APPLICATION ON SPIRAL DESIGN

S. H. YAHAYA, J. M. ALI

School of Mathematical Sciences, University Science of Malaysia, Penang, Malaysia.
saifudinhz@hotmail.com

School of Mathematical Sciences, University Science of Malaysia, Penang, Malaysia
jamaluma@cs.usm.my

Abstract: In this paper, we introduce a G2 planar cubic transition curve by using Habib's approach. This approach purposed in 2002 where the parameters are more complete and easier to read. We presented very short algorithm same as Habib method and it's comfortable in practical, for example highways or railways designing. Baass [1] identified five transition curve cases in highway design, one of the cases is circle to circle where one circle lies inside the other with a C transition and we shown this completely by numerical example.

Keywords: G2 transition, cubic Bézier curve, spiral, curvature

1. Introduction

Spiral is used to design of highways or railways. Spiral has several advantages of containing neither inflection points, singularities nor curvature extreme. Such curves are useful for transition between two circles. Transition curves are used in several Computer Aided Geometric Design (CAGD) or Computer Aided Design (CAD) applications. The continuity in the transition curves usually G^2 continuity because curves have considered their position and curvature continuity. Parametric cubic curves are popular in CAD or CAGD because they are the lowest degree polynomial curves that allow the inflection points where curvature is zero and also suitable for G^2 continuity. The Bézier form of parametric cubic curve is usually used in CAGD or CAD because of their geometric and numerical properties. The planar parametric cubic curve has eight degrees of freedom that can be match with the eight values of G^2 Hermite data. Meek and Walton [3] have considered planar G^2 transition between two circles with a fair curve. They showed that there is no curvature extremum in the case S-shaped and a single curvature extremum exists in the C-shaped.

The objectives in this paper are to find the connection between two circles when the circle lies inside the other circle by using Habib's method where the connection is a fair curve and single spiral, next to examine the shapes of curves by using Mathematica and finally, to determine equations of S-transition and C-transition with theoretical analysis and also shows some application of spiral design.

The reminder of this paper is organized as follows. Section 2 explains about discussion of notation and convention used in this paper followed by section 3 with theoretical analysis background. Section 4 and section 5 with a new result of S-shaped and C-shaped transitions. Then, section 6 with a special case where circle lies inside the other circle followed by section 7 numerical examples and its applications. Section 8 includes conclusions and future works.

2. Notation and Convections

Consider Cartesian coordinate system with x and y-axes. The angles measured by clockwise. Points and vectors may also be indicated using the ordered pair notation, example (x, y). In additions, the components of a vector **A** may be denoted (A_x, A_y), or in the case as (A_{0x}, A_{0y}). The dot product of two vectors, **A** and **B** is denoted as **A.B** and norm of vector **A** is denoted as $\|\mathbf{A}\| = \sqrt{A.A}$. A planar parametric curve is defined by the set of points $Z(t) = (x(t), y(t))$ for t in some given interval of real line. The cross product of two vectors, **A** and **B** is denoted as **A×B** or **A^B** as Juhász [2], by **A×B** = $A_x B_y - A_y B_x$. The tangent vector of a plane curve $Z(t)$ is given by $Z'(t)$. If $Z'(t) \neq 0$, and then curvature is defined as [3]:

$$\kappa(t) = \frac{Z'(t) \times Z''(t)}{\|Z'(t)\|^3} \tag{1}$$

Differential of Eq. (1) yields

$$\kappa'(t) = \frac{v(t)}{\|Z'(t)\|^5} \tag{2}$$

where

$$v(t) = \{Z'(t).Z'(t)\} \tfrac{d}{dt}\{Z'(t) \times Z''(t)\} - 3\{Z'(t) \times Z''(t)\}\{Z'(t).Z''(t)\} \tag{3}$$

3. Background

First, we consider a parametric cubic curve; the first and second derivatives are, represented in Bézier form as [3]:

$$Z(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3, 0 \le t \le 1 \tag{4}$$

and

$$Z'(t) = 3(P_1 - P_0)(1-t)^2 + 6(P_2 - P_1)(1-t)t + 3(P_3 - P_2)t^2, 0 \le t \le 1 \tag{5}$$

$$Z''(t) = (6P_2 - 12P_1 + 6P_0)(1-t) + (6P_3 - 12P_2 + 6P_1)t, 0 \le t \le 1 \tag{6}$$

where $P_i, i = 0,......3$ are the control points as refer [3].

4. S-Shaped Transition Curve

Now, we consider an S-shaped transition curve between two circles. We assume two non-enclosing circles Ω_0, Ω_1 with centers C_0, C_1 and radii r_0, r_1. These circles do not intersect where $r_0 + r_1 < r(=\| c \|)$ with $c = C_1 - C_0$ and $r_1 = \lambda^2 r_0, 0 \le \lambda \le 1$ follow in [4]. We use (4) and properties of G^1 continuity require that $t_0 = (\cos\alpha, \sin\alpha)$ and $t_1 = (-\sin\alpha, \cos\alpha)$ with:

$$P_0 = C_0 + r_0 t_0, P_1 = P_0 + h t_1, P_3 = C_1 - r_1 t_0, P_2 = P_3 - k t_1 \tag{7}$$

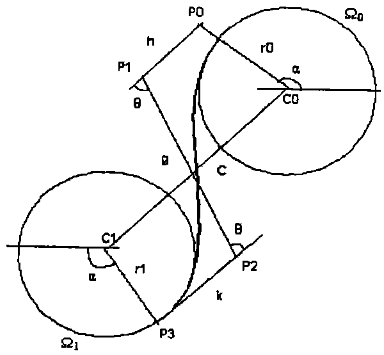

Figure 1. S-shaped cubic Bézier transition curve.

Let $\|P_2 - P_1\| = g$ to get $P_2 - P_1 = -g(\sin(\alpha+\theta), -\cos(\alpha+\theta))$ (8)

We know that $P_2 - P_1 = c - (r_0 + r_1)t_0 - (h+k)t_1$ from (7). (9)

From (8) and (9), we get an orthogonal matrix of order 2 and named as:

$$A = \begin{bmatrix} r_0 + r_1 - g\sin\theta & -(h+k+g\cos\theta) \\ h+k+g\cos\theta & r_0 + r_1 - g\sin\theta \end{bmatrix} \tag{10}$$

then (8), (9) and (10) can be written as:

$$At_0' = c \text{ or } t_0' = \frac{A'c}{r^2} \tag{11}$$

When $\|t_0\| = 1$, its norms in (11) to obtain:

$$g^2 + (h+k)^2 + (r_0 + r_1)^2 + 2g(h+k)\cos\theta - 2g(r_0 + r_1)\sin\theta = r^2 \tag{12}$$

For G^2 continuity, require $\kappa(0) = \frac{1}{r_0}$ and $\kappa(1) = \frac{1}{r_1}$, we use (1), (5), (6) and its requirements to have:

$$h = \sqrt{\frac{2gr_0\sin\theta}{3}},\ k = \sqrt{\frac{2gr_1\sin\theta}{3}} \tag{13}$$

Then, choose $h = \frac{4r_0\tan\theta}{9}$ as in [3] where reduces (12) to the quadratic $f(q) = 0$ which is $q = \tan^2\theta$:

$$f(q) = 64r_0^2q^2 - 32r_0^2(9\lambda^2 - 15\lambda + 1)q + 729(r_0^2(\lambda^2+1)^2 - r^2) \tag{14}$$

This has just one positive root since $r_0 + r_1 < r$. Note the quadratic $\mu(\tau)$ in [3]:

$$\mu(\tau) = \{6(3\lambda^2 - 5\lambda + 1) - \frac{729d}{16r_0^2}\}\tau^2 - 2(9\lambda^2 - 15\lambda + 5)\tau + 4$$
$$\tau = \cos^2\theta, d = r^2 - r_0^2(\lambda^2 + 1)^2 > 0 \quad (15)$$

We can see in (15), $\mu(0) = 4 > 0$ and $\mu(1) = -\frac{729d}{16r_0^2} < 0$ which means that $\mu(\tau)$ has exactly one zero in the interval $0 < \tau < 1$ because $\mu(\tau)$ is a quadratic [3]. Next, the transition curve is spiral shown in [3] and [4].

Theorem 1. *If $r_1 \le r_0 \le 36r_1$, then G^2cubic S-shaped transition curve of the form (4) with (7) between two circles is a unique spiral where θ and α are determined by (14), (11) and g, h, k are given by:*

$$h = \frac{4r_0 \tan\theta}{9}, k = \lambda h, g = \frac{3h^2}{2r_0 \sin\theta} \quad (16)$$

5. C-Shaped Transition Curve

Here, we consider a C-shaped transition curve $Z(t)$ of the form (4) from circles Ω_0 to Ω_1, centers C_0, C_1 and radii r_0, r_1. This shaped has $(\alpha, \beta) = (\frac{\pi}{2} - \theta + \mu, \frac{\pi}{2} - \theta - \mu)$ with:

$$P_0 = C_0 + r_0(\cos\alpha, \sin\alpha), P_1 = P_0 + h(-\sin\alpha, \cos\alpha)$$
$$P_3 = C_1 - r_1(\cos\beta, -\sin\beta), P_2 = P_3 + k(\sin\beta, \cos\beta) \quad (17)$$

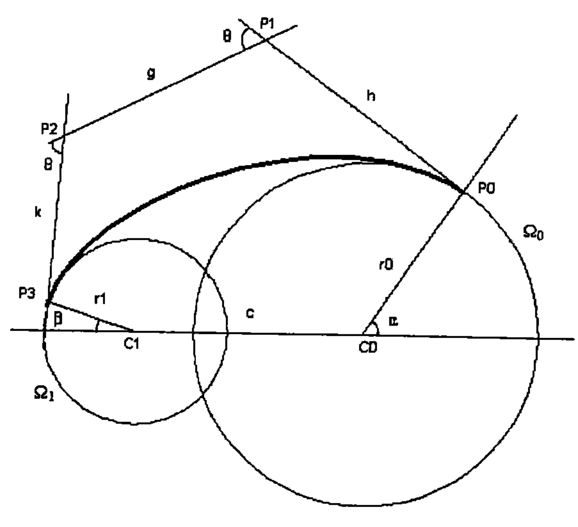

Figure 2. C-shaped cubic Bézier transition curve.

Let $||P_2 - P_1|| = g$ to get $P_2 - P_1 = -g(\cos\mu, \sin\mu)$ (18)

From (17) and (18), we get an orthogonal matrix of order 2 named as:

$$B=\begin{bmatrix}(r_0+r_1)\sin\theta-(h+k)\cos\theta-g & -((r_0-r_1)\cos\theta+(h-k)\sin\theta)\\ (r_0-r_1)\cos\theta+(h-k)\sin\theta & (r_0+r_1)\sin\theta-(h+k)\cos\theta-g\end{bmatrix} \tag{19}$$

Rewrite (19) as:

$$B(\cos\mu,\sin\mu)'=c \text{ or } (\cos\mu,\sin\mu)'=\frac{B'c}{r^2} \tag{20}$$

When $\|(\cos\mu,\sin\mu)\|=1$, it's a norm in (20) to obtain:

$$g^2+h^2+k^2+r_0^2+r_1^2+2g(h+k)\cos\theta-2g(r_0+r_1)\sin\theta+2(hk-r_0r_1)\cos2\theta \\ -2(kr_0+hr_1)\sin2\theta=r^2 \tag{21}$$

For G^2 continuity, require $\kappa(0)=\frac{1}{r_0}$ and $\kappa(1)=\frac{1}{r_1}$, we use (1), (5), (6) and the requirements to have:

$$h=\sqrt{\frac{2gr_0\sin\theta}{3}},\ k=\sqrt{\frac{2gr_1\sin\theta}{3}} \tag{22}$$

Then, a purposed value of $h=\frac{2r_0\tan\theta}{3}$ [3], reduces (21) to cubic $f(q)=0$ where $q=\tan^2\theta$:

$$f(q)=4r_0^2q^3+8r_0^2(1-\lambda^2)q^2+(r_0^2(1-\lambda)^2(13+18\lambda+19\lambda^2)-9r^2)q \\ +9(r_0^2(1-\lambda^2)^2-r^2) \tag{23}$$

Note that the constant term of $f(q)$ is negative since the smaller circle Ω_1 is not contained in the larger circle Ω_0. The analysis in [3] for $\frac{2}{3}\le\lambda\le1$ is insufficient on the unique positive zero of their cubic equation $\mu(\tau)$:

$$\mu(\tau)=4\lambda(2-3\lambda)\tau^3+(-\frac{9d}{r_0^2}-8\lambda+20\lambda^2)\tau^2-4(1+2\lambda^2)\tau+4 \\ d=r^2-r_0^2(1-\lambda^2)^2>0 \\ \tau=\cos^2\theta \tag{24}$$

Now, $\mu(0)=4>0$ and $\mu(1)=-\frac{9d}{r_0^2}<0$, so (24) has an odd number (at least one) solution in [0,1]. In additions, the transition curve is spiral shown in [3].

Theorem 2. *The G^2 cubic C-shaped transition curve of form (4) with (17) between two circles is a unique cubic curve with the single interior curvature extremum where θ and μ are determined by (23), (20) and g, h, k are given by:*

$$h = \frac{2r_0 \tan\theta}{3}, k = \lambda h, g = \frac{3h^2}{2r_0 \sin\theta} \tag{25}$$

6. Circle to Circle (One circle inside the other)

This case purposed by Walton and Meek [5] where a part of highways design. But, this case not always has a solution. Here, we want get a connection between two circles by a fair curve and single spiral. We know that, these case same as C-shaped transition curve but the control points, P_i where i = 0…3, θ and μ are different. Walton and Meek [5], one of control points, P_0 and θ are assumed. This is not a good condition because we don't know that our assumption is corrected. We must try one by one to get a solution, so a lot of times we need here. By the way, Habib and Sakai [4] purposed a method that can solve our problem just now. Now, we use this method to find a way for our problems.

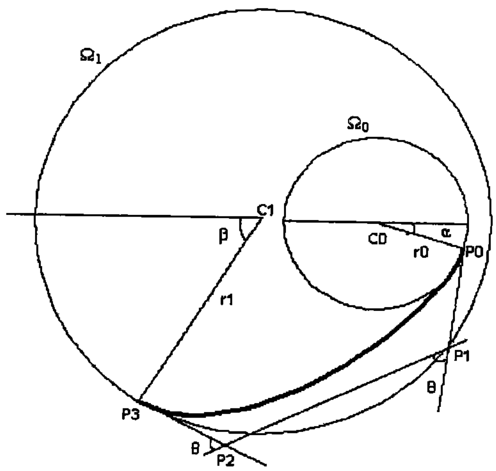

Figure 3. Circle to circle with fair curve.

In Figure 3, we use same rules as section 5. Firstly, we use (23) to determine θ, and then we compute g, h, and k from Theorem 2. Finally, we find μ in (20). Notice that, μ an important value because α and β are depended on it. Fair curve means that $Z(t)$ is self-intersecting in the interval $(-\infty,\infty)$, has no inflection points, its curvature has either a single extremum or three extrema (maximum, minimum, maximum) in the interval $(-\infty,\infty)$. Since the curvature of $Z(t)$ approaches zero as t tends to $\pm\infty$, a single curvature extremum, namely a minimum is guaranteed in the interval [0, 1] by requiring that curvature decreases at $t=0$ and increases at $t=1$. For examples, $\kappa'(0) \le 0$ and $\kappa'(1) \ge 0$ shown in[3].

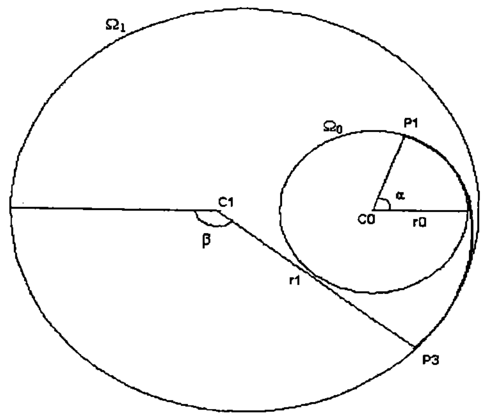

Figure 4. Circle to circle with single spiral.

Figure 4 shows about circle into circle where circle inside other circle by connection a single spiral. Here, we use same equation as before, just we control a value of μ because μ gave an effort for α and β. At control point, P_3 the curvature is zero because curvature is not exists at circle. Single spiral means that the curvatures either monotonically decreasing or increasing at $t = 0$ and $t = 1$. For example, $\kappa'(0) \leq 0$ and $\kappa'(1) \leq 0$ mean no extremum here.

Theorem 3. *The G^2 cubic C-shaped transition curve of form (4) with (17) between two circles (circle inside other circle) by single spiral where θ and μ are determined by (23), (20) and $\mu > 90^0$ and next g, h, k are given by:*

$$h = \frac{2r_0 \tan\theta}{3}, k = \lambda h, g = \frac{3h^2}{2r_0 \sin\theta} \tag{26}$$

7. Numerical Examples

This section gives two numerical examples to proof our theoretical analysis.
Example 1 (Figure 5): $r_0 = 1$, $r_1 = 0.2$, $C_0 = (1, 0)$, $C_1 = (-1, 0)$

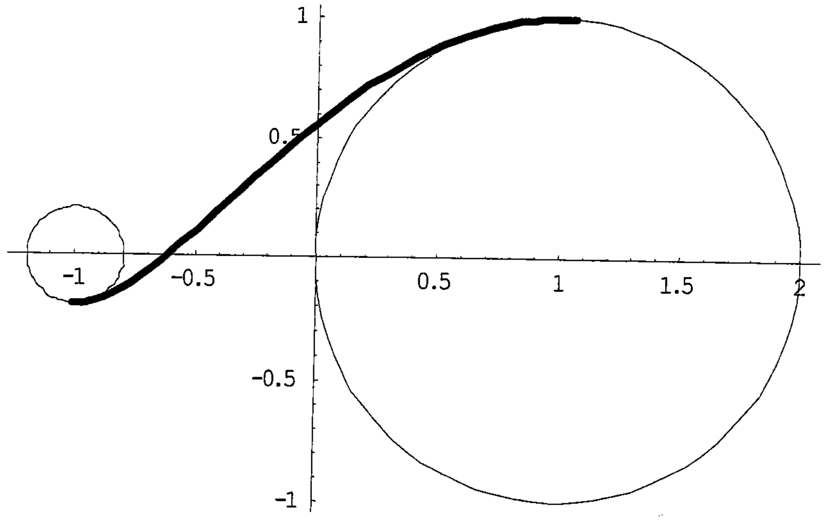

Figure 5. Graph of $Z(t)$.

Example 2 (Figure 6): $r_0 = 1.5$, $r_1 = 0.8$, $C_0 = (1, 0)$, $C_1 = (-1, 0)$

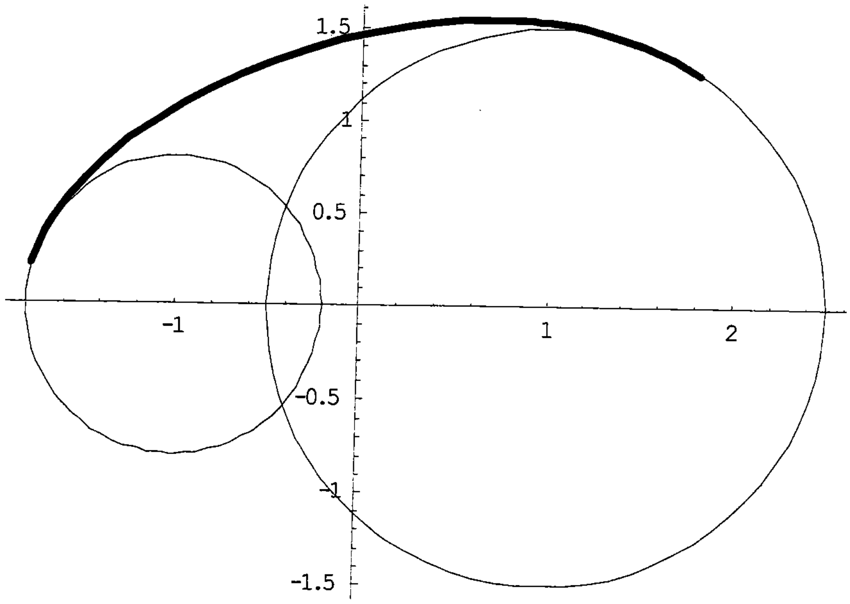

Figure 6. Graph of $Z(t)$.

8. Application

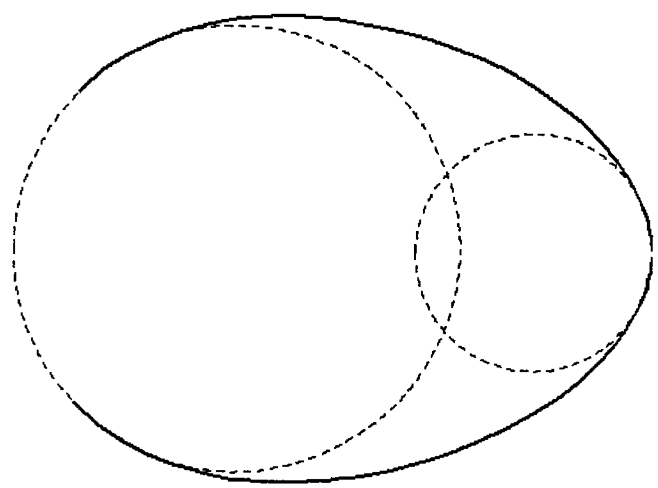

Figure 7. Cam-like cross section by Habib's method.

Figure 7 shows about cam-like cross section by Habib's method. This cam-cross is composed of two circular arcs joined by an upper and a lower fair C-shaped cubic curve. Actually, this figure already shown by Walton and Meek [3].

9. Conclusion

Combination Habib and Walton method give us the solution for circle inside other circle. This combination is very good because it's simpler than before. Idea from Habib and concepts from Walton and Meek gave our spiral design more reasonable and practical. In the future, researchers can analyze by using Mat lab with friendly user. It's works very well because can shows a spiral design anytime.

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