A stronger Bell argument for quantum non-locality

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It is widely accepted that the violation of Bell inequalities excludes local theories of the quantum realm. In this paper I present a stronger Bell argument which even forbids certain non-local theories. The remaining non-local theories, which can violate Bell inequalities, are characterised by the fact that at least one of the outcomes in some sense probabilistically depends both on its distant as well as on its local parameter. While this is not to say that parameter dependence in the usual sense necessarily holds, it shows that the received analysis of quantum non-locality as "outcome dependence or parameter dependence" is deeply misleading about what the violation of Bell inequalities implies.

Contents

1 Introduction

Quantum non-locality is the fact that some features of certain microscopic objects fundamentally depend on another although they are space-like separated. The dependence in question can be understood in a probabilistic or in a metaphysical sense. The received view of quantum non-locality involves both senses and is established in a four-step argument:

(i) The basis of the reasoning is the Bell argument (Bell, 1964, 1971, 1975), which is a mathematical no-go theorem. It proves that the results of experiments with entangled quantum objects (EPR/B correlations) violate Bell inequalities and that this violation excludes all local theories of the quantum realm. In its most general version the Bell argument is completely formulated in probabilistic terms and its result is that the total probability distribution of the experiments does not factorise into local terms. So whatever the exact dynamics and state description of a theory might be: as long as it implies a local probability distribution it cannot be the true physical theory of our world. In this sense all local theories are excluded. So the quantum non-locality which *directly* follows from the violation of Bell inequalities, the failure of local factorisation, is first of all a *probabilistic* fact.

(ii) In a second step, Jarrett (1984) tries to make explicit what the probabilistic notion of quantum non-locality amounts to, still on a probabilistic level. He analysed the failure of local factorisation as the disjunction of two pairwise probabilistic dependencies, "outcome dependence or parameter dependence". This result is assumed to state the probabilistic nature of quantum non-locality: there must be either a dependence between the outcomes or between at least one of the outcomes and its distant parameter (or both). Note that so far in the argument purely mathematical consequences have been drawn from the violation of Bell inequalities.

(iii) Many philosophers of science continue to argue that it is outcome dependence and not parameter dependence which holds because parameter dependence is incompatible with relativity, while outcome dependence is not (Jarrett, 1984; Shimony, 1984, 1990; Arntzenius, 1994). The true theory about the quantum world, they maintain, very likely is "outcome dependent and parameter *in*dependent", just as quantum theory is. This step invokes a physical assumption, namely the compatibility with relativity.

(iv) Finally, the probabilistic dependencies are given a metaphysically interpretation. While parameter dependence is mostly seen as constituting a causal relation, outcome dependence is interpreted as a non-causal influence ("passion at-a-distance": Shimony, 1984; Redhead, 1986, 1987, 1989) or a kind of non-separability / physical holism (Howard, 1985, 1989; Teller, 1986, 1989; Jarrett, 1989, 2009; Healey, 1991, 1994; Fogel, 2007). This establishes a *metaphysical* notion of quantum non-locality. (Note that some authors swap the last two steps: they first give the metaphysical interpretation, (iv), and then argue for outcome dependence and parameter independence, (iii), because relativity is assumed to be incompatible with space-like causal relations.)

Steps (iii) and (iv) are not uncontroversial. They have been criticized mainly for the fact that regardless of whether parameter dependence or outcome dependence holds, there must be a causal relation between the two wings (Butterfield, 1992; Jones and Clifton, 1993; Maudlin, 2011; Berkovitz, 1998a,b). However, we shall not engage in this discussion here. In this paper we shall focus on the so far widely uncontroversial mathematical consequences of the violations of Bell inequalities, i.e. the probabilistic concept of quantum non-locality (i) and its analysis (ii). My main claim is not that steps (i) and (ii) are false but that the conclusions are weaker than they could be. This might be important because the total argument from (i) to (iv) shows that the mathematical consequences, which follow from the violation of Bell inequalities, are the basis for discussing the metaphysical implications; they are the material which is interpreted in order to reach metaphysical conclusions. So if we want to get the philosophical conclusions right, we first have to infer appropriate mathematical results—and the latter is what we shall attempt in this paper. We investigate what exactly the violation of Bell inequalities implies on the probabilistic level: *which variables have to depend probabilistically on another given the results of experiments with entangled quantum objects*? From now on, when I use the term "quantum non-locality" I always mean—if not otherwise stated—the *probabilistic* consequences of the violation of Bell inequalities.

The present paper is divided into two parts. First, I shall show how the usual Bell argument for quantum non-locality can be made stronger to give a new, tightly fitting concept of quantum non-locality (section 2). The basic idea is that the violation of Bell inequalities excludes even more than just the local theories, for certain kinds of *non*-local theories turn out to be too weak to violate Bell inequalities as well. Accordingly, the new result requires to redefine the probabilistic notion quantum non-locality from the failure of locality to the failure of locality *and* of those weak forms of non-locality, which is a considerably more informative concept.

In a second part, I analyse this new concept in terms of pairwise independencies, in a similar manner as Jarrett analysed the failure of local factorisation (section 3). The result of the analysis will be that, regardless of whether the outcomes depend on another or not, there must be a certain dependence between at least one of the outcomes and both its local and its distant parameter.

This will bring out that, though being logically correct, the received analysis of quantum non-locality as "outcome dependence or parameter dependence" paints a fatally deceptive picture of the probabilistic dependencies which are implied by the violation of Bell inequalities. For in this catchy short form it is liable to be misunderstood to say that one can avoid *any* probabilistic dependence of an outcome on its distant parameter if one accepts that outcomes depend on each other. (Moreover, step (iii) in the above argument seems to say that this is *in fact* the case.) In this wrong reading there is a contradiction to my analysis of the new stronger concept, which says that in some sense at least one of the outcomes *must* depend on its distant parameter. The tension resolves if one realises that the standard analysis only says that you can avoid a *certain* dependence of the outcome on the distant parameter, namely parameter dependence, if you accept a certain dependence between the outcomes. But there are other kinds of probabilistic dependencies between an outcome and its distant parameter, which differ from parameter dependence by the conditional variables! So accepting the new analysis is not to say that one has to accept parameter dependence in the usual sense of the word. This will become clearer in the course of this paper, when we have clearly defined the corresponding mathematical expressions. In any case, it is important to note that quantum mechanics is *not* ruled out by my new analysis.

As the procedure suggests, my new analysis strongly relies on the standard analysis but attempts to improve it towards a stronger result. As usual in the Bell-Jarrett approach, the characterizations of probability distributions are *qualitative*: we shall mainly be concerned with the question *whether two variables are independent or not*, and not how strong a possible correlation would have to be. The only quantitative fact which will be used are the empirically measured EPR/B correlations.

2 Two concepts of quantum non-locality

2.1 EPR/B experiments and correlations

Many arguments for a quantum non-locality consider an EPR/B setup with polarisation measurements of photons (fig. 1; Einstein, Podolsky, and Rosen 1935; Bohm 1951; Clauser and Horne 1974). One run of the experiment goes as follows: a suitable source C (e.g. a calcium atom) is excited and emits a pair of photons whose quantum mechanical polarisation state $\boldsymbol{\psi}$ is entangled. Possible hidden variables of this state are called $\boldsymbol{\lambda}$, so that the *complete* state of the particle at the source is $(\boldsymbol{\psi}, \boldsymbol{\lambda})$. Since the preparation procedure is usually the same in all runs, the quantum mechanical state $\boldsymbol{\psi}$ is the same in all runs and will not explicitly be noted in the following. (One may think of any probability being conditional on one fixed state $\boldsymbol{\psi} = \psi_0$.) After the emission, the photons move in opposite direction towards two polarisation measurement devices A and B, whose measurement directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are randomly chosen among two of three possible settings ($\boldsymbol{a} = 1, 2$; $\boldsymbol{b} = 2, 3$) while the photons are on their flight. A photon either passes the polariser (and is detected) or is absorbed by it (and is not detected), so that at each measuring device there are two possible measurement outcomes $\boldsymbol{\alpha} = \pm$ and $\boldsymbol{\beta} = \pm$.

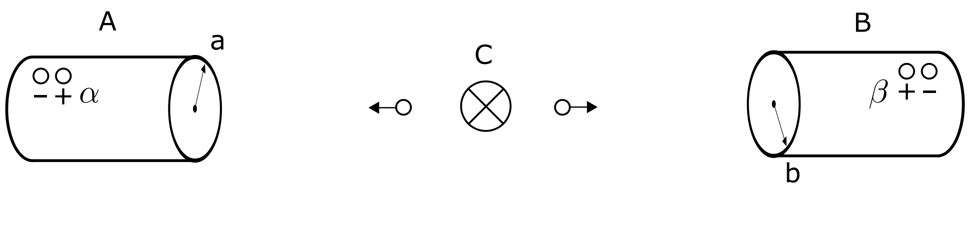

Fig. 1: EPR/B setup

On a probabilistic level, the experiment is described by the joint probability distribution $P(\alpha\beta ab\lambda) := P(\boldsymbol{\alpha} = \alpha, \boldsymbol{\beta} = \beta, \boldsymbol{a} = a, \boldsymbol{b} = b, \boldsymbol{\lambda} = \lambda)$ of the five random variables just defined.[1] We shall consistently use bold symbols $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{a}, \dots)$ for random variables and normal font symbols $(\alpha, \beta, a, \dots)$ for the corresponding values of these variables. We use indices to refer to *specific* values of variables, e.g. $\alpha_- = -$ or $a_1 = 1$, which provides useful shorthands, e.g. $P(\alpha_-\beta_+a_1b_2\lambda) := P(\boldsymbol{\alpha} = -, \boldsymbol{\beta} = +, \boldsymbol{a} = 1, \boldsymbol{b} = 2, \boldsymbol{\lambda} = \lambda)$. Expressions including probabilities with non-specific values of variables, e.g. $P(\alpha|a) = P(\alpha)$, are meant to hold for all values of these variables (if not otherwise stated).

Containing the hidden states $\boldsymbol{\lambda}$, which are by definition not measurable, the total distribution is empirically not accessible ("hidden level"), i.e. purely theoretical. Only the marginal distribution which does not involve $\boldsymbol{\lambda}$, $P(\alpha\beta ab)$, *is* empirically accessible and determined by the results of actual measurements in EPR/B experiments ("observable level").[2]

Although the EPR/B setup is constructed in order to weaken and minimize correlations between the involved variables,[3] a statistical evaluation of a series of many runs with similar preparation procedures yields that there *are* observable correlations: the outcomes are correlated given the parameters,[4]

$$P(\alpha\beta|ab) = P(\alpha|\beta ab)P(\beta) = \begin{cases} \cos^2\phi_{ab} \cdot \frac{1}{2} & \text{if } \alpha = \beta \\ \sin^2\phi_{ab} \cdot \frac{1}{2} & \text{if } \alpha \neq \beta \end{cases} \qquad \text{(Corr)}$$

(where ϕ_{ab} is the angle between the measurement directions a and b). These famous EPR/B correlations between space-like separated measurement outcomes have first been measured by Aspect et al. (1982) and are correctly predicted by quantum mechanics.

2.2 The standard Bell argument for quantum non-locality

Since according to (Corr), one outcome depends on both the other space-like separated outcome as well as on the distant (and local) parameter, the observable part of the probability distribution is clearly non-local. Bell (1964) could show that EPR/B correlations are so extraordinary that even if one allows for hidden states $\boldsymbol{\lambda}$ one cannot restore locality: given EPR/B correlations

[1] While the outcomes and settings are discrete variables, the hidden state may be continuous or discrete. In the following I assume λ to be discrete, but all considerations can be generalized to the continuous case.

[2] The (theoretical) transition from the total probability distribution to the observable marginal distribution is given by a marginalisation over λ, $P(\alpha\beta ab) = \sum_\lambda P(\alpha\beta ab\lambda)$. In order to be empirically adequate, any theoretical distribution must in this way yield the distribution which describes the statistics of EPR/B measurements.

[3] First, the settings are random and statistically independent. Second, the parameters are set *after* the emission, so that the setting may not influence the state of the particles at the emission. And, finally, but most importantly, the wings of the experiment are space-like separated, so that according to the First Signal Principle of relativity there cannot be any influence from one outcome to the other or from one setting to the outcome on the other wing.

[4] A correlation of the outcomes means that the joint probability $P(\alpha\beta|ab)$ is in general not equal to the product $P(\alpha|ab)P(\beta|ab) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

the *theoretical* probability distribution (including possible hidden states) must be non-local as well. Hence, *any possible* probability distribution which might correctly describe the experiment must be non-local.

This "Bell argument for quantum non-locality", as I shall call it, runs as follows. Bell realized the mathematical fact that EPR/B correlations have the remarkable feature to violate Bell inequalities. Since Bell then did not know that suitable measurements indeed yield the correlations, the violation was merely hypothetical, but today the violation of Bell inequalities is an empirically confirmed fact. It follows that at least one of the assumptions in the derivation of the inequalities must be false. Indeterministic generalizations (Bell, 1971; Clauser and Horne, 1974; Bell, 1975) of Bell's original deterministic derivation (1964) employ two probabilistic assumptions, "local factorisation"[5]

$$P(\alpha\beta|ab\lambda) = P(\alpha|a\lambda)P(\beta|b\lambda) \tag{ℓF}$$

and "autonomy"

$$P(\lambda|ab) = P(\lambda). \tag{A}$$

Another type of derivation (Wigner, 1970; van Fraassen, 1989; Graßhoff et al., 2005) additionally requires the empirical fact that there are perfect correlations (PCorr) between the outcomes if the measurement settings are equal. For both types of derivation we have the dilemma that any empirically correct probability distribution of the quantum realm must either violate autonomy or local factorisation (or both). Since giving up autonomy seems to be ad hoc and implausible ("cosmic conspiracy"), most philosophers conclude that the empirical violation of Bell inequalities implies that local factorisation fails. And since local factorisation states the factorisation of the hidden joint probability distribution into *local* terms, the *failure* of local factorisation indicates a certain kind of *non*-locality, which is specific to the quantum realm—hence "quantum non-locality".

In order to make the logical structure clear let me note the Bell argument in an explicit logical form (where (I1), (I2), ... indicate intermediate conclusions). Here and in the following I shall use the Wigner-type derivation of Bell inequalities because, as we will see, it is the most powerful one allowing to derive Bell inequalities from the widest range of probability distributions:

(P1) There are EPR/B correlations: (Corr)

(P2) EPR/B correlations violate Bell inequalities: (Corr) $\rightarrow \neg$(BI)

  (I1) Bell inequalities are violated: $\neg$(BI)   (from P1 & P2, MP)

(P3) EPR/B correlations include perfect correlations: (Corr) $\rightarrow$ (PCorr)

[5] "Local factorisation" is my term. Bell calls (ℓF) "local causality", some call it "Bell-locality", but most often it is simply called "factorisation" or "factorisability". Bell's terminology already suggests a metaphysical interpretation, which I would like to avoid in this paper, and the latter two names are too general since, as I shall show, there are other forms of the hidden joint probability which can be said to factorise; hence "*local* factorisation".

(I2) There are perfect correlations: (PCorr) (from P1 & P3, MP)

(P4) Bell inequalities can be derived from autonomy, perfect correlations and local factorisation: $(\mathrm{A}) \wedge (\mathrm{PCorr}) \wedge (\ell\mathrm{F}) \rightarrow (\mathrm{BI})$

(I3) Autonomy or local factorisation has to fail: $\neg(\mathrm{A}) \vee \neg(\ell\mathrm{F})$ (from I1 & I2 & P4, MT)

(P5) Autonomy holds: (A)

(C1) Local factorisation fails: $\neg(\ell\mathrm{F})$ (from I3 & P5)

(P6) Quantum non-locality is the failure of local factorisation: $(\mathrm{QNL}) :\leftrightarrow \neg(\ell\mathrm{F})$ (definition)

It is obvious that the argument from (P1)–(P5) to (C1) is valid. It shows that if autonomy holds, EPR/B correlations mathematically imply a non-locality which is called quantum non-locality (P6). (P6) is *not* a premise of the Bell argument but labels its result with an appropriate name; it determines what *quantum non-locality according to the standard view* amounts to an a probabilistic level. It is clear that if the Bell argument could be modified to have a stronger conclusion, the definition (P6) would have to be adapted. What we call "quantum non-locality" depends on the result of the Bell argument. In this sense the *analysis* of quantum non-locality, in which (P6) functions as a premise (it determines the analysandum, see section 3), is based on the Bell argument. Note that defining quantum non-locality as the conclusion of the Bell argument, the logical structure of the argument is such that quantum non-locality only provides *necessary conditions* for EPR/B correlations, i.e. for being empirically adequate. So we have to keep in mind that the analysis of quantum non-locality is *not* an analysis of EPR/B correlations but of a necessary condition for them.

The core idea of my critique concerning the standard view of quantum non-locality is that the result of the Bell argument is weaker as it could be. I do not say that the argument is invalid nor do I say that one of its premises is not sound, I just say that the argument can be made considerably stronger and that the stronger conclusion will allow us to define a tighter, more informative concept of quantum non-locality: one can be much more precise about what EPR/B correlations imply (if we assume that autonomy holds) than just saying that local factorisation has to fail. I shall show that besides the local classes EPR/B correlations also exclude certain non-local classes. Given this new result, the standard definition of quantum non-locality (P6) will turn out to be inappropriately weak, because it includes those non-local classes which can be shown to be forbidden.

Specifically, I shall show that it is premise (P4) which can be made stronger (while leaving the other premises basically at work). Being an implication from autonomy, perfect correlations and local factorisation to Bell inequalities, it is clear that we can make (P4) the stronger the weaker we can formulate the antecedens, i.e. the assumptions to derive the inequalities. This idea is not essentially a new one. Since Bell's original proof (1964) considerable efforts have been made to find derivations with weaker and weaker assumptions. For example, one of the milestones was to show that one can derive Bell inequalities without the original assumption of determinism. Currently, autonomy and local factorisation seem to constitute the weakest set of probabilistic assumptions which allow a derivation. What will be new about my approach is to try to find *alternatives to local factorisation*, which (given autonomy and perfect correlations) also imply that Bell inequalities hold. Since local factorization is the weakest possible form of local distributions, it is clear that such alternatives have to involve a kind of *non-locality*, i.e. what I am trying to show in the following is that we can derive Bell inequalities from certain non-local probability distributions.

2.3 Classes of probability distributions

We can find potential alternatives to local factorisation if we consider what it is: a particular feature of the *hidden joint probability*, as I shall call $P(\alpha\beta|ab\lambda)$. According to the product rule of probability theory, for *any* of the possible hidden probability distributions the joint probability of the outcomes (given the other variables) can be written as a product,

$$\begin{aligned} P(\alpha\beta|ab\lambda) &= P(\alpha|\beta ba\lambda)P(\beta|ab\lambda) && (1) \\ &= P(\beta|\alpha ab\lambda)P(\alpha|ba\lambda). && (2) \end{aligned}$$

Since there are two product forms, one whose first factor is a conditional probability of $\boldsymbol{\alpha}$ and one whose first factor is a conditional probability of $\boldsymbol{\beta}$, for the time being, let us restrict our considerations to the product form (1), until at the end of this section I shall generalize the results to the other form (2).

The product form (1) of the hidden joint probability holds in general, i.e. for all probability distributions. According to probability distributions with appropriate independencies, however, the factors on the right-hand side of the equation reduce in that certain variables in the conditionals can be left out. If, for instance, outcome independence holds, $\boldsymbol{\beta}$ can disappear from the first factor, and the joint probability is said to "factorise". *Local* factorisation further requires that the distant parameters in both factors disappear, i.e. that parameter independence holds. Prima facie, any combination of variables in the two conditionals in (1) seems to constitute a distinct product form of the hidden joint probability. Restricting ourselves to *irreducibly hidden* joint probabilities, i.e. requiring $\boldsymbol{\lambda}$ to appear in both factors, there are $2^5 = 32$ *combinatorially* possible forms (for any of the three variables in the first conditional and any of the two variables in the second conditional *besides* $\boldsymbol{\lambda}$ can or cannot appear). Table 1 shows these conceivable forms which I label by (H_1^α) to (H_{32}^α) (the superscript α is due to the fact that we have used (1) instead of (2)).

The specific product form of the hidden joint probability is *the essential feature* of the probability distributions of EPR/B experiments. For, as we shall see, it not only determines whether a probability distribution can violate Bell inequalites but also carries unambiguous information about which variables of the

Table 1: Classes of probability distributions

	I	II	III	IV		V	VI		VII	VIII	IX
	(H$_i^\alpha$):	$P(\alpha\beta\|ab\lambda) = \ldots$									
	i	$P(\alpha\|$ β	b	a	$\lambda)\ \cdot P(\beta\|$	a	b	$\lambda)$	$\square$(BI)	Group	Notes
strong non-locality	1	1	1	1		1	1		0	iv	
	2	1	1	1		1	0		0	iv	
	3	1	1	1		0	1		0	iv	
	4	1	1	0		1	1		0	iv	
	5	1	0	1		1	1		0	iv	
	6	0	1	1		1	1		0	iv	Bohm
	7	1	1	1		0	0		0	iv	QM
	8	0	1	1		1	0		0	iv	
	9	0	1	1		0	1		0	iv	
	10	1	0	0		1	1		0	iv	
	11	0	1	0		1	1		0	iv	
	12	0	0	1		1	1		0	iv	
	13	0	1	1		0	0		0	iv	
	14	0	0	0		1	1		0	iv	
weak non-locality	15	1	1	0		1	0		1	iii	
	16	1	0	1		0	1		1	iii	
	17	1	0	1		1	0		1	i	
	18	1	1	0		0	1		1	i	
	19	1	1	0		0	0		1	i	
	20	1	0	1		0	0		1	i	
	21	1	0	0		1	0		1	i	
	22	0	1	0		1	0		1	ii	
	23	0	0	1		1	0		1	i	
	24	1	0	0		0	1		1	i	
	25	0	1	0		0	1		1	i	
	26	1	0	0		0	0		1	i	
	27	0	1	0		0	0		1	i	
	28	0	0	0		1	0		1	i	
locality	29	0	0	1		0	1		1	ii	local fact.
	30	0	0	1		0	0		1	i	
	31	0	0	0		0	1		1	i	
	32	0	0	0		0	0		1	i	
Analysis:		$\neg$(OI$_1$)	$\neg$(PI$_1^\alpha$)	$\neg$(ℓPI$_1^\alpha$)		$\neg$(PI$_2^\beta$)	$\neg$(ℓPI$_2^\beta$)				

experiment are probabilistically independent of another. Virtually any interesting philosophical question involving probabilistic facts of EPR/B experiments depends on the specific product form of the hidden joint probability. Hence, it is natural to use the product form of the hidden joint probability in order to classify the probability distributions. We can say that each product form of the hidden joint probability constitutes a *class of probability distributions* in the sense that probability distributions with the same form (but different numerical weights of the factors) belong to the same class. In order to make the assignment of probability distributions to classes unambiguous let us require that each probability distribution belongs only to that class which corresponds to its *simplest* product form, i.e. to the form with the minimal number of variables appearing in the conditionals (according to the distribution in question).

This scheme of classes is comprehensive: Any probability distribution of the EPR/B experiment must belong to one of these 32 classes. In this systematic overview, the class constituted by local factorisation is (H^{α}_{29}) (see table 1, column IX), and if we allow that there might be no hidden states $\boldsymbol{\lambda}$, we can assign the quantum mechanical distribution to class (H^{α}_{7}). The de-Broglie-Bohm theory falls under class (H^{α}_{6}), and similarly any other theory of the quantum realm has its unique place in one of the classes. The advantage of this classification is that it simplifies matters insofar we can now derive features of *classes* of probability distributions and can be sure that these features hold for all members of the class, i.e. for all theories whose probability distributions fall under the class in question. The feature that we are most interested in is, of course, which of these classes (given autonomy) make Bell inequalities hold.

2.4 Bell inequalities from local and non-local distributions

In order to discern those classes which imply Bell inequalities (if autonomy and perfect correlations hold) from those which do not, it will provide useful to partition the classes into four groups depending on which variables appear in their constituting product forms (see table 1, column VIII):

(i) At least one of the parameters does not appear at all: (H^{α}_{17})–(H^{α}_{21}), (H^{α}_{23})–(H^{α}_{28}), (H^{α}_{30})–(H^{α}_{32})

(ii) Both parameters appear separately, one in each factor: (H^{α}_{22}), (H^{α}_{29})

(iii) As (ii) but the first factor additionally involves the outcome $\boldsymbol{\beta}$: (H^{α}_{15}), (H^{α}_{16})

(iv) Both parameters appear together in at least one of the factors: (H^{α}_{1})–(H^{α}_{14})

The claim that I shall attempt to prove in this section is that, given autonomy and perfect (anti-)correlations, the classes belonging to groups (i), (ii) and (iii) imply Bell inequalities while those in (iv) do not. In other words, probability distributions in classes belonging to group (iv) *can* violate Bell inequalities while classes in groups (i), (ii) and (iii) *have to* obey them.

2.4.1 Group (i)

Consider one version of the Wigner-Bell inequality (Wigner, 1970; van Fraassen, 1989),

$$P(\alpha_- \beta_+ | a_1 b_3) \leq P(\alpha_- \beta_+ | a_1 b_2) + P(\alpha_- \beta_+ | a_2 b_3). \tag{3}$$

We can write the probabilities in terms of the hidden probability distribution if we sum over λ,

$$P(\alpha_- \beta_+ | ab) = \sum_\lambda P(\alpha_- \beta_+ | ab\lambda) P(\lambda | ab), \tag{4}$$

and assuming autonomy (A), we can further rewrite it as

$$P(\alpha_- \beta_+ | ab) = \sum_\lambda P(\alpha_- \beta_+ | ab\lambda) P(\lambda). \tag{5}$$

It is obvious that in this form the empirical joint probability $P(\alpha_- \beta_+ | ab)$ depends on the parameters only via the hidden joint probability $P(\alpha_- \beta_+ | ab\lambda)$. Hence, if a certain parameter does not appear in a specific product form of the hidden joint probability (group (i)), the empirical joint probability becomes independent of this parameter. Consider, for instance, how class (H^{α}_{17}), the product form of which does not involve the parameter $\boldsymbol{b}$,

$$P(\alpha\beta | ab\lambda) = P(\alpha | \beta a \lambda) P(\beta | a\lambda), \tag{6}$$

makes the empirical joint probability independent of $\boldsymbol{b}$:

$$P(\alpha_- \beta_+ | ab) = \sum_\lambda P(\alpha_- | \beta_+ a\lambda) P(\beta_+ | a\lambda) P(\lambda) = P(\alpha_- \beta_+ | a). \tag{7}$$

Inserting this empirical joint probability, which does not depend on $\boldsymbol{b}$, into the Bell-Wigner inequality, reveals that in this case the inequality holds trivially, just because it has lost its functional dependence on $\boldsymbol{b}$:[6]

$$P(\alpha_- \beta_+ | a_1) \leq P(\alpha_- \beta_+ | a_1) + P(\alpha_- \beta_+ | a_2) \tag{8}$$

(H^{α}_{17}) implying that Bell inequalities hold is surprising because its constituting product form is both *non-local* and *non-factorising*: $\boldsymbol{\beta}$ depends on the distant parameter $\boldsymbol{a}$ in the second factor, $P(\beta|a\lambda)$, and $\boldsymbol{\alpha}$ depends on $\boldsymbol{\beta}$ in the first, $P(\alpha|\beta a\lambda)$, i.e. $\boldsymbol{\lambda}$ and $\boldsymbol{a}$ do *not* screen-off the outcomes from another. However, very similarly, we can show that all other classes in group (i) meet the requirements of Bell inequalities: no matter what kind of non-localities they involve, if at least one of the parameters does not appear in the product form, Bell inequalities hold trivially. Hence, we can conclude that if autonomy holds (which we have used to simplify the expectation value in (5)) distributions in

[6] Note that (7) even *directly* contradicts the empirical distribution (not only indirectly by making Bell inequalities true), because it states that the empirical joint probability does not depend on one of the parameters, which is wrong.

group (i) imply that Bell inequalities hold:[7]

$$\Big[(\mathrm{A}) \wedge \Big(\bigvee_{\substack{i=17-21\\23-28\\30-32}} (\mathrm{H}_i^{\alpha}) \Big)\Big] \rightarrow (\mathrm{BI}) \tag{9}$$

2.4.2 **Group** (ii)

Let us now turn to distributions in group (ii). Since according to this group both parameters appear in the product form (one in each factor), it is clear that, contrary to group (i), here Bell inequalities do not hold just because of the functional dependencies. However, local factorisation (H_{29}^{α}) belongs to this group and we know how we can derive Bell inequalities with this product form of the hidden joint probability. Since the derivation from the other class in this group, (H_{22}^{α}), is very similar, let me first sketch a derivation with local factorisation, which is based on the ideas of Wigner (1970) and van Fraassen (1989).

We proceed from the empirical fact that there are perfect correlations between the measurement outcomes if the settings equal another:

$$P(\alpha_{\pm}\beta_{\mp}|a_i b_i) = 0 \tag{10}$$

Similarly to (5), using autonomy and local factorisation we can rewrite the empirical joint probability in terms of the hidden joint probability,

$$P(\alpha_{\pm}\beta_{\mp}|a_i b_i) = 0 = \sum_{\lambda} P(\lambda) P(\alpha_{\pm}|a_i\lambda) P(\beta_{\mp}|b_i\lambda). \tag{11}$$

Since probabilities are non-negative (and we assume $P(\lambda) > 0$ for all λ), at least one of the two remaining factors in each summand must be zero, i.e. for all values of i and λ we must have:

$$\begin{aligned} & \big[P(\alpha_+|a_i\lambda) = 0 \quad \vee \quad P(\beta_-|b_i\lambda) = 0\big] && (12)\\ \wedge \quad & \big[P(\alpha_-|a_i\lambda) = 0 \quad \vee \quad P(\beta_+|b_i\lambda) = 0\big] && (13) \end{aligned}$$

There are two cases. Suppose first that $P(\alpha_+|a_i\lambda) = 0$. From there all other probabilities follow as either 0 or 1:

$$\overset{\text{(CE)}}{\Rightarrow} P(\alpha_-|a_i\lambda) = 1 \qquad \overset{(13)}{\Rightarrow} P(\beta_+|b_i\lambda) = 0 \qquad \overset{\text{(CE)}}{\Rightarrow} P(\beta_-|b_i\lambda) = 1$$

Here, "(CE)" stands for "complementary event" and refers to a theorem of probability theory that the sum of the probability of an event A and of its complementary event $\overline{A}$ is 1, e.g. $P(\alpha_+|a_i\lambda) + P(\alpha_-|a_i\lambda) = 1$.

Assume, second, that $P(\beta_-|b_i, \lambda) = 0$. Again all other probabilities are

[7] The sign "$\bigvee$" denotes a multiple disjunction, e.g. $\bigvee_{i=1..n}(\mathrm{H}_i^{\alpha}) := (\mathrm{H}_1^{\alpha}) \vee (\mathrm{H}_2^{\alpha}) \vee \ldots \vee (\mathrm{H}_n^{\alpha})$.

determined to be either 0 or 1:

$$\overset{\text{(CE)}}{\Rightarrow} P(\beta_+|b_i\lambda) = 1 \qquad \overset{(13)}{\Rightarrow} P(\alpha_-|a_i\lambda) = 0 \qquad \overset{\text{(CE)}}{\Rightarrow} P(\alpha_+|a_i\lambda) = 1$$

In order to avoid contradiction the two cases have to be disjunct. So given a certain measurement direction i, the two cases define a partition of the values of $\boldsymbol{\lambda}$: all values of $\boldsymbol{\lambda}$ for which $P(\alpha_+|a_i\lambda) = 0$ belong to the set $\Lambda(i)$, while all other values, for which $P(\alpha_-|a_i\lambda) = 0$, belong to $\overline{\Lambda(i)}$. Note that each value of i defines a different partition.

We can use the fact that the $\boldsymbol{\lambda}$-partitions depend on just *one* parameter i to calculate the hidden joint probability $P(\alpha\beta|ab\lambda)$ for *any* choice of measurement directions a_ib_j by forming intersections of partitions for different parameters (see table 2). Since all values are either 0 or 1 we have shown that determinism holds on the hidden level.

Table 2: Values of the hidden joint probability

	$\lambda \in$			
	$\Lambda(i) \cap \Lambda(j)$	$\Lambda(i) \cap \overline{\Lambda(j)}$	$\overline{\Lambda(i)} \cap \Lambda(j)$	$\overline{\Lambda(i)} \cap \overline{\Lambda(j)}$
$P(\alpha_+\beta_+|a_ib_j\lambda) =$	0	0	0	1
$P(\alpha_+\beta_-|a_ib_j\lambda) =$	0	0	1	0
$P(\alpha_-\beta_+|a_ib_j\lambda) =$	0	1	0	0
$P(\alpha_-\beta_-|a_ib_j\lambda) =$	1	0	0	0

Given table 2, i.e. determinism and the composability of the $\boldsymbol{\lambda}$-partitions (each of which depends on just one parameter), it is easy to show that Wigner-Bell inequalities must hold. Consider the inequality

$$P(X \cap \overline{Z}) \leq P(X \cap \overline{Y}) + P(Y \cap \overline{Z}), \tag{14}$$

which in general holds for any events X, Y, Z of a measurable space (the validity of the inequality is obvious if one draws a Venn diagram, see (Neapolitan and Jiang, 2006)). Assuming $X = \Lambda(1)$, $Y = \Lambda(2)$ and $Z = \Lambda(3)$ gives the inequality

$$P(\Lambda(1) \cap \overline{\Lambda(3)}) \leq P(\Lambda(1) \cap \overline{\Lambda(2)}) + P(\Lambda(2) \cap \overline{\Lambda(3)}). \tag{15}$$

We can now express the probabilities in the inequality by the empirical probability distribution if we use the hidden joint probability from table 2, e.g.:

$$\begin{aligned} P(\Lambda(1) \cap \overline{\Lambda(2)}) \overset{(\sigma\text{-additivity})}{=} & \sum_{\lambda \in \Lambda(1) \cap \overline{\Lambda(2)}} P(\lambda) = \\ \overset{(\text{table 2})}{=} & \sum_{\lambda} P(\lambda) P(\alpha_- \beta_+ | a_1 b_2 \lambda) = \\ \overset{(\text{A})}{=} & P(\alpha_- \beta_+ | a_1 b_2) \end{aligned} \tag{16}$$

The resulting inequality is the Wigner-Bell inequality (3).

This derivation reminds us how local factorisation together with autonomy and perfect correlations implies Bell inequalities. The other class in group (ii), (H^{α}_{22}), differs from local factorisation in that the parameters are swapped: instead of a dependence of each outcome on the local parameters it involves a dependence on the *distant* ones. Regardless of the implicit non-locality it can be used to derive a Bell inequality in a very similar way: given (H^{α}_{22}), instead of (11) we have

$$P(\alpha_\pm \beta_\mp | a_i b_i) = 0 = \sum_{\lambda} P(\lambda) P(\alpha_\pm | b_i \lambda) P(\beta_\mp | a_i \lambda). \tag{17}$$

and by very similar arguments we arrive at a similar partition of the values of $\boldsymbol{\lambda}$: $\Lambda(i)$ denotes all values of lambda for which $P(\alpha_+ | b_i \lambda) = 0$, while the complementary set $\overline{\Lambda(i)}$ is defined by $P(\alpha_- | b_i \lambda) = 0$. If we then calculate the values of the hidden joint probability we arrive at the very same result as in table 2—and the rest of the derivation runs identically up to the Bell-Wigner inequality (3). Hereby we have found another *non-local* hidden joint probability which implies Bell inequalities and the result for group (ii) reads

$$\Big[(\text{A}) \wedge (\text{PCorr}) \wedge \Big(\bigvee_{i=22,29} (\text{H}^{\alpha}_{i}) \Big) \Big] \rightarrow (\text{BI}). \tag{18}$$

2.4.3 Group (iii)

Up to this point you might have been surprised about how easy one can derive Bell inequalities from product forms other than local factorisation, but that one can do it even from classes in group (iii) is my strongest claim. These classes include both parameters, one in each factor, so they do not trivially imply Bell inequalities as classes in group (i). Neither, it seems, can they imply Bell inequalities in the same way as classes in group (ii) because they additionally involve $\boldsymbol{\beta}$ in the first factor. However, they do imply Bell inequalities, and they do it in a very similar (however slightly more complicated) way than classes in group (ii), if besides perfect correlations for equal settings (10) we also assume perfect anti-correlations (PACorr) for perpendicular settings ($a_i b_{i_\perp}$):

$$P(\alpha_\pm, \beta_\pm | a_i, b_{i_\perp}) = 0 \tag{19}$$

Let me sketch the proof for class (H^{α}_{16}) which follows along the lines of that for local factorisation. By autonomy and the product form of (H^{α}_{16}) we rewrite (10) and (19) as

$$P(\alpha_\pm\beta_\mp|a_ib_i) = 0 = \sum_\lambda P(\lambda)P(\alpha_\pm|\beta_\mp a_i\lambda)P(\beta_\mp|b_i\lambda) \tag{20}$$

$$P(\alpha_\pm\beta_\pm|a_ib_{i_\perp}) = 0 = \sum_\lambda P(\lambda)P(\alpha_\pm|\beta_\pm a_i\lambda)P(\beta_\pm|b_{i_\perp}\lambda), \tag{21}$$

and again, at least one of the factors in each summand must vanish, i.e. for all values of i and $\boldsymbol{\lambda}$ (assuming $P(\lambda) > 0$) we must have:

$$\begin{aligned}
&\big[P(\alpha_+|\beta_- a_i\lambda) = 0 \quad \vee \quad P(\beta_-|b_i\lambda) = 0\big] && (22)\\
\wedge\;&\big[P(\alpha_-|\beta_+ a_i\lambda) = 0 \quad \vee \quad P(\beta_+|b_i\lambda) = 0\big] && (23)\\
\wedge\;&\big[P(\alpha_+|\beta_+ a_i\lambda) = 0 \quad \vee \quad P(\beta_+|b_{i_\perp}\lambda) = 0\big] && (24)\\
\wedge\;&\big[P(\alpha_-|\beta_- a_i\lambda) = 0 \quad \vee \quad P(\beta_-|b_{i_\perp}\lambda) = 0\big] && (25)
\end{aligned}$$

As above, from these conditions we can infer that all involved probabilities must be 0 or 1, depending on which of the following two cases holds.

If $P(\alpha_+|\beta_- a_i\lambda) = 0$:

$$\begin{aligned}
&\overset{\text{(CE)}}{\Rightarrow} P(\alpha_-|\beta_- a_i\lambda) = 1 && \overset{(25)}{\Rightarrow} P(\beta_-|b_{i_\perp}\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\beta_+|b_{i_\perp}\lambda) = 1 && \overset{(24)}{\Rightarrow} P(\alpha_+|\beta_+ a_i\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\alpha_-|\beta_+ a_i\lambda) = 1 && \overset{(23)}{\Rightarrow} P(\beta_+|b_i\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\beta_-|b_i\lambda) = 1
\end{aligned}$$

If $P(\beta_-|b_i\lambda) = 0$:

$$\begin{aligned}
&\overset{\text{(CE)}}{\Rightarrow} P(\beta_+|b_i\lambda) = 1 && \overset{(23)}{\Rightarrow} P(\alpha_-|\beta_+ a_i\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\alpha_+|\beta_+ a_i\lambda) = 1 && \overset{(24)}{\Rightarrow} P(\beta_+|b_{i_\perp}\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\beta_-|b_{i_\perp}\lambda) = 1 && \overset{(25)}{\Rightarrow} P(\alpha_-|\beta_- a_i\lambda) = 0\\
&\overset{\text{(CE)}}{\Rightarrow} P(\alpha_+|\beta_- a_i\lambda) = 1
\end{aligned}$$

The cases are disjunct and, hence, define a partition for the values of $\boldsymbol{\lambda}$ for each measurement direction i: $\Lambda(i)$ includes all values of $\boldsymbol{\lambda}$ for which $P(\beta_+|b_i\lambda) = 0$, while $\overline{\Lambda(i)}$ includes the complementary values, which make $P(\beta_-|b_i\lambda) = 0$. Calculating the hidden joint probability $P(\alpha\beta|ab\lambda)$ for an arbitrary choice of measurement directions a_ib_j gives us the very same result as in table 2—and again a Wigner Bell inequality follows.

Since the derivation for class (H_{15}) runs mutatis mutandis, we have shown:

$$\Big[(\mathrm{A}) \wedge (\mathrm{PCorr}) \wedge (\mathrm{PACorr}) \wedge \Big(\bigvee_{i=15,16} (\mathrm{H}^{\alpha}_{i})\Big)\Big] \rightarrow (\mathrm{BI}) \tag{26}$$

2.4.4 Group (iv)

Finally, classes of group (iv) do *not* imply Bell inequalities. Involving both parameters in at least one of the factors, they neither fulfill Bell inequalities by their functional dependencies nor do they admit of deriving a Bell inequality in the manner of classes in group (ii) or (iii). In order to rule out that there are other kinds of derivations one has to find explicit examples of probability distributions for each class in the group which *violate* Bell inequalities. Requiring just *any* example we can assume a toy model with only two possible hidden states ($\boldsymbol{\lambda} = 1, 2$). Then the probability distribution $P(\alpha\beta ab\lambda)$ is determined by assigning a value to each of the $2^5 = 32$ probabilities which conform to the laws of probability theory (each value lies in the interval $[0, 1]$ and all values sum to 1). Furthermore, the values have to be chosen such that autonomy and the specific product form of the class in question hold and that Bell inequalities are violated. I have found appropriate distributions for each class (H_1^α)–(H_{14}^α) by solving numerically a corresponding set of equations. This fact, that some probability distributions of these classes violate Bell inequalities, means that none of these classes implies Bell inequalities in general, i.e. by its constituting product form. Of course, this does *not* mean that *all* probability distributions in these classes *violate* Bell inequalities: in fact one can as well find examples of probability distributions in each class (H_1^α)–(H_{14}^α) which *fulfill* Bell inequalities. This means that for these classes the product form alone does not determine whether Bell inequalities hold or fail; whether they do depends on the numerical values of the specific distribution. On the general level of the classes we can only say that classes in group (iv) neither imply Bell inequalities nor do they imply their failure. They *can* violate Bell inequalities.

2.4.5 Result of the derivations

Reflecting on this result, that probability distributions in group (i), (ii) and (iii) entail that Bell inequalities hold, while distributions in group (iv) do not, it now becomes clear what features a product form of the hidden joint probability must have in order to imply Bell inequalities. *Each of the two factors of the hidden joint probability may at most involve one parameter.* If both involve no parameter or the same, Bell inequalities hold just because of the functional independencies (group (i)), and if each involves exactly one parameter, one can explicitly derive Bell inequalities (group (ii) and (iii)). It does not matter whether the parameters are local or non-local, and neither does it matter whether there is a dependence on the outcome $\boldsymbol{\beta}$ or not. (In discerning product forms which allow and which do not allow deriving Bell inequalities, we have not had to make reference to the outcome $\boldsymbol{\beta}$ at all.) But as soon as there is a dependence on *both* parameters in at least one of the factors of the product form, one can*not* derive Bell inequalities and, on the contrary, can easily find examples of distributions which violate them (group (iv)).

So as opposed to what the standard discussion suggests, it is *not* true that local factorisation is the *only* product form which allows deriving Bell inequalities. Rather, we have found that 18 (!) of the 32 classes imply Bell inequalities if autonomy (and perfect (anti-)correlations) hold (see column VII of table 1; "$\Box$(BI)" means *necessarily, Bell inequalities hold*), among them 14 *non-local* classes. In order to separate these non-local classes from those which do not imply Bell inequalities, I introduce the following names:

local$^\alpha$ classes: (H_{29}^α)–(H_{32}^α) (imply Bell inequalities)

weakly non-local$^\alpha$ classes: (H_{15}^α)–(H_{28}^α) (imply Bell inequalities)

strongly non-local$^\alpha$ classes: (H_1^α)–(H_{14}^α) (do not imply Bell inequalities)

Note the superscript α, which indicates that we refer to classes deriving from (1) (instead of from (2)). Note further that the set of local$^\alpha$ and weakly non-local$^\alpha$ classes is just the union of the classes in group (i), (ii) and (iii) while strongly non-local$^\alpha$ classes correspond to group (iv). With the new terminology, we can formulate our result that, *given autonomy and perfect (anti-)correlations, local*$^\alpha$ *or weakly non-local*$^\alpha$ *classes imply Bell inequalities, while strongly non-local*$^\alpha$ *classes do not.*

2.5 A stronger Bell argument for quantum non-locality

This result, that one can derive Bell inequalities from non-local product forms enables us to strengthen premise (P4) in the Bell argument. We can now write:

(P4') Bell inequalities can be derived from autonomy, perfect correlations, perfect anti-correlations and any local$^\alpha$ or weakly non-local$^\alpha$ class of probability distributions:

$$\left[(\mathrm{A}) \wedge (\mathrm{PCorr}) \wedge (\mathrm{PACorr}) \wedge \Big(\bigvee_{i=15}^{32} (\mathrm{H}_i^\alpha) \Big)\right] \rightarrow (\mathrm{BI})$$

Compared to (P4), we have made two changes. First, we have replaced local factorisation in the antecedent by the *disjunction* of the local$^\alpha$ and weakly non-local$^\alpha$ classes (including local factorisation (H_{29}^α)). This makes the antecedent of (P4') weaker than that in (P4) and, hence, the argument stronger. Second, we have added the condition that there are perfect anti-correlations (PACorr), since classes in group (iii) require them for the derivation. This additional assumption however does not weaken the argument since the perfect anti-correlations follow from the EPR/B correlations (as the perfect correlations do). We just have to modify premise (P3) to

(P3') EPR/B correlations include perfect correlations and perfect anti-correlations:

$(\mathrm{Corr}) \rightarrow (\mathrm{PCorr}) \wedge (\mathrm{PACorr})$

Changing these two premises has a considerable effect on the Bell argument. Instead of the standard conclusion (C1), that the violation implies the failure of local factorisation, by the modified argument from (P1), (P2), (P3'), (P4') and (P5), we arrive at the essentially stronger conclusion:

(C1') Both local$^{\alpha}$ and weakly non-local$^{\alpha}$ classes fail:

$$\Big(\bigwedge_{i=15}^{32} \neg(\mathrm{H}_i^{\alpha})\Big)$$

While the original result, the failure of local factorisation, implied that all local$^{\alpha}$ classes fail (because the other local classes are specializations of local factorisation), the new result additionally excludes all weakly non-local$^{\alpha}$ classes.

Our considerations leading to this new result of the Bell argument rest on the fact that we have found alternatives to local factorisation from writing the hidden joint probability according to the product rule (1) and conceiving different possible product forms (table 1). However, we can as well write the hidden joint probability according to the second product rule (2), and similar arguments as above lead us to a similar table as table 1, whose classes, (H_1^{β})–$(\mathrm{H}_{32}^{\beta})$, differ to those in table 1 in that the outcomes and the parameters are swapped. For instance, class $(\mathrm{H}_{16}^{\beta})$ is defined by the product form $P(\alpha\beta|ab\lambda) = P(\beta|\alpha b\lambda)P(\alpha|a\lambda)$ in contrast to $(\mathrm{H}_{16}^{\alpha})$, which is constituted by $P(\alpha\beta|ab\lambda) = P(\alpha|\beta a\lambda)P(\beta|b\lambda)$. Note that this new classification is a *different partition* of the possible probability distributions. Any probability distribution must fall in exactly one of the classes (H_1^{α})–$(\mathrm{H}_{32}^{\alpha})$ *and* in exactly one of the classes (H_1^{β})–$(\mathrm{H}_{32}^{\beta})$. Following along the lines of section 2.4 we find for the new partition that the local$^{\beta}$ and weakly non-local$^{\beta}$ classes imply Bell inequalities as well, so that we can reformulate (P4') as:

(P4") Bell inequalities can be derived from autonomy, perfect correlations, perfect anti-correlations and any local$^{\alpha}$, weakly non-local$^{\alpha}$, local$^{\beta}$ or weakly non-local$^{\beta}$ class of probability distributions:

$$\Big[(\mathrm{A}) \wedge (\mathrm{PCorr}) \wedge (\mathrm{PACorr}) \wedge \Big(\bigvee_{i=15}^{32} (\mathrm{H}_i^{\alpha}) \vee \bigvee_{i=15}^{32} (\mathrm{H}_i^{\beta})\Big)\Big] \rightarrow (\mathrm{BI})$$

Then we can formulate an even stronger Bell argument from (P1), (P2), (P3'), (P4") and (P5) to

(C1") All local$^{\alpha}$, weakly non-local$^{\alpha}$, local$^{\beta}$ and weakly non-local$^{\beta}$ classes fail:

$$\Big(\bigwedge_{i=15}^{32} \neg(\mathrm{H}_i^{\alpha}) \wedge \bigwedge_{i=15}^{32} \neg(\mathrm{H}_i^{\beta})\Big)$$

Since in section 2.2 we have seen that the result of the original Bell argument defines what we call quantum non-locality, it is clear that given this new result we have to adapt the definition appropriately. It simply becomes implausible to stick to the old, looser definition *including* weakly non-local classes, given that we now know that we have to *exclude* them. With the new result of the Bell argument we can be much more precise about what quantum non-locality amounts to on a probabilistic level and re-define it as:

(P6') Quantum non-locality is the failure of the disjunction of all local$^\alpha$, weakly non-local$^\alpha$, local$^\beta$ and weakly non-local$^\beta$ classes (i.e. it is not the case that both factors in each product form involve at most one parameter):

$$\text{(QNL')} :\leftrightarrow \left(\bigwedge_{i=15}^{32} \neg(\mathrm{H}_i^\alpha) \wedge \bigwedge_{i=15}^{32} \neg(\mathrm{H}_i^\beta) \right) \qquad \text{(definition)}$$

This is the first main result of my investigation. (P6') takes the notion of quantum non-locality from *any* kind of non-locality (the mere failure of local factorisation) to a more *specific* one (namely exclusive the weakly non-local$^\alpha$ and weakly non-local$^\beta$ classes). Since our scheme of logically possible classes is comprehensive, the failure of all local$^\alpha$ and weakly non-local$^\alpha$ classes is equivalent to the fact that one of the strongly non-local$^\alpha$ classes, (H_1^α)–(H_{14}^α), has to hold (and mutatis mutandis for the classes (H_i^β), see (P6")). Therefore, equivalently to (P6') we can say:

(P6") Quantum non-locality is strong non-locality$^\alpha$ and strong non-locality$^\beta$ (i.e. at least one of the factors in each product form must involve both parameters):

$$\text{(QNL')} \leftrightarrow \left(\bigvee_{i=1}^{14} (\mathrm{H}_i^\alpha) \wedge \bigvee_{i=1}^{14} (\mathrm{H}_i^\beta) \right)$$

2.6 Discussion I

According to the logic of the Bell argument, we have noted in section 2.2, quantum non-locality is a *necessary* condition for EPR/B correlations and their violation of Bell inequalities. This still holds for the new result and definition (since the logical structure of the argument has not essentially changed). However, we now know that the new concept of quantum non-locality is *not sufficient* for EPR/B correlations because we have seen that there are strongly non-local distributions which do *not* violate Bell inequalities. On the other hand, since we have also found that all strongly non-local classes include distributions which violate Bell inequalities and reproduce EPR/B correlations, we can say that on a qualitative level, which only considers the product forms of the hidden joint probability, i.e. probabilistic dependencies and independencies, we cannot improve the argument any more. It is impossible to reach a stronger conclusion than (C1") by showing that we can derive Bell inequalities from still further classes. For if my argument and counterexamples are correct, there are no classes left which in general may imply Bell inequalities. Any future characterization of quantum non-locality which is more detailed must involve reference to *numerical* features of the strongly non-local classes. In this sense, my new definition of quantum non-locality, although not being sufficient for EPR/B correlations, captures their strongest possible consequences on a qualitative probabilistic level.

Why has this stronger consequence of the Bell argument, that we have derived, been overlooked so far? Obviously, it has wrongly been assumed that local factorisation is the *only* basis to derive Bell inequalities, and the main reason for neglecting other product forms of hidden joint probabilities might have been the fact that, originally, Bell inequalities were derived to capture consequences of a *local* worldview. The main result of former considerations was that locality has consequences which are in conflict with the quantum mechanical distribution. Given this historical background, the idea to derive Bell inequalities from non-local assumptions maybe was beyond interest because the conflict with locality was considered to be the crucial point; or maybe it was neglected because Bell inequalities were so tightly associated with locality that a derivation from non-locality sounded totally implausible. Systematically, however, since it is now clear that the quantum mechanical distribution is empirically correct and Bell inequalities are violated, it is desirable to draw as strong consequences as possible, which requires to check without prejudice whether some non-local classes allow a derivation of Bell inequalities as well—and this is what we have done here.

Before we go on with the argument let me shortly digress on how my argument dissolves two common misunderstandings in the debate about quantum non-locality. First, in the discussion Bell inequalities are so closely linked to locality that one could have the impression that Bell inequalities are locality conditions in the sense that, if a probability distribution obeys a Bell inequality, it must be local. Of course, Bell's argument never really justified that view, for the logic of the standard Bell argument is that local factorisation (given autonomy and perfect (anti-)correlations) is merely *sufficient* (and not necessary) for Bell inequalities. Maybe the association between Bell inequalities and locality might have arisen from the the fact that up to now local factorisation has been the *only* product form which was shown to imply Bell inequalities. Given only this information, it was at least possible (though unproven) that the holding of Bell inequalities implies locality. However, since we have shown that weakly non-local classes in general imply Bell inequalities and since the simulations show that even some strongly non-local distributions can conform to Bell inequalities, it has become explicit that this is not true. If a probability distribution obeys a Bell inequality it does not have to be local because *not all* probability distributions obeying Bell inequalities are *local*.

Second, sometimes it has been said that the violation of Bell inequalities by EPR/B correlations implies that there *cannot* be a screener-off for the correlations (e.g. van Fraassen, 1989). Table 1 shows that this is not true either. Among the strongly non-local classes there are classes according to which $\boldsymbol{\alpha}$ is screened-off from $\boldsymbol{\beta}$ (namely all classes with the value 0 in column II). Consider for example class (H_6^α) for which the product form reads $P(\alpha\beta|ab\lambda) = P(\alpha|ab\lambda)P(\beta|ab\lambda)$. Here we have it that the conjunction of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{\lambda}$ screens the correlation off. Including the distant parameter for both outcomes, this is of course *not* a *local* screener-off. But that there can be a screener-off for the correlations, although non-local, shows that it is not true saying that EPR/B correlations disprove Reichenbachs common cause principle. This claim would only be true if you exclude non-local screener-offs by adding the premise of locality, which, however, would be odd, because the violation of Bell inequalities shows that we cannot avoid a non-locality anyway. Of course, the true theory of the quantum realm might *in fact* have a distribution which does not screen-off, but to argue for that one needs further assumptions; this claim is not in general implied by the violation of Bell inequalities.

Finally, and maybe most importantly, my argument points out that the discussion so far has been based on an inappropriate concept of quantum non-locality. Capturing all non-local classes the standard concept of quantum non-locality (the failure of local factorisation (P6)) includes classes which we have found to be compatible with Bell inequalities (weakly non-local classes). In this sense *the standard concept is inappropriately weak*, i.e. weaker as it should be. Therefore, if we analyse this concept, as Jarrett did, and take it as informing us about the consequences of EPR/B correlations violating Bell inequalities, we must expect to be misled, either in that some of the options we arrive at are not really available or that the analysantia do not at all cut the problem at its natural joints (in section 3.5 we will see that the latter is the case). This is the core of my critique concerning the standard view.

Excluding local *and* weakly non-local classes, my new concept (P6') picks out a proper subset of the classes which are included in the standard concept, and comprises only those classes that do *not* imply Bell inequalities (strongly non-local classes): it is considerably stronger and much more informative than the standard concept. Moreover, I have just argued that it cannot be made stronger on a qualitative probabilistic level, because it comprehensively describes the qualitative probabilistic consequences of EPR/B correlations violating Bell inequalities. In this sense, my new concept is an appropriate basis for an analysis (while the standard concept is not). Analysing the new stronger concept will be the subject of the following section, and it will turn out that the result of *this* analysis differs significantly from Jarrett's.

3 Analysing quantum non-locality

The aim of this section is to provide an analysis of the *new* concept of quantum non-locality (P6'), which was the result of the modified Bell argument in the last section. Jarrett proved that the standard concept of quantum non-locality (P6) is equivalent to the disjunction of outcome dependence and parameter dependence. The *idea* of Jarrett's analysis is that a specific product form of the hidden joint probability (such as local factorisation), which is a *complex* independence condition, can be analysed by *pairwise* independencies (such as outcome independence or parameter independence). Our new concept of quantum non-locality (P6') is a conjunction of two disjunctions of several product forms and, hence, itself a complex independence condition. So we can apply Jarrett's idea to our new case and understand "analysis" as providing an expression in terms of pairwise probabilistic independencies which is equivalent to the new concept. I first recall shortly Jarrett's analysis and introduce an appropriate set of independencies, which will serve as analysantia. Then I develop an analysis for each of the classes (H_i^α) and subsequently of the disjunction $\bigvee_{i=15}^{32}(\mathrm{H}_i^\alpha)$ (the first part of the disjunction of classes which imply Bell inequalities). Finally, we can transfer our results from the first to the second part $\bigvee_{i=15}^{32}(\mathrm{H}_i^{\beta})$, and the negation of the disjunction of the two parts will yield the analysis of quantum non-locality (P6').

3.1 Jarrett's analysis of quantum non-locality

Jarrett (1984) had the idea that one can be more explicit about the probabilistic nature of quantum non-locality (P6) by analysing the probabilistic statement local factorisation (ℓF) in terms of pairwise conditional probabilistic independencies. By a "pairwise conditional probabilistic independence" I mean the fact that a random variable $\boldsymbol{x}$ is independent of another $\boldsymbol{y}$ given a conjunction of further variables $\boldsymbol{z}$. This is said to be true iff for all values of the variables the joint probability over the variables makes the following equation true:

$$P(x|yz) = P(x|z) \tag{27}$$

The independence is noted as $I(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{z})$. If, however, there is at least one set of values for which (27) does not hold, the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are called dependent given $\boldsymbol{z}$, and this probabilistic dependence is noted as $\neg I(\boldsymbol{x}, \boldsymbol{y}|\boldsymbol{z})$.

Jarrett uses three pairwise independencies: "outcome independence" is defined as $I(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{ab\lambda})$ and "parameter independence" as a conjunction of two independencies, $I(\boldsymbol{\alpha}, \boldsymbol{b}|\boldsymbol{a\lambda}) \wedge I(\boldsymbol{\beta}, \boldsymbol{a}|\boldsymbol{b\lambda})$. (Originally, Jarrett denotes these independencies as "completeness" and "locality" respectively, but we shall use the now established names.) Jarrett proved mathematically that

(P7) Local factorisation is equivalent to the conjunction of outcome independence and parameter independence:

$$(\ell\mathrm{F}) \leftrightarrow I(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{ab\lambda}) \wedge I(\boldsymbol{\alpha}, \boldsymbol{b}|\boldsymbol{a\lambda}) \wedge I(\boldsymbol{\beta}, \boldsymbol{a}|\boldsymbol{b\lambda}) \tag{28}$$

From (P6) and (P7) he concluded that

(C2) Quantum non-locality is equivalent to the disjunction of outcome dependence or parameter dependence:

$$(\mathrm{QNL}) \leftrightarrow \neg I(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{ab\lambda}) \vee \neg I(\boldsymbol{\alpha}, \boldsymbol{b}|\boldsymbol{a\lambda}) \vee \neg I(\boldsymbol{\beta}, \boldsymbol{a}|\boldsymbol{b\lambda}) \tag{29}$$

which is the *probabilistic analysis of quantum non-locality according to the standard view* ("Jarrett's analysis"). The analysis is correct, but since (as we have seen) the analysandum (P6) is inappropriately weak, its result is not as informative as it could be and, as we will see, in fact misleading about the nature of quantum non-locality. Therefore, what we aim to do is to analyse the new stronger concept of quantum non-locality (P6') in terms of pairwise conditional independencies, which will give us a clearer picture of what quantum non-locality amounts to.

3.2 Pairwise independencies

The first step towards an analysis is to get an overview which concepts can play the role of the analysantia. In table 3 I introduce those nine pairwise independencies which will be relevant. Among the relevant independencies we find the normal outcome independence, $I(\boldsymbol{\alpha}, \boldsymbol{\beta}|\boldsymbol{ab\lambda})$, as well as $I(\boldsymbol{\alpha}, \boldsymbol{b}|\boldsymbol{a\lambda})$, one independence of the disjunction which is usually called parameter independence. Here we see a first problem with the standard names: how shall we call the latter if its disjunction with $I(\boldsymbol{\beta}, \boldsymbol{a}|\boldsymbol{b\lambda})$ is called parameter independence? I have tried to stay as close to the standard names as possible, but obviously further qualifications are needed. My suggestion is to continue to use the name "parameter independence" for all independencies between an outcome and its distant parameter, but to add the outcome in question, namely "α-parameter independence" or "β-parameter independence" respectively. Further differentiation in the nomenclature is required by the fact that there is another α-parameter independence in the table, $I(\boldsymbol{\alpha}, \boldsymbol{b}|\boldsymbol{\beta a\lambda})$, which differs from the one already mentioned in the conditional variables (it additionally includes the outcome $\boldsymbol{\beta}$). Such independencies of the same type but with different conditional variables are different independencies and are in general logically independent of another: one can hold or not irrespective of whether the other does or does not. (One can show that only if one involves more than two independencies logical restrictions appear.) I discern them by indices, e.g. the former is called "α-parameter independence$_2$", the latter "α-parameter independence$_1$". Of course, there are further α-parameter independencies (namely those conditional on $\boldsymbol{\beta\lambda}$ and $\boldsymbol{\lambda}$) which, however, do not play any role for the analysis.

Similarly to the parameter independencies I define local parameter independencies (see table 3), which instead of the independence of an outcome on its distant parameter $(\boldsymbol{\alpha}, \boldsymbol{b})$ claim the independence of an outcome on its *local* parameter $(\boldsymbol{\alpha}, \boldsymbol{a})$. Besides these new names I have also introduced short labels for each independence, which we will mainly use in the following.

Given these new concepts we are now in a position to see one of the sources of confusion in the standard discussion. "Outcome dependence or parameter dependence" does *not necessarily* mean that if you accept outcome dependence you can avoid parameter dependence in the sense of *any kind* of dependence of an outcome on its distant parameter (conditional on whatever variables). The slogan just says that in this case you can avoid parameter dependence in the usual sense of $\neg(\mathrm{PI}_2^\alpha) \vee \neg(\mathrm{PI}_2^\beta)$, while other kinds of parameter dependencies like $\neg(\mathrm{PI}_1^\alpha)$ might still hold! And indeed the analysis of the new concept will yield that at least one of the two parameter dependencies $\neg(\mathrm{PI}_1^\alpha)$ and $\neg(\mathrm{PI}_2^\beta)$ *must* hold. Parameter dependence in this broader sense cannot be avoided but will turn out to be a necessary condition for quantum non-locality.

3.3 Analysing the classes

With these pairwise independencies we can now attempt to analyse each class of probability distributions. For the analysis of the classes (H_i^α) in table 1 we shall need the first five independencies in table 3 (the other four independencies

Table 3: Definition of conditional independencies

independence	standard name	new name	label
$I(\boldsymbol{\alpha}, \boldsymbol{\beta}\|\boldsymbol{ab\lambda})$	outcome independence	outcome independence$_1$	(OI_1)
$I(\boldsymbol{\alpha}, \boldsymbol{b}\|\boldsymbol{\beta a\lambda})$	–	α-parameter independence$_1$	(PI_1^α)
$I(\boldsymbol{\alpha}, \boldsymbol{b}\|\boldsymbol{a\lambda})$	[part of] parameter ind.	α-parameter independence$_2$	(PI_2^α)
$I(\boldsymbol{\beta}, \boldsymbol{a}\|\boldsymbol{\alpha b\lambda})$	–	β-parameter independence$_1$	(PI_1^β)
$I(\boldsymbol{\beta}, \boldsymbol{a}\|\boldsymbol{b\lambda})$	[part of] parameter ind.	β-parameter independence$_2$	(PI_2^β)
$I(\boldsymbol{\alpha}, \boldsymbol{a}\|\boldsymbol{\beta b\lambda})$	–	α-local parameter independence$_1$	$(\ell\mathrm{PI}_1^\alpha)$
$I(\boldsymbol{\alpha}, \boldsymbol{a}\|\boldsymbol{b\lambda})$	–	α-local parameter independence$_2$	$(\ell\mathrm{PI}_2^\alpha)$
$I(\boldsymbol{\beta}, \boldsymbol{b}\|\boldsymbol{\alpha a\lambda})$	–	β-local parameter independence$_1$	$(\ell\mathrm{PI}_1^\beta)$
$I(\boldsymbol{\beta}, \boldsymbol{b}\|\boldsymbol{a\lambda})$	–	β-local parameter independence$_2$	$(\ell\mathrm{PI}_2^\beta)$

plus outcome independence$_1$ are only used for the analysis of the classes (H_1^β); see below). The result of the analysis will be that *each class in table 1 is equivalent to the conjunction of the specific pattern of independencies indicated in columns II–VI (0's indicate* in*dependencies, see the bottom line!)*, i.e. each pattern corresponds to exactly one of the classes, e.g.

$$(\mathrm{H}_7^\alpha) \leftrightarrow (\mathrm{PI}_2^\alpha) \wedge (\ell\mathrm{PI}_2^\alpha). \tag{30}$$

One can see from the table that *each of the five independencies corresponds to exactly one of the five variables in the conditionals of the factors*: if a certain independence *holds*, the corresponding variable does *not* appear (and vice versa), and if a certain independence *fails*, the corresponding variable does *appear* (and vice versa). Specifically, if (OI_1) holds, the first factor of the hidden joint probability does not involve the other outcome $\boldsymbol{\beta}$ (and vice versa), and if it does not, the first factor includes it (and vice versa). Similarly, (PI_1^α) and $(\ell\mathrm{PI}_1^\alpha)$ correspond to the distant and the local parameter in the *first* factor respectively, while (PI_2^α) and $(\ell\mathrm{PI}_2^\alpha)$ are linked to the distant and the local parameter in the *second* factor respectively. So the holding or failure of each of the five independencies has a very well defined impact on the product form of the hidden joint probability (and vice versa), and the conjunction of *all* independencies which hold according to a certain probability distribution determines its product form, i.e. its class (and vice versa).

The proof of the equivalences of product forms and conjunctions of independencies involves only some basic laws of probability theory. Providing an analysis for *each* product form of the hidden joint probability, this claim is an extension of Jarrett's analysis of local factorisation (P7). One can demonstrate the equivalence for each hidden joint probability separately (analogous to how

Jarrett derived (P7)), but the following constructive method is more elegant: in the case that the hidden joint probability factorises according to the product rule, (H_1^α), none of the relevant independencies holds (and vice versa). Then we consider the five cases in which exactly *one* independence holds (H_2^α)–(H_6^α). Here is the proof of $(\mathrm{H}_2^\alpha) \leftrightarrow (\ell\mathrm{PI}_2^\beta)$:

$$\boxed{\leftarrow} \quad P(\alpha\beta|ab\lambda) = P(\alpha|\beta ba\lambda)P(\beta|ab\lambda) \overset{(\ell\mathrm{PI}_2^\beta)}{=} P(\alpha|\beta ba\lambda)P(\beta|a\lambda) \quad (31)$$

$$\boxed{\rightarrow} \quad P(\beta|ab\lambda) = \sum_\alpha P(\alpha\beta|ab\lambda) \overset{(\mathrm{H}_2^\alpha)}{=} P(\beta|a\lambda)\sum_\alpha P(\alpha|\beta ba\lambda) = P(\beta|a\lambda) \quad (32)$$

The equivalence $(\mathrm{H}_3^\alpha) \leftrightarrow (\mathrm{PI}_2^\beta)$ can be shown mutatis mutandis (just swap the local with the distant parameter). $(\mathrm{H}_4^\alpha) \leftrightarrow (\ell\mathrm{PI}_1^\alpha)$ can be derived as follows:

$$\boxed{\leftarrow} \quad P(\alpha\beta|ab\lambda) = P(\alpha|\beta ba\lambda)P(\beta|ab\lambda) \overset{(\ell\mathrm{PI}_1^\alpha)}{=} P(\alpha|b\beta\lambda)P(\beta|ab\lambda) \quad (33)$$

$$\boxed{\rightarrow} \quad P(\alpha|\beta ba\lambda) = \frac{P(\alpha\beta|ab\lambda)}{P(\beta|ab\lambda)} \overset{(\mathrm{H}_4^\alpha)}{=} \frac{P(\alpha|\beta b\lambda)\cancel{P(\beta|ab\lambda)}}{\cancel{P(\beta|ab\lambda)}} = P(\alpha|\beta b\lambda) \quad (34)$$

The equivalences $(\mathrm{H}_5^\alpha) \leftrightarrow (\mathrm{PI}_1^\alpha)$ and $(\mathrm{H}_6) \leftrightarrow (\mathrm{OI}_1)$ are proved similarly. Then, by pairs of these five equivalences involving one independence, we prove equivalences with two independencies, and subsequently, equivalences with three independencies, and so on. Here is an example how to derive an equivalence with two independencies, $(\mathrm{H}_7^\alpha) \leftrightarrow (\mathrm{PI}_2^\beta) \wedge (\ell\mathrm{PI}_2^\beta)$, on the basis of the corresponding equivalences with one independence respectively:

$$\boxed{\leftarrow} \quad (\mathrm{PI}_2^\beta) \wedge (\ell\mathrm{PI}_2^\beta) \overset{(31),\,(32)}{\longleftrightarrow} (\mathrm{PI}_2^\beta) \wedge (\mathrm{H}_2^\alpha) \overset{(35)}{\rightarrow} (\mathrm{H}_7^\alpha)$$

$$P(\alpha\beta|ab\lambda) \overset{(\mathrm{H}_2^\alpha)}{=} P(\alpha|\beta ba\lambda)P(\beta|ab'\lambda) \overset{(\mathrm{PI}_2^\beta)}{=} P(\alpha|\beta ba\lambda)P(\beta|a'b'\lambda) \quad (35)$$

$$\boxed{\rightarrow} \quad (\mathrm{H}_7^\alpha) \overset{(*)}{\rightarrow} (\mathrm{H}_3^\alpha) \leftrightarrow (\mathrm{PI}_2^\beta); \quad (\mathrm{H}_7^\alpha) \overset{(*)}{\rightarrow} (\mathrm{H}_2^\alpha) \leftrightarrow (\ell\mathrm{PI}_2^\beta)$$

$(*)$: (H_7^α) is a common special case of (H_2^α) and (H_3^α); if (H_7^α) holds, then a forteriori (H_2^α) and (H_3^α):

$$\forall a, a', b, b': \quad P(\alpha\beta|ab\lambda) = P(\alpha|\beta ba\lambda)P(\beta|a'b'\lambda) \qquad (\mathrm{H}_7^\alpha)$$
$$\forall a = a', b, b': \quad P(\alpha\beta|ab\lambda) = P(\alpha|\beta ba\lambda)P(\beta|a'b'\lambda) \qquad (\mathrm{H}_2^\alpha)$$
$$\forall a, a', b = b': \quad P(\alpha\beta|ab\lambda) = P(\alpha|\beta ba\lambda)P(\beta|a'b'\lambda) \qquad (\mathrm{H}_3^\alpha)$$

As one can see by this constructive method the proofs remain basic and short, even for the more complex equivalences. Similarly, with some patience, we can derive step by step the other equivalences between product forms and independencies in table 1.

Note that according to table 1 local factorisation is analysed as $(\mathrm{H}^{\alpha}_{29}) \leftrightarrow (\mathrm{OI}_1) \wedge (\mathrm{PI}^{\alpha}_1) \wedge (\mathrm{PI}^{\beta}_2)$, while according to Jarrett it is $(\mathrm{H}^{\alpha}_{29}) \leftrightarrow (\mathrm{OI}_1) \wedge (\mathrm{PI}^{\alpha}_2) \wedge (\mathrm{PI}^{\beta}_2)$, i.e. in Jarrett's claim (PI^{α}_1) is replaced by (PI^{α}_2). Given that (OI_1) holds, the replacement is correct, because one can show that $(\mathrm{OI}_1) \wedge (\mathrm{PI}^{\alpha}_1) \leftrightarrow (\mathrm{OI}_1) \wedge (\mathrm{PI}^{\alpha}_2)$.

3.4 Quantum non-locality as double parameter dependence

Finally, with the analysis of the single classes we shall now formulate the second main result of this paper, the analysis of the new, stronger concept of quantum non-locality (P6'). We had found that quantum non-locality is the failure of all local$^{\alpha}$, weakly non-local$^{\alpha}$, local$^{\beta}$ and weakly non-local$^{\beta}$ classes and that these classes are characterized by the fact that their constituting product forms involve *at most one parameter in each of its factors.* Let us first give an analysis of the local$^{\alpha}$ and weakly non-local$^{\alpha}$ classes. Our analysis of the single classes (H^{α}_1)–$(\mathrm{H}^{\alpha}_{32})$ has revealed that each variable in the conditionals of the factors corresponds to exactly one of the five independencies in table 1. The distant parameter in the first factor corresponds to α-parameter independence$_1$, (PI^{α}_1), and the local parameter to α-local parameter independence$_1$, $(\ell\mathrm{PI}^{\alpha}_1)$. So the first factor involves at most one parameter if and only if at least one of these independencies holds, $(\mathrm{PI}^{\alpha}_1) \vee (\ell\mathrm{PI}^{\alpha}_1)$. Similarly, at most one parameter appears in the second factor iff β-parameter independence$_2$ or β-local parameter indepence$_2$ hold, $(\mathrm{PI}^{\beta}) \vee (\ell\mathrm{PI}^{\beta})$. So we have found the following equivalence:

(P7'a) The disjunction of local$^{\alpha}$ and weakly non-local$^{\alpha}$ classes is equivalent to the fact that $\boldsymbol{\alpha}$ is independent$_1$ of at least one parameter and $\boldsymbol{\beta}$ is independent$_2$ of at least one parameter:

$$\Big(\bigvee_{i=15}^{32} (\mathrm{H}^{\alpha}_i)\Big) \leftrightarrow \Big[\Big((\mathrm{PI}^{\alpha}_1) \vee (\ell\mathrm{PI}^{\alpha}_1)\Big) \wedge \Big((\mathrm{PI}^{\beta}_2) \vee (\ell\mathrm{PI}^{\beta}_2)\Big)\Big]$$

In a very similar way as we have proceeded for the classes (H^{α}_1)–$(\mathrm{H}^{\alpha}_{32})$ one can find an analysis for the classes (H^{β}_1)–$(\mathrm{H}^{\beta}_{32})$ (remember the table which is symmetric to table 1 in swapping the outcomes and the parameters and apply all considerations mutatis mutandis):

(P7'b) The disjunction of local$^{\beta}$ and weakly non-local$^{\beta}$ classes is equivalent to the fact that $\boldsymbol{\beta}$ is independent$_1$ of at least one parameter and $\boldsymbol{\alpha}$ is independent$_2$ of at least one parameter:

$$\Big(\bigvee_{i=15}^{32} (\mathrm{H}^{\beta}_i)\Big) \leftrightarrow \Big[\Big((\mathrm{PI}^{\beta}_1) \vee (\ell\mathrm{PI}^{\beta}_1)\Big) \wedge \Big((\mathrm{PI}^{\alpha}_2) \vee (\ell\mathrm{PI}^{\alpha}_2)\Big)\Big]$$

Since (P6') defined quantum non-locality as the *failure* of the disjunction of all local$^{\alpha}$, weakly non-local$^{\alpha}$, local$^{\beta}$ and weakly non-local$^{\beta}$ classes, the negation of the disjunction of (P7'a) and (P7'b) finally yields the analysis of quantum non-locality:

(C2') Quantum non-locality is equivalent to the fact that $\boldsymbol{\alpha}$ depends$_1$ on both parameters or $\boldsymbol{\beta}$ depends$_2$ on both parameters and $\boldsymbol{\beta}$ depends$_1$ on both parameters or $\boldsymbol{\alpha}$ depends$_2$ on both parameters:

$$\begin{aligned}\text{(QNL')} \quad \leftrightarrow \quad & \Bigg\{ \bigg[\Big(\neg(\mathrm{PI}_1^\alpha) \wedge \neg(\ell\mathrm{PI}_1^\alpha) \Big) \vee \Big(\neg(\mathrm{PI}_2^\beta) \wedge \neg(\ell\mathrm{PI}_2^\beta) \Big) \bigg] \wedge \\ \wedge \quad & \bigg[\Big(\neg(\mathrm{PI}_1^\beta) \wedge \neg(\ell\mathrm{PI}_1^\beta) \Big) \vee \Big(\neg(\mathrm{PI}_2^\alpha) \wedge \neg(\ell\mathrm{PI}_2^\alpha) \Big) \bigg] \Bigg\}\end{aligned}$$

While the definition of quantum non-locality in (P6') was in terms of classes of probability distributions, here we have the equivalent expression, the analysis, in terms of pairwise independencies. It is a rather complex logical expression whose meaning and implications are not at all easy to grasp. A first understanding might be attained by making explicit how this analysis of quantum non-locality is also an analysis of the conjunction of strongly non-local$^\alpha$ and strongly non-local$^\beta$ classes (which is necessarily so, see (P6")). These classes were characterized by the fact that at least one of the factors in each product form must involve both parameters and this is exactly what (C2') says: The first term in the first disjunction, $\neg(\mathrm{PI}_1^\alpha) \wedge \neg(\ell\mathrm{PI}_1^\alpha)$ ("α-double parameter dependence$_1$"), guarantees a dependence on both parameters in the first factor of the product forms (H_i^α), the second term in the first disjunction, $\neg(\mathrm{PI}_2^\beta) \wedge \neg(\ell\mathrm{PI}_2^\beta)$ ("β-double parameter dependence$_2$"), implies a similar fact for the second factor of these forms, and analogously, the second disjunction entails dependence on both parameters in at least one of the factors of the product forms (H_i^β) (and vice versa).

So the analysis involves double parameter dependencies for each outcome in two different forms, either conditional on all other variables (double parameter dependence$_1$) or conditional on all other variables excluding the other outcome (double parameter dependence$_2$). The logic of the expression has it that these can hold in different combinations, but whichever combination does, there is one thing that necessarily follows if (C2') is true:

(C3) Double parameter dependence: at least one of the outcomes depends probabilistically on both parameters (in at least one of the forms double parameter dependence$_1$ or double parameter dependence$_2$).

For one *can* avoid that *one* of the outcomes is double parameter dependent$_1$ and double parameter dependent$_2$, but then it follows that the respective other outcome must be double parameter dependent$_1$ as well as double parameter dependent$_2$. Of course, you can also have mixed cases in which both outcomes are double parameter dependent (in one or both of the two forms), but in any case you have double parameter dependence of at least one of the outcomes.

So we have found two results: the precise probabilistic analysis of quantum non-locality (C2') and a general feature of and deriving from that analysis (C3), that at least one of the outcomes must be double parameter dependent. Since quantum non-locality is a necessary condition for EPR/B correlations (if autonomy holds) double parameter dependence of at least one of the outcomes, which is implied by quantum non-locality, is a *necessary* condition for EPR/B correlations as well: whenever we find that EPR/B correlations hold double parameter dependence (C3) must hold as well. So given that measurement results in our world yield EPR/B correlations (and assuming autonomy), we can be sure that at least one of the outcomes depends both on the local as well as on the distant parameter. Note, however, that we have *not* shown that quantum non-locality, and hence neither its analysis (C2') nor double parameter dependence (C3), is *sufficient* for the violation of Bell inequalities. If an outcome depends on both parameters in the sense of (C2') the correlations between the two wings *might* be sophisticated enough to violate Bell inequalities—but they need not be. We have noted above that the product form and, hence, the dependencies alone cannot guarantee a violation because, additionally, the probability distribution has to fulfill certain *numerical* conditions.

3.5 Discussion II

We shall now compare our new results (C2') and (C3) with that of Jarrett's analysis (C2). Here is a summary of the two different arguments and their results:

$$
\begin{array}{l}
\text{(A)}\wedge \\ \text{(Corr)}
\end{array}
\overset{\substack{\text{standard}\\ \text{Bell}\\ \text{argument}}}{\Rightarrow} \quad \neg\left(\bigvee_{i=29}^{32}(\mathrm{H}_i^\alpha)\vee\bigvee_{i=29}^{32}(\mathrm{H}_i^\beta)\right)
$$

$$
\overset{\substack{\text{Jarrett's}\\ \text{analysis}}}{\Leftrightarrow} \quad \neg(\mathrm{OI}_1)\vee\neg(\mathrm{PI}_2^\alpha)\vee\neg(\mathrm{PI}_2^\beta) \tag{C2}
$$

$$
\begin{array}{l}
\text{(A)}\wedge \\ \text{(Corr)}
\end{array}
\overset{\substack{\text{new Bell}\\ \text{argument}}}{\Rightarrow} \quad \neg\left(\bigvee_{i=15}^{32}(\mathrm{H}_i^\alpha)\vee\bigvee_{i=15}^{32}(\mathrm{H}_i^\beta)\right)
$$

$$
\overset{\substack{\text{new}\\ \text{analysis}}}{\Leftrightarrow} \quad
\begin{array}{r}
\Bigg\{\Bigg[\Big(\neg(\mathrm{PI}_1^\alpha)\wedge\neg(\ell\mathrm{PI}_1^\alpha)\Big)\vee\Big(\neg(\mathrm{PI}_2^\beta)\wedge\neg(\ell\mathrm{PI}_2^\beta)\Big)\Bigg]\wedge \\
\wedge\Bigg[\Big(\neg(\mathrm{PI}_1^\beta)\wedge\neg(\ell\mathrm{PI}_1^\beta)\Big)\vee\Big(\neg(\mathrm{PI}_2^\alpha)\wedge\neg(\ell\mathrm{PI}_2^\alpha)\Big)\Bigg]\Bigg\}
\end{array} \tag{C2'}
$$

$$
\Rightarrow \quad \text{At least one of the outcomes depends on both parameters.} \tag{C3}
$$

(1) At the end of the first part of this paper I have made precise in what sense the standard concept of quantum non-locality, the result of the Bell argument, is too weak (it picks out both weakly and strongly non-local classes instead of just the latter), and it is clear that the analysis of the new concept is superior just because its *analysandum is much more informative.* This is the fundamental difference between the two analyses. But what exactly changes from the one to the other? In what sense does the result of Jarrett's analysis paint a different picture of the probabilistic dependencies in EPR/B experiments?

(2) The main message of my result is that given EPR/B correlations and autonomy *one cannot avoid some kind of dependence between at least one of the outcomes and both parameters (C3)* (either double parameter dependence$_1$ or double parameter dependence$_2$). This is the unambiguous probabilistic requirement of quantum non-locality according to the new analysis. Jarrett's analysis, however, does not bring out this necessary condition: from his result "outcome dependence or parameter dependence" one just cannot see that, necessarily, there must be some kind of double parameter dependence.

(3) Actually, *Jarrett's result is liable to be misunderstood to the contrary meaning*, for it might be interpreted to say that if we opt for outcome dependent$_1$ theories we could avoid *any* dependence of an outcome on its distant parameter—which is wrong in two respects. First this *understanding* of the analysis is wrong because this is not what it says (it only says that you can avoid parameter dependence$_2$). Second, my analysis shows that it is *in fact* wrong that we can avoid any kind of parameter dependence. My result reveals that there is no choice to make between whether an outcome depends on the other outcome or on its distant parameter, for we have found that any theory correctly reproducing the EPR/B correlations *must* have some kind of dependence between each outcome and both its local and distant parameter.

But is quantum mechanics not a counterexample to my result? Is it not a theory which violates Bell inequalities although there is no dependence on the distant parameter? It is true, quantum mechanics is well known to be "outcome dependent and parameter independent", but again this is not to be understood that according to quantum mechanics there is no probabilistic dependence of an outcome on the distant parameter at all. In fact, it is easy to check, which independencies hold according to quantum mechanics: one can calculate all relevant conditional probabilities from the quantum mechanical probability distribution for the EPR/B experiment (Corr). A simple comparison of these probabilities then shows which of the independencies hold and which do not, and it turns out that quantum mechanics is parameter dependent$_1$, $\neg(\mathrm{PI}_1^\alpha)$ and $\neg(\mathrm{PI}_1^\beta)$, so according to quantum mechanics each outcome *does* depend on its distant parameter! This parameter dependence in quantum mechanics is not as surprising as it may seem since, according to the formalism, the measurement direction at A determines the possible collapsed states at B and the actual outcome at A only determines in which of the (two) possible states the photon state at B collapses. So contrary to what the standard talk suggests, quantum mechanics is parameter dependent, and it is important to see that it is as well local parameter dependent$_1$, $\neg(\ell\mathrm{PI}_1^\alpha)$ and $\neg(\ell\mathrm{PI}_1^\beta)$ (while it is local parameter independent$_2$, $(\ell\mathrm{PI}_2^\alpha)$ and $(\ell\mathrm{PI}_2^\beta)$), because then, the quantum mechanical distribution fulfills the requirement of a quantum non-locality by rendering the first terms of the two disjunctions in (C2') true. If my argument in this paper is true, it cannot be otherwise. For if quantum mechanics were not parameter dependent in this double sense, it could not (as it does) violate Bell inequalities.

(4) How then do Jarrets's analysantia outcome dependence$_1$, α-parameter dependence$_2$ and β-parameter dependence$_2$ relate to the new concept of quantum non-locality? Here is, first, how they do *not* relate: Jarrett's analysis of

the weaker concept is a disjunction of these three dependencies and it could have been that the analysis of the stronger concept just cancels one or two of the elements in the disjunction, revealing them as options which are not really available. For instance it might have been that the new analysis yields just $\neg(\mathrm{PI}_2^\alpha) \vee \neg(\mathrm{PI}_2^\beta)$, canceling $\neg(\mathrm{OI}_1)$. However, it turns out that this is not the case. The logical structure of the new analysis is not just a simplification of the former, but, in fact, is much more complicated involving new concepts (parameter dependence$_1$, local parameter dependence$_1$ and local parameter dependence$_2$) and not involving others (outcome dependence$_1$). This suggests that *Jarrett's categories outcome dependence$_1$ and parameter dependence$_2$ cannot capture the new concept of quantum non-locality.*

Table 4: Jarrett's classes of possible probability distributions

Label	$\neg(\mathrm{OI}_1)$	$\neg(\mathrm{PI}_2^\alpha) \vee\neg(\mathrm{PI}_2^\beta)$	Notes	
(J_1)	1	1		
(J_2)	0	1	Bohm	quantum non-locality
(J_3)	1	0	QM	
(J_4)	0	0		locality

To make this explicit, consider the partition of the probability distributions according to the dependencies in Jarrett's analysis (table 4). There are four classes, which I call "Jarrett's classes" and label as (J_1)–(J_4). Any of the 32 possible classes from table 1 must fall into one of Jarrett's coarse-grained classes. While the local classes belong to (J4), any of the classes (J1)–(J3) includes both weakly and strongly non-local classes. So Jarrett's non-local classes, which are assumed to be able to violate Bell inequalities, *mix probability distributions which can with such which cannot* (see fig. 2). *They do not cut the probability distributions at their natural joints!* This means that neither outcome dependence$_1$ nor parameter dependence$_2$ are necessary or (contrary to Jarrett's analysis) sufficient for the new concept of quantum non-locality. Providing, for instance, the information that a certain probability distribution is outcome dependent$_1$ does not tell you whether it can violate Bell inequalities or not. The crucial fact is whether double parameter dependence of a certain kind holds, and α-parameter dependence$_2$ and β-parameter dependence$_2$ at least play a *certain* role in this complex condition. Outcome dependence$_1$, however, does *not* play any *essential* role for the general concept of quantum non-locality. Not being able to capture the new concept, we conclude that the partition according to Jarrett's categories outcome dependence$_1$ and parameter dependence$_2$ is inappropriate or unnatural for the analysis of quantum non-locality.

So it seems that a significant amount of the debate after Jarrett's paper which has focused on the question of the formal, physical and metaphysical differences between outcome dependence$_1$ and parameter dependence$_2$, in order

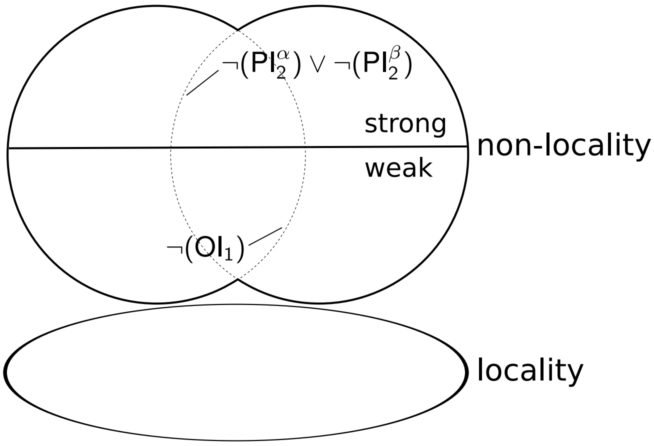

Fig. 2: Outcome dependence$_1$ and parameter dependence$_2$ vs. weak and strong non-locality. "Strong non-locality" means *strong$^\alpha$* <u>*and*</u> *strong$^\beta$ non-locality* (i.e. those distributions which can violate Bell inequalities), while "weak non-locality" means *weak$^\alpha$* <u>*or*</u> *weak$^\beta$ non-locality* (i.e. those non-local distributions which imply Bell inequalities).

to decide which of the two does hold, is misguided. *"Outcome dependence or parameter dependence?" is just the wrong question if you want to explore deeper into the nature of quantum non-locality*, for each of the two options subsumes probability distributions which can and such which cannot violate Bell inequalities. Making this question a guide to quantum non-locality is like asking whether those humans which can get pregnant have dark or fair hair. Rather, the natural question, the new analysis shows, is which of the outcomes is double parameter dependent and whether it is double parameter dependent$_1$ or double parameter dependent$_2$.

(5) Finally, it might be noted that there is one particularly outstanding investigation of quantum non-locality which does not agree with the received view but rather seems to be in accordance with my new analysis: instead of analysing local factorisation in probabilistic terms, Maudlin (2011, ch. 6) develops an information theoretic account of how the EPR/B correlations come about. He shows that quantum non-locality *necessarily* involves a transmission of information about the parameter on one side to the measurement outcome on the other side:

> *Bell's inequality can reliably be violated only when the response of one of the particles depends (at least sometimes) on the question asked its partner.* [...] [D]ependence on the distant polariser setting is crucial. Jarrett's division of theories into those that violate outcome independence and those that violate parameter independence is again seen to be misleading: *any* successful theory must postulate some influence of a distant "parameter" (i.e. the polariser angle) on the response of a local photon. Without such dependence the quantum statistics cannot be recovered. (p. 167)

Assuming that the outcome ("response") on one side depends on its local parameter, Maudlin's claim that one cannot avoid a certain dependence on the distant parameter amounts to saying that there must be a dependence on *both* parameters! This sounds very similar to my result (C3) that one cannot avoid double parameter dependence for at least one of the outcomes. However, while Maudlin's investigation is in terms of *information* about the parameters which is needed to reproduce the EPR/B correlations, my analysis is in terms of *probabilistic dependencies*, so the two accounts are not easy to compare. To assess how they agree and in what they differ would require to have a probabilistic account of information flow—a question which is left open for future work. But *if* information flow between variables in an EPR/B experiment can roughly be associated with probabilistic dependence of these variables (which might be false in general), then there was a tension between Jarrett's probabilistic analysis and Maudlin's information theoretic one, and I have shown exactly where Jarrett's argument fails and provided a new probabilistic account, which very much seems to be in accordance with Maudlin's. Moreover, the similarity of the results is striking, especially because they stem from very different approaches: EPR/B correlations imply that at least one of the outcomes depends on both parameters, both probabilistically and information theoretically.

4 Conclusion

In this paper we have investigated what EPR/B correlations, which violate Bell inequalities, imply on a qualitative probabilistic level. We started our considerations by giving a comprehensive overview of the possible types of probability distributions, which can describe EPR/B experiments (table 1). The overview has revealed that one can derive Bell inequalities not only from local theories but also from a large range of non-local ones, which we have called weakly non-local. This has enabled us to formulate a stronger Bell argument than usual to exclude local *and* weakly non-local theories of the quantum world. Since the result of the Bell argument defines what we appropriately call quantum non-locality, this new result yielded a tighter, more informative concept of quantum non-locality, which describes more precisely what the violation of Bell inequalities implies on a probabilistic level. In fact, my new concept of quantum non-locality, although not being sufficient for EPR/B correlations, captures the strongest possible consequences of EPR/B correlations on a qualitative probabilistic level. In this sense, my new concept is appropriate, while the standard concept is too weak. Furthermore, the argument shows that Bell inequalities are *not* locality conditions (because weakly non-local theories obey them) and that their violation does *not* necessarily imply that there is no screener-off for the correlations.

In a second part, we have given an analysis of the new concept of quantum non-locality, similar to how Jarrett analysed the failure of local factorisation. We have provided an exact logical expression in terms of pairwise independencies, which is equivalent to the new concept. (It includes two types of parameter dependencies with respect to each outcome, namely parameter dependence$_2$, which is the usual parameter dependence, and parameter dependence$_1$, which differs from the usual parameter dependence in that the conditional variables include the other outcome respectively.) A general feature of the result is that at least one of the outcomes must depend on both parameters, either including parameter dependence$_1$ or parameter dependence$_2$ (while outcome dependence is neither necessary nor sufficient for the new concept quantum non-locality). Jarrett's analysis, however, does not bring out this necessary requirement of EPR/B correlations. Rather, it is liable to be misunderstood to the contrary meaning that one can avoid any dependence between the outcomes and their distant parameters if one accepts a dependence between the outcomes. (This understanding, however, is not what it says.) A closer examination revealed that the result of Jarrett's analysis, "outcome dependence or parameter dependence", does not cut the probability distributions at their natural joints (it mixes theories which can violate Bell inequalities with such which cannot). So it turns out that asking whether outcome dependence or parameter dependence holds is a deeply misleading question if one aspires to understand quantum non-locality.

These deficiencies of Jarrett's analysis require that the debate based on the inappropriate disjunction "outcome dependence or parameter dependence" needs a fundamental revision. There are two main issues. First, regarding the question whether EPR/B correlations are compatible with relativity, many have argued that parameter dependent$_2$ theories are forbidden by relativity, while outcome dependent (and parameter independent$_2$) theories might peacefully coexist with relativity. However, we now know that theories in this latter class which correctly describe the EPR/B correlations, e.g. quantum mechanics, must be parameter dependent$_1$. As we cannot avoid parameter dependence in some sense, we have to check the alleged inconsistency of parameter dependence and relativity again. Is the so far widely neglected parameter dependence$_1$ as problematic for relativity as parameter dependence$_2$ has been said to be? Do the arguments which have been adduced against parameter dependence$_2$ apply to parameter dependence$_1$ as well? Either at least one of them can live in harmony with Lorentz invariance or no theory correctly reproducing the EPR/B correlations by a quantum non-locality can be compatible with relativity on a fundamental level.

Second, concerning the metaphysical implications of quantum non-locality it has been argued that while parameter dependence$_2$ requires a causal relation (action at-a-distance), outcome dependence is best understood as a non-causal connection (non-separability / holism). Since one cannot take refuge in outcome dependence any more: does that mean that we necessarily have to accept action at-a-distance? If yes, between which variables? Or can the idea of a non-separability be made intelligible even for parameter dependent theories? These and similar questions have to be addressed. Some old arguments might be transferred to the new situation, some might not. In any case, the debate will need a fresh look. It seems that we are far from being finished with our enquiries into the nature of quantum non-locality, since it has just received new probabilistic foundations.

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