G^1 Scattered Data Interpolation with Minimized Sum of Squares of Principal Curvatures

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Abstract

One of the main focus of scattered data interpolation is fitting a smooth surface to a set of non-uniformly distributed data points which extends to all positions in a prescribed domain In this paper, given a set of scattered data $V=\{(x_i, y_i), i=1, \ldots, n\} \in R^2$ over a polygonal domain and a corresponding set of real numbers $\{z_i\}_{i=1}^{n}$ we wish to construct a surface S which has continuous varying tangent plane everywhere (G^1) such that $S(x_i, y_i) = z_i$ Specifically, the polynomial being considered belong to G^1 quartic Bézier functions over a triangulated domain In order to construct the surface, we need to construct the triangular mesh spanning over the unorganized set of points, V which will then have to be covered with Bezier patches with coefficients satisfying the G^1 continuity between patches and the minimized sum of squares of principal curvatures Examples are also presented to show the effectiveness of our proposed method

1. Introduction

Scattered data interpolation refers to the problem of fitting smooth surfaces through a non-uniform distribution of data points In practice, this subject is very important in various sciences and engineering where data are often measured or generated at sparse and irregular positions The goal of interpolation is to construct underlying functions which may be evaluated at certain set of positions

There are 3 principal sources of scattered data [10] measured value of physical quantities (such as in geology and meteorology), experimental results (in sciences and engineering) and computational values (in various applications of computer graphics and vision with functional data)

For the purpose of this paper, we will only focus on the scattered data from functions to be used by our proposed method We will construct a G^1 continuous surface which interpolates these given functional data The geometric continuity needed between adjacent parametric patches especially in the case of Bézier patches has been given attention due to the free shape parameters which it provides and can be used to construct and modify very complicated geometric objects In practice, geometric continuity avoids the usual dependence on the parameterisation of the constructed patches The conditions required for patches to be G^n-continuous and subsequent construction of the corresponding G^n-continuous surfaces are amongst important topics in Computer Aided Geometric Design [17]

Although several papers have dealt with G^n interpolating surfaces (e g see [5] and [12]), but very few touched on the scattered data interpolation We thus propose a method to construct an interpolating G^1 surface using a set of scattered data in R^3 as well as provide examples to test the ability of this proposed scheme

If we simply use G^1 continuity conditions, the resulting surfaces might have an undesired undulation or over flatness To overcome this problem, several global optimal fairness criteria of the constructed surfaces have been introduced in various literatures Amongst the earliest surface fairing aimed at minimizing fairness criteria as an anology to the one for curve was based on the strained energy in a thin elastic plate which is related to the total curvature in a surface Thus, the resulting surface can then be approximated using its geometric and curvature related properties ([4],[14])

In this paper, we consider the method of minimized sum of squares of principle curvatures ([6], [9]) The objective function can then be approximated using a corresponding quadratic form The extremum of the integral function is then calculated with respect to the G^1 continuity conditions along the shared edges as a constrained function

Quadratic functions can generate good surface fit to the data points The range of applications can be extended by using iterative procedures which will successively improve the parameter values assigned to these data points.

We have chosen quartic Bézier triangular patches since degree 4 is necessary and sufficient to satisfy the patch boundary interpolation constraints while at the same time allows some control points in the interior of the patch to have certain freedom to be used for surface fairing.

This paper is organized as follows: Relevant background materials including the definition of quartic Bézier form of triangular patches and G^1 continuity conditions between two adjacent patches at vertices of triangle meshes are given in section 2, a method to construct surfaces using the minimized sum of squares of principle curvature is covered in section 3, examples are given in section 4 and finally in section 5, the concluding remarks will be given.

2. Background

The n^{th} degree Bernstein polynomials over a triangle are defined by

$$B^n_{ijk}(u,v,w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k \qquad (1)$$

where u,v,w are barycentric coordinates such that $u + v + w = 1$ and $i+j+k = n$

The n^{th} degree Bézier triangular patch is defined by

$$P(u,v,w) = \sum_{i+j+k=n} b_{ijk} B^n_{ijk}(u,v,w) \qquad (2)$$

where b_{ijk} are called Bézier ordinates of P We will use quartic Bézier triangular patches (with n = 4). Each patch is defined by a control net of 15 vertices as shown in Figure 1.

With b_{ijk} as the height of a control point, a quartic Bézier triangular patch can be represented by

$$S(u,v,w) = \sum_{i+j+k=4} B_{ijk} B^4_{ijk}(u,v,w) \qquad (3)$$

where $B_{ijk} = (i/4, j/4, k/4, b_{ijk})$

Let T be a triangle with vertices V_1, V_2, V_3 and barycentric coordinates u, v and w such that any point V on T can be expressed as

$$V = uV_1 + vV_2 + wV_3\,, \qquad (4)$$

where $u + v + w = 1$ and $u, v, w \geq 0$

S interpolates the Bézier ordinates B_{400}, B_{040} and B_{004} at the vertices V_1, V_2, V_3 of T respectively The normal at each vertex is estimated using the surrounding triangles with a quadratic approximation function,

$$F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f \qquad (5)$$

where a, b, c, d, e and f are unknown coefficients.

Let vertex P_0 of the triangular mesh surrounded by vertices $\{P_1, P_2, \quad ,P_k\}$ and the height of the vertices $\{P_0, P_1, \quad ,P_k\}$ be represented by $\{z_0, z_1, \ldots, z_k\}$ respectively We turn our attention to the vertex represented by (1,0,0) which corresponds to $P_0(x_0,y_0)$. We can derive the first order partial derivative at P_0 as

$$F_x(x_0,y_0) = 2ax_0 + by_0 + d \text{ and}$$
$$F_y = bx_0 + 2cy_0 + e \qquad (6)$$

If we substitute the height of vertices $z_0, z_1, \ldots, z_k$ into (5), we obtain a linear system

$$Uc = z \qquad (7)$$

where

$$U = \begin{bmatrix} x_0^2 & x_0y_0 & y_0^2 & x_0 & y_0 & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ x_k^2 & x_ky_k & y_k^2 & x_k & y_k & 1 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ \\ \cdot \\ \cdot \\ z_k \end{bmatrix}$$

Matrix U in (7) is always in the form of rank deficient matrix (that is, rank of U is less than number of unknowns) and we can solve the systems by using the simple linear algebra [7] using pseudo inverse of U in the least square method.

Theorem 1: Consider the (m x n) system $\boldsymbol{Ax=b}$

(a) The associated system $\boldsymbol{A^TAx = A^Tb}$ is always consistent

(b) The least-squares solutions of $\boldsymbol{Ax=b}$ are precisely the solutions of $\boldsymbol{A^TAx = A^Tb}$

(c) The least square solution is unique if and only if rank A = n

Proof of the Theorem 1 can be found in [7] Thus, the values of a, b, c, d, e and f can be obtained, using the least square method,

$$C = (U^TU)^{-1}U^Tz, \qquad (8)$$

and the values of F_x and F_y at vertex P_0 can then be calculated

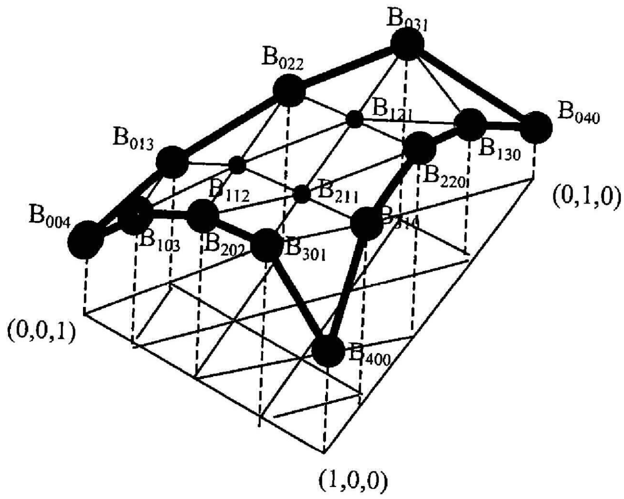


Figure 1. Control net of quartic Bézier triangular patch

Let V, V_1, V_2 and V_3 be located as shown in Figure 2. The vertices V_1, V_2 and V_3 correspond to the barycentric coordinates (1,0,0), (0,1,0) and (0,0,1) respectively.

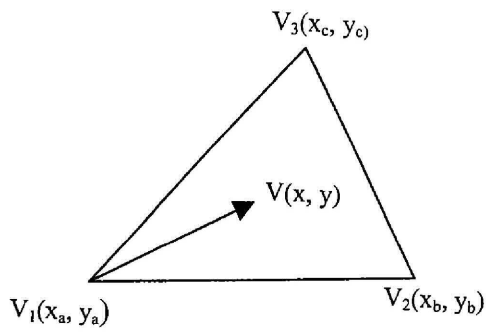


Figure 2. Vertices of a triangle element

Let e_{12} = (-1,1,0) and e_{13}= (-1,0,1) represent the direction of edges V_1V_2 and V_1V_3 respectively. The directional derivatives along e_{12} and e_{13} at V_1 are

$$D_{e_{12}}S(1,0,0)=(\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u})F_x(x_a)+(\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u})F_y(x_a)$$
$$=(x_b-x_a)F_x(x_a)+(y_b-y_a)F_y(x_a) \quad (9)$$

$$D_{e_{13}}S(1,0,0)=(\frac{\partial x}{\partial w}-\frac{\partial x}{\partial u})F_x(x_a)+(\frac{\partial y}{\partial w}-\frac{\partial y}{\partial u})F_y(x_a)$$
$$=(x_c-x_a)F_x(x_a)+(y_c-y_a)F_y(x_a) \quad (10)$$

Thus, we have

$$D_{e_{12}}S(1,0,0)=4(-B_{400}+B_{310}) \quad (11)$$

$$D_{e_{13}}S(1,0,0)=4(-B_{400}+B_{301}) \quad (12)$$

or

$$B_{310}=\frac{1}{4}D_{e_{12}}S(1,0,0)+B_{400} \quad (13)$$

$$B_{301}=\frac{1}{4}D_{e_{13}}S(1,0,0)+B_{400} \quad (14)$$

Similarly, B_{130}, B_{031}, B_{103} and B_{013} can be determined from B_{040} and B_{004} respectively, with 6 more control points left to be determined.

2.1 G[1] continuity between adjacent patches

Two patches with a common boundary curve satisfy G[1] continuity if both have continuously varying tangent plane along the common curve. Figure 3 shows an example of Bézier control points of two adjacent quartic Bézier triangular patches. H_0 and H_4 are the given vertices of the patches. G_0, F_0, H_1, G_3, H_3 and F_3 are obtained from the patch gradients while G_1, F_1, H_2, G_2 and F_2 are points to be determined.

We only have to consider $\{H_i, i = 0, 1, \ldots 4\}$ as the common boundary curve and $\{G_i, F_i, i = 0, 1, \ldots 3\}$ which consist of the control points in each patch. Details of derivation with regard to the G[1] conditions can be found in [3].

If the heights of $\{G_i\}$, $\{H_i\}$, $\{F_i\}$ are denoted by $\{g_i\}$, $\{h_i\}$ and $\{f_i\}$ respectively, the conditions satisfying G[1] continuity between the two adjacent patches can be written as

$$\alpha g_0 + (1-\alpha)f_0 = \beta h_0 + (1-\beta)h_1 \quad (15)$$
$$\alpha g_1 + (1-\alpha)f_1 = \beta h_1 + (1-\beta)h_2 \quad (16)$$
$$\alpha g_2 + (1-\alpha)f_2 = \beta h_2 + (1-\beta)h_3 \quad (17)$$
$$\alpha g_3 + (1-\alpha)f_3 = \beta h_3 + (1-\beta)h_4 \quad (18)$$

where α and β are constants.

Since the values of g_0, f_0, h_0, h_1, g_3, h_3, h_4 and f_3 are already known, α and β can thus be determined from (15) and (18). (16) and (17) can also be written as

$$A_1x_1 = b - A_2x_2 \quad (19)$$

where

$A_1=\begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix}$ is a 2x2 scalar matrix, $\mathbf{x}_1=\begin{bmatrix}g_1\\ g_2\end{bmatrix}$ is the unknown vector, $A_2=\begin{bmatrix}1-\alpha & 0 & \beta-1\\ 0 & 1-\alpha & -\beta\end{bmatrix}$ is a 2 x3 matrix, $\mathbf{x}_2=\begin{bmatrix}f_1\\ f_2\\ h_2\end{bmatrix}$ is a vector of free parameters and

$\mathbf{b}=\begin{bmatrix}\beta h_1\\ (1-\beta)h_2\end{bmatrix}$ is a constant vector.

Since A_1 is a non-singular matrix for $\alpha \neq 0$, we can always solve (19) for x_1, that is,

$$\mathbf{x}_1 = A_1^{-1}(\mathbf{b}-A_2\mathbf{x}_2) \quad (20)$$

If x_2 can be determined adequately, all the control points B_{ijk} in Bézier control net will be known for the surface to be G^1 continuous

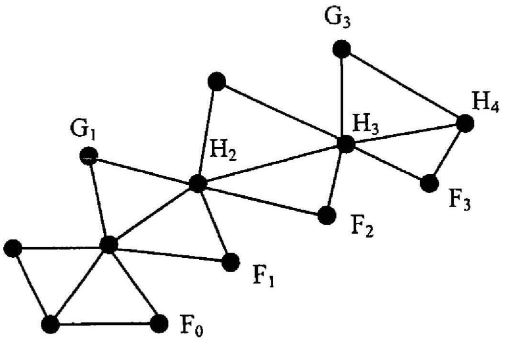


Figure 3. Control points of adjacent quartic Bézier triangular patches

3. Surface with minimized sum of squares of principle curvartures

A surface fairing objective will be the integral of the sum of squares of principle curvartures

$$I(S(x,y)) = \iint_D k_1^2 + k_2^2 \, dxdy \qquad (21)$$

where k_1 and k_2 are the principle curvartures and D represents the surface to be constructed (21) is not widely used to fair composite surfaces with many free parameters, perhaps because of the complexity of its parametric form and computational difficulties A much more popular method [19] is the linearized form of (21) which uses second-order parametric derivatives in place of the principal curvartures and can be written as,

$$I(S(x,y)) = \iint_D S_{xx}^2 + 2S_{xy}^2 + S_{yy}^2 \, dxdy \qquad (22)$$

Now, our aim is then to find the function S(x,y) which will minimize the integral I(S). Assume that the whole surface can be constructed by a collection of Bézier triangular patches with each patch defined as in (3). We can represent each patch as a convex combination of 15 control points,

$$S^t(u,v,w) = \sum_{k=1}^{15} B_k^t \phi_k(u,v,w), \qquad (23)$$

where $t = 1,2 \ldots m$ (m is the number of triangles in a mesh),

$\{B_1^t, B_2^t, B_3^t, \ldots, B_{15}^t\} = \{B_{400}^t, B_{310}^t, B_{130}^t, B_{040}^t, B_{301}^t, B_{031}^t, B_{103}^t, B_{013}^t, B_{004}^t, B_{220}^t, B_{211}^t, B_{121}^t, B_{202}^t, B_{112}^t, B_{022}^t\}$

$\{\phi_1, \phi_2, \ldots, \phi_{15}\} = \{u^4, 4u^3v, 4uv^3, v^4, 4u^3w, 4v^3w, 4uw^3, 4vw^3, w^4, 6u^2v^2, 12u^2vw, 12uv^2w, 6u^2w^2, 12uvw^2, 6v^2w^2\}$

Let $I(S^t(u,v,w))$ be defined in a triangle element D_t such that we can write (22) as

$$I(S(x,y)) = \sum_{t=1}^{m} I(S^t(u,v,w))$$

$$= \sum_{t=1}^{m} \iint_{D^t} (S_{xx}^t)^2 + 2(S_{xy}^t)^2 + (S_{yy}^t)^2 \, dx\,dy \qquad (24)$$

Using the parametric transformation from (x,y) to the local coordinates (u,v), we can write the double integral of second order partial derivatives of S with respect to u and v, S_{uu}^t, S_{uv}^t and S_{vv}^t and thus (24) can be expressed as

$$I(S(x,y)) = \sum_{t=1}^{m} \int_0^1 \int_0^{1-v} G^t(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \qquad (25)$$

where $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$ and

$$G^t(u,v) = c_1^t {S_{uu}^t}^2 + c_2^t S_{uu}^t S_{uv}^t + c_3^t S_{uu}^t S_{vv}^t + c_4^t {S_{uv}^t}^2 + c_5^t S_{uv}^t S_{vv}^t + c_6^t {S_{vv}^t}^2,$$

with $(\{c_j^t\}, j = 1, 2, \ldots, 6$ as constants)

In matrix form, the functional $I(S^t(u,v,w)) = \int_0^1 \int_0^{1-v} G^t(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$ can be represented as $B^t Q_t (B^t)^T$ where B^t is a (1 x 15) matrix represented by $[B_1^t \; B_2^t \; B_3^t \ldots B_{15}^t]$ and Q_t is a (15 x 15) matrix with its (i,j) entry defined as

$$[Q_t]_{ij} = c_1^t (\phi_i)_{uu} (\phi_j)_{uu} + c_2^t (\phi_i)_{uu} (\phi_j)_{uv} + c_3^t (\phi_i)_{uu} (\phi_j)_{vv} + c_4^t (\phi_i)_{uv} (\phi_j)_{uv} + c_5^t (\phi_i)_{uv} (\phi_j)_{vv} + c_6^t (\phi_i)_{vv} (\phi_j)_{vv},$$

Thus, (25) can be written as,

$$I(S(x,y)) = \sum_{t=1}^{M} B^t Q_t (B^t)^T . \qquad (26)$$

The right hand side of (26) is in quadratic form with 6 Bézier coefficients unknown We can also rewrite $I(S(x,y))$ in terms of a matrix-vector representation

$$I(S(x,y)) = \mathbf{z}^T \mathbf{M}\mathbf{z} + \mathbf{e}\mathbf{z} + \mathrm{c}, \qquad (27)$$

where M is a real (n x n) symmetric matrix, **e** is a (1x n) row vector, **z** is a (n x 1) column vector representing the unknown Bézier points for the entire triangular mesh and c as a real constant

In order to find a function S(x,y) which will minimize I(S(x,y)) lead us to an optimisation problem of $\mathbf{z}^T \mathbf{M}\mathbf{z} + \mathbf{e}\mathbf{z} + \mathrm{c}$ subject to the G^1 continuity constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$

To solve for the required values of **z**, we can use an optimization toolbox in MATLAB software and obtain the coefficients of $S'(u,v,w)$ in (23) The interpolated G^1 surface with the minimized sum of squares of principle curvartures optimized can then be constructed

4. Examples

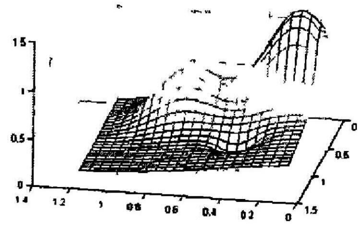
To test the accuracy of our method, we choose 36 data points from the 3 well-known functions in [15],

$$F1(x,y) = 0\ 75exp(-((9x-2)^2 + (9y-2)^2)/4 + 0\ 75exp(-(9x+1)^2/49 - (9y+1)/10) + 0\ 50exp(-((9x-7)^2 + (9y-3)^2)/4) -0\ 20exp(-(9x-4)^2 - (9y-7)^2, \quad (x,y) \in [0,1\ 2]x[0,1\ 2]$$

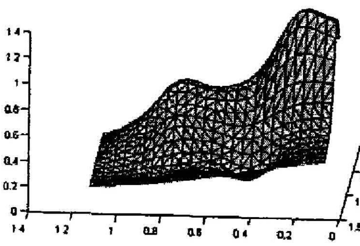
$$F2(x,y)= (1\ 25 +cos(5\ 4y))/(6+6(3x-1)^2, \quad (x,y) \in [0,1]x[0,1]$$

$$F3(x,y)=exp(-20\ 25((x-0\ 5)^2+(y-0\ 5)^2)), \quad (x,y) \in [0,1]x[0,1]$$

For each data set, the two-dimensional region (x,y) is divided into triangular elements using Delaunay triangulation For each patch, the corresponding surface was constructed using MATLAB 7 sofware The measure of error norm [15] was taken to be SSE/SSM, where SSE is the sum of squared errors (deviation from test function values), and SSM is the sum of squared of deviation ot the 1296 test function values (36x36 grid points) from their mean We also compute the coefficient of determination r^2=1-SSE/SSM as shown in Table 1 According to [15], the values of r^2 show that our method is a very good fit for the functions, F1 and F2 and a good fit for F3 These results are shown in Figures 4, 5 and 6 respectively



(a)



(b)

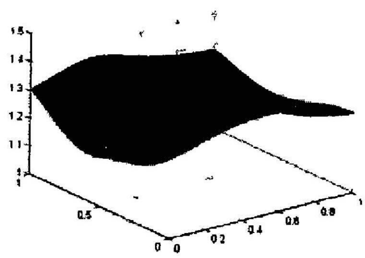
Figure 4. (a) Test function F1 (b) Proposed method

5. Conclusions

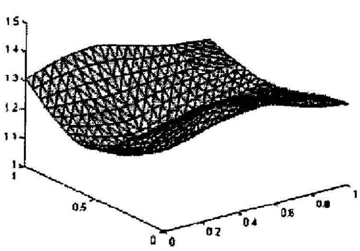
This paper describes an approach to construct smooth interpolating surfaces using a combining method of the minimized sum of squares of principle curvarture with the G^1 continuity conditions between adjacent patches The examples show that the surface obtained are fairer compared with surfaces which are constructed based on just G^1 continuity conditions We will focus on G^2 continuity conditions in future research

Table 1. Error norms and coefficient of determination ,r^2

Fcn	Error norms	r^2
F1	0 007873	0 992127
F2	0 002942	0 997058
F3	0 01681	0 98318

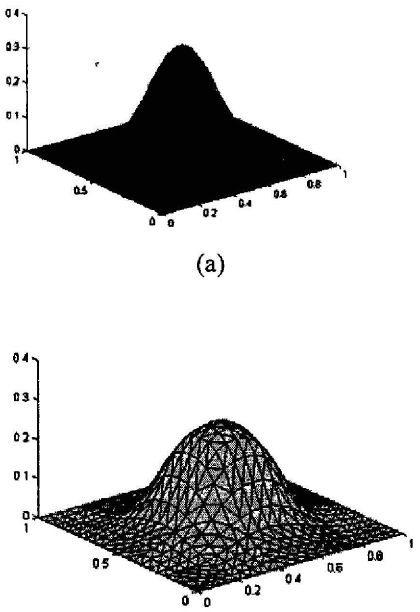


(a)



(b)

Figure 5. (a) Test function F2 (b) Proposed method



(a)

(b)

Figure 6. (a) Test function F3 (b) Proposed method

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