Far East J. Math. Sci. (FJMS) 15(1) (2004), 87-94

DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN ANALYTIC FUNCTIONS

ROSIHAN M. ALI, V. RAVICHANDRAN, M. HUSSAIN KHAN

and

K. G. SUBRAMANIAN

(Received June 30, 2004)

Submitted by Ravi P. Agarwal

Abstract

Let q_1, q_2 be univalent in $\Delta := \{z : |z| < 1\}$. We give some applications of first order differential superordinations to obtain sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z).$$

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\Delta := \{z : |z| < 1\}$ ar $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z)$ $a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}$ be the class of all analytic functio $f(z) = z + a_2 z^2 + \ldots$ $(z \in \Delta)$. Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times$ $\to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfi the second order superordination

2000 Mathematics Subject Classification: Primary 30C80; Secondary 30C45.

Key words and phrases: differential subordinations, differential superordinatio subordinant.

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z),$$

then p is a solution of the differential superordination (1.1). subordinate to F, then F is superordinate to f.) An analytic funct called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A ur subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.1) is be *best subordinant*. Recently Miller and Mocanu [3] obtained cor on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [3], Bulboaca [2] have con certain classes of first order differential superordinations as superordination-preserving integral operators [1]. In the present we give some applications of first order differential superordinati functions in $\mathcal{A}$.

In our present investigation, we shall need the following:

Definition 1.1 [3, Definition 2, p. 817]. Denote by Q, the se functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1.2 [2]. *Let $q(z)$ be univalent in the unit disk Δ and ϑ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re[\vartheta'(q(z))/\varphi(q(z))] \geq 0$ *for* $z \in \Delta$,

(2) $zq'(z)\varphi(q(z))$ *is starlike univalent in* Δ.

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, *with* $p(\Delta) \subseteq D$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p$ *univalent in* Δ, *then*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

2. Sandwich Theorems

By making use of Lemma 1.2, we obtain the following results.

Lemma 2.1. *Let $q(z)$ be convex univalent in Δ and $\alpha, \beta, \gamma \in \mathbb{C}$. Further assume that*

$$\Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right] \geq 0.$$

If $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $\alpha p(z) + \beta p^2(z) + \gamma z p'(z)$ is univalent in Δ, then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \alpha p(z) + \beta p^2(z) + \gamma z p'(z)$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

Proof. Define the functions ϑ and φ by

$$\vartheta(w) := \alpha w + \beta w^2 \text{ and } \varphi(w) := \gamma.$$

Clearly, $\vartheta(w)$ and $\varphi(w)$ are analytic in $\mathbb{C}$. Also

$$\Re\frac{\vartheta'(q(z))}{\varphi(q(z))} = \Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right] \geq 0$$

and the function $\gamma z q'(z)$ is starlike univalent in Δ. Lemma 2.1 now follows by an application of Lemma 1.2.

Remark 1. When $\alpha = 1$ and $\beta = 0$, Lemma 2.1 reduces to [3, Theorem 8, p. 822]. When $\alpha = \beta = 0$ and $\gamma = 1$ Lemma 2.1 reduces to [3, Theorem 9, p. 823].

By making use of Lemma 2.1, we now prove the following:

Theorem 2.2. *Let $\alpha \in \mathbb{C}$. Let $q(z)$ be convex univalent in Δ and $\Re q(z) \geq \Re\frac{\alpha - 1}{2\alpha}$. If $f \in \mathcal{A}$, $zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap Q$, $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}$ is univalent in Δ, then*

$$(1-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}$$

and $q(z)$ *is the best subordinant.*

Proof. Define the function $p(z)$ by

$$p(z) := \frac{zf'(z)}{f(z)}.$$

Then a computation shows that

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} = (1 - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z).$$

By using Lemma 2.1, we have the result.

Together with the corresponding result for differential subo
(see Ravichandran [4]), we obtain the following "sandwich result"

Corollary 2.3. *Let* $q_1(z)$ *and* $q_2(z)$ *be convex univ t*
$\alpha \in \mathbb{C}$. *Assume that* $\Re q_i(z) \geq \Re \frac{\alpha - 1}{2\alpha}$ *for* $i = 1, 2$. *If* $f \in \mathcal{A}$, $zf'($
$\mathcal{H}(1, 1) \cap Q$, $\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$ *is univalent in* Δ, *then*

$$(1 - \alpha)q_1(z) + \alpha q_1^2(z) + \alpha z q_1'(z) \prec \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}$$

$$\prec (1 - \alpha)q_2(z) + \alpha q_2^2(z) + \alpha z q_2'$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant*
dominant.

Lemma 2.4. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* α, β
assume that $\Re[\alpha \bar{\beta} q(z)] \geq 0$ *and* $zq'(z)/q(z)$ *is starlike univalent*
$p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $p(z) \neq 0$, $\alpha p(z) + \beta \frac{zp'(z)}{p(z)}$ *is univalent in* Δ,

$$\alpha q(z) + \beta \frac{zq'(z)}{q(z)} \prec \alpha p(z) + \beta \frac{zp'(z)}{p(z)}$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. The Lemma 2.4 follows from Lemma 1.2 when the functions ϑ and φ are given by $\vartheta(w) := \alpha w$ and $\varphi(w) := \beta/w$.

By making use of Lemma 2.4, we now prove the following:

Theorem 2.5. *Let* $\alpha \in \mathbb{C}$. *Let* $q(z) \neq 0$ *be univalent in* Δ. *Further assume that* $\Re[\bar{\alpha} q(z)] \geq 0$ *and* $zq'(z)/q(z)$ *is starlike univalent in* Δ. *If* $f \in \mathcal{A}, 0 \neq zf'(z)/f(z) \in \mathcal{H}(1,1) \cap Q, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$ *is univalent in* Δ, *then*

$$q(z) + \alpha \frac{zq'(z)}{q(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

implies

$$q(z) \prec \frac{zf'(z)}{f(z)}$$

and $q(z)$ *is the best subordinant.*

Proof. Theorem 2.5 follows from Lemma 2.4 by taking $p(z)$ to be the function given by $p(z) := zf'(z)/f(z)$.

Together with the corresponding result for differential subordination (see Ravichandran and Darus [6]), we obtain the following:

Corollary 2.6. *Let* $\alpha \in \mathbb{C}$. *Let* $q_i(z) \neq 0$ $(i = 1, 2)$ *be univalent in* Δ. *Further assume that* $\Re[\bar{\alpha} q_i(z)] \geq 0$ *for* $i = 1, 2$ *and* $zq_i'(z)/q_i(z)$ $(i = 1, 2)$ *is starlike univalent in* Δ. *If* $f \in \mathcal{A}, 0 \neq zf'(z)/f(z) \in \mathcal{H}(1,1) \cap Q, (1-\alpha)\frac{zf'(z)}{f(z)}$ $+ \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$ *is univalent in* Δ, *then*

$$q_1(z) + \alpha \frac{zq_1'(z)}{q_1(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q_2(z) + \alpha \frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and dominant.*

By making use of Lemma 2.4, we obtain the following:

Theorem 2.7. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q(z)$ *be univalent in* Δ. *If* $f \in \mathcal{A}$, $0 \neq z^2 f'(z)/f^2(z) \in \mathcal{H}(1, 1) \cap Q, \frac{(zf)''(z)}{f'(z)} -$ *is univalent in* Δ, *then*

$$\frac{zq'(z)}{q(z)} \prec \frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$$

implies

$$q(z) \prec \frac{z^2 f'(z)}{f^2(z)}$$

and $q(z)$ *is the best subordinant.*

Together with the corresponding result for differential subor (see Ravichandran [4]), we obtain the following:

Corollary 2.8. *Let* $q_i(z) \neq 0$ *be univalent in* Δ *and* $zq_i'(z)/$ *starlike univalent in* Δ *for* $i = 1, 2$. *If* $f \in \mathcal{A}, 0 \neq z^2 f'(z)/f^2(z) \in \mathcal{H}($ $\frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}$ *is univalent in* Δ, *then*

$$\frac{zq_1'(z)}{q_1(z)} \prec \frac{(zf)''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec \frac{zq_2'(z)}{q_2(z)},$$

implies

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and b*
dominant.

Lemma 2.9. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q^2(z)$ *be starl*
univalent in Δ. *If* $p(z) \in \mathcal{H}(q(0), 1) \cap Q$, $p(z) \neq 0$, $zp'(z)/p^2(z)$ *is univale*
in Δ, *then*

$$\frac{zq'(z)}{q^2(z)} \prec \frac{zp'(z)}{p^2(z)}$$

implies $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. Lemma 2.9 follows from Lemma 1.2 when $\vartheta(w) := 0$ a
$\varphi(w) := 1/w^2$.

Theorem 2.10. *Let* $q(z) \neq 0$ *be univalent in* Δ *and* $zq'(z)/q^2(z)$
starlike univalent in Δ. *If* $f \in \mathcal{A}, 0 \neq zf'(z)/f(z) \in \mathcal{H}(1,1) \cap Q$, $\frac{1+zf''(z)/f'(}{zf'(z)/f(z)}$
is univalent in Δ, *then*

$$1 + \frac{zq'(z)}{q^2(z)} \prec \frac{1 + z''f(z)/f'(z)}{zf'(z)/f(z)}$$

implies $q(z) \prec zf'(z)/f(z)$ *and* $q(z)$ *is the best subordinant.*

Proof. The result follows from Lemma 2.9 by taking $p(z) = zf'(z)/f($

Together with the corresponding result for differential subordinati
(see Ravichandran and Darus [5]), we obtain the following:

Theorem 2.11. *Let* $q_i(z) \neq 0$ *be univalent in* Δ *and* $zq_i'(z)/q_i^2(z)$
starlike univalent in Δ *for* $i = 1, 2$. *If* $f \in \mathcal{A}$, $0 \neq zf'(z)/f(z) \in \mathcal{H}(1, 1) \cap$
$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)}$ *is univalent in* Δ, *then*

$$1 + \frac{zq_1'(z)}{q_1^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq_2'(z)}{q_2^2(z)}$$

implies $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$ *and* $q_1(z)$ *and* $q_2(z)$ *are respectiv*
the best subordinant and the best dominant.

Acknowledgement

Research of R. M. Ali and V. Ravichandran are respectively supp by a Universiti Sains Malaysia Fundamental Research Grant and a doctoral fellowship.

References

[1] T. Bulboaca, A class of superordination-preserving integral operators, Indag. New Ser. 13(3) (2002), 301-311.

[2] T. Bulboaca, Classes of first-order differential superordinations, Demonstr. 35(2) (2002), 287-292.

[3] S. S. Miller and P. T. Mocanu, Subordinants of differential superordin Complex Variables 48(10) (2003), 815-826.

[4] V. Ravichandran, Certain applications of first order differential subordinati East J. Math. Sci. (FJMS) 12(1) (2004), 41-51.

[5] V. Ravichandran and Maslina Darus, On a criteria for starlikeness, Internat J. 4(2) (2003), 119-125.

[6] V. Ravichandran and Maslina Darus, On a class of α-convex func , J Appl. 2(1) (2004), 17-25.

Rosihan M. Ali and V. Ravichandran
School of Mathematical Sciences
Universiti Sains Malaysia
11800 USM Penang, Malaysia
e-mail: rosihan@cs.usm.my; vravi@cs.usm.my

M. Hussain Khan
Department of Mathematics
Islamiah College, Vaniambadi 635 751, India
e-mail: khanhussaff@yahoo.co.in

K. G. Subramanian
Department of Mathematics
Madras Christian College
Tambaram, Chennai 600 059, India
e-mail: kgsmani@vsnl.net