FUNCTIONAL SCATTERED DATA G^1 INTERPOLATION WITH SUM OF SQUARES OF PRINCIPAL CURVATURES

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Scattered data interpolation problem deals with fitting of a smooth surface to a set of non-uniformly distributed data points which extends to all positions in a domain. For instance, given a set of scattered data $V = \{(x_i, y_i), i=1,...,n\} \in R^2$ over a polygonal domain φ and a corresponding set of real numbers $\{z_i\}_{i=1}^{n}$, we wish to construct a surface S which has continuous varying tangent plane everywhere (typically G^1) such that $S(x_i,y_i) = z_i$. Specifically in this paper, the polynomial surfaces being considered belong to G^1 quartic Bézier functions over a triangulated domain. In order to construct the surface, we first need to construct the triangular mesh spanning over the unorganized set of points, V which will then have to be covered with Bézier patches with coefficients satisfying the G^1 continuity between patches and the sum of squares of principal curvatures. Examples are also presented to show the extend of our proposed method.

Keywords : Scattered data interpolation, G^1 continuity, Quartic triangular Bézier patch, Sum of squares of principal curvatures

1. Introduction

Scattered data interpolation refers to the problem of fitting smooth surfaces through a non-uniform distribution of data points. In practise, this subject is very important in various sciences and engineering where data are often measured or generated at sparse and irregular positions. The goal of interpolation is to construct underlying functions which may be evaluated at certain set of positions.

There are three principal sources of scattered data [Lee et al.(1997)] : measured value of physical quantities (such as in geology and meteorology), experimental results (in sciences and engineering) and computational values (in various applications of computer graphics and vision with functional data).

For the purpose of this paper, we will only focus on the scattered data from functions to be used by our proposed method. We will construct a G^1 continuous surface which interpolates these given functional data. The geometric continuity needed between adjacent parametric patches especially in the case of Bézier patches has been given attention due to the free shape parameters which it provide and can be used to construct and modify very complicated geometric objects. In practice, geometric continuity avoids the usual dependence on the parameterisation of the constructed patches. The conditions required for patches to be G^n-continuous and subsequent construction of the corresponding G^n-continuous surfaces are amongst important topics in Computer Aided Geometric Design [Ye et al.(1996)].

Although several papers have dealt with G^n interpolating surfaces [e.g. Loop(1994); Hahmann & Bonneau(1999)], but very few are concerned with the scattered data interpolation as compared to the one on C^n-surfaces. We thus propose a method to construct an interpolating G^1 surface using a set of scattered data in R^3 as well as provide examples to test the ability of this proposed scheme.

If we simply use G^1 continuity conditions, the resulting surfaces might have an undesired undulation or over flatness. To overcome this problem, several global optimal fairness criteria of surfaces have been introduced in various literatures. Amongst the earliest surface fairing aimed at minimizing fairness criteria as an anology to the one for curve was based on the strained energy in a thin elastic plate which is related to the total curvature in a surface. Thus, the resulting surface can then be approximated using its geometric and curvature related properties [Nowacki & Reese(1983); Hagen & Schulze(1987)].

In this paper, we consider the method of sum of squares of principle curvatures, $k_1^2 + k_2^2$ where k_1 and k_2 are the principle curvatures [Halstead et al.(1993); Kobbelt(1997)]. We can approximate the objective function, $k_1^2 + k_2^2$ using a corresponding quadratic form and find the extremum of the integral function with respect to the G^1 continuity conditions along the shared edges as a constrained function.

Quadratic functions can generate good surface fit to the data points. The range of applications can be extended by using iterative procedures which will successively improve the parameter values assigned to these data points.

We have chosen quartic Bézier triangular patches since degree 4 is necessary and sufficient to satisfy the patch boundary interpolation constraints while at the same time allows some control points in the interior of the patch to have certain freedom to be used for surface fairing.

This paper is organized as follows: Relevant background materials including the definition of quartic Bézier form of triangular patches and G^1 continuity conditions between two adjacent patches at vertices of triangle meshes are given in section 2; a method to construct surfaces using the sum of squares of principle curvature is covered in section 3; examples are given in section 4 and finally in section 5, the concluding remarks will be given.

2. Background

Let D be a two-dimensional region with (x,y) as the global coordinates system and z_i, the corresponding height at (x_i,y_i) for i=1,2,...n. (x_i, y_i) will be known as the data site while (x_i, y_i, z_i) as the data point. D is divided into N triangle elements, D_e, e = 1, 2, ..., N and (u,v), u,v ε [0,1] are the local coordinates in each triangle element.

The following known facts on triangular Bézier surfaces may be found in the literatures on curves and surfaces for computer aided geometric design [see Farin(1992)].

2.1 *Bernstein polynomial*

The n^{th} degree Bernstein polynomials over a triangle are defined by

$$B^n_{ijk}(u,v,w) = \frac{n!}{i!\,j!\,k!}u^i v^j w^k \qquad (1)$$

where u,v,w are barycentric coordinates such that $u + v + w = 1$ and $i+j+k = n$.

2.2 *Bézier triangular patches*

The n^{th} degree Bézier triangular patch is defined by

$$P(u,v,w) = \sum_{i+j+k=n} b_{ijk} B^n_{ijk}(u,v,w) \qquad (2)$$

where b_{ijk} are called Bézier ordinates of P. In this paper, we will use quartic Bézier triangular patches (with n = 4). Each patch is defined by a control net of 15 vertices as shown in Figure 1.

With B_{ijk} as the height of a control point, a quartic Bézier triangular patch can be represented by

$$S(u,v,w) = \sum_{i+j+k=4} B_{ijk} B^4_{ijk}(u,v,w) \qquad (3)$$

where $B_{ijk} = (i/4, j/4, k/4, b_{ijk})$.

2.3 *G^1 continuity at vertices of triangular patches*

Let T be a triangle on the x-y plane with vertices $V_1(x_a, y_a)$, $V_2(x_b,y_b)$, $V_3(x_c,y_c)$ and barycentric coordinates u, v and w such that any point V on T can be expressed as

$$V = uV_1 + vV_2 + wV_3, \text{ where } u+v+w = 1 \text{ and } u,v,w \geq 0. \qquad (4)$$

S interpolates the Bézier ordinates B_{400}, B_{040} and B_{004} at the vertices of V_1, V_2, V_3 of T respectively since barycentric coordinates of these vertices are (1,0,0), (0,1,0) and (0,0,1).

We estimate the normal at each vertex using the surrounding triangles with a quadratic approximation function,

$$F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f \qquad (5)$$

where a, b, c d, e and f are unknown coefficients.

Let the vertex P_0 of the triangular mesh surrounded by vertices $\{P_1, P_2, \ldots, P_k\}$ and the height of the vertices $\{P_0, P_1, \ldots, P_k\}$ be represented by $\{z_0, z_1, \ldots, z_k\}$. We turn our attention to the vertex represented by (1,0,0) which corresponds to $P_0(x_0,y_0)$. Other vertices can be dealt with in a similar way. We can derive the first order partial derivative at P_0 as

$$F_x(x_0,y_0) = 2ax_0 + by_0 + d \quad and \quad F_y = bx_0 + 2cy_0 + e \tag{6}$$

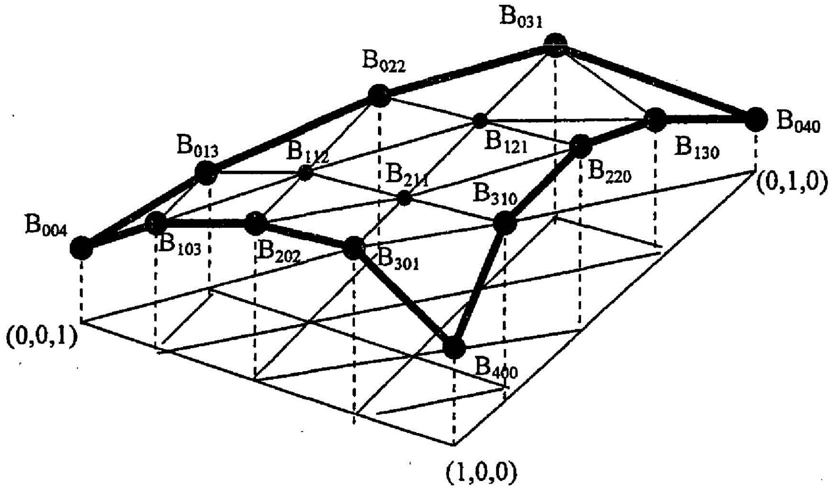

Figure 1: Quartic Bézier triangular patch and its control net

If we substitute the height of vertices $z_0, z_1, \ldots, z_k$ into (5), we obtain a linear system

$$UC = Z \tag{7}$$

where

$$U = \begin{bmatrix} x_0^2 & x_0y_0 & y_0^2 & x_0 & y_0 & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_k^2 & x_ky_k & y_k^2 & x_k & y_k & 1 \end{bmatrix}, \quad C = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} z_0 \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_k \end{bmatrix}.$$

The values of a, b, c, d, e and f can be obtained by using the least square method

$$C = (U^TU)^{-1}U^TZ, \tag{8}$$

and the values of F_x and F_y at vertex P_0 can then be calculated.

Next, G^1 continuity conditions at the vertices of a triangular patch can be determined by using the directional derivatives along the edges at vertex (1,0,0). Assuming that V, V_1, V_2 and V_3 are located as shown in Figure 2. The vertices V_1, V_2 and V_3 correspond to the barycentric coordinates (1,0,0), (0,1,0) and (0,0,1) respectively.

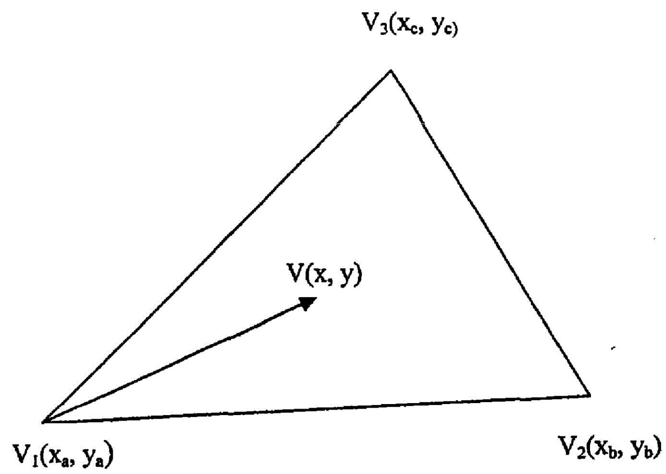

Figure 2 : Vertices of a triangle element

V can be represented as in (4), that is,

$$(x,y) = u(x_a, y_a) + v(x_b,y_b) + w(x_c, y_c).$$

Let e_{12} = (-1,1,0) and e_{13}= (-1,0,1) represent the direction of edges V_1V_2 and V_1V_3 respectively. The directional derivatives along e_{12} and e_{13} at V_1 are

$$D_{e_{12}}S(1,0,0) = (\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u})F_x(x_a) + (\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u})F_y(x_a)$$
$$= (x_b - x_a)F_x(x_a) + (y_b - y_a)F_y(x_a) \tag{9}$$

$$D_{e_{13}}S(1,0,0) = (\frac{\partial x}{\partial w} - \frac{\partial x}{\partial u})F_x(x_a) + (\frac{\partial y}{\partial w} - \frac{\partial y}{\partial u})F_y(x_a)$$
$$= (x_c - x_a)F_x(x_a) + (y_c - y_a)F_y(x_a) \tag{10}$$

From the same Bézier triangular patch, we also have

$$D_{e_{12}}S(1,0,0) = 4(-B_{400} + B_{310}) \tag{11}$$
$$D_{e_{13}}S(1,0,0) = 4(-B_{400} + B_{301}) \tag{12}$$

From (9), (10), (11) and (12), we obtain

$$B_{310} = \frac{1}{4}D_{e_{12}}S(1,0,0) + B_{400} \tag{13}$$
$$B_{301} = \frac{1}{4}D_{e_{13}}S(1,0,0) + B_{400} \tag{14}$$

Hence, using a similar method with the other vertices of the same triangle, we can determine B_{130}, B_{031}, B_{103} and B_{013} by the heights of data sites B_{040} and B_{004}.. Thus, 9 control points B_{400}, B_{040}, B_{004}, B_{310}, B_{301}, B_{130}, B_{031}, B_{103} and B_{013} are obtained with 6 more control points left to be determined.

2.4 *G^1 continuity between adjacent patches*

Two patches with a common boundary curve satisfy G^1 continuity if both have continuously varying tangent plane along the common curve. Figure 3 shows Bézier control points of two adjacent quartic Bézier triangular patches. H_0 and H_4 are given points (i.e. vertices of the triangular patches). G_0, F_0, H_1,

G_3, H_3 and F_3 are points which are obtained by the patch gradients while G_1, F_1, H_2, G_2 and F_2 are the unknown points.

We only have to consider the control polygon of $\{H_i\}$, i = 0, 1,...4 as the common boundary curve and the two rows $\{G_i\}$,$\{F_i\}$, i= 0, 1...3 which consist of the control points in each patch. Details of derivation with regard to the G^1 conditions can be found in (Farin, 1992).

If the heights of $\{G_i\}$, $\{H_i\}$, $\{F_i\}$ are $\{g_i\}$, $\{h_i\}$ and $\{f_i\}$ respectively, the conditions satisfying G^1 continuity between the two adjacent patches can be written as

$$\alpha g_0 + (1-\alpha)f_0 = \beta h_0 + (1-\beta)h_1 \quad (15)$$

$$\alpha g_1 + (1-\alpha)f_1 = \beta h_1 + (1-\beta)h_2 \quad (16)$$

$$\alpha g_2 + (1-\alpha)f_2 = \beta h_2 + (1-\beta)h_3 \quad (17)$$

$$\alpha g_3 + (1-\alpha)f_3 = \beta h_3 + (1-\beta)h_4 \quad (18)$$

where α and β are constants.

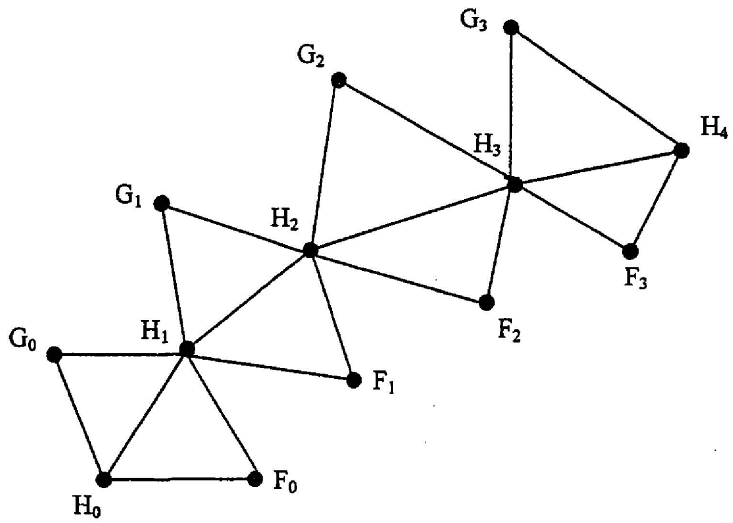

Figure 3 : Bézier control points of two adjacent quartic Bézier triangular patches with a common boundary curve

Since the values of g_0, f_0, h_0, h_1, g_3, h_3, h_4 and f_3 are already known, α and β can thus be determined from (15) and (18). (16) and (17) can also be written as

$$\begin{bmatrix} \alpha & 1-\alpha & 0 & 0 & -1+\beta \\ 0 & 0 & \alpha & 1-\alpha & -\beta \end{bmatrix} \begin{bmatrix} g_1 \\ f_1 \\ g_2 \\ f_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} \beta h_1 \\ (1-\beta)h_3 \end{bmatrix}. \quad (19)$$

We obtain a linear system Ax = b where A is a (2 x 5) matrix with rank(A) = 2 and we can solve this system in term of 3 free parameters, f_1, f_2 and h_2, that is,

$$g_1 = \frac{1}{\alpha}[\beta h_1 + (1-\beta)h_2 - (1-\alpha)f_1],$$

$$g_2 = = \frac{1}{\alpha}[\beta h_2 + (1-\beta)h_3 - (1-\alpha)f_2]. \tag{20}$$

If these free parameters can be determined adequately, all the control points B_{ijk} in Bézier control net will be known for the surface to be G^1 continuous.

In general, we can obtain (16) and (17) on all shared edges over the whole triangular mesh. We can form a linear system **Ax=b** where A is a (M x N) matrix (with M<H) and rank(A) = M.

3. Surface With Sum Of Squares Of Principle Curvartures

We will take as a surface fairing objective, the integral of the sum of squares of principle curvartures over a smooth surface S

$$I(S(x,y)) = \iint_D k_1^2 + k_2^2 \; dxdy \tag{21}$$

where k_1 and k_2 are the principle curvartures and D represents the whole surface.

We can approximate (21) by

$$I(S(x,y)) = \iint_D S_{xx}^2 + 2S_{xy}^2 + S_{yy}^2 \; dxdy \; . \tag{22}$$

Our goal is to find the function S(x,y) which will minimize the integral I(S). We will assume that the whole surface S is constructed by a collection of Bézier triangular patches with each patch defined as in (3). We can represent each patch as a convex combination of 15 control points

$$S^t(u,v,w) = \sum_{k=1}^{15} B_k^t \phi_k(u,v,w), \tag{23}$$

where $t = 1, 2 \ldots M$ (M is the number of triangles in a mesh),

$\{B_1^t, B_2^t, B_3^t, \ldots, B_{15}^t\} = \{B_{400}^t, B_{310}^t, B_{130}^t, B_{040}^t, B_{301}^t, B_{031}^t, B_{103}^t, B_{013}^t, B_{004}^t, B_{220}^t, B_{211}^t, B_{121}^t, B_{202}^t, B_{112}^t, B_{022}^t\}$

$\{\phi_1, \phi_2, \ldots, \phi_{15}\} = \{u^4, 4u^3v, 4uv^3, v^4, 4u^3w, 4v^3w, 4uw^3, 4vw^3, w^4, 6u^2v^2, 12u^2vw, 12uv^2w, 6u^2w^2, 12uvw^2, 6v^2w^2\}$.

Let $I(S^t(u,v,w))$ be a functional defined in a triangle element D_t, such that we can rewrite (22) as

$$I(S(x,y)) = \sum_{t=1}^{M} I(S^t(u,v,w))$$

$$= \sum_{t=1}^{M} \iint_{D^t} (S_{xx}^t)^2 + 2(S_{xy}^t)^2 + (S_{yy}^t)^2 \; dx\,dy \tag{24}$$

Using the parametric transformation from global coordinates (x,y) to the local coordinates (u,v), we can transform the double integral terms of second order partial derivatives of S with respect to u and v, S_{uu}^t, S_{uv}^t and S_{vv}^t and rewrite (24) as

$$I(S(x,y)) = \sum_{t=1}^{M} \int_0^1 \int_0^{1-v} G^t(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{25}$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$ and

$$G^t(u,v) = c_1^t {S_{uu}^t}^2 + c_2^t S_{uu}^t S_{uv}^t + c_3^t S_{uu}^t S_{vv}^t + c_4^t {S_{uv}^t}^2$$
$$+ c_5^t S_{uv}^t S_{vv}^t + c_6^t {S_{vv}^t}^2, \text{ with } (\{c_j^t\}, j = 1, 2, ..., 6 \text{ as constants}).$$

In matrix form, the functional $I(S^t(u,v,w))$ = $\int_0^1 \int_0^{1-v} G^t(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$ can be represented as $B^t Q_t (B^t)^T$ where B^t is a (1 x 15) matrix represented by $[B_1^t \; B_2^t \; B_3^t \; ... \; B_{15}^t]$ and Q_t is a (15 x 15) matrix with its (i,j) entry defined as

$$[Q_t]_{ij} = c_1^t(\phi_i)_{uu}(\phi_j)_{uu} + c_2^t(\phi_i)_{uu}(\phi_j)_{uv} + c_3^t(\phi_i)_{uu}(\phi_j)_{vv} + c_4^t(\phi_i)_{uv}(\phi_j)_{uv}$$
$$+ c_5^t(\phi_i)_{uv}(\phi_j)_{vv} + c_6^t(\phi_i)_{vv}(\phi_j)_{vv}, \; (\{c_j^t\}, j = 1,2,...,6 \text{ are constants}).$$

Therefore, (25) can now be written as,

$$I(S(x,y)) = \sum_{t=1}^{M} B^t Q_t (B^t)^T \; . \tag{26}$$

The right hand side of (26) is in quadratic form with n Bézier coefficients unknown. We can also rewrite $I(S(x,y))$ using a matrix-vector representation

$$I(S(x,y)) = \mathbf{z}^T D\mathbf{z} + \mathbf{e}\mathbf{z} + c, \tag{27}$$

where D is a real (n x n) symmetric matrix, **e** is a (1x n) row vector, **z** is a (n x 1) column vector representing the unknown Bézier points for the entire triangular mesh and c as a real constant.

In order to find a function S(x,y) which will minimize I(S(x,y)) lead us to an optimisation problem of $\mathbf{z}^T D\mathbf{z} + \mathbf{e}\mathbf{z} + c$ subject to the G^1 continuity constraints $A\mathbf{x} = \mathbf{b}$.

To solve for the required values of **z**, we can use an optimization toolbox in MATLAB software and obtain the coefficients of $S^t(u,v,w)$ in (23). The interpolated G^1 surface with the sum of squares of principle curvartures optimized can then be constructed.

4. Examples

The proposed method has been tested using the following two test functions,
$f(x,y) = -8-x^2-y^2$, $(x,y) \in [-5,5]\text{x}[-5,5]$, $g(x,y) = x^2 + xy + 3y^2$, $(x,y) \in [-10,10]\text{x}[-5,5]$.
For simplicity, we choose only 4 points from each test function and the two-dimensional region (x,y) is divided into triangular elements using the Delaunay triangulation method. For each triangular patch, the corresponding surface was constructed based on the proposed method using MATLAB 7 sofware. The test functions are illustrated in Figures 4 and 5 respectively together with the generated G^1 continuous surfaces just for comparisons to the one which we have obtained using both method of sum of squares of principle curvature and G^1 continuity conditions.

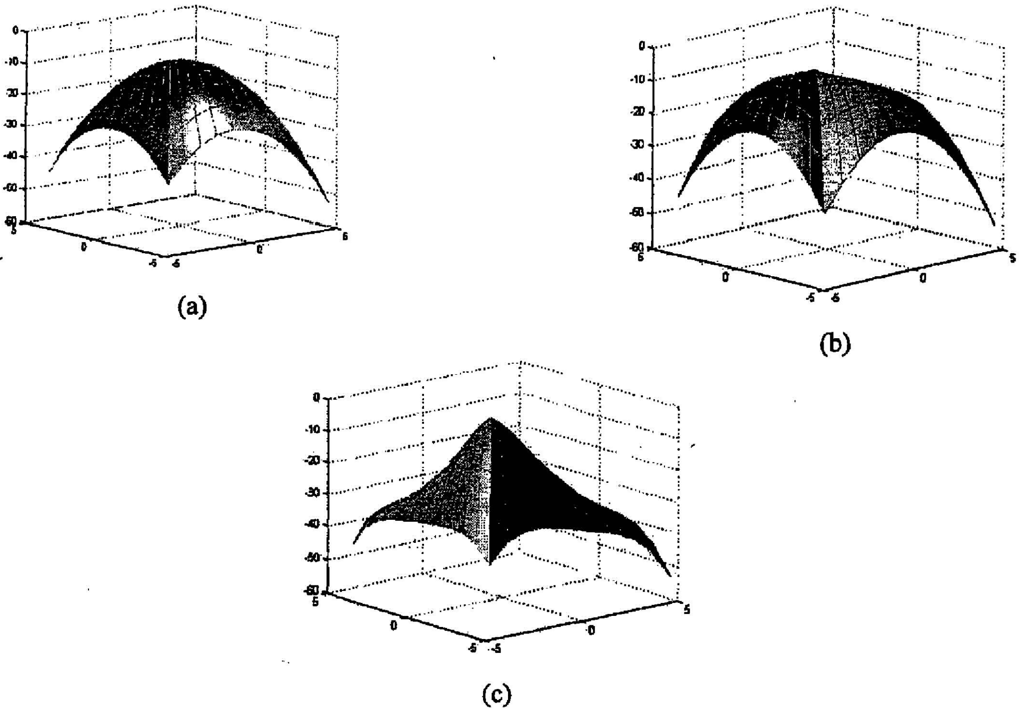

Figure 4 : (a) Test function f (b) G^1 surface interpolation with sum of squares of principle curvarture
(c) G^1 surface interpolation only

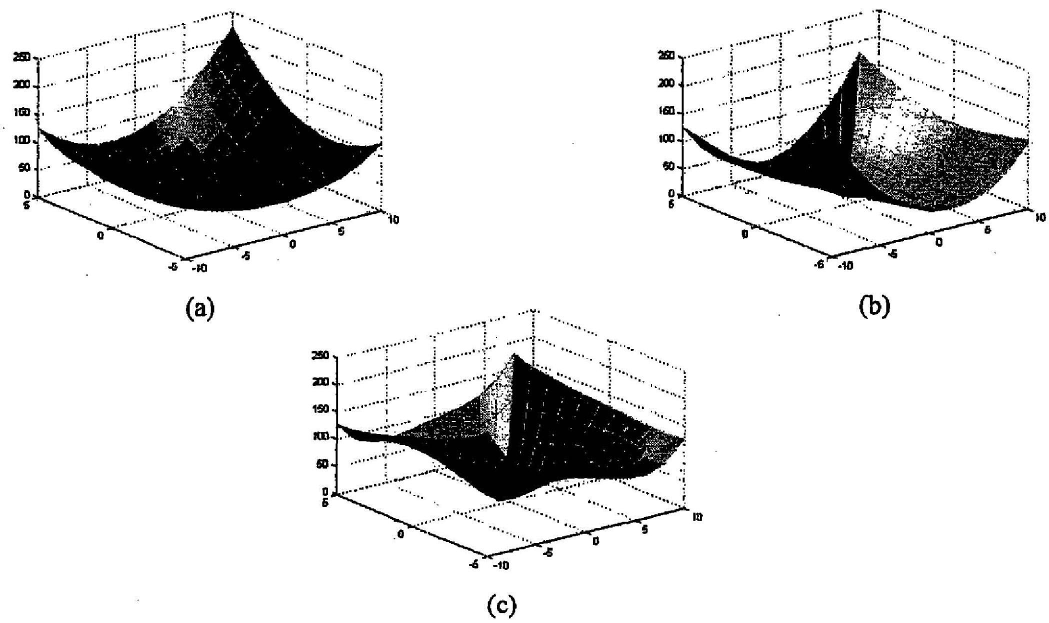

Figure 5 : (a) Test function g (b) G^1 surface interpolation with sum of squares of principle curvarture
(c) G^1 surface interpolation only

5. Conclusions

This paper describes an approach to construct smooth interpolating surfaces using a combining method of the sum of squares of principle curvarture with the G^1 continuity conditions between adjacent patches. The examples show that the surfaces obtained are fairer compared with surfaces which are constructed based on just G^1 continuity conditions.

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