\{at^2+bt+c : -1\leq t\leq 1\} = \begin{cases} \frac{4ac-b^2}{4a}, & \text{if } a>0 \text{ and } |b|<2a, \\ a-|b|+c, & \text{otherwise,} \end{cases}$$

the condition (2.2.1) shows that $\varphi(t)$ is a decreasing function of $t=\cos\theta$. Thus

$$\varphi(t)\geq\varphi(1) = \left[\frac{L+G}{I+J+H}\right]^2.$$

Consider the function

$$\begin{aligned}
\psi(k) &:= \frac{L+G}{I+J+H} \\
&= \frac{(1+A)\beta+(1+B)\gamma+k}{[(1+B)D-(1+A)E][(1+A)\beta+(1+B)\gamma]-kE(A-B)}.
\end{aligned}$$

Since

$$\psi'(k) = \frac{[(1+A)\beta+(1+B)\gamma](1+B)(D-E)}{[[(1+B)D-(1+A)E][(1+A)\beta+(1+B)\gamma]-kE(A-B)]^2},$$

clearly $\psi'(k) > 0$ and hence $\psi(k)$ is an increasing function of k. Since $k \geq 1$, we have $\psi(k) \geq \psi(1)$ and therefore

$$\left|\frac{P(z_0)-1}{D-EP(z_0)}\right| \geq \frac{[\beta(1+A)+\gamma(1+B)+1](A-B)}{[\beta(1+A)+\gamma(1+B)][D(1+B)-E(1+A)]-E(A-B)},$$

which by (2.2.2) is greater than or equal to 1. This contradicts that $P(z) \prec (1+Dz)/(1+Ez)$ and completes the proof. $\square$

2.3. APPLICATION TO THE BERNARDI'S INTEGRAL OPERATOR

THEOREM 2.3.1. *Let the conditions of Theorem 2.2.1 hold with $\beta = 1$ and $\gamma = c > -1$. If $f \in \mathcal{S}^*[D,E]$, then the function $F(z)$ given by the Bernardi's integral (1.2.8) is in $\mathcal{S}^*[A,B]$.*

PROOF. Differentiation of the Bernardi's integral (1.2.8) yields

$$(c+1)f(z) = zF'(z) + cF(z).$$

Logarithmic differentiation now yields

$$\frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+c},$$

with $p(z) = zF'(z)/F(z)$. The result now follows from Theorem 2.2.1. $\square$

Observe that when $J = 0$, the condition (2.2.1) reduces to the equivalent form

(2.3.1) $$(LI - GH)(LH - GI) \geq 0.$$

REMARK 2.3.1. If $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \leq 1)$, then $G = \alpha(1-c)$, $H = 2\delta(1+c)$, $I = 2\alpha\delta(1+k-c)$, $J = 0$ and $L = 1+c+k$. In this case, $LI - GH = 2\alpha\delta k(2+k) > 0$. In addition, $LH - GI \geq 0$ becomes $(1+c)(1+c+k) \geq \alpha^2(1-c)(1-c+k)$. Clearly this condition holds when $c \geq 0$. In the case $-1 < c < 0$, since

$$\frac{(1+c)(2+c)}{(1-c)(2-c)} \leq \frac{(1+c)(1+c+k)}{(1-c)(1-c+k)},$$

condition (2.3.1) holds provided $\alpha^2 \leq (1+c)(2+c)/(1-c)(2-c)$. Thus Theorem 2.3.1 not only reduces to Theorem 2.1.1 for $c \geq 0$, but also extends it for the case $-1 < c < 0$.

Corollary 2.3.1. *Let* $-1 < c < 0$, $0 < \alpha \leq \sqrt{(1+c)(2+c)/(1-c)(2-c)}$, *and* δ *be as in Theorem 2.1.1. If* $f \in \mathcal{S}^*[\delta]$, *then the function* $F(z)$ *given by the Bernardi's integral (1.2.8) belongs to* $\mathcal{S}^*[\alpha]$.

Remark 2.3.2. Let $A = 1-\alpha$, $B = 0$, $D = 1-\delta$ and $E = 0$ $(0 \leq \alpha, \delta < 1)$. Then $G = 1-\alpha$, $H = (1-\delta)(1+c)$, $I = (1-\alpha)(1-\delta)$, $J = 0$ and $L = 1+c+k$. Since $J = 0$, the condition (2.3.1) reduces to

$$(1+c)(1+c+k) - (1-\alpha)^2 \geq 0. \tag{2.3.2}$$

Since $(1+c)(1+c+k)-(1-\alpha)^2 \geq (1+c)(2+c)-(1-\alpha)^2$, the inequality (2.3.2) holds provided $\alpha \geq 1 - \sqrt{(1+c)(2+c)}$. This condition holds for $c \geq \left[\sqrt{4(\alpha-1)^2+1} - 3\right]/2$. This yields the following result for the class $\mathcal{S}^*(\delta)$.

Corollary 2.3.2. *Let* $\delta := \alpha - (1-\alpha)/(2+c-\alpha)$, $f(z) \in \mathcal{S}^*(\delta)$ *and* $F(z)$ *be given by the Bernardi's integral (1.2.8). If* $\alpha_0 \leq \alpha < 1$, *then* $F(z) \in \mathcal{S}^*(\alpha)$ *for all* $c > -1$. *Here* $\alpha_0 := (3+c-\sqrt{(3+c)^2-4})/2$.

Theorem 2.3.2. *Under the conditions stated in Theorem 2.2.1 with* $\beta = 0$ *and* $\gamma = c+1$, *if* $f \in R[D,E]$, *then the function* $F(z)$ *given by the Bernardi's integral (1.2.8) is in* $R[A,B]$.

Proof. Since

$$(c+1)f(z) = zF'(z) + cF(z),$$

it follows that

$$f'(z) = \frac{zF''(z)}{c+1} + F'(z). \tag{2.3.3}$$

The result now follows from Theorem 2.2.1 with $p(z) = F'(z)$, $\beta = 0$ and $\gamma = c+1$. □

Remark 2.3.3. For $A = \alpha$, $B = -\alpha$, $D = \delta$ and $E = -\delta$ $(0 < \alpha, \delta \leq 1)$, then $G = -\alpha(1+c)$, $H = 2\delta(1+c)$, $I = 2\alpha\delta(k-1-c)$, $J = 0$ and $L = 1+c+k$. The condition (2.3.1) becomes

$$4\alpha\delta^2k^2(1+c)[(1+c)(1-\alpha^2) + k(1+\alpha^2)] \geq 0$$

which holds for any $c > -1$. This shows that Theorem 2.3.2 reduces to Theorem 2.1.2 and that the assertion even holds in the case $-1 < c < 0$.

REMARK 2.3.4. For $A = \delta$, $B = 0$, $D = \alpha$ and $E = 0$ $(0 < \alpha, \delta \leq 1)$, then $G = I = J = 0$, $H = \alpha(1+c)$, and $L = 1 + c + k$. In this case the condition (2.3.1) holds for any $c > -1$. Thus Theorem 2.3.2 extends the earlier result of Anbudurai [**6**, Theorem 2.1, p. 20] even in the case $-1 < c < 0$.

REMARK 2.3.5. For $A = 1 - \alpha$, $B = 0$, $D = 1 - \delta$ and $E = 0$ $(0 \leq \alpha, \delta < 1)$, then $G = 0$, $H = (1-\delta)(1+c)$, $I = 0$, $J = 0$ and $L = 1 + c + k$. Theorem 2.3.2 yields the following:

COROLLARY 2.3.3. *Let $c > -1$, $1/(2+c) \leq \alpha < 1$ and $\delta := \alpha - (1-\alpha)/(1+c)$. If $f(z) \in R_\delta$, then $F(z) \in R_\alpha$.*

2.4. ANOTHER DIFFERENTIAL SUBORDINATION

LEMMA 2.4.1. *Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ and $\beta \neq 0$. Assume that*

$$(A-B)|\beta| \geq (D-E)(1+B^2) + |2B(D-E) - E\beta(A-B)|\,. \tag{2.4.1}$$

If $p(z)$ is analytic in U with $p(0) = 1$ and

$$1 + \beta z p'(z) \prec \frac{1+Dz}{1+Ez},$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

PROOF. Define the function $P(z)$ by

$$P(z) := 1 + \beta z p'(z) \tag{2.4.2}$$

and the function $w(z)$ by (2.2.4). Then $w(z)$ is meromorphic in U and $w(0) = 0$. Using (2.2.4) in (2.4.2), we get

$$P(z) = \frac{(1+Bw(z))^2 + (A-B)\beta z w'(z)}{(1+Bw(z))^2},$$