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# AN IMPROVED MULTIVARIATE SHORT RUN CONTROL CHART BASED ON THE CUSUM STATISTIC


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## Abstract

Short run control charting is necessary at the start-up of a process and at the initiation of a new process for which prior information is unavailable. Numerous univariate short run control charts have been introduced to overcome the problems faced by conventional charts in a short run environment. Most practical scenarios involve several related variables. The multivariate short run charts for individual measurements and subgrouped data were proposed so that a joint monitoring of several correlated variables can be made simultaneously in a short run environment (see [5]). This paper aims at improving the performance of the multivariate short run chart for individual measurements by using its statistics to construct a cumulative sum (CUSUM) chart.



Keywords: short run, cumulative sum (CUSUM), Type-I error, non-centrality parameter, in-control, out-of-control (o.o.c.)


## 1. Introduction

Among the problems encountered by conventional charts in a short run environment is a frequent paucity of relevant data available for estimating process parameters and establishing control limits as well as the existence of many different types of measurements which require a large number of conventional charts. Research on short run charts to address these problems include the $Q$ charts (see [6] – [10]), Zed chart (see [12]), DNOM chart (see [2]), two-stage $\bar{X}-R$ charts (see [1], [3] and [11]). The multivariate short run charts were proposed to enable a simultaneous monitoring of several correlated variables in a short run environment (see [5]). The main advantages of these charts are they allow control charting to be implemented after the first few units of production without the availability of a historical data set and enabling different variables of a process to be plotted on the same chart because all statistics are charted on a standard scale. Section 2 gives a brief discussion on the multivariate short run control chart for individual measurements.

## 2. Multivariate Short Run Control Chart for Individual Measurements

Let $X_n=(X_{n1},X_{n2},\ldots,X_{np})'$, $n=1,2,\ldots$ denotes the $p\times 1$ vector of quality characteristic where $X_{nj}$ is the observation on quality characteristic $j$ at time $n$. It is assumed that $X_1,X_2,\ldots$ are independently and identically distributed (i.i.d.) vectors with a multivariate $N_p(\mu,\Sigma)$ distribution. The estimated mean vector obtained from a sequence of random vectors, $X_1,X_2,\ldots,X_n$ is defined as $\overline{X}_n=(\overline{X}_1,\overline{X}_2,\ldots,\overline{X}_p)'$ where $\overline{X}_j=\sum_{i=1}^{n}X_{ij}\Big/n$ is the estimated mean for quality characteristic $j$ made from the first $n$ observations. The required notations are as follow (see Table 1).

**Table 1:** Notations for Cumulative Distribution Functions

| | |
|---|---|
| $\Phi(\cdot)$ | *The standard normal cumulative distribution function* |
| $\Phi^{-1}(\cdot)$ | *The inverse of the standard normal cumulative distribution function* |
| $H_v(\cdot)$ | *The chi-squared cumulative distribution function with v degrees of freedom* |
| $F_{v_1,v_2}(\cdot)$ | *The Snedecor-F cumulative distribution function with $(v_1,v_2)$ degrees of freedom* |

The following formulas give the $V$ statistics of the multivariate short run chart for the two cases of both $\mu$ and $\Sigma$ known and unknown (see [5]).

*Case KK* $\mu=\mu_0$, $\Sigma=\Sigma_0$, *both known*

$$T_n^2=(X_n-\mu_0)'\Sigma_0^{-1}(X_n-\mu_0)$$

and

$$V_n=\Phi^{-1}\{H_p(T_n^2)\},\quad n=1,2,\ldots \tag{1}$$

Case UU $\mu$ and $\Sigma$ both unknown

$$T_n^2=(X_n-\overline{X}_{n-1})'S_{n-1}^{-1}(X_n-\overline{X}_{n-1})$$

where

$$S_n=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X}_n)(X_i-\overline{X}_n)'$$

and

$$V_n=\Phi^{-1}\left\{F_{p,n-p-1}\left[\left(\frac{(n-1)(n-p-1)}{np(n-2)}\right)T_n^2\right]\right\}\quad n=p+2,p+3,\ldots \tag{2}$$

Note that $p$ (see Eqs 1 and 2) is the number of quality characteristics monitored simultaneously, hence $p \geq 2$, while $S_n$ is the unbiased estimator of $\Sigma$ The $V$ statistics in Eqs 1 and 2 are all i i d $N(0,1)$ variables (see [5])

## 3. An Improved Short Run Multivariate Control Chart

Since from the previous section the $V_n$ statistics in eqs (1) and (2) are all i i d $N(0,1)$ random variables, a cumulative sum (CUSUM) chart can be constructed from the sequence of the $V$ statistics to enhance the performance of the short run multivariate chart because the CUSUM is a memory control chart The following equation gives the upper-sided $(C^+)$ CUSUM statistic (see [7])

$$C_i^+ = \max\{0, C_{i-1}^+ + V_i - k_s\}, \quad i = 1, 2, \tag{3}$$

The starting value is $C_0^+ = 0$ while the reference value, $k_s$ and decision interval, $h_s$ are selected based on a desired Type-I error Note that only the upper-sided CUSUM statistic (Eq 3) is considered because the short run multivariate chart is directionally invariant (see [5]), i e its performance depends only on the magnitude of a shift in the mean vector given by the square root of the non-centrality parameter (Eq 4) and not in the direction of the shift

$$\lambda^2 = (\mu_s - \mu_0)' \Sigma_0^{-1} (\mu_s - \mu_0) \tag{4}$$

Here, $\mu_0$ and $\mu_s$ represent the in-control and out-of-control mean vectors respectively while $\Sigma_0$ denotes the covariance matrix

## 4 An Evaluation of the Performance of the Improved Chart

A simulation study using SAS version 8 is conducted to study the performance of the improved chart based on the CUSUM statistic for a bivariate case where the number of quality characteristics, $p = 2$ An in-control process is assumed to follow a bivariate normal $N(\mu_0, \Sigma_0)$ distribution where $\mu_0 = (0,0)'$ and $\Sigma_0 = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ A shift in the mean vector from $\mu_0$ to $\mu_s$ for the following three cases are considered

Case 1: $\mu_s = (\delta,0)', (0,\delta)', (-\delta,0)'$ or $(0,-\delta)'$

Case 2: $\mu_s = (\delta,\delta)'$ or $(-\delta,-\delta)'$

Case 3: $\mu_s = (\delta,-\delta)'$ or $(-\delta,\delta)'$

Note that all of the above o o c mean vectors, $\mu_s$, for the same case are of the same distance away from the in-control mean vector, $\mu_0$ (see [5]) Since the improved chart and its basic counterpart are directionally invariant, only the first $\mu_s$ for each of the above cases will be considered Correlation coefficients of $\rho = 0$, 0 2, 0 5 and 0 8 are considered

For each value of $c \in \{10, 20, 50\}$, $c$ in-control observations are generated from $N_2(\mu_0, \Sigma_0)$ distribution followed by 30 o o c observations from a $N_2(\mu_s, \Sigma_0)$ distribution. This is repeated 5000 times and the proportion of times an o o c signal observed from $c$+1 to $c$+30 for the first time is recorded

Because of space constraint, only the results of $\mu_s = (\delta, 0)'$ and $\rho$ = 0 5 are given (see Table 2) The results for the other cases also have similar trend The results are given for the 1-of-1, 3-of-3, 4-of-5, EWMA (see [4]) and the CUSUM tests Note that the results of the improved chart denoted by CUSUM (see Table 2) are obtained from a simulation study using Eq 3 with $k_s$ = 0 75 and $h_s$ = 3 34 where these two values are chosen to obtain Type-I error similar to that of the 1-of-1 test Given a sequence of $V$ statistics $V_a, V_{a+2}, \ldots, V_m$, $a \geq 0$, the 1-of-1, 3-of-3, 4-of-5 and EWMA tests are defined as follow (se [5])

The 1-of-1 test When $V_m$ is plotted, the test signals a shift in $\mu$ if $V_m > 3$

The 3-of-3 test When $V_m$ is plotted, the test signals a shift in $\mu$ if $V_m, V_{m-1}$ and $V_{m-2}$ all exceed 1

The 4-of-5 test When $V_m$ is plotted, the test signals a shift in $\mu$ if at least four of the five value $V_m, V_{m-1}, \ldots, V_{m-4}$ exceed 1

The EWMA test The EWMA statistic $Z_m$ is defined as $Z_m = \alpha V_m + (1-\alpha) Z_{m-1}$, $m$=$b$, $b$+1, where $b \geq 1$ and $Z_{b-1} = 0$ The EWMA chart is constructed based on $UCL = K\sqrt{\alpha/(2-\alpha)}$ where the values of ($\alpha$, $K$) used are (0 25, 2 90) which gives $UCL$=1 096 These values are chosen to give an in-control average run length (ARL) of 372 6 and an ARL of 5 18 to detect a shift of 1 5 standard deviations in a normal mean (see [5]) An o o c signal is given at time $m$ if $Z_m > 1$ 096

The results show that Case KK has a superior performance to Case UU For example, for $c$=10 and $\delta$=1, the o o c proportions of Case KK for the 1-of-1, 3-of-3, 4-of-5, EWMA and CUSUM tests are 0 319, 0 583, 0 506, 0 673 and 0 628 respectively while the corresponding values of Case UU are 0 042, 0 115, 0 068, 0 050 and 0 080 respectively, where the values of Case KK are always greater than that of Case UU Note also that for Case UU with $\delta$>0, the o o c proportion increases as the value of $c$ increases For Case KK, the performance of the CUSUM test is comparable to that of the EWMA test However, the CUSUM test outperforms that of the EWMA for Case UU The CUSUM test for Case UU has the best performance for moderate and large values of $\delta$ Although for small values of $\delta$>0 the 3-of-3 test seems to outperform the other tests, its false alarm rate is somewhat too high for the case of $\delta$=0 Overall, the CUSUM test gives a good performance as it has the lowest rate of false alarm (similar to the 1-of-1 and EWMA tests) and the highest rate of detecting an o o c signal for moderate and large values of $\delta$ for Case UU If the 3-of-3 test is excluded in the comparison because of its high and unacceptable false alarm rate, then the CUSUM test performs best for Case UU even for small values of $\delta$>0 The above discussion shows that the improved short run multivariate chart based on the CUSUM statistic has provided superior results to its standard counterpart

## 5. Conclusion

We have shown in this paper that the improved short run multivariate chart outperforms its standard counterpart which makes it a good alternative to the existing approach. The new pproach addresses the need for a quicker detection of o o c signals in a short run nvironment Further research can be made to enable process monitoring to be carried out sing data from skewed and non-normal distributions

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| | Case | $c$=10 | | | | | $c$=20 | | | | | $c$=50 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1-of-1 | 3-of-3 | 4-of-5 | EWMA | [redacted] | 1-of-1 | 3-of-3 | 4-of-5 | EWMA | [redacted] | 1-of-1 | 3-of-3 | 4-of-5 | EWMA | [redacted] |
| 0 0 | KK | 0 038 | 0 095 | 0 055 | 0 042 | 0 037 | 0 038 | 0 100 | 0 055 | 0 043 | 0 038 | 0 038 | 0 102 | 0 055 | 0 039 | 0 034 |
| | UU | 0 040 | 0 103 | 0 052 | 0 039 | 0 037 | 0 039 | 0 100 | 0 049 | 0 041 | 0 038 | 0 038 | 0 101 | 0 050 | 0 043 | 0 043 |
| 0 5 | KK | 0 091 | 0 196 | 0 127 | 0 131 | 0 125 | 0 094 | 0 196 | 0 126 | 0 133 | 0 124 | 0 092 | 0 196 | 0 123 | 0 138 | 0 122 |
| | UU | 0 042 | 0 102 | 0 055 | 0 041 | 0 046 | 0 041 | 0 115 | 0 063 | 0 049 | 0 066 | 0 054 | 0 144 | 0 079 | 0 079 | 0 093 |
| 1 0 | KK | 0 319 | 0 583 | 0 506 | 0 673 | 0 628 | 0 311 | 0 583 | 0 500 | 0 669 | 0 627 | 0 315 | 0 583 | 0 499 | 0 670 | 0 623 |
| | UU | 0 042 | 0 115 | 0 068 | 0 050 | 0 080 | 0 056 | 0 165 | 0 098 | 0 097 | 0 158 | 0 098 | 0 281 | 0 197 | 0 217 | 0 311 |
| 1 5 | KK | 0 740 | 0 952 | 0 945 | 0 995 | 0 992 | 0 742 | 0 949 | 0 940 | 0 994 | 0 992 | 0 742 | 0 944 | 0 942 | 0 994 | 0 992 |
| | UU | 0 047 | 0 139 | 0 087 | 0 077 | 0 125 | 0 079 | 0 266 | 0 181 | 0 199 | 0 337 | 0 181 | 0 527 | 0 440 | 0 553 | 0 723 |
| 2 0 | KK | 0 980 | 1 000 | 0 999 | 1 000 | 1 000 | 0 982 | 1 000 | 1 000 | 1 000 | 1 000 | 0 981 | 1 000 | 1 000 | 1 000 | 1 000 |
| | UU | 0 062 | 0 182 | 0 119 | 0 121 | 0 209 | 0 126 | 0 416 | 0 325 | 0 364 | 0 607 | 0 314 | 0 785 | 0 744 | 0 870 | 0 967 |
| 2 5 | KK | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 |
| | UU | 0 091 | 0 250 | 0 167 | 0 173 | 0 318 | 0 218 | 0 578 | 0 495 | 0 564 | 0 853 | 0 508 | 0 927 | 0 922 | 0 983 | 1 000 |
| 3 0 | KK | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 |
| | UU | 0 139 | 0 317 | 0 217 | 0 229 | 0 453 | 0 341 | 0 717 | 0 645 | 0 733 | 0 966 | 0 719 | 0 979 | 0 983 | 0 999 | 1 000 |
| 4 0 | KK | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 |
| | UU | 0 293 | 0 424 | 0 288 | 0 354 | 0 708 | 0 678 | 0 883 | [illegible] | 0 935 | 1 000 | 0 965 | 1 000 | 1 000 | 1 000 | 1 000 |
| 5 0 | KK | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 | 1 000 |
| | UU | 0 518 | 0 489 | 0 325 | 0 473 | 0 869 | 0 909 | 0 949 | 0 911 | 0 989 | 1 000 | 0 999 | 1 000 | 1 000 | 1 000 | 1 000 |

**Table 2:** Simulation Results for the Basic and Improved Short