Lautman and the Reality of Mathematics

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It was with great delight that I accepted the invitation to contribute to this book. Many years ago, while training in Paris for a possible career in psychoanalysis, I became interested in the philosophy of mathematics, not such a surprising event if one recalls how in France psychoanalytic theory grapples with a vast range of intellectual endeavours. Well, the name of Albert Lautman was mentioned and I found a volume of his, the 1977 edition of his works with 10/18, in the library of the George Pompidou Centre. At the time, much of the philosophical setting of his work passed me by, but I was entranced by his varied examples of structure similarity, which reminded me of what I had enjoyed most from an undergraduate mathematics degree.

At the same time, inspired by Colin McLarty's *Uses and Abuses of the History of Topos Theory* (1990), I was learning some category theory, in particular through Lambek and Scott's book, *Introduction to Higher-Order Categorical Logic*, and it wasn't hard to figure out that had he survived the war Lautman would have embraced category theory whole-heartedly. Shortly after, I returned to London to study for a Masters and PhD in philosophy of mathematics, and it was only then that I began to appreciate the gulf that separated Lautman's treatment of mathematics from the dominant analytic philosophical approach prevalent in the USA and UK. Where Lautman could talk about Galois connections and theta functions, Poincaré duality and class field theory, these remained as probable topics for Anglophone philosophical discussion as the monsoon season in southern India, or the reproductive cycle of the lesser horseshoe bat. The best opportunity I could conceive to talk about real mathematics was provided by Lakatos's invitation to transfer his research programmme construction, devised for science, to mathematics. Heeding his call, case study material could be chosen at will from those parts of mathematics found most attractive. Lakatos's somewhat rigid framework of stark rivalry between parallel research programmes made it very hard, however, to treat the interweaving of theories in twentieth century mathematics so dear to Lautman.

Here, then, is an opportunity to return to Lautman and to the origin of theses I have held for twenty years:

1. Rather than accord logic philosophical priority over other parts of mathematics, we should consider it as any other branch, a place where key concepts recurrently manifest themselves.

2. The reality of mathematics is to be addressed through these recurrent manifestations, the 'realisation of dialectical ideas' as Lautman put it, and not through Quinean notions of ontological commitment.

In this paper I want to take a look at the second of these theses in the context of a couple of examples Lautman provided for us, and to raise some questions about the mathematical nature of his 'dialectical ideas'.

1   La Montée vers L'Absolu

If there's one chapter I remember well twenty years later it is Lautman's *La Montée vers L'Absolu*. Here we find brilliantly associated the manifestations within philosophy and within mathematics of an idea relating imperfection and perfection. This two-part idea maintains that any case of imperfection presupposes a corresponding perfection, and that it is possible to understand the attributes of the associated perfection through the defects of the imperfect entity. This idea's realisation in philosophy occurs in Descartes' argument that we may know the existence of a perfect being, and its attributes, through the awareness of our own imperfections. For example, we sometimes doubt rather than know, this is an imperfection, therefore we can say that a perfect being is omniscient.

Longer is spent by Lautman on mathematical examples from algebraic number theory and from algebraic topology. An imperfection of $\mathbb{Q}$ is that the polynomial $x^2 - 2$ doesn't split over it. It lacks an element squaring to 2, something a perfected $\mathbb{Q}$ must then have. The field $\mathbb{Q}(\sqrt{2})$ is a step towards perfection, but we can tell the same story equally for other polynomials, so that the perfected $\mathbb{Q}$ must in fact be its algebraic closure. Turning now to topology, an imperfection of the circle is that a loop in it may not be contractible to a point. A single circuit of the circle, for example, cannot be contracted to a point. On the other hand, the path sitting over this circuit in a circle wound twice around the original circle *is* contractible. Again, this doubled circle suffers from less imperfection, but still any path in it lifted from a path in the original circle winding an even nonzero number of turns will not be contractible. In this situation the perfected circle is a helix. These two mathematical examples share much in common, where a lattice of intermediate fields, in the one case, and a lattice of intermediate covering spaces, in the other, may be associated to certain lattices of subgroups.

As Jean Dieudonné observed in his preface to the 10/18 edition, in selecting this material, Lautman showed great sensitivity to the kind of structure similarity so prevalent in contemporary mathematics, and which is extremely well captured by the language of category theory. Indeed, a very important moment in the unification of the topological and number theoretic constructions occurred with Grothendieck's introduction of the notion of a fibre functor. Through this notion, mathematicians can try to understand the number theoretic manifestations of the noncommutativity of the fundamental group. Remember that in the topological case, the fundamental group is the group of based loops up to deformation. In topology itself, we can construct a Galoisian account of other imperfections represented by higher homotopy groups.

Certainly, this extraordinarily rich seam of mathematics is far from exhausted. Indeed, an updated version of Lautman's chapter would cover, as Yves André (2008) does, differential equations, motives, periods and renormalisation. But what we do have, and which Lautman did not, is the notion that the heart of the matter, his 'idea', is capable of being framed by a piece of

mathematical language, at least to the extent that the idea manifests itself in mathematics. Mathematics can address the idea at a level of abstraction above its manifestations. Indeed, when Saunders Mac Lane in his article *The Protean Character of Mathematics* (Mac Lane 1992) use the Galoisian idea to provide evidence for his claim that "the same mathematical structure has many different empirical realizations" (p. 3), he ends his list of major contributions with

> Janelidze, 1988 Categorical formulation of Galois structure
> Various, 1990 One adjunction handles Galois and much more. (p. 13)

So now here's a question: What work does the term 'dialectic' do? Why aren't these primitive ideas which manifest themselves so clearly in mathematics just simply 'mathematical'? While there was no language to capture the commonality of Galoisian field extensions and Poincaréan deck transformations, it might seem plausible to take this commonality to be something beyond mathematics. But does this make sense today?

Perhaps the solution is to be found by recalling that the perfect/imperfect idea does not manifest itself solely in mathematics. Remember the example of a non-mathematical realisation given by Lautman was Descartes' argument for God from his own imperfections. We could argue that something of what is manifest here is not mathematical, so that the idea common to both this philosophical situation and the Galoisian situation is not itself mathematical. Certainly there would seem to be little point in mathematising Descartes' argument. But we might wonder whether it is significant that the manifestation in philosophy is rather, shall we say, thin, while manifestations in mathematics are enormously richer. Moreover, if we want to capture the essence of the mathematical situation it is arguable whether perfection and imperfection are the best terms. Galois himself seems to have more inclined to think in terms of 'ambiguity' rather than imperfection. In his final letter he writes,

> Mes principales méditations depuis quelque temps étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguïté.
>
> Il s'agissait de voir a priori dans une relation entre quantités ou fonctions transcendantes quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître tout de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher.
>
> Mais je n'ai pas le temps et mes idées ne sont pas encore bien développées sur ce terrain qui est immense... (quoted in André 2008)

Of course, ambiguity could be taken as a form of imperfection, but unless the cartesian case can be understood as a form of ambiguity, the commonality must appear to us as lessened.

Now, although Lautman steers clear of the term, Michael Polanyi's treatment of universals as the 'joint meaning of things forming a class' is relevant here. This meaning, he claims, is "something real since it is capable of manifesting itself indefinitely in the future". He continues,

> It has, indeed, an heuristic power that is usually twofold. (1) A universal concept usually anticipates the occurrence of further instances of itself in the future, and if the concept is true, it will

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validly subsume these future instances in spite of the fact that they will unpredictably differ in every particular from all the instances subsumed in the past. (2) A true universal concept, designating a natural class, for example a species of animals, anticipates that the members of the class will yet be found to share an indefinite range of uncovenanted properties; i.e., that the class will be found to have a yet unrevealed range of intension. (Polanyi 1969, 170-1)

To illustrate these powers, at a stage when it had been proposed that mice and elephants belong to a class, named 'mammalia', we might have expected (1) very different animals would turn out to be mammals, and (2) that an indefinite range of commonalities between mice and elephants would be discovered. In this case, in the light of the subsumption of legless, aquatic dolphins under the class, and the discovery of major physiological and ultimately genetic similarities between mice and elephants, these powers were admirably displayed.

I think we can risk overlooking the differences between universals and ideas here, and ask whether the commonality manifested in Decartes' and Galois' thoughts has these same heuristic powers. Along the lines of (1), we might look for another rich manifestion of the imperfection/perfection dialectic elsewhere. Certainly we can find Galois theory employed in physics, for example, Gepner 2006, but it is not obvious that perfection/imperfection is relevant to this work on rational conformal field theory. Elsewhere, homotopy theory accounts for *defects* in nematic liquid crystals (Nash and Sen 1983, Ch. 9). Although with a connotation of imperfection, it is not clear that from the existence of a crystal with defects we arrive at the corresponding notion of a perfected defectless crystal. Furthermore, if this is to count as a third manifestation of Lautman's dialectical idea, we should note the close relation to mathematics it bears. This is hardly a third point in a triangle of manifestations of the dialectical idea, but rather a point a whisker away from the mathematical ones.

On the other hand, along the lines of (2), perhaps the appearance of thinness in the philosophical example is illusory. Perhaps it could reveal itself to be more Galoisian then we thought. Well, there is an obvious way to make the philosophical story richer, it seems to me, one suggested strongly by our later mathematical understanding. Had the full Galoisian idea appeared in Descartes' thought, he would have had to put into association all the hierarchy of different substructures of the complex of man's imperfections with the hierarchical of different kinds of angelic being interposed between Man and God. Apparently, in his *Summa Theologica*, Aquinas describes a hierarchy of angels arranged in three ranks of three levels each. It would not be at all surprising to me if this or some other elaborate angelogical theory of the Middle Ages could be drafted into something like a Galoisian form.

But perhaps the mathematical example was rather special in any case. Surely not all examples of common realisation in mathematics are so well addressed by a mathematical theory as this one. Let us look to another of Lautman's case studies.

2   Reciprocity and Duality

In 'Nouvelles Recherches sur la Structure Dialectique des Mathématiques' (1939), Lautman discusses the use of analysis in number theory. He notes that some

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have felt uncomfortable with this use and have sought to eliminate it. But Lautman sees no metaphysical necessity for this 'purification'. Rather than take arithmetic as metaphysically prior to analysis, instead he proposes that we consider them equally as realisations of the same dialectical structures.

He gives the example of reciprocal entities. In arithmetic we have quadratic reciprocity, where the Legendre symbols are acting as a kind of inverse to each other.

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}.$$

Here, for odd primes p and q, the first Legendre symbol takes the value $+1$, if p is a square modulo q, and otherwise it takes the value -1.

He goes on to note that it has been possible to generalise reciprocity in two different ways. First, to algebraic integers in any field. Second, to allow more general congruences, not just to a square, but to other powers. This has been achieved algebraically he notes, but then adds that Hecke has also provided analytic means of deriving general quadratic reciprocity results using theta functions.

Here we define

$$\theta(\tau) = \sum_{m=-\infty}^{m=+\infty} e^{-\pi \tau m^2},$$

noting that singular points are at $\tau = 2ir$, r a rational, but that for any such r, $\sqrt{\tau}\theta(\tau + 2ir)$ takes a finite value which is, up to factors, the Gauss sum $C(-r)$. Now we have the transformational property of the theta function

$$\theta(1/\tau) = \sqrt{\tau}\theta(\tau),$$

and this tells us that there is a reciprocal relation between $C(r)$ and $C(-1/4r)$, from which ordinary quadratic reciprocity follows.

Lautman claims,

> This dialectical idea of reciprocity between elements can be so clearly distinguished from its realisations in arithmetic and in in analysis that it is possible to find a certain number of other mathematical theories in which it realises itself similarly.

Now, before we look to see where else Lautman sees reciprocity realised, let us note that it is important for Lautman that the dialectical idea be clearly distinguished from its realisations. Again we may wonder what hangs on the inability of 1930s mathematics to capture, or at least approach, the idea of reciprocity itself. At the time it must have seemed a distant prospect to have a framework which could cover all realisations, especially as Lautman goes on to invoke some very recent work of André Weil indicating that there was a relationship between reciprocity laws and Poincaré duality. This duality relates homology in dimension m and cohomology in dimension $n - m$ in a manifold of dimension n, which in certain kinds of manifold entails a simple relationship between homology in complementary dimensions. Once again, Lautman had chosen his case study brilliantly. There is indeed a commonality between quadratic reciprocity

and Poincaré duality, one which relates closely to the Galoisian idea of the last section, which I shall briefly sketch.

Later, in the 1970s, Barry Mazur and David Mumford noted a powerful analogy between ideals in algebraic number fields and links in 3-manifolds as follows. From the perspective of étale-cohomology, the integers $\mathbb{Z}$, or rather the scheme $Spec(\mathbb{Z})$, appears to be a kind of three-dimensional sphere. A prime number p has an associated scheme $K_p = Spec(\mathbb{Z}/p\mathbb{Z})$ which from the same perspective appears to be a 1-dimensional submanifold of this sphere. A huge amount of progress can now be made by transferring ideas from knot theory over to algebraic number theory. Poincaré duality can then be used to define the linking number of two 'knots' K_p and K_q. Quadratic reciprocity, in this framework, amounts to little more than the fact that the linking number of knots A and B is the same as the linking number of knots B and A (Waldspurger 1976).

The analytic part of the story has also been enormously developed since Lautman's time to relate L-function reciprocity with Poincaré duality, and so much more. Here we are touching on the hottest areas of contemporary mathematics, the monumental Langland's Program and Grothendieck's motives. Lautman's sense of the unity of mathematics was extremely acute. He had a phenomenal ability to point us to mathematical reality. But again, with the success in indicating mathematical reality comes the question as to what is not mathematical about the ideas manifesting themselves here. Although we are not yet at the point where there's a comprehensive mathematical theory of all forms of duality, there is at least a strong sense that the idea of duality itself, rather than its specific manifestations, may be addressed mathematically. Again the language of category theory, and its higher-dimensional cousin, is involved, dualities often being expressible as equivalences.

One useful distinction is that Lawvere and Rosebrugh introduce in chapter 7 of their book *Sets for Mathematics* (2003) between 'formal' and 'concrete' duality. *Formal* duality concerns mere arrow reversal in the relevant diagrams, so

> of course if the original diagrams had been given specific interpretation in terms of specific sets and mappings, such interpretation is lost when we pass to this formal dual in that the formal dualization process in itself does not determine specific sets and specific mappings that interpret the dualized statement. (p. 121)

Concrete duality, on the other hand, occurs in situations where a new diagram is formed from an old one by exponentiating each object with respect to a given dualizing object, e.g., X becomes V^X, with V the dualizing object. The arrows are naturally reversed in the new diagram. Now,

> Not every statement will be taken into its formal dual by the process of dualizing with respect to V, and indeed a large part of the study of mathematics

> > space vs. quantity

> and of logic

> > theory vs. example

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may be considered as the detailed study of the extent to which formal duality and concrete duality into a favorite V correspond or fail to correspond. (p. 122)

Now, unlike in the case of the perfection/imperfection dialectic, we do not learn from Lautman of a non-mathematical realisation of the idea of reciprocity/duality, but if the Langlands Program is to be made of philosophical importance for the rest of philosophy due to its being a case of the manifestation of an idea which equally manifests itself in other spheres of life, we would need to come up with some interesting examples. We could think up some suggestions, but a worry would surely be that they paled into insignificance beside the mathematical one. Our best chance again would be to look to physics, e.g., the duality between the electric field and the magnetic field. However, again physical examples share an enormous amount in common with mathematical examples. Indeed they share a whole interlocking history of mutual influence and development, from Hodge theory and de Rham cohomology, leading right up to mirror symmetry and the S- and T-dualities of string theory. Kapustin and Witten (2006) relate electric-magnetic duality to a part of the Langlands Program. There are many indications again of category theory playing a key role.

But perhaps Lautman's problem is that he instinctively points us to manifestations of the same idea in mathematics which is itself ultimately approachable by mathematical theory. There are suggestions that not all of mathematics goes this way.

3   The Two Cultures of Mathematics

While Lautman's aesthetic sense regarding mathematics drove him to examples which have ultimately shown themselves to be addressable by category theory, other such aesthetic senses are current. While I believe Lautman would have thoroughly enjoyed the material we discuss on the blog I jointly run, *The n-Category Café*, the blog written by Terence Tao reveals a rather different sensibility. The best effort to capture this difference is, I believe, Timothy Gowers essay *The Two Cultures of Mathematics*, in which the distinction is made between 'theory-builders' and 'problem-solvers'. I think we have to be very careful with these labels, as Gowers himself is.

> ...when I say that mathematicians can be classified into theory-builders and problem-solvers, I am talking about their priorities, rather than making the ridiculous claim that they are exclusively devoted to only one sort of mathematical activity. (p. 2)

To avoid misunderstanding, then, perhaps it is best to give straight away paradigmatic examples of work from each culture.

- Theory-builders: Grothendieck's algebraic geometry, Langlands Program, mirror symmetry, elliptic cohomology.

- Problem-solvers: Combinatorial graph theory, Ramsey's theorem, Szemerédi's theorem, arithmetic progressions among the primes.

Gowers mentions Sir Michael Atiyah as a prime example of a theory builder, and recommends his informal essays, the 'General papers' of Volume 1 of his Collected Works. Indeed, they convey an aesthetic which I came to admire enormously as a PhD student in philosophy. On the other hand, Paul Erdös is mentioned as a consummate problem-solver. What then of the corresponding aesthetic?

One of the attractions of problem-solving subjects, which Gowers collects under the loose mantle of 'combinatorics', is the easy accessibility of the problems.

> One of the great satisfactions of mathematics is that, by standing on giants' shoulders, as the saying goes, we can reach heights undreamt of by earlier generations. However, most papers in combinatorics are self-contained, or demand at most a small amount of background knowledge on the part of the reader. Contrast that with a theorem in algebraic number theory, which might take years to understand if one begins with the knowledge of a typical undergraduate syllabus. (p. 12)

For someone who had recently won a Fields' Medal, it would seem strange to feel the need to defend ones interests, but after describing a problem involving the Ramsey numbers, Gowers writes:

> I consider this to be one of the major problems in combinatorics and have devoted many months of my life unsuccessfully trying to solve it. And yet I feel almost embarrassed to write this, conscious as I am that many mathematicians would regard the question as more of a puzzle than a serious mathematical problem. (p. 11)

Two types of appeal which are commonly made to warrant the importance of one's field are its connections to other fields and its applicability. Now,

> As for connections with other subjects, there are applications of combinatorics to probability, set theory, cryptography, communication theory, the geometry of Banach spaces, harmonic analysis, number theory ... the list goes on and on. However, I am aware as I write this that many of these applications would fail to impress a differential geometer, for example, who might regard all of them as belonging somehow to that rather foreign part of mathematics that can be safely disregarded. Even the applications to number theory are to the "wrong sort" of number theory. (p. 13)

The Green-Tao theorem, concerning the lengths of arithmetic progressions amongst the primes, might be a good candidate to illustrate this "wrong sort" of number theory. Indeed, this provided Minhyong Kim with a nice way to represent the difference between cultures. Which of the following attracts us more?:

- The theorem about primes in arithmetic progressions.

- The theorem about arithmetic progressions in primes.

The first of these was a result by Dirichlet, an early step in the theory-builders' topic algebraic number theory.

Now, it's not that, on the theory-building side, all number theoretic results emerging from the "right sort" of number theory are deemed important. Indeed, Fermat's Last Theorem has come in for plenty of abuse over the years. Rather, it was the successful activity behind the scenes leading to the proof of the Taniyama-Shimura conjecture that is generally regarded as the major achievement. So, even were results about the existence of arithmetic progressions amongst the primes to be judged similarly abused, there might again be some general result lurking behind the scenes. However, according to Gowers, in combinatorics one deals not so much with general theorems, but rather broad principles. For example,

> ...if one is trying to maximize the size of some structure under certain constraints, and if the constraints seem to force the extremal examples to be spread about in a uniform sort of way, then choosing an example randomly is likely to give a good answer. (p. 6)

It is no accident that category theory does not come into play in Gower's 'combinatorics'. Thus, if one is according importance to mathematical activity in terms of its impact on mathematics as a whole, then rather than the transfer of theoretical results and apparatus between fields, made so much easier by category theory, it may be necessary to look to more subtle relationships, such as when:

> Area A is sufficiently close in spirit to area B, that anybody who is good at area A is likely to be good at area B. Moreover, many mathematicians make contributions to both areas. (p. 14)

Now, it certainly seems that the common 'spirit' may be expressed as an idea. For example, in his paper, *The dichotomy between structure and randomness, arithmetic progressions, and the primes* (Tao 2005), Terence Tao tells us about manifestations in much of his work of the structure/randomness idea. Is this territory more promising for Lautman?

First, we should say that the current mathematical state of play now should not mislead us. It is possible that a more explicit general theory will emerge to account for the results of Tao and Gowers, just as an abstract theory of the Galoisian dialectical idea has emerged since Lautman's day. For thoughts of a possible reconciliation with regard to the two approaches to number theory, see Kim 2007. But, even if no such explicit theory captures the structure/randomness pair, as the category theoretic formulation does the Galoisian idea, it seems to me that we can do no other than take Tao's paper as mathematical. If there is no alternative access to an idea than a mathematical one, I don't see why the idea shouldn't be seen as mathematical.

4 Conclusion

Lautman's study of reciprocity is placed after an account of Heidegger's ideas on the difference between 'ontological' and 'ontic', 'essence' and 'existence'. I have not mentioned this discussion here, finding its language to be rather foreign to

my own. Instead I have dwelt on the Lautman's attempts to situate the source of the *reality* of mathematics in a place which lies beyond mathematics.

> Nous voudrions montrer, avant de conclure, comment cette conception d'une réalité idéale, supérieure aux mathématiques et pourtant si prête à s'incarner dans leur movement, vient s'intégrer dans les intérpretations les plus autorisées du platonisme. (Lautman, p. 230)

I have argued that this 'réalité idéale' does not lie beyond mathematics but rather is the core of mathematics itself, and may be approached mathematically in many cases on a more abstract level than via its instantiations in specific mathematical theories.

An alternative position *would* want to count the treatment of ideas at this level as philosophical. William Lawvere in his 'Categories of Space and of Quantity' (Lawvere 1992) writes:

> It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attentin by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy. (p. 16)

But it is really of little importance how we designate the kind of work done in common by Lautman and Lawvere. What *is* important, on the other hand, is that we recognise how central this work should be to the philosophical understanding of mathematics. Even if I am right to want to name as *mathematics* the extraction of the Galoisian idea and other such ideas, Lautman has still done us an enormous service in being such a reliable guide to mathematical reality. If people want to reflect on this reality they can do no better than look at theoretical developments, parts of whose courses appear in his case studies. For example, the ideas encapsulated in the Langlands Program are quintessential pieces of mathematical reality. But rather than point us to a reality superior to mathematics, I believe we can best see the situation as a case of a discipline dealing with its own specific reality. In other words, mathematical reality is an instance of reality *tout court*.

The interest of mathematics to philosophy, then, is that it provides another example of the notion of reality. This notion manifests itself in mathematics one way, biology another, politics another and art yet another. Lautman's gift was to tap in extraordinarily early to the aesthetic sensibility which was to come to dominate large sections of mathematics of the twentieth century. This sensibility is nothing less than mankind's most powerful access to mathematical reality to date. With it the mind has become more adequate to its object.

Another philosopher who took our struggles to expand our knowledge and the movement of our theories as indices of the real was Michael Polanyi. His idea of reality is "that which may yet inexhaustibly manifest itself" (Polanyi 1969, p. 141). In the context of mathematics he writes,

...while in the natural sciences the feeling of making contact with reality is an augury of as yet undreamed of future empirical confirmations of an immanent discovery, in mathematics it betokens an indeterminate range of future germinations within mathematics itself. (Polanyi 1958: 189)

This is a description of mathematical reality with which I can rest happy. In some people, notaly Galois, Poincaré and Grothendieck, this feeling has shown itself to be have been correct by the extraordinarily rich range of future germinations following from their work.

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