The Logic of Partitions: Introduction to the Dual of the Logic of Subsets

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Abstract

Partitions on a set are dual to subsets of a set in the sense of the category-theoretic duality of epimorphisms and monomorphisms–which is reflected in the duality between quotient objects and subobjects throughout algebra. Modern categorical logic as well as the Kripke models of intuitionistic logic suggest that the interpretation of ordinary "propositional" logic might be the logic of subsets of a given universe set. If "propositional" logic is thus seen as the logic of subsets of a universe set, then the question naturally arises of a dual logic of partitions on a universe set. This paper is an introduction to that logic of partitions dual to classical subset logic. The paper goes from basic concepts up through the correctness and completeness theorems for a tableau system of partition logic.

1 Introduction to partition logic

1.1 The idea of a dual logic of partitions

In ordinary propositional logic, the atomic variables and compound formulas are usually interpreted as representing propositions. But in terms of mathematical entities, the variables and formulas may be taken as representing subsets of some fixed universe set U (with the propositional interpretation being isomorphic to the special case of a one element set U with subsets 0 and 1).

> The propositional calculus considers "Propositions" p, q, r,... combined under the operations "and","or", "implies", and "not", often written as $p \wedge q$, $p \vee q$, $p \Rightarrow q$, and $\neg p$. Alternatively, if P, Q, R,... are subsets of some fixed set U with elements u, each proposition p may be replaced by the proposition $u \in P$ for some subset $P \subset U$; the propositional connectives above then become operations on subsets; intersection $\wedge$, union $\vee$, implication ($P \Rightarrow Q$ is $\neg P \vee Q$), and complement of subsets. (Mac Lane and Moerdijk 1992, p. 48)

The view of propositional logic as being about subsets is an old view that goes back to Boole himself but it also follows forcefully from the recent treatment of logic using category theory. Largely due to the efforts of William Lawvere, the modern treatment of logic was reformulated and generalized in what is now called *categorical logic*.[1] Subsets were generalized to subobjects or "parts" (equivalence classes of monomorphisms) so that logic has become the logic of subobjects or parts in a topos (such as the category of sets).[2]

There is a duality between subsets of a set and partitions on a set which can be generalized in categories. "The dual notion (obtained by reversing the arrows) of 'part' is the notion of *partition*." (Lawvere and Rosebrugh 2003, p. 85) In category theory, this reverse-the-arrows duality gives the duality between monomorphisms, e.g., injective set functions, and epimorphisms, e.g., surjective set functions, and between subobjects and quotient objects.

Quite aside from category theory duality, Gian-Carlo Rota emphasized the seminal analogies between the subsets of a set and the partitions on a set. Just as subsets of a set are partially ordered by inclusion, so partitions on a set are partially ordered by refinement. Moreover, both partial orderings are lattices (i.e., have meets and joins) with a top element and a bottom element.

This work on partition logic was inspired by both Rota's program to develop the subset-partition analogies and by the category-theoretic treatment of logic together with the reverse-the-arrows duality between subsets and partitions. If modern logic is formulated as the logic of subsets (or more generally, subobjects or "parts"), then the question naturally arises of a dual logic of partitions. This paper is an introduction to the "propositional" (i.e., non-quantifier) part of partition logic.

1.2 Duality of elements of a subset and distinctions of a partition

The set-of-blocks definition of a *partition* on a set U is a set of non-empty subsets ("blocks") of U where the blocks are mutually exclusive (the intersection of distinct blocks is empty) and jointly exhaustive (the union of the blocks is U). If subsets are dual to partitions (in the sense of monomorphisms being dual to epimorphisms), then what is the dual concept that corresponds to the notion of *elements of a subset*? The dual notion is the notion of a *distinction of a partition* which is a pair of elements in distinct blocks of the partition. The duality between elements of a subset and distinctions of a partition already appears in the very notion of a function between sets. What binary relations, i.e., subsets $R \subseteq X \times Y$, specify functions $f : X \to Y$?

A binary relation $R \subseteq X \times Y$ *transmits elements* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.

[1] See Lawvere and Rosebrugh (2003, Appendix A) for a good treatment. For the generalization to topos theory see Mac Lane and Moerdijk (1992) and for the category theoretic background, the best references are Mac Lane (1971) and Awodey (2006).

[2] Sometimes the propositional and subset interpretations are "connected" by interpreting U as the set of possible worlds and a subset as the set of possible worlds where a proposition is true. While this interpretation may be pedagogically useful, it is conceptually misleading since U is simply an abstract set. None of the philosophical problems involved in "possible worlds" semantics have anything to do with the subset interpretation of ordinary logic.

A binary relation $R \subseteq X \times Y$ *reflects elements* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.

A binary relation $R \subseteq X \times Y$ *transmits distinctions* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.

A binary relation $R \subseteq X \times Y$ *reflects distinctions* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.

The dual role of elements and distinctions can be seen if we translate the usual characterization of the binary relations that define functions into the elements-and-distinctions language. A binary relation $R \subseteq X \times Y$ defines a *function* $X \to Y$ if it is defined everywhere on X and is single-valued. But "being defined everywhere" is the same as transmitting elements, and being single-valued is the same as reflecting distinctions:

a binary relation R is a *function* if it transmits elements and reflects distinctions.

What about the other two special types of relations, i.e., those which transmit distinctions or reflect elements? The two important special types of functions are the injections and surjections, and they are defined by the other two notions:

an *injective function* is a function that transmits distinctions, and a *surjective function* is a function that reflects elements.

In view of the dual role of subsets and partitions (and elements and distinctions), it is interesting to note that many basic ideas expressed using subsets such as the notion of a "function" could just as well be expressed in a dual manner using partitions. The dual to the product $X \times Y$ is the coproduct $X \uplus Y$ which in the category of sets is the disjoint union. If a binary relation is defined as a subset R of the product $X \times Y$, then a *binary corelation* would be a partition π on the coproduct $X \uplus Y$. Instead of defining a function as a certain type of binary relation (i.e., which transmits elements and reflects distinctions), a function could just as well be defined as a certain type of binary corelation. Let $[u]_\pi$ denote the block of a partition π containing an element u from the universe set of the partition. Then a binary corelation π (a partition on $X \uplus Y$) is *functional* if 1) every element $x \in X$ is transmitted to some y-block, i.e., $\exists y \in Y, x \in [y]_\pi$, and 2) distinctions on Y are reflected as distinctions of π, i.e., if $y \neq y'$ for $y, y' \in Y$, then $[y]_\pi \neq [y']_\pi$.

Moreover, this definition of a function is quite familiar (with different terminology) in combinatorics. For a functional corelation π, there is one and only one block of the partition for each element $y \in Y$ so the blocks $[y]_\pi$ can be thought of as "boxes." Then the elements of X can be thought of as "balls" and then a function is just a distribution of the balls into the boxes. Thus the functional corelation definition of a function is just a "disguised" version of the balls-in-boxes definition of a function used in combinatorial theory (Stanley 1997, p. 31). A functional corelation is *injective* if distinctions between balls are transmitted as distinctions between boxes ("different balls to different boxes"), i.e., $x \neq x'$ implies $[x]_\pi \neq [x']_\pi$, and is *surjective* if each box contains at least one ball (i.e., each y is reflected as an x). Although functions were historically defined as functional binary relations, from the mathematical viewpoint, functions could just as well be defined as functional binary corelations.

The duality between the two definitions of functions is clear in category theory. Given the diagram $f : X \to Y$ in the category of sets, its limit is the functional relation corresponding to f and its colimit is the functional corelation corresponding to f. The functional relation corresponding to a function is its *graph* and the functional corelation corresponding to a function is its *cograph* (Lawvere and Rosebrugh 2003, p. 29).

1.3 Partitions and equivalence relations

An *equivalence relation* on a set U is a subset $E \subseteq U \times U$ that is reflexive, symmetric, and transitive. Every equivalence relation on a set U determines a partition on U where the equivalence classes are the mutually exclusive and jointly exhaustive blocks of the partition. Conversely, every partition on a set determines an equivalence relation on the set (two elements are equivalent if they are in the same block of the partition). The notions of a partition on a set and an equivalence relation on a set are thus interdefinable ("cryptomorphic" as Gian-Carlo Rota would say). Indeed, equivalence relations and partitions are often considered as the "same." But for our purposes it is important to keep the notions distinct (as in the above definitions) so that we may consider the complementary type of binary relation. A *partition relation* $R \subseteq U \times U$ is irreflexive (i.e., $(u,u) \notin R$ for any $u \in U$), symmetric [i.e., $(u,u') \in R$ implies $(u',u) \in R$], and *anti-transitive* in the sense that if $(u,u') \in R$, then for any $a \in U$, either $(u,a) \in R$ or $(a,u') \in R$ [i.e., $U \times U - R = R^c$ is transitive]. Thus as binary relations, equivalence relations and partition relations are complementary. That is, $E \subseteq U \times U$ is an equivalence relation if and only if (iff) $E^c \subseteq U \times U$ is a partition relation. A partition relation is the set of distinctions of a partition.

In a similar manner, the closed and open sets of a topological space can each be defined in terms of the other and are complementary as subsets of the space. Indeed, this is a useful analogy. There is a natural ("built-in") closure operation on $U \times U = U^2$ which makes it a closure space. A subset $C \subseteq U^2$ is *closed* (1) if C contains the diagonal $\Delta = \{(u,u) \mid u \in U\}$ (reflexivity), (2) if $(u,u') \in C$, then $(u',u) \in C$ (symmetry), and (3) if (u,u') and (u',u'') are in C, then (u,u'') is in C (transitivity). Thus the closed sets of U^2 are the reflexive, symmetric, and transitive relations, i.e., the equivalence relations on U. The intersection of any number of closed sets is closed. Given a subset $S \subseteq U^2$, the *closure* $\overline{S}$ is the reflexive, symmetric, and transitive closure of S. The formation of the closure $\overline{S}$ can be divided into two steps. First S^* is formed from S by adding any diagonal pairs (u,u) not already in S and by symmetrizing S, i.e., adding (u',u) if $(u,u') \in S$. To form the transitive closure of S^*, for any finite sequence $u = u_1, u_2, ..., u_n = u'$ with $(u_i, u_{i+1}) \in S^*$ for $i = 1, ..., n-1$, add (u,u') and (u',u) to the closure. The result is the reflexive, symmetric, and transitive closure $\overline{S}$ of S. The complements of the closed sets in $U \times U$ are defined as the *open* sets which are the partition relations on U. As usual, the *interior* $\text{int}(S)$ of any subset S is defined as the complement of the closure of its complement: $\text{int}(S) = \left(\overline{S^c}\right)^c$.

It should, however, be carefully noted that the closure space $U \times U$ is not a topological space, i.e., the closure operation on U^2 is not a *topological* closure operation in the sense that the union of two closed set is not necessarily closed (or, equivalently, the intersection of two open sets is not necessarily open). Since the lattice of open sets (or of closed sets) of a topological space is distributive, this failure of the closure operation on $U \times U$ to be topological is behind the non-distributivity of the lattice of partitions (or of equivalence relations) on a set U.

The set-of-blocks definition of a *partition* π on a set U is a set $\{B\}_{B\in\pi}$ of non-empty subsets or "blocks" $B \subseteq U$ that are disjoint and whose union is U.[3] A pair $(u, u') \in U \times U$ is a *distinction* or *dit* (from DIsTinction) of the partition π if there are distinct blocks $B, B' \in \pi$ with $u \in B$ and $u' \in B'$. The set of distinctions of a partition π, its *dit set* denoted $\text{dit}(\pi) \subseteq U \times U$, is the partition seen as a partition relation:

$$\text{dit}(\pi) = \bigcup_{B,B'\in\pi, B\neq B'} B \times B'$$

(where it is understood that the union includes both the cartesian products $B \times B'$ and $B' \times B$ for $B \neq B'$).[4]

A pair $(u, u') \in U \times U$ is an *indistinction* or *indit* (from INDIsTinction) of a partition π if u and u' belong to the same block of π. The set of indistinctions of a partition π, its *indit set* denoted $\text{indit}(\pi) = U \times U - \text{dit}(\pi)$, is the complementary equivalence relation:

$$\text{indit}(\pi) = \bigcup_{B\in\pi} B \times B = U \times U - \text{dit}(\pi) = \text{dit}(\pi)^c.$$

In terms of the closure space structure on $U \times U$, let $\mathcal{O}(U \times U)$ be the open sets (partition relations) which are the dit sets $\text{dit}(\pi)$ of partitions while the complementary closed sets (equivalence relations) are the indit sets $\text{indit}(\pi)$ of partitions.

Partitions on U are partially ordered by the *refinement* relation: given two partitions $\pi = \{B\}_{B\in\pi}$ and $\sigma = \{C\}_{C\in\sigma}$,

[3] Just as the usual treatment of the Boolean algebra of all subsets of a universe U assumes that U has one or more elements, so our treatment of the lattice of all partitions on U will assume that U has *two* or more elements. This avoids the "degenerate" special cases of there being only one subset of an empty U and only one partition on a singleton U.

[4] Strictly speaking, one could argue that a "distinction" should be an unordered pair $\{u, u'\}$ but it is analytically more convenient to deal with ordered pairs. In finite probability theory with equiprobable elements in the sample space, the relative count of elements in a subset (or event) defines the probability $\text{Prob}(S)$ of the subset S. Dualizing, the count of the distinctions of a partition relative to the total number of ordered pairs with a finite universe U defines the "logical entropy" $h(\pi)$ of a partition π (Ellerman 2009). In this "logical" information theory, it is also analytically better to deal with ordered pairs. Then the logical entropy $h(\pi)$ of a partition π is simply the probability that a random draw of a pair (with replacement) is a distinction of the partition just as $\text{Prob}(S)$ is the probability that a random draw is an element of the subset.

$\sigma \preceq \pi$ (read "π refines σ" or "σ is refined by π") if for any block $B \in \pi$, there is a block $C \in \sigma$ with $B \subseteq C$.[5]

The equivalent definition using dit sets (i.e., partition relations) is just inclusion:

$$\sigma \preceq \pi \text{ iff } \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi).$$

Partitions might be represented by surjections $U \rightarrow \pi$ and every refinement relation $\sigma \preceq \pi$ is realized by the unique map $\pi \rightarrow \sigma$ that takes each block $B \in \pi$ to the block $C \in \sigma$ containing it. The refinement map makes the following triangle commute:

$$\begin{array}{ccc} U & \rightarrow & \pi \\ \downarrow & \swarrow & \\ \sigma & & \end{array}$$

Refinement as a map

and thus it gives a morphism in the ("coslice") category of sets under U (Awodey 2006, p. 15).

The partial ordering of partitions on U has a least element or bottom which is the indiscrete partition $0 = \{U\}$ (nicknamed the "blob") with the null dit set $\operatorname{dit}(0) = \emptyset$ (no distinctions). The blob distinguishes nothing and is refined by all partitions on U. The partial ordering also has a greatest element or top which is the discrete partition $1 = \{\{u\} : u \in U\}$ where all blocks are singletons and whose dit set is all ordered pairs off the diagonal, i.e., $\operatorname{dit}(1) = U \times U - \Delta$ where $\Delta = \{(u,u) : u \in U\}$. The discrete partition refines all partitions on U.

In any partial order with a least element 0, an element α is an *atom* in the partial ordering if there is no element between it and the bottom 0, i.e., if $0 \leq \pi \leq \alpha$ implies $\pi = 0$ or $\pi = \alpha$. In the inclusion partial order of subsets of U, the atoms are the singleton subsets. In the refinement partial order of partitions, the atomic partitions are the binary partitions, the partitions with two blocks. Any partition less refined than a partition π must fuse two or more blocks of π. Hence the binary partitions are the partitions so that any less refined partition has to be the blob.

1.4 Category-theoretic duality of subsets and partitions

In addition to the basic monomorphism-epimorphism duality between subsets and partitions, a set of dual relationships between subset and partition concepts as well as between element and distinction concepts will be described in this section using basic category-theoretic notions in the category of sets. This duality in the category of sets extends beyond the basic reverse-the-arrows duality that

[5] Note that the opposite partial order is called the "refinement" ordering in the customary "upside down" treatment of the lattice of partitions. Gian-Carlo Rota used to joke that it should be called the "unrefinement" relation. Indeed, in a recent book on Rota-style combinatorial theory, that relation is sensibly called "reverse refinement" (Kung, Rota, and Yan 2009, p. 30). It could also be called the "coarsening" (Lawvere and Rosebrugh 2003, p. 38) relation.

holds in all categories, and it underlies the duality between subset logic and partition logic.

In the category of sets, the singleton 1 might be thought of as the generic element. We have seen that functions preserve (or transmit) elements and reflect (or transmit in the backwards direction) distinctions. The basic property of the generic element 1 is that for every element $u \in U$, there is a function $1 \stackrel{u}{\rightarrow} U$ that transmits "elementness" from the generic element to $u \in U$. The partition-dual to the generic element 1 is $2 = \{0, 1\}$ which might be thought of as the generic distinction. The basic property of the generic distinction 2 is that for any pair u, u' of distinct elements of U, there is a function $\alpha : U \rightarrow 2$ that reflects or backwards-transmits "distinctness" from the generic distinction 2 to the pair u, u'.

Given two parallel functions $f, g : X \rightarrow Y$, if they are different, $f \neq g$, then there is an element $x \in X$ such that the two functions carry x to a distinction $f(x) \neq g(x)$ of Y. By the basic property of the generic element 1, there is a function $1 \stackrel{x}{\rightarrow} X$ that transmits the generic element to that element x. Thus the generic element 1 is a *separator* in the sense that given two set functions $f, g : X \rightarrow Y$, if $f \neq g$, then $\exists x : 1 \rightarrow X$ (an injection) such that $1 \stackrel{x}{\rightarrow} X \stackrel{f}{\rightarrow} Y \neq 1 \stackrel{x}{\rightarrow} X \stackrel{g}{\rightarrow} Y$. Dually, by the basic property of the generic distinction, there is a function $\alpha : Y \rightarrow 2$ that reflects the generic distinction to the distinction $f(x) \neq g(x)$ of Y. Thus the generic distinction 2 is a *coseparator* (Lawvere and Rosebrugh 2003, pp. 18-19) in the sense that given two set functions $f, g : X \rightarrow Y$, if $f \neq g$, then $\exists \alpha : Y \rightarrow 2$ (a surjection) such that $X \stackrel{f}{\rightarrow} Y \stackrel{\alpha}{\rightarrow} 2 \neq X \stackrel{g}{\rightarrow} Y \stackrel{\alpha}{\rightarrow} 2$.

Other dual roles of the generic element 1 and generic distinction 2 follow from the dual basic properties. Consider the product of X and Y in the category of sets. A set P with maps $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ is the *product*, denoted $X \times Y$, if for any set Z and pair of maps $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ with domain Z, there is a unique map $\langle f, g \rangle : Z \rightarrow P$ such that $p_1 \langle f, g \rangle = g$ and $p_2 \langle f, g \rangle = g$. The generic element 1 has the property that it suffices as the test set $Z = 1$. That is, if the set P with its pair of maps had the universal mapping property for pairs of maps with domain 1, then it has the universal mapping property for any pairs of maps with a common domain Z, i.e., it is the product. This property of the generic element 1 extends to all limits in the category of sets.

The dual construction is the coproduct, denoted $X \uplus Y$ or $X + Y$, which can be constructed as the disjoint union of X and Y with the two insertion maps. A set C with maps $i_1 : X \rightarrow C$ and $i_2 : Y \rightarrow C$ is the *coproduct* $X \uplus Y$ if for any set Z and pair of maps $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ with codomain Z, there is a unique map (which we will denote) $\rangle f, g \langle : C \rightarrow Z$ such that: $\rangle f, g \langle\, i_1 = f$ and $\rangle f, g \langle\, i_2 = g$.[6] The generic distinction 2 has the property that it suffices as the test set $Z = 2$. That is, if the set C and its pair of maps had the universal

[6]There seems to be no standard notation for the coproduct factor map so we have just reversed the angle brackets from the product factor map.

mapping property for pairs of maps with codomain 2, then it has the universal mapping property for any pair of maps with a common codomain Z, i.e., it is the coproduct.(Lawvere and Schanuel 1997, p. 272) This property of the generic distinction 2 extends to all colimits in the category of sets.

The dual properties also show up in the respective partial orders (and lattices). The images of injections $1 \xrightarrow{u} U$ are the atoms $\{u\}$ in the inclusion partial order of subsets of U and in the powerset Boolean algebra $\mathcal{P}(U)$. The inverse images of surjections $U \xrightarrow{\alpha} 2$ are the atoms (binary partitions) in the refinement partial order of partitions on U and in the partition lattice $\Pi(U)$ defined below.

Given a subset S of U and a partition π on U, there is the associated injection $S \longrightarrow U$ and the associated surjection $U \longrightarrow \pi$ (taking π as a set of blocks). The atom $\{u\}$ given by $1 \xrightarrow{u} U$ is contained in S, iff $1 \xrightarrow{u} U$ uniquely factors through $S \longrightarrow U$. Analogously, an atomic partition $U \xrightarrow{\alpha} 2$ is refined by π $[\operatorname{dit}(\alpha) \subseteq \operatorname{dit}(\pi)]$ iff $U \xrightarrow{\alpha} 2$ uniquely factors through $U \longrightarrow \pi$.

$$\begin{array}{ccc} 1 & & \\ \downarrow^{\exists !} & \searrow^{u} & \\ S & \longrightarrow & U \end{array} \qquad\qquad \begin{array}{ccc} U & \longrightarrow & \pi \\ & \searrow^{\alpha} & \downarrow^{\exists !} \\ & & 2 \end{array}$$

Analogous diagrams showing which atoms contained in an object (subset or partition)

The dual pullback and pushout constructions allow us to represent each partition as a subset of a product and to represent each subset as a partition on a coproduct.

Given a partition as a surjection $U \rightarrow \pi$, the pullback of the surjection with itself, i.e., the *kernel pair* (Mac Lane 1971, p. 71) of $U \rightarrow \pi$, gives the indit set $\operatorname{indit}(\pi)$ as a subset of the product $U \times U$, i.e., as a binary (equivalence) relation on U:

$$\begin{array}{ccc} \operatorname{indit}(\pi) & \xrightarrow{p_2} & U \\ {}^{p_1}\downarrow & & \downarrow \\ U & \longrightarrow & \pi \end{array}$$

Pullback for equivalence relation $\operatorname{indit}(\pi)$.

Given a subset as an injection $S \rightarrow U$, the pushout of the injection with itself, i.e., the *cokernel pair* (Mac Lane 1971, p. 66) of $S \rightarrow U$, gives a partition $\Delta(S)$ on the coproduct (disjoint union) $U \uplus U$, i.e., a binary corelation which might be called a *subset corelation*:

$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & & \downarrow^{[u^*]} \\ U & \xrightarrow{[u]} & \Delta(S) \end{array}$$

Pushout for subset corelation $\Delta(S)$.

The disjoint union $U \uplus U$ consists of the elements $u \in U$ and the copies u^* of the elements $u \in U$. The subset corelation $\Delta(S)$ is constructed by identifying any u and its copy u^* for $u \in S$ so $\Delta(S)$ is the partition on $U \uplus U$ whose only non-singleton blocks are the pairs $\{u, u^*\}$ for $u \in S$.

The constructions can also be reversed by viewing the pullback square as a pushout square, and by viewing the pushout square as a pullback square. Equivalently, we can reconstruct π as the coequalizer of the two projection maps p_1, p_2 from $\mathrm{indit}(\pi) \subseteq U \times U$ to U (Lawvere and Rosebrugh 2003, p. 89).

$$\begin{array}{ccccc} \mathrm{indit}(\pi) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & U & \longrightarrow & \pi \\ & & & \searrow^{\alpha} & \downarrow^{\exists!} \\ & & & & 2 \end{array}$$

Partition π as coequalizer of $\mathrm{indit}(\pi) \xrightarrow{p_1} U$ and $\mathrm{indit}(\pi) \xrightarrow{p_2} U$.

Dually, we have the two maps $U \to \Delta(S)$ given by $u \mapsto [u]_{\Delta(S)}$ and $u \mapsto [u^*]_{\Delta(S)}$, and the subset S is reconstructed as their equalizer:

$$\begin{array}{ccccc} 1 & & & & \\ {}^{\exists!}\downarrow & \searrow^{u} & & & \\ S & \rightarrow & U & \rightrightarrows & \Delta(S) \end{array}$$

Subset S as equalizer of $[u] : U \to \Delta(S)$ and $[u^*] : U \to \Delta(S)$.

In general, the equalizer (in the category of sets) of two set functions $f, g : X \to Y$ is the largest subset S of the domain X so that no element of S goes via the functions to a distinction $(f(x), g(x))$ of the codomain Y.

Dually, the coequalizer of two set functions $f, g : X \to Y$ is the largest (most refined) partition π on the codomain Y so that no distinction of π comes via the functions from an element of the domain X (i.e., has the form $(f(x), g(x))$ for some $x \in X$).

Then the functions $[u], [u^*] : U \rightrightarrows \Delta(S)$ are such that S is the largest subset of the domain U so that no element of the subset goes via those functions to a distinction of the codomain $\Delta(S)$.

The functions $p_1, p_2 : \mathrm{indit}(\pi) \rightrightarrows U$ are such that π is the largest partition on the codomain U so that no distinction of the partition comes via those functions from an element of the domain $\mathrm{indit}(\pi)$ (Lawvere and Rosebrugh 2003, p. 89).

Dualities	Subsets	Partitions
Generics	Generic element 1	Generic distinction 2
Basic generic property	Each element $u \in U$ realized by some $1 \to U$	Each distinction $u \neq u'$ realized by some $U \to 2$
Separating functions	1 is a separator	2 is a coseparator
Sufficient test set	1 is a test set for limits	2 is a test set for colimits
Objects	Subsets: monos $S \longrightarrow U$	Partitions: epis $U \longrightarrow \pi$
Atoms in partial orders	Images of monos $1 \xrightarrow{u} U$	Inv. images of epis $U \xrightarrow{\alpha} 2$
Inclusion of atoms	$1 \xrightarrow{u} U$ uniquely factors through $S \longrightarrow U$	$U \xrightarrow{\alpha} 2$ uniquely factors through $U \longrightarrow \pi$
Subsets $\leftrightarrow$ Partitions	Partition $\Delta(S)$ on $U \uplus U$ is cokernal pair of $S \to U$	Subset $\mathrm{indit}(\pi)$ of $U \times U$ is kernel pair of $U \to \pi$
Inverse operation	Subset S is equalizer of $[u], [u^*] : U \rightrightarrows \Delta(S)$	Partition π is coequalizer of $p_1, p_2 : \mathrm{indit}(\pi) \rightrightarrows U$

Summary of dual relationships

1.5 Lattice of partitions

Traditionally the "lattice of partitions," e.g., (Birkhoff 1948) or (Grätzer 2003), was defined as isomorphic to the lattice of equivalence relations where the partial order was inclusion between the equivalence relations as subsets of $U \times U$. But since equivalence relations and partition relations are complementary subsets of the closure space $U \times U$, we have two anti-isomorphic lattices with opposite partial orders.

Which lattice should be used in partition logic? For the purposes of comparing formulas with ordinary logic (interpreted as applying to subsets of elements), it is crucial to take the lattice of partitions as (isomorphic to) the lattice $\mathcal{O}(U \times U)$ of partition relations (sets of distinctions), the opposite of the lattice of equivalence relations.

The *lattice of partitions* $\Pi(U)$ on U adds the operations of join and meet to the partial ordering of partitions on U with the top 1 and the bottom 0.[7] There are at least four ways that partitions and operations on partitions might be defined:

1. the basic set-of-blocks definition of partitions and their operations;

2. the closure space approach using open subsets or dit sets and the interior operator on $U \times U$;

3. the graph-theoretic approach where the blocks of a partition on U are the nodes in the connected components of a simple (at most one arc between two nodes and no loops at a node) undirected graph;[8] and

[7] For a survey of what is known about partition lattices, see (Grätzer 2003) where the usual opposite presentation is used.

[8] See any introduction to graph theory such as Wilson (1972) for the basic notions.

4. the approach where the blocks of a partition on U are the atoms of a complete Boolean subalgebra of the powerset Boolean algebra $\mathcal{P}(U)$ of subsets of U (Ore 1942).

The lattice of partitions $\Pi(U)$ is the partition analogue of the powerset Boolean lattice $\mathcal{P}(U)$. In the powerset lattice, the partial order is inclusion of elements, and in the partition lattice, it is inclusion of distinctions.

The join $\pi \vee \sigma$ in $\Pi(U)$ is the partition whose blocks are the non-empty intersections $B \cap C$ of the blocks of the two partitions. The equivalent dit-set definition in $\mathcal{O}(U \times U)$ is simply the union: $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$.

Recall that the closure operator on the closure space was not topological in the sense that the union of two closed sets is not necessarily closed and thus the intersection of two open sets (i.e., two dit sets) is not necessarily open. Hence the definition of the meet of two partitions requires some more complication. The dit-set definition in $\mathcal{O}(U \times U)$ is the easiest: the dit set of the *meet* of two partitions is the interior of the intersection of the two dit sets, i.e.,

$$\operatorname{dit}(\sigma \wedge \pi) = \operatorname{int}(\operatorname{dit}(\sigma) \cap \operatorname{dit}(\pi)).$$

In the older literature, this meet of two partitions is what is defined as the join of the two equivalence relations. Given the two partitions as sets of blocks $\{B\}_{B \in \pi}$ and $\{C\}_{C \in \sigma}$ in $\Pi(U)$, two elements u and u' are directly equated, $u \sim u'$ if u and u' are in the same block of π or σ so the set of directly equated pairs is: $\operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi)$. Then u and u^* are in the same block of the join in $\Pi(U)$ if there is a finite sequence $u = u_1 \sim u_2 \sim \ldots \sim u_n = u^*$ that indirectly equates u and u^*. The operation of indirectly equating two elements is just the closure operation in the closure space so the set of pairs indirectly equated, i.e., equated in the join $\sigma \wedge \pi$ in $\Pi(U)$, is:

$$\operatorname{indit}(\sigma \wedge \pi) = \overline{(\operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi))}.$$

The complementary subset of $U \times U$ is the dit set of the meet of the partitions in $\mathcal{O}(U \times U)$:

$$\operatorname{dit}(\sigma \wedge \pi) = \operatorname{indit}(\sigma \wedge \pi)^c = \overline{(\operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi))}^c = \operatorname{int}(\operatorname{dit}(\sigma) \cap \operatorname{dit}(\pi)).$$

This defines the lattice of partitions $\Pi(U)$ and isomorphic lattice $\mathcal{O}(U \times U)$ which represents the partitions as open subsets of the product $U \times U$:

$$\boxed{\Pi(U) \cong \mathcal{O}(U \times U)}.$$

Representation of the lattice of partitions $\Pi(U)$
as the lattice of open subsets $\mathcal{O}(U \times U)$.

The analogies between the lattice of subsets $\mathcal{P}(U)$ and the lattice of partitions $\Pi(U)$ are summarized in the following table.

Analogies	Boolean lattice of subsets	Lattice of partitions
"Elements"	Elements of subsets	Distinctions of partitions
Partial order	Inclusion of elements	Inclusion of distinctions
Join	Elements of join are union of elements	Distinctions of join are union of distinctions
Meet	Largest subset of only common elements	Largest partition of only common distinctions
Top	Subset U with all elements	Partition 1 with all distinctions
Bottom	Subset $\emptyset$ with no elements	Partition 0 with no distinctions

Elements-distinctions analogies between the Boolean lattice of subsets and the lattice of partitions

With this definition of the lattice of partitions $\Pi(U)$, the usual lattice of equivalence relations is $\Pi(U)^{op}$ where the top is $\widehat{1} = U \times U = \text{indit}(0)$ and the bottom is $\widehat{0} = \Delta = \text{indit}(1)$.[9]

1.6 Two other definitions of the partition meet operation

Since the partition meet is the first non-trivial definition of a partition operation, we might also give the equivalent definitions using the graph-theoretic method and the complete-Boolean-subalgebras method.

The power of the dit-set approach to defining partition operations is that it allows us to mimic subset operations using dit sets and the interior operations as needed. The power of the graph-theoretic approach is that it allows a very intuitive connection back to the truth tables of classical propositional logic. The truth tables for the classical Boolean propositional connectives can be stated in an abbreviated form using signed formulas such as $T(\pi \wedge \sigma)$ or $F\sigma$. The truth table for the Boolean meet $\pi \wedge \sigma$ is abbreviated by saying the *Boolean conditions* for $T(\pi \wedge \sigma)$ are "$T\pi$ and $T\sigma$" while the *Boolean conditions* for $F(\pi \wedge \sigma)$ are "$F\pi$ or $F\sigma$". Thus for the four Boolean operations of join $\pi \vee \sigma$, meet $\pi \wedge \sigma$, implication $\sigma \Rightarrow \pi$, and Sheffer stroke, not-and or nand $\sigma \mid \pi$, the table of Boolean conditions is as follows:

Signed Formula	$T(\pi \vee \sigma)$	$F(\pi \vee \sigma)$	$T(\sigma \Rightarrow \pi)$	$F(\sigma \Rightarrow \pi)$
Boolean Conditions	$T\pi$ or $T\sigma$	$F\pi$ and $F\sigma$	$F\sigma$ or $T\pi$	$T\sigma$ and $F\pi$

Boolean conditions for $\vee$ and $\Rightarrow$,

and

Signed Formula	$T(\pi \wedge \sigma)$	$F(\pi \wedge \sigma)$	$T(\sigma \mid \pi)$	$F(\sigma \mid \pi)$
Boolean Conditions	$T\pi$ and $T\sigma$	$F\pi$ or $F\sigma$	$F\sigma$ or $F\pi$	$T\sigma$ and $T\pi$

Boolean conditions for $\wedge$ and $\mid$.

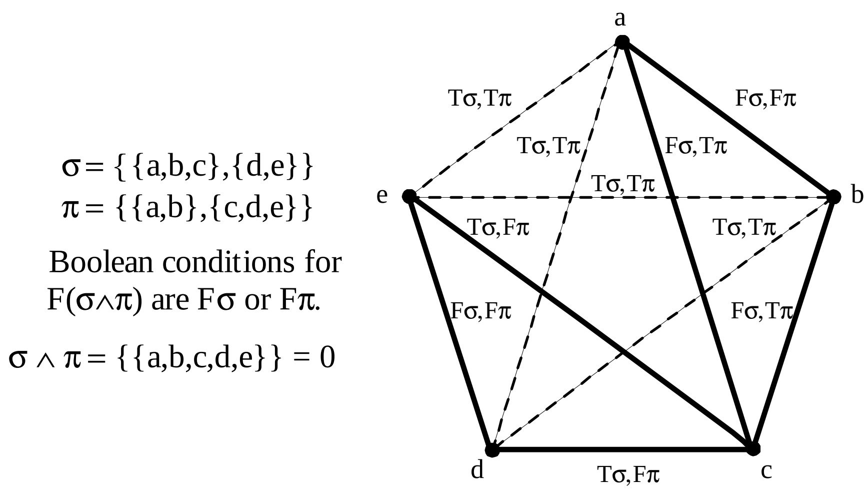
[9] Inevitably notational conflicts arise for such common symbols as "0" and "1" so where there is less risk of confusion, different uses of these symbols will be clear from the context. In other cases, the symbols are modified as in using $\widehat{1}$ and $\widehat{0}$ for the top and bottom of the opposite lattice of equivalence relations.

Given any partition π on U, and any pair of elements (u, u'), we say that $T\pi$ holds at (u, u') if (u, u') is a distinction of π, and that $F\pi$ holds at (u, u') if (u, u') is not a distinction of π, i.e., if u and u' are in the same block of π. Given any two partitions π and σ on U, we can define the partition version of *any* Boolean connective $\pi * \sigma$ by putting an arc between any two nodes u and u' if the Boolean conditions for $F(\pi * \sigma)$ hold at (u, u'). Then the blocks of the partition operation $\pi * \sigma$ are the nodes in the connected components of that graph. Thus two elements u and u' are in the same block of the partition $\pi * \sigma$ if there is a chain or finite sequence $u = u_1, u_2, ..., u_{n-1}, u_n = u'$ such that for each $i = 1, ..., n-1$, the Boolean conditions for $F(\pi * \sigma)$ hold at (u_i, u_{i+1}).

In order for $\pi * \sigma$ to distinguish u and u', it has to "cut" them apart in the sense of the graph-theoretic notion of a "cut" which is the graph-theoretic dual to the notion of a chain (Rockafellar 1984, p. 31). A set of arcs in a graph form a *cut between the nodes u and u'* if every chain connecting u and u' contains an arc from the set–so that the set of arcs cut every chain connecting the two points. The complementation-duality between chains and cuts is brought out by the fact that if we arbitrarily color the arcs of any simple undirected graph by either black or white, then for any two nodes, there is either a white cut between the nodes or a black chain connecting the nodes. The above graph-theoretic definition of $\pi * \sigma$, i.e., two points are not distinguished if there is chain connecting the points with the Boolean conditions for $F(\pi * \sigma)$ holding at each arc (i.e., a black chain), can be stated in an equivalent dual form. Two points are distinguished in $\pi * \sigma$ if the set of arcs where the Boolean conditions for $T(\pi * \sigma)$ hold form a (white) cut between the two points.

This graph-theoretic approach can be used to uniformly define all the partition logical operations in terms of the corresponding Boolean logical operations, but the case at hand is the meet. The graph constructed for the meet would have an arc between u and u' if the Boolean conditions for $F(\pi \wedge \sigma)$ held at (u, u'), i.e., if $F\pi$ or $F\sigma$ held at (u, u'). But this just means that $(u, u') \in \operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi)$, and the nodes in the connected components of that graph are the nodes u and u' connected by a finite sequence $u = u_1, u_2, ..., u_{n-1}, u_n = u'$ where for each $i = 1, ..., n-1$, $(u_i, u_{i+1}) \in \operatorname{indit}(\sigma) \cup \operatorname{indit}(\pi)$, which is the closure space definition of the meet given above.

Example 1 *Let $\sigma = \{\{a, b, c\}, \{d, e\}\}$ and $\pi = \{\{a, b\}, \{c, d, e\}\}$. In the graph below, all the arcs in the complete graph K_5 on five nodes are labelled according to the status of the two endpoints in the two partitions. The Boolean conditions for $F(\sigma \wedge \pi)$ are "$F\sigma$ or $F\pi$" . The arcs where those conditions hold are the solid lines. In the graph with only the solid arcs, there is only one connected component so $\sigma \wedge \pi = \{\{a, b, c, d, e\}\} = 0$. Equivalently, the set of arcs where the Boolean conditions for $T(\sigma \wedge \pi)$ hold, i.e., the dashed arcs, do not "cut" apart any pair of points.*



Graph for meet $\sigma \wedge \pi$

For the Boolean subalgebra approach, given a partition π on U, define $\mathcal{B}(\pi) \subseteq \mathcal{P}(U)$ as the complete subalgebra generated by the blocks of π as the atoms so that all the elements of $\mathcal{B}(\pi)$ are formed as the arbitrary unions and intersections of blocks of π. Conversely, given any complete subalgebra $\mathcal{B}$ of $\mathcal{P}(U)$, the intersection of all elements of $\mathcal{B}$ containing an element $u \in U$ will provide the atoms of $\mathcal{B}$ which are the blocks in a partition π on U so that $\mathcal{B} = \mathcal{B}(\pi)$. Thus an operation on complete subalgebras of the powerset Boolean algebra will define a partition operation. Since the blocks of the partition meet $\pi \wedge \sigma$ are minimal under the property of being the exact union of π-blocks and also the exact union of σ-blocks, a nice feature of this approach to partitions is that:

$$\mathcal{B}(\pi \wedge \sigma) = \mathcal{B}(\pi) \cap \mathcal{B}(\sigma).$$

The powerset Boolean algebra (BA) $\mathcal{P}(U)$ is not just a lattice; it has additional structure which can be defined using the binary connective of the set implication: $A \Rightarrow B = (U - A) \cup B = A^c \cup B$, for $A, B \subseteq U$. The lattice structure on $\Pi(U)$ needs to be enriched with other operations such as the binary operation of implication on partitions.

1.7 Partition implication operation

Boolean algebras, or more generally, Heyting algebras are not just lattices; there is another operation $A \Rightarrow B$, the implication operation. In a Heyting algebra, the implication can be introduced by an adjunction (treating the partial order as the morphisms in a category) that can be written in the Gentzen style[10] which in this case is an "if and only if" statement:

[10] Sometimes the Gentzen-style statement $\frac{x \to Gy}{Fx \to y}$ of an adjunction, $\mathrm{Hom}_Y(Fx, y) \cong \mathrm{Hom}_X(x, Gy)$, has the top and bottom reversed. But there is a theory showing how adjoints

$$\frac{C \leq A \Rightarrow B}{C \wedge A \leq B}$$

Implication as the right adjoint to meet in a Heyting algebra.

In the standard model of the Heyting algebra of open sets of a topological space, the implication is defined for open sets A and B as:

$$A \Rightarrow B = \operatorname{int}(A^c \cup B).$$

A co-Heyting algebra is also a lattice with top and bottom but with the dual adjunction where the *difference* $B^c \backslash A^c$ is left adjoint to the join:

$$\frac{B^c \leq A^c \vee C^c}{B^c \backslash A^c \leq C^c}$$

Difference as the left adjoint to join in a co-Heyting algebra.

In the standard model of the co-Heyting algebra of closed sets of a topological space, the difference is defined for closed sets A^c and B^c (where A and B are open sets) as:

$$B^c \backslash A^c = \overline{(B^c \cap A^{cc})} = \overline{(B^c \cap A)} = (A \Rightarrow B)^c.$$

Neither of these adjunctions holds in the lattice of partitions $\Pi(U)$ (or its opposite). The adjunctions imply distributivity for Heyting and co-Heyting algebras, and lattices of partitions (usually viewed in the opposite presentation as the lattice of equivalence relations) are standard examples of non-distributive lattices.

How might the implication partition $\sigma \Rightarrow \pi$ of two partitions (or the difference between two equivalence relations) be defined? Some motivation might be extracted from Heyting algebras, or, equivalently, intuitionistic propositional logic. The subset version of intuitionistic propositional logic is explicit in its topological interpretation where the variables are interpreted as open subsets of a topological space U and the valid formulas are those that evaluate to the whole space U regardless of what open subsets are assigned to the atomic variables. The implication is then defined as: $A \Rightarrow B = \operatorname{int}(A^c \cup B)$ for open subsets A and B which gives the classical definition if the topology is discrete. Since we have an interior operator on the (non-topological) closure space $U \times U$, this suggests that the implication partition $\sigma \Rightarrow \pi$ might be defined by the dit-set definition:

$$\operatorname{dit}(\sigma \Rightarrow \pi) = \operatorname{int}\left(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi)\right) = \overline{\left(\operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma)^c\right)}^c.$$

The equivalence relation that corresponds to a partition is its indit set so the corresponding notion of the *difference* $\operatorname{indit}(\pi) - \operatorname{indit}(\sigma)$ between two equivalence relations would be the equivalence relation:

arise out of representations of heteromorphisms (Ellerman 2006), and that theory suggests that the Gentzen-style statement should be written as above since there are "behind the scenes" heteromorphisms (dashed arrows) as vertical downward maps $Gy \dashrightarrow y$ and $x \dashrightarrow Fx$ so that the square commutes, i.e., $x \to Gy \dashrightarrow y = x \dashrightarrow Fx \to y$.

$$\text{indit}(\pi) - \text{indit}(\sigma) = \overline{\left(\text{indit}(\pi) \cap \text{indit}(\sigma)^c\right)} = \text{indit}(\sigma \Rightarrow \pi) = \text{dit}(\sigma \Rightarrow \pi)^c.$$

The dit set $\text{dit}(\sigma \Rightarrow \pi)$ and its complement, the indit set $\text{indit}(\sigma \Rightarrow \pi) = \text{indit}(\pi) - \text{indit}(\sigma)$, define the *same* partition which is denoted $\sigma \Rightarrow \pi$ rather than say "$\pi - \sigma$" since we have made the symmetry-breaking decision to define the lattice of partitions to be isomorphic to the lattice of partition relations rather than the opposite lattice of equivalence relations.

Since the dit-set definition of $\sigma \Rightarrow \pi$ involves the interior operator on the closure space $U \times U$, it would be very convenient to have a direct set-of-blocks definition of the implication partition $\sigma \Rightarrow \pi$. From Boolean algebras and Heyting algebras, we can extract one desideratum for the implication $\sigma \Rightarrow \pi$: if $\sigma \leq \pi$ in the partial order of the Boolean or Heyting algebra, then and only then $\sigma \Rightarrow \pi = 1$. Hence for any partitions σ and π on U, if σ is refined by π, i.e., $\sigma \preceq \pi$ in $\Pi(U)$, then and only then we should have $\sigma \Rightarrow \pi = 1$ (the discrete partition).[11] The property is realized by the simple set-of-blocks definition of the implication, temporarily denoted as $\sigma \overset{*}{\Rightarrow} \pi$, that if a block $B \in \pi$ is contained in a block $C \in \sigma$, then B is "discretized," i.e., replaced by singleton blocks $\{u\}$ for all $u \in B$, in the implication $\sigma \overset{*}{\Rightarrow} \pi$ and otherwise the block B remains the same. The following proposition says that the dit-set definition is the same as the set-of-blocks definition so that either may be used to define the partition implication $\sigma \Rightarrow \pi$.

Proposition 1 *$\sigma \Rightarrow \pi = \sigma \overset{*}{\Rightarrow} \pi$.*

Proof: By the two definitions, $\text{dit}(\pi) \subseteq \text{dit}(\sigma \Rightarrow \pi)$ and $\text{dit}(\pi) \subseteq \text{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$ with the reverse inclusions holding between the indit sets. We prove the proposition by showing $\text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \text{indit}(\sigma \Rightarrow \pi)$ and $\text{indit}(\sigma \Rightarrow \pi) \subseteq \text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$ where $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))} = \overline{[\text{dit}(\sigma) \cap \text{indit}(\pi)]}$. Let $(u, u') \in \text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$ where $\text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \text{indit}(\pi)$ so that $u, u' \in B$ for some block $B \in \pi$. Moreover if B were contained in any block $C \in \sigma$, then $(u, u') \in \text{dit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) = \text{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)^c$ contrary to assumption so B is not contained in any $C \in \sigma$. If u and u' were in different blocks of σ then $(u, u') \notin \text{indit}(\sigma)$ so that (u, u') would not be subtracted off in the formation of $\text{indit}(\sigma \Rightarrow \pi) = \overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$ and thus would be in $\text{indit}(\sigma \Rightarrow \pi)$ which was to be shown. Hence we may assume that u and u' are in the same block $C \in \sigma$. Thus (u, u') was subtracted off in $\text{indit}(\pi) - \text{indit}(\sigma)$ and we need to show that it is restored in the closure $\overline{(\text{indit}(\pi) - \text{indit}(\sigma))}$. Since $u, u' \in B \cap C$ but B is not contained in any one block of σ, there is another σ-block C' such that $B \cap C' \neq \emptyset$. Let $u'' \in B \cap C'$. Then (u, u'') and (u', u'') are not in $\text{indit}(\sigma)$ since $u, u' \in C$ and $u'' \in C'$ but those two pairs are in $\text{indit}(\pi)$ since $u, u', u'' \in B$. Hence the pairs $(u, u''), (u', u'') \in \text{indit}(\pi) - \text{indit}(\sigma) = \text{indit}(\pi) \cap \text{dit}(\sigma)$ which implies

[11] The equality sign "=" is not a sign in the formal language of partition logic so "$\sigma \Rightarrow \pi = 1$" is not a formula in that language. It simply says that the formulas "$\sigma \Rightarrow \pi$" and "1" denote the same partitions in $\Pi(U)$.

that (u,u') must be in the closure $\operatorname{indit}(\sigma \Rightarrow \pi) = \overline{(\operatorname{indit}(\pi) - \operatorname{indit}(\sigma))}$. That establishes $\operatorname{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right) \subseteq \operatorname{indit}(\sigma \Rightarrow \pi)$.

To prove the converse $\operatorname{indit}(\sigma \Rightarrow \pi) \subseteq \operatorname{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$, if $(u,u') \in \operatorname{indit}(\sigma \Rightarrow \pi) = \overline{[\operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi)]}$, then there is a sequence $u = u_1, u_2, ..., u_n = u'$ with every pair $(u_i, u_{i+1}) \in \operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi)$. Now $(u_i, u_{i+1}) \in \operatorname{indit}(\pi)$ implies there exists a block $B_i \in \pi$ with $u_i, u_{i+1} \in B_i$ for $i = 1, ..., n-1$. But $u_i, u_{i+1} \in B_i$ and $u_{i+1}, u_{i+2} \in B_{i+1}$ implies $B_i = B_{i+1}$ so all the elements u_i belong to the same block $B \in \pi$ and in particular, $u, u' \in B$. Now if there was a $C \in \sigma$ with $B \subseteq C$, then, contrary to assumption, we could not have any $(u_i, u_{i+1}) \in \operatorname{dit}(\sigma)$ since all the $u_i \in B \subseteq C$. Hence there is no $C \in \sigma$ containing B so B would not be discretized in $\sigma \overset{*}{\Rightarrow} \pi$ and thus $(u,u') \in \operatorname{indit}\left(\sigma \overset{*}{\Rightarrow} \pi\right)$. ■

Hence we may drop the temporary notation $\sigma \overset{*}{\Rightarrow} \pi$ and consider the partition implication $\sigma \Rightarrow \pi$ as characterized by the set-of-blocks definition: form $\sigma \Rightarrow \pi$ from π by discretizing any block $B \in \pi$ contained in a block $C \in \sigma$.[12]

Another way to characterize the partition implication $\sigma \Rightarrow \pi$ is by using an adjunction.[13] In a Heyting algebra, the implication is characterized by the adjunction which in our notation would be:

$$\frac{\tau \preceq \sigma \Rightarrow \pi}{\tau \wedge \sigma \preceq \pi}.$$

For partitions, the top implies the bottom, but the bottom does not imply the top. The simplest non-trivial partition algebra is that on the three element set $U = \{a,b,c\}$ where we may take $\tau = \{\{a,b\},\{c\}\}$, $\sigma = \{\{a,c\},\{b\}\}$, and $\pi = \{\{a\},\{b,c\}\}$. Then $\tau \wedge \sigma = 0$ so the bottom $0 \preceq \pi$ is true. But $\sigma \Rightarrow \pi = \pi$ (since no non-singleton block of π is contained in a block of σ), so the top is $\tau \preceq \pi$ which is false.

However, on the closure space $U \times U$, for any $S \subseteq U \times U$, there is the usual adjunction $\mathcal{P}(U \times U) \rightleftarrows \mathcal{P}(U \times U)$ defining the set implication:

$$\frac{T \subseteq S \Rightarrow P}{T \cap S \subseteq P}$$

(where $S \Rightarrow P$ is just $S^c \cup P$) for any subsets $T, P \in \mathcal{P}(U \times U)$. Moreover, the dit-set representation $\Pi(U) \to \mathcal{P}(U \times U)$ where $\tau \longmapsto \operatorname{dit}(\tau)$ has a right adjoint where $P \in \mathcal{P}(U \times U)$ is taken to the partition $G(P)$ whose dit set is $\operatorname{int}(P)$:

$$\frac{\tau \preceq G(P)}{\operatorname{dit}(\tau) \subseteq P}.$$

[12] For the analogy with subsets, the set difference $X - Y = X \cap Y^c$ is obtained from X by deleting any $u \in X$ that is contained in Y, i.e., $\{u\}$ is locally replaced by the null set, the zero element of the Boolean algebra of subsets of U. Similarly, in the difference $\operatorname{indit}(\pi) - \operatorname{indit}(\sigma)$ of equivalence relations, any equivalence class B of π contained in an equivalence class C of σ is locally replaced by the zero in the lattice of equivalence relations, i.e., is discretized.

[13] See Mac Lane (1971, p. 93) for the notion of an adjunction or covariant Galois connection between partial orders.

Composing the two right adjoints $\mathcal{P}(U \times U) \to \mathcal{P}(U \times U) \to \Pi(U)$ gives a functor taking $P \in \mathcal{P}(U \times U)$ to $G_S(P)$ which is the partition whose dit set is $\operatorname{int}(S^c \cup P)$. Its left adjoint is obtained by composing the two left adjoints $\Pi(U) \to \mathcal{P}(U \times U) \to \mathcal{P}(U \times U)$ to obtain a functor taking a partition τ to $F_S(\tau) = \operatorname{dit}(\tau) \cap S$:

$$\frac{\tau \preceq G_S(P)}{F_S(\tau) \subseteq P}.$$ [14]

Specializing $S = \operatorname{dit}(\sigma)$ and $P = \operatorname{dit}(\pi)$ gives $G_{\operatorname{dit}(\sigma)}(\operatorname{dit}(\pi))$ as the partition whose dit set is $\operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\pi))$ which we know from above is the partition implication $\sigma \Rightarrow \pi$, i.e., $G_{\operatorname{dit}(\sigma)}(\operatorname{dit}(\pi)) = \sigma \Rightarrow \pi$. Using these restrictions, the adjunction gives the iff statement characterizing the partition implication.

$$\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi) \text{ iff } \tau \preceq \sigma \Rightarrow \pi.$$
Characterization of $\sigma \Rightarrow \pi$

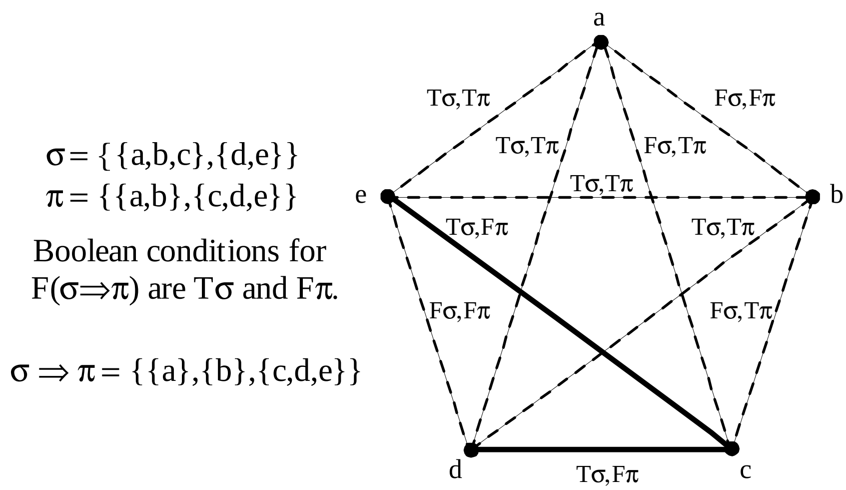
Thus $\sigma \Rightarrow \pi$ is the most refined partition τ such that $\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. The arbitrary intersection of equivalence relations (indit sets) is an equivalence relation so the arbitrary union of dit sets is a dit set, i.e., the dit set of the join of the partitions whose dit sets were in the union. Moreover, distributivity in $\mathcal{P}(U \times U)$ implies that the arbitrary union of dit sets $\operatorname{dit}(\tau)$ such that $\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$ will also satisfy that same condition. Hence we may construct the most refined partition τ such that $\operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$ by taking the join of those partitions:

$$\sigma \Rightarrow \pi = \bigvee \{\tau : \operatorname{dit}(\tau) \cap \operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)\}.$$

The equivalent graph-theoretic definition of the partition implication can be illustrated using the previous example.

Example 2 *Let $\sigma = \{\{a,b,c\},\{d,e\}\}$ and $\pi = \{\{a,b\},\{c,d,e\}\}$ as before. In the graph below, all the arcs in the complete graph K_5 on five nodes are again labelled according to the status of the two endpoints in the two partitions. The Boolean conditions for $F(\sigma \Rightarrow \pi)$ are "$T\sigma$ and $F\pi$" . The arcs where those conditions hold are the solid lines. In the graph with only the solid arcs, there are three connected components giving the blocks of the implication: $\sigma \Rightarrow \pi = \{\{a\},\{b\},\{c,d,e\}\}$. Note that only the π-block $\{a,b\}$ is contained in a σ-block so $\sigma \Rightarrow \pi$ is like π except that $\{a,b\}$ is discretized.*

[14] Thanks to the referee for suggesting the simpler presentation of this adjunction (as the composition of two adjunctions) as well as for other helpful comments and suggestions.



Graph for $\sigma \Rightarrow \pi$

1.8 Partition negation operation

In intuitionistic logic, the negation $\neg\sigma$ would be defined as the implication $\sigma \Rightarrow 0$ with the consequent taken as the zero element 0, i.e., $\neg\sigma = \sigma \Rightarrow 0$. In the topological interpretation using open subsets, σ would be an open subset and $\neg\sigma$ would be the interior of its complement. Adapted to partitions, these give the same dit-set definition of the partition negation (since $\operatorname{dit}(0) = \emptyset$):

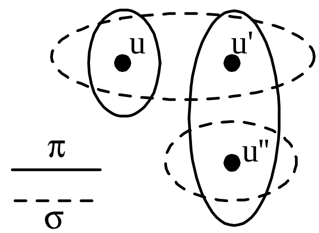
$$\operatorname{dit}(\neg\sigma) = \operatorname{int}(\operatorname{dit}(\sigma)^c) = \operatorname{dit}(\sigma \Rightarrow 0).$$

It is a perhaps surprising fact that this dit set is always empty (so that $\neg\sigma = 0$) except in the singular case where $\sigma = 0$ in which case we, of course, have $\neg 0 = (0 \Rightarrow 0) = 1$.[15] The key fact is that any two partitions (aside from the blob) must have some dits in common.

Theorem 3 (Common-dits theorem) *Any two non-empty dit sets have some dits in common.*

Proof: Let π and σ be any two partitions on U with non-empty dit sets, i.e., $\pi \neq 0 \neq \sigma$. We need to show that $\operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma) \neq \emptyset$. Since π is not the blob 0, consider two elements u and u' distinguished by π but identified by σ [otherwise $(u, u') \in \operatorname{dit}(\pi) \cap \operatorname{dit}(\sigma)$]. Since σ is also not the blob, there must be a third element u'' not in the same block of σ as u and u'.

[15] In graph theory, this is the result that given any disconnected (simple) graph G, its complement G^c (set of all links not in G) is connected (Wilson 1972, p. 30).



(u,u'') as a common dit

But since u and u' are in different blocks of π, the third element u'' must be distinguished from one or the other or both in π. Hence (u,u'') or (u',u'') must be distinguished by both partitions and thus must be in $\operatorname{dit}(\pi)\cap\operatorname{dit}(\sigma)$. ■

It should be noted that the interior of the intersection $\operatorname{dit}(\pi)\cap\operatorname{dit}(\sigma)$ could be empty, i.e., $\sigma\wedge\pi=0$, even when the intersection is non-empty. It might also be useful to consider the contrapositive form of the common-dits theorem which is about equivalence relations. If the union of two equivalence relations is the universal equivalence relation, i.e., $\operatorname{indit}(\pi)\cup\operatorname{indit}(\sigma)=U\times U$, then one of the equivalence relations is the universal one, i.e., $\operatorname{indit}(\pi)=U\times U$ or $\operatorname{indit}(\sigma)=U\times U$.

For any non-blob partition σ, $\operatorname{dit}(\neg\sigma)=\operatorname{int}(\operatorname{dit}(\sigma)^c)$ is a dit set disjoint from the non-empty $\operatorname{dit}(\sigma)$ so by the common-dits theorem, it has to be empty and thus $\neg\sigma=0$. Negation becomes more useful if we generalize by replacing the blob in the definition $\neg\sigma=\sigma\Rightarrow 0$ by an arbitrary but fixed partition π. This leads to the notion of the *π-negation* of a partition σ which is just the implication $\sigma\Rightarrow\pi$ with the fixed partition π as the consequent. We added a π to the negation symbol to represent this negation relative to π:

$$\pi\text{-negation: } \overset{\pi}{\neg}\sigma=\sigma\Rightarrow\pi.$$

The unadorned negation $\neg\sigma$ is the 0-negation, i.e., $\neg\sigma=\sigma\Rightarrow 0$. Using this suggestive notation, the partition tautology that internalizes modus ponens, $(\sigma\wedge(\sigma\Rightarrow\pi))\Rightarrow\pi$, is the law of non-contradiction, $\overset{\pi}{\neg}\left(\sigma\wedge\overset{\pi}{\neg}\sigma\right)$, for π-negation. While it is useful to establish the notion of partition negation, it need not be taken as a primitive operation.

1.9 Partition stroke, not-and, or nand operation

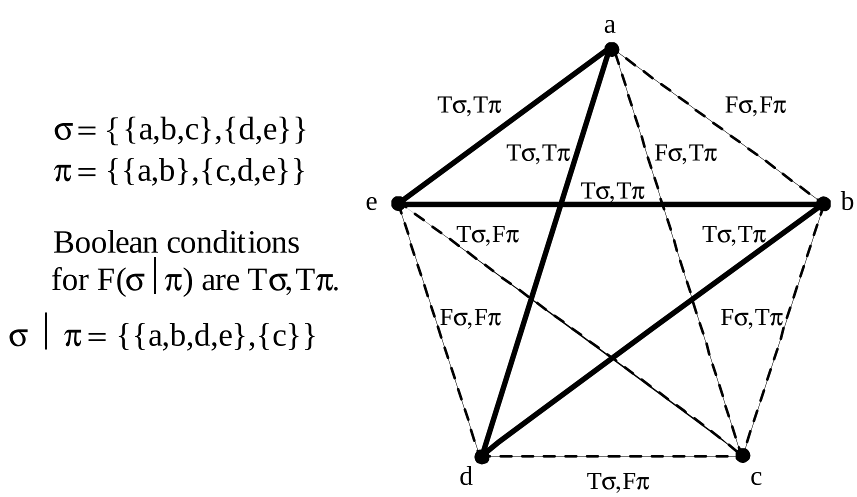
In addition to the lattice operations of the join and meet, and the implication operation, we introduce the *Sheffer stroke*, *not-and*, or *nand* operation $\sigma\mid\tau$, with the dit-set definition:

$$\operatorname{dit}(\sigma\mid\tau)=\operatorname{int}[\operatorname{indit}(\sigma)\cup\operatorname{indit}(\tau)].$$

For a set-of-blocks definition consider a graph whose nodes are the elements $u\in U$. Given $\sigma=\{C\}$ and $\tau=\{D\}$, each element u is in a unique block $C\cap D$ of the join $\sigma\vee\tau$. Given elements $u\in C\cap D$ and $u'\in C'\cap D'$, u is connected by an arc or link in the graph, i.e., $u \sim u'$, if $C \neq C'$ and $D \neq D'$, i.e., if $(C \cap D) \times (C' \cap D') \subseteq \text{dit}(\sigma) \cap \text{dit}(\tau) = [\text{indit}(\sigma) \cup \text{indit}(\tau)]^c$. Then the nodes in each connected component of the graph are the blocks of $\sigma \mid \tau$. Two nodes u, u' are connected in this graph if and only if the ordered pair (u, u') is in the closure $\overline{(\text{dit}(\sigma) \cap \text{dit}(\tau))} = \overline{[\text{indit}(\sigma) \cup \text{indit}(\tau)]^c}$, and thus they are a distinction if and only if they are in the complement of the closure which is the interior: $\text{int}[\text{indit}(\sigma) \cup \text{indit}(\tau)]$. Hence this graph-theoretic definition of the nand operation is the same as the dit-set definition.

To turn it into a set-of-blocks definition, note that when $u \sim u'$ because $C \neq C'$ and $D \neq D'$ then all the elements of $C \cap D$ and $C' \cap D'$ are in the same block of the nand $\sigma \mid \tau$. But if for any non-empty $C \cap D$, there is no other block $C' \cap D'$ of the join with $C \neq C'$ and $D \neq D'$, then the elements of $C \cap D$ would not even be connected with each other so they would be singletons in the nand. Hence for the set-of-blocks definition of the nand $\sigma \mid \tau$, the blocks of the nand partition are formed by taking the unions of any join blocks $C \cap D$ and $C' \cap D'$ which differ in both "components" but by taking as singletons the elements of any $C \cap D$ which does not differ from any other join block in both components.

Example 4 *Let $\sigma = \{\{a,b,c\},\{d,e\}\}$ and $\pi = \{\{a,b\},\{c,d,e\}\}$ as before. In the graph below, all the arcs in the complete graph K_5 on five nodes are again labelled according to the status of the two endpoints in the two partitions. The Boolean conditions for $F(\sigma \mid \pi)$ are "$T\sigma$ and $T\pi$". The arcs where those conditions hold are the solid lines. In the graph with only the solid arcs, there are two connected components giving the blocks of the nand: $\sigma \mid \pi = \{\{a,b,d,e\},\{c\}\}$.*



Example 5 *If $\sigma = \{C, C'\}$ where $C = \{u\}$ and $C' = U - \{u\}$ and $\tau = \{D, D'\}$ where $D = U - \{u'\}$ and $D' = \{u'\}$, then $\sigma \vee \tau = \{\{u\},\{u'\},U - \{u,u'\}\}$. Hence $u \in C \cap D = \{u\} \cap (U - \{u'\})$ and $u' \in C' \cap D' = (U - \{u\}) \cap \{u'\}$ so $u \sim u'$ in the graph for $\sigma \mid \tau$. But the elements $u'' \in C' \cap D = U - \{u,u'\}$ are not connected to any other elements since $C' \cup D = (U - \{u\}) \cup (U - \{u'\}) = U$ so they are all singletons in the nand. Hence $\sigma \mid \tau = \{\{u,u'\},\{u''\},...\}$.*

This example can be stated in more general terms. A *modular* partition is a partition with at most one non-singleton block. A non-zero partition φ is an *atom* in the lattice of partitions $\Pi(U)$ if $0 \preceq \pi \preceq \varphi$ implies $\pi = 0$ or $\pi = \varphi$. A non-unitary partition φ is a *coatom* if $\varphi \preceq \pi \preceq 1$ implies $\pi = \varphi$ or $\pi = 1$. All coatoms are modular where the non-singleton block is some pair $\{u, u'\}$. The example then shows that the nand of any two distinct modular atoms is a coatom.

For subsets $S, T \subseteq U$, the nand subset $S \mid T = S^c \cup T^c = (S \cap T)^c$ has as elements those elements $u \in U$ which are not elements of both S and T. Using the relationship between elements of a subset and distinctions of a partition, the nand partition $\sigma \mid \tau$ has as distinctions those distinctions $(u, u') \in U \times U - \Delta$ which are, directly or indirectly, not distinctions of both σ and τ. In the example above, (u, u') is a distinction of both σ and τ so it is not a distinction of $\sigma \mid \tau$. For any third element $u'' \in U$, then u'' paired with any other element of U is not a dit of both σ and τ so the pair is a distinction of $\sigma \mid \tau$, i.e., $\{u''\}$ is a singleton in the nand partition.

A number of the relations which we are accustomed to in subset logic also hold in partition logic. For instance, negation can be defined using the nand: $\sigma \mid \sigma = \neg\sigma$. In fact, if $\sigma \preceq \tau$, then $\sigma \mid \tau = \neg\sigma$. For example, since σ is always refined by $\tau \Rightarrow \sigma$ for any τ, $\sigma \mid (\tau \Rightarrow \sigma) = \neg\sigma$. The formula $\sigma \mid \sigma = \neg\sigma$ is also a special case of the formula $(\sigma \mid \tau) \wedge (\sigma \Rightarrow \tau) = \neg\sigma$ derived in the next section.

In subset logic, the "and" and the nand subsets would be complements of one another but the relationship is more subtle in partition logic. We say that two partitions φ and φ' which refine a partition π, i.e., $\pi \preceq \varphi, \varphi'$, are π-*orthogonal* if $\overset{\pi}{\neg}\varphi \vee \overset{\pi}{\neg}\varphi' = 1$. Since all partitions refine 0, two partitions φ and φ' are 0-*orthogonal* or, simply, *orthogonal* if $\neg\varphi \vee \neg\varphi' = 1$. This may look odd as a criterion for orthogonality but it is classically equivalent to the more familiar $\varphi \wedge \varphi' = 0$.

Lemma 6 *φ and φ' are orthogonal, i.e., $\neg\varphi \vee \neg\varphi' = 1$, iff $\varphi \mid \varphi' = 1$.*

Proof: If $\neg\varphi \vee \neg\varphi' = 1$, then $\operatorname{int}(\operatorname{indit}(\varphi)) \cup \operatorname{int}(\operatorname{indit}(\varphi')) = \operatorname{dit}(1) = U^2 - \Delta$. By the monotonicity of the interior operator, $\operatorname{int}(\operatorname{indit}(\varphi)) \cup \operatorname{int}(\operatorname{indit}(\varphi')) \subseteq \operatorname{int}(\operatorname{indit}(\varphi) \cup \operatorname{indit}(\varphi')) = \operatorname{dit}(\varphi \mid \varphi')$ so $\varphi \mid \varphi' = 1$. Conversely if $\varphi \mid \varphi' = 1$, then $\operatorname{int}(\operatorname{indit}(\varphi) \cup \operatorname{indit}(\varphi')) = \operatorname{dit}(1) = U^2 - \Delta$. Since $\Delta \subseteq \operatorname{indit}(\varphi), \operatorname{indit}(\varphi')$ (so that only Δ is removed by the interior operator), $\operatorname{indit}(\varphi) \cup \operatorname{indit}(\varphi') = U^2$. It was previously noted that if the union of two equivalence relations is the universal equivalence relation U^2, then one of the equivalence relations must be the universal one. Hence either $\varphi = 0$ or $\varphi' = 0$ and since $\neg 0 = 1$, we have either way, $\neg\varphi \vee \neg\varphi' = 1$. ■

Just as the unary negation operation $\neg\varphi$ is usefully generalized by the binary operation $\overset{\pi}{\neg}\varphi = \varphi \Rightarrow \pi$, so the binary nand operation $\sigma \mid \tau$ is usefully generalized by the ternary operation of π-*nand* defined by:

$$\operatorname{dit}(\sigma \mid_\pi \tau) = \operatorname{int}(\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)).$$

Then a similar argument shows that for $\pi \preceq \varphi, \varphi'$:

$$\varphi \text{ and } \varphi' \text{ are } \pi\text{-orthogonal iff } \varphi \mid_\pi \varphi' = 1.$$

Thus two partitions are orthogonal when if one of the partitions is non-zero, then the other partition must be zero (i.e., at least one is zero). If φ and φ' are orthogonal, i.e., $\varphi \mid \varphi' = 1$, then $\varphi \wedge \varphi' = 0$ follows but not vice-versa (see next example).

Every partition σ and its 0-negation $\neg\sigma$ are orthogonal since $\neg\sigma \vee \neg\neg\sigma = 1$. In the example above, the meet of $\sigma = \{\{u\}, U - \{u\}\}$ and $\tau = \{\{u'\}, U - \{u'\}\}$ is $\sigma \wedge \tau = 0$ and $\neg 0 = 1$ but $\sigma \mid \tau \neq 1$ so the negation $\neg(\sigma \wedge \tau)$ is not necessarily the same as the nand $\sigma \mid \tau$. However, the "and" or meet $\sigma \wedge \tau$ and the "not-and" or nand $\sigma \mid \tau$ are orthogonal; if one is non-zero, the other must be zero. Thus no pair (u, u') can be a dit of both and hence $(\sigma \mid \tau) \mid (\sigma \wedge \tau) = 1$ is a partition tautology. The same example above shows that the nand $\sigma \mid \tau$ is also not the same as $\neg\sigma \vee \neg\tau$ (which equals 0 in the example). Although the three formulas are equal in subset logic, in partition logic we only have the following refinement relations holding in general:

$$\neg\sigma \vee \neg\tau \preceq \sigma \mid \tau \preceq \neg(\sigma \wedge \tau).$$

Thus only one direction $\neg\sigma \vee \neg\tau \preceq \neg(\sigma \wedge \tau)$ holds in general so the "strong" DeMorgan law $\neg\sigma \vee \neg\tau = \neg(\sigma \wedge \tau)$ does not hold in partition logic. However, the other "weak" DeMorgan law holds in partition logic even for π-negation, i.e., $\overset{\pi}{\neg}(\sigma \vee \tau) = \overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau$.

Example 7 *The universe set $U = \{Tom, John, Jim\}$ consists of three people and there are two partitions: α which distinguishes people according to the first letter of their name so that $\alpha = \{\{Tom\}, \{John, Jim\}\}$, and ω which distinguishes people according to the last letter of their name so that $\omega = \{\{Tom, Jim\}, \{John\}\}$. Then the meet $\alpha \wedge \omega$ would identify people who are directly and indirectly identified by the two partitions. Tom and John are not directly identified but are indirectly identified: Tom $\overset{\omega}{\sim}$ Jim $\overset{\alpha}{\sim}$ John so that $\sigma \wedge \omega = 0$. But since the meet is 0, the 0-orthogonal nand of the two partitions could be non-zero, and in fact $\alpha \mid \omega = \{\{Tom, John\}, \{Jim\}\}$. Thus the fact that Tom and John are directly distinguished by both the first and last letters of their names results in them not being distinguished by the not-and partition.*

In any dit-set definition of a partition φ as $\operatorname{dit}(\varphi) = \operatorname{int}(P)$ for some $P \subseteq U \times U$, two elements u and u' will be in the same block of φ if and only if they are in the closure $\overline{(P^c)}$, i.e., if there is a finite sequence of links $(u_i, u_{i+1}) \in P^c$ connecting u and u'. The question arises of there being an upper bound on the number of links required to put two elements in the same block. In the simple case of the join $\sigma \vee \tau$ where $\operatorname{dit}(\sigma \vee \tau) = \operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)$, no interior operator is needed since the union of open subsets of the closure space $U \times U$ is open. Thus the complement $(\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau))^c = \operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau)$ is already closed (i.e., the intersection of two equivalence relations is an equivalence relation) so one link $(u, u') \in \operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau)$ suffices to put u and u' into the same block of the join $\sigma \vee \tau$. Thus for the join, one link suffices.

For the implication $\sigma \Rightarrow \tau$, $(u,u') \in \text{indit}(\sigma \Rightarrow \tau)$ if and only if $(u,u') \in \text{indit}(\tau)$, say, $u,u' \in D \in \tau$, and there is no $C \in \sigma$ such that $D \subseteq C$ so the block D remains whole in the implication $\sigma \Rightarrow \tau$. But that means there is another block $C' \in \sigma$ such that $D \cap C' \neq \emptyset$, i.e., there is an $a \in D \cap C'$ such that (u,a) and (a,u') are both dits of σ but indits of τ. Thus there is at most a two link chain connecting u and u' where each link is in $\text{dit}(\sigma) \cap \text{indit}(\tau) = (\text{dit}(\sigma)^c \cup \text{dit}(\tau))^c$. Thus for the implication, two links suffice.

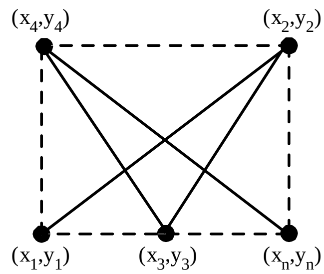
For the meet of two partitions, it is well-known that there is no upper bound on the finite number of links needed to connect two elements which are in the same block. For instance on the natural numbers, take $\sigma = \{\{0,1\},\{2,3\},...\}$ and $\tau = \{\{0\},\{1,2\},\{3,4\},...\}$ so that $\sigma \wedge \tau$ is the blob and thus any two elements are connected. But clearly there is no upper bound on the number of links needed to connect any two elements.

For the nand operation, it is perhaps interesting that four links suffice. To show this, we first exhibit an example where four links are required, i.e., no shorter set of links would suffice. Then we show that in general, longer chains can always be shortened to four or fewer links..

For an example where four links are required, consider the four-link chain u,a,b,c,u' connecting u and u' in the nand $\sigma \mid \tau$ where $\sigma = \{\{u,u',b\},\{a,c\}\}$ and $\tau = \{\{u,c\},\{u',a\},\{b\}\}$. Each link (u,a), (a,b), (b,c), and (c,u') in the four-link chain is in the set $(\text{indit}(\sigma) \cup \text{indit}(\tau))^c = \text{dit}(\sigma) \cap \text{dit}(\tau)$ so (u,u') is in its closure, i.e., u and u' are in the same block of $\sigma \mid \tau = 0$. And there are no short-cuts. By placing the five points on the vertices of a pentagon, then it is easy to see that none of the short-cutting chords are in $\text{dit}(\sigma) \cap \text{dit}(\tau)$.

Lemma 8 *Four links suffice to put any two elements in the same block of any nand $\sigma \mid \tau$.*

Proof: The proof can be formulated abstractly using sequences of ordered pairs which can be pictured as points on the plane. Suppose we have a chain of ordered pairs (x_1,y_1), (x_2,y_2),..., (x_n,y_n) where each pair differs from the previous one on both coordinates. However if any pair differs on both coordinates with a previous pair, then all intermediate pairs could be cut out thus shortening the chain. We want to construct a subchain with four or less links. Since we cannot just directly connect the end points they must agree on one coordinate such as the x coordinate. Then (x_2,y_2) must also agree on one coordinate with (x_n,y_n) or we would just connect them and be finished with a two-link chain. But they cannot agree on the x coordinate since it has to differ on both coordinates from the first point (x_1,y_1). Hence it has to agree on the y coordinate with (x_n,y_n).



Four links suffice

The third point (x_3, y_3) must agree with the first and last which means on the x coordinate as pictured above. Then the fourth point (x_4, y_4) must differ from (x_3, y_3) on both coordinates but must agree with the first and second points on some coordinates. Thus it must agree with the first point on the y coordinate and with the second point on the x coordinate. But then it will differ from the last point (x_n, y_n) on both coordinates so it can be directly connected giving a four link subchain where each successive pair differs on both coordinates.

To map this abstract proof into the case at hand, recall that $\operatorname{indit}(\sigma \mid \tau) = \overline{[\operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)]}$ so that (u, u') is an indit of $\sigma \mid \tau$ if there is a finite sequence $u = u_1, u_2, ..., u_n = u'$ with each pair $(u_i, u_{i+1}) \in \operatorname{dit}(\sigma) \cap \operatorname{dit}(\tau)$. Different horizontal coordinates correspond to different σ blocks and different vertical coordinates correspond to different τ blocks where only a finite number of points are needed to model the finite sequence. The first link (u_1, u_2) then maps to the first line segment from (x_1, y_1) (the pair of coordinates representing the blocks $C_1 \in \sigma$ and $D_1 \in \tau$ containing u_1) to (x_2, y_2) (the pair representing the blocks $C_2 \in \sigma$ and $D_2 \in \tau$ containing u_2). The second link (u_2, u_3) maps to the second line segment from (x_2, y_2) to (x_3, y_3), and so forth. ■

1.10 Sixteen binary operations on partitions

What other partition operations might be defined? For binary operations $\sigma * \tau$ on Boolean $0, 1$ variables σ and τ, there are four combinations of values for σ and τ, and thus there are $2^4 = 16$ possible binary Boolean operations: $2 \times 2 \to 2$. Thinking in terms of subsets $S, T \subseteq U$ instead of Boolean propositional variables, there are the four basic disjoint regions in the general position Venn diagram for S and T, namely $S \cap T$, $S \cap T^c$, $S^c \cap T$, and $S^c \cap T^c$. Then there are again $2^4 = 16$ subsets of U defined by including or not including each of these four basic regions. That defines the 16 binary *logical* operations on subsets of U.

Now take $S = \operatorname{dit}(\sigma)$ and $T = \operatorname{dit}(\tau)$ as subsets of $U \times U$ and define the 16 subsets of $U \times U$ in the same way. Some of these such as $S \cup T = \operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau) = \operatorname{dit}(\sigma \vee \tau)$ will be open and thus will be the dit sets of partitions on U. For those which are not already open, we must apply the interior operator to get the dit set of a partition on U. This gives 16 binary operations on partitions that would naturally be called *logical* since they are immediately paired with the corresponding 16 binary logical operations on subsets. We will use the same notation for the partition operations. For instance, for subsets $S, T \subseteq U$, the conditional or implication subset is $S^c \cup T = S \Rightarrow T$. When $S = \text{dit}(\sigma)$ and $T = \text{dit}(\tau)$ as subsets of $U \times U$, the subset $S^c \cup T$ is not necessarily open so we must apply the interior operator to get the dit set defining the corresponding implication operation on partitions, i.e., $\text{int}[\text{dit}(\sigma)^c \cup \text{dit}(\tau)] = \text{dit}(\sigma \Rightarrow \tau)$.

In both subset and partition logic, there are only two nullary operations (constants), 0 and 1. With unary operations, the situation is still straightforward. There are only four subset logical unary operations: identity and negation (or complementation) in addition to the two nullary operations (seen as constant unary operations). These immediately yield the partition operations of identity σ and negation $\neg\sigma$ in addition to the two partition constant operations 0 and 1. If these partition operations are compounded using the logical operations such as negation, implication, join, meet, and nand, then two other distinct unary operations are generated: the double negation $\neg\neg\sigma$ and the excluded middle formula $\sigma \vee \neg\sigma$ (which is also equal to $\neg\neg\sigma \Rightarrow \sigma$, the direction of the usual law of double negation that is not a partition tautology)–to make six logical unary partition operations.

The situation for binary partition operations is considerably more complicated. If the sixteen binary operations on subsets are compounded, then the result is always one of the sixteen binary operations, e.g., $S \cap (S \Rightarrow T) = S \cap T$. But the presence of the interior operator significantly changes the partition case. Compounding gives many new binary operations on partitions, e.g., $\neg(\sigma \wedge \tau)$ and $\neg\sigma \vee \neg\tau$ (noted in the analysis of $\sigma \mid \tau$), and they could just as well be called "logical" operations.[16] For our purposes here, we will settle for being able to define the sixteen binary logical operations on partitions that correspond to the sixteen logical binary subset operations. But which binary operations suffice to define all those sixteen operations?

1.11 Conjunctive normal form in partition logic

The four operations, the join, meet, implication, and nand, suffice to define the sixteen binary logical partition operations by using the partition version of conjunctive normal form–which, in turn, is based on the following result.

Lemma 9 *For any subsets* $A, B \subseteq U \times U$, $\text{int}[A \cap B] = \text{int}[\text{int}(A) \cap \text{int}(B)]$.

Proof: Since $\text{int}(A) \subseteq A$ and $\text{int}(B) \subseteq B$, $\text{int}[\text{int}(A) \cap \text{int}(B)] \subseteq \text{int}[A \cap B]$. Conversely, $A \cap B \subseteq A, B$ so $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$ and since $\text{int}(A \cap B)$ is open, $\text{int}[A \cap B] \subseteq \text{int}[\text{int}(A) \cap \text{int}(B)]$. ■

In the treatment of the 16 subsets defined from four basic regions $S \cap T$, $S \cap T^c$, $S^c \cap T$, and $S^c \cap T^c$, we were in effect using disjunctive normal form to define the 15 non-empty subsets by taking the unions of the 15 combinations of those four basic regions. But the above lemma shows that the conjunctive

[16] Although beyond the scope of this paper there are, for example, over a hundred logical binary operations definable just with formulas using only the implication $\Rightarrow$ and 0.

normal form will be more useful in partition logic (since the corresponding result for the union and the interior operator does not hold).

In the subset version of the conjunctive normal form, the 15 non-universal subsets are obtained by taking the intersections of 15 combinations of the four regions: $S\cup T$, $S\cup T^c$, $S^c\cup T$, and $S^c\cup T^c$. Taking $S=\operatorname{dit}(\sigma)$ and $T=\operatorname{dit}(\tau)$, the interiors of these four basic "conjuncts" are, respectively, the dit sets of: $\sigma\vee\tau$, $\tau\Rightarrow\sigma$, $\sigma\Rightarrow\tau$, and $\sigma\mid\tau$. By expressing each of the 15 non-universal subsets of $U\times U$ in conjunctive normal form, applying the interior operator, and then using the lemma to distribute the interior operator across the intersections, we express each of the 15 partition operations (aside from the constant 1) as a meet of some combination of the join $\sigma\vee\tau$, the implications $\tau\Rightarrow\sigma$ and $\sigma\Rightarrow\tau$, and the nand $\sigma\mid\tau$. The constant operation 1 can be obtained using just the implication $\sigma\Rightarrow\sigma$ or $\tau\Rightarrow\tau$. These results and some other easy reductions are given in the following tables. In the first table, the interior of the subset of $U\times U$ in the first column yields the dit set of the binary operation given in the second column.[17]

15 regions Conjunctive Normal Form	Binary operation on partitions
$(S\cup T)\cap(S^c\cup T)\cap(S\cup T^c)\cap(S^c\cup T^c)$	0
$(S^c\cup T)\cap(S\cup T^c)\cap(S^c\cup T^c)$	$\sigma\overline{\vee}\tau=\neg\sigma\wedge\neg\tau$
$(S\cup T)\cap(S\cup T^c)\cap(S^c\cup T^c)$	$\tau\not\Leftarrow\sigma=\sigma\wedge\neg\tau$
$(S\cup T^c)\cap(S^c\cup T^c)$	$\neg\tau=\tau\Rightarrow 0$
$(S\cup T)\cap(S^c\cup T)\cap(S^c\cup T^c)$	$\sigma\not\Leftarrow\tau=\neg\sigma\wedge\tau$
$(S^c\cup T)\cap(S^c\cup T^c)$	$\neg\sigma=\sigma\Rightarrow 0$
$(S\cup T)\cap(S^c\cup T^c)$	$\sigma\not\equiv\tau$
$S^c\cup T^c$	$\sigma\mid\tau$
$(S\cup T)\cap(S^c\cup T)\cap(S\cup T^c)$	$\sigma\wedge\tau$
$(S^c\cup T)\cap(T^c\cup S)$	$\sigma\equiv\tau$
$(S\cup T)\cap(S\cup T^c)$	σ
$S\cup T^c$	$\tau\Rightarrow\sigma$
$(S\cup T)\cap(S^c\cup T)$	τ
$S^c\cup T$	$\sigma\Rightarrow\tau$
$S\cup T$	$\sigma\vee\tau$

Interior of column 1 gives partition operation in column 2

Using the lemma, the interior is distributed across the intersections of the subset CNF to give the partition CNF in the following table.

[17] For notation, we have followed, for the most part, Church (1956).

Binary operation	Partition CNF for 15 binary operations
0	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \mid \tau)$
$\sigma \overline{\vee} \tau = \neg\sigma \wedge \neg\tau$	$= (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \mid \tau)$
$\tau \not\Leftarrow \sigma = \sigma \wedge \neg\tau$	$= (\sigma \vee \tau) \wedge (\tau \Rightarrow \sigma) \wedge (\sigma \mid \tau)$
$\neg\tau = \tau \Rightarrow 0$	$= (\tau \Rightarrow \sigma) \wedge (\sigma \mid \tau)$
$\sigma \not\Leftarrow \tau = \neg\sigma \wedge \tau$	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\sigma \mid \tau)$
$\neg\sigma = \sigma \Rightarrow 0$	$= (\sigma \Rightarrow \tau) \wedge (\sigma \mid \tau)$
$\sigma \not\equiv \tau$	$= (\sigma \vee \tau) \wedge (\sigma \mid \tau)$
$\sigma \mid \tau$	$= \sigma \mid \tau$
$\sigma \wedge \tau$	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$
$\sigma \equiv \tau$	$= (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$
σ	$= (\sigma \vee \tau) \wedge (\tau \Rightarrow \sigma)$
$\tau \Rightarrow \sigma$	$= \tau \Rightarrow \sigma$
τ	$= (\sigma \vee \tau) \wedge (\sigma \Rightarrow \tau)$
$\sigma \Rightarrow \tau$	$= \sigma \Rightarrow \tau$
$\sigma \vee \tau$	$= \sigma \vee \tau$

Distributing interior across intersections gives partition CNF

In classical subset logic, these 15 binary operations on subsets plus the universe set would be closed under combining the operations so we would have the reduction of all formulas in two variables to conjunctive normal form. But in partition logic, these functions are not at all closed under combinations, so we have only derived the conjunctive normal form for the 15 binary operations. The point was to show that the 15 functions, and thus all their further combinations, could be defined in terms of the four primitive operations of join, meet, implication, and nand.[18]

The fourteen non-zero operations occur in natural pairs: $\Rightarrow$ and $\not\Rightarrow$, $\Leftarrow$ and $\not\Leftarrow$, $\equiv$ and $\not\equiv$, $\vee$ and $\overline{\vee}$, and $\wedge$ and $\mid$ in addition to σ and $\neg\sigma$, and τ and $\neg\tau$. Except in the case of the join $\vee$ (and, of course, σ and τ), the second operation in the pair is not the negation of the first. The relationship is not negation but 0-orthogonality. The pairs of formulas $\sigma \Rightarrow \tau$ and $\sigma \not\Rightarrow \tau$ (and similarly for the other pairs) are 0-orthogonal; if one is non-zero, the other must be zero. Later we see a different pairing of the operations by duality.

1.12 Partition algebra $\Pi(U)$ on U and its dual $\Pi(U)^{op}$

The partition lattice of all partitions on U with the top 1 and bottom 0 enriched with the binary operations of implication and nand is the *partition algebra* $\Pi(U)$ of U. It plays the role for partition logic that the Boolean algebra $\mathcal{P}(U)$ of all subsets of U plays in ordinary subset logic. Dualization in classical propositional logic–when expressed in terms of subsets–amounts to reformulating the operations as operations on subset complements. But since the complements

[18] There are other combinations which can be taken as primitive since the *inequivalence*, *symmetric difference*, *exclusive-or*, or *xor* $\sigma \not\equiv \tau$ can be used to define the nand operation: $((\sigma \vee \tau) \Rightarrow (\sigma \not\equiv \tau)) = \sigma \mid \tau$.

are in the same Boolean algebra, Boolean or classical duality can be expressed as a theorem about a Boolean algebra. We have defined the lattice of partitions (sets of disjoint and mutually exhaustive non-empty subsets of a set) as being isomorphic to the lattice of partition relations on $U \times U$ (anti-reflexive, symmetric, and anti-transitive relations). Rather than multiply notations, we have used $\Pi(U)$ to refer ambiguously to both those isomorphic lattices. The complement of a partition relation is an equivalence relation (reflexive, symmetric, and transitive relations) which is not an element in the same lattice. Hence in partition logic, duality is naturally expressed as a relationship between the partition algebra $\Pi(U)$ (seen as the algebra of partition relations) and the dual algebra $\Pi(U)^{op}$ of equivalence relations.

Given a formula φ in Boolean propositional logic, the dual formula φ^d is obtained by interchanging 0 and 1, and by interchanging each of the following pairs of operations: $\Rightarrow$ and $\not\Leftarrow$, $\vee$ and $\wedge$, $\equiv$ and $\not\equiv$, $\Leftarrow$ and $\not\Rightarrow$, and $\overline{\vee}$ and $|$, while leaving the atomic variables and negation $\neg$ unchanged (Church 1956, p. 106). In partition logic, we may use exactly the same general definition of dualization except that the atomic variables (and constants) will now stand for equivalence relations rather than partitions so we will indicate this by adding the superscript "d" to the atomic variables. However the partition formulas may be assumed to involve only $\vee$, $\wedge$, $\Rightarrow$, and $|$ along with 0 and 1. Hence the dual of modus ponens $\varphi = (\sigma \wedge (\sigma \Rightarrow \tau)) \Rightarrow \tau$ is $\varphi^d = \left(\sigma^d \vee \left(\sigma^d \not\Leftarrow \tau^d\right) \not\Leftarrow \tau^d\right)$. The *converse non-implication* $\not\Leftarrow$ (to use Church's terminology) is the difference operation (Lawvere and Rosebrugh 2003, p. 201), i.e., $\sigma^d \not\Leftarrow \tau^d$ is the result of subtracting σ^d from τ^d so it might otherwise be symbolized as $\tau^d - \sigma^d$ (or $\tau^d \backslash \sigma^d$). Then the dual to the modus ponens formula would be: $\varphi^d = \tau^d - \left(\sigma^d \vee \left(\tau^d - \sigma^d\right)\right)$. This, incidentally, is the formula that would have been compared to modus ponens $(\sigma \wedge (\sigma \Rightarrow \tau)) \Rightarrow \tau$ in classical and intuitionistic logic if the lattice of partitions had been written upside down instead of just comparing the *same* formulas in classical, intuitionistic, and partition logic (a benefit of writing that lattice right side up). Similarly the *non-implication* $\sigma^d \not\Rightarrow \tau^d$, dual to the reverse implication $\sigma \Leftarrow \tau$, might otherwise by symbolized as the difference $\sigma^d - \tau^d$ (or $\sigma^d \backslash \tau^d$).[19] The difference $\tau^d - \sigma^d$ and nor $\sigma^d \overline{\vee} \tau^d$ will be taken as primitive operations on equivalence relations. Those operations on partitions are not primitive: $\sigma \not\Leftarrow \tau = \tau - \sigma = \tau \wedge \neg\sigma$ and $\sigma \overline{\vee} \tau = \neg\sigma \wedge \neg\tau$. The equivalence and inequivalence operations on partitions are also not taken as primitive: $\sigma \equiv \tau = (\sigma \Rightarrow \tau) \wedge (\tau \Rightarrow \sigma)$ and $\sigma \not\equiv \tau = (\sigma \vee \tau) \wedge (\sigma \mid \tau)$.

The process of dualization is reversible. Starting with a formula φ^d with superscript "d" on all atomic variables (to indicate they refer to equivalence relations instead of partitions), dualizing means making the same interchanges of operation symbols and constants, and erasing the "d" superscripts so that the dual of the dual is the original formula.

[19] Church's usual attention to detail, e.g., treating the implication $\Rightarrow$ and the reverse implication $\Leftarrow$ as different operations, has been followed for its clarity in dualizing a formula φ to obtain φ^d. But in the semantic operations on partitions, only one implication need be defined since the reverse implication is obtained by reversing the two partitions, and similarly for the non-implications.

We have used the lower case Greek letters π, σ, ... to stand for set-of-blocks partitions while the corresponding binary partition relations were the dit sets $\operatorname{dit}(\pi)$, $\operatorname{dit}(\sigma)$, ... The Greek letters with the superscript "d" stand for binary equivalence relations which take the form $\operatorname{indit}(\pi)$, $\operatorname{indit}(\sigma)$, ... Thus atomic variables such as π dualize to π^d and would be interpreted as denoting indit sets $\operatorname{indit}(\pi)$.

The operations of the dual algebra $\Pi(U)^{op}$ of equivalence relations on U could be defined directly but it is more convenient to define them using duality from the partition operations. The top of the dual algebra, usually denoted $\widehat{1}$, is $0^d = \operatorname{indit}(0) = U \times U$, the universal equivalence relation that identifies everything. The bottom of the dual algebra, usually denoted $\widehat{0}$, is $1^d = \operatorname{indit}(1) = \Delta$, the diagonal where each element of U is only identified with itself. Given any equivalence relations $\operatorname{indit}(\pi)$ and $\operatorname{indit}(\sigma)$ on U, their meet $\wedge$ is defined via duality as the indit set of the join of the two corresponding partitions: $\operatorname{indit}(\pi) \wedge \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \vee \sigma) = \operatorname{indit}(\pi) \cap \operatorname{indit}(\sigma)$. Using the superscript-$d$ notation, this is: $\pi^d \wedge \sigma^d = (\pi \vee \sigma)^d = \operatorname{indit}(\pi \vee \sigma)$. Similarly the join of two equivalence relations is defined via duality as: $\operatorname{indit}(\pi) \vee \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \wedge \sigma) = \overline{\{\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)\}}$, so that using the superscript-d notation: $\pi^d \vee \sigma^d = (\pi \wedge \sigma)^d = \operatorname{indit}(\pi \wedge \sigma)$. The same pattern is applied to the duals of the other two primitive operations of implication and nand. The difference of two equivalence relations is defined via duality as: $\operatorname{indit}(\pi) - \operatorname{indit}(\sigma) = \operatorname{indit}(\sigma \Rightarrow \pi) = \overline{\{\operatorname{dit}(\sigma) \cap \operatorname{indit}(\pi)\}}$, which in the other notation is: $\pi^d - \sigma^d = (\sigma \Rightarrow \pi)^d = \operatorname{indit}(\sigma \Rightarrow \pi)$. And finally, the not-or or nor operation on equivalence relations is defined via duality as: $\operatorname{indit}(\pi) \bigtriangledown \operatorname{indit}(\sigma) = \operatorname{indit}(\pi \mid \sigma) = \overline{\{(\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma))^c\}}$, which gives: $\pi^d \bigtriangledown \sigma^d = (\pi \mid \sigma)^d = \operatorname{indit}(\pi \mid \sigma)$. That completes the definition of the dual algebra $\Pi(U)^{op}$ of equivalence relations on U with the top $\widehat{1}$, bottom $\widehat{0}$, and the four primitive operations of meet, join, difference, and nor.

The dualization operation $\varphi \longmapsto \varphi^d$ is a purely syntactic operation, but in the partition algebra $\Pi(U)$ and equivalence relation algebra $\Pi(U)^{op}$ we reason semantically about partitions and equivalence relations on U. Given a compound formula φ in the language of the partition algebra, it would be interpreted by interpreting its atomic variables as denoting partitions on U and then applying the partition operations (join, meet, implication, and nand) to arrive at an interpretation of φ. Such an interpretation automatically supplies an interpretation of the dual formula φ^d. If α was an atomic variable of φ and was interpreted as denoting a partition on U, then α^d is interpreted as denoting the equivalence relation $\operatorname{indit}(\alpha)$. Then the equivalence relation operations (meet, join, difference, and nor) are applied to arrive at an equivalence relation interpretation of the formula φ^d. The relationship between the two interpretations is very simple.

Proposition 2 $\varphi^d = \operatorname{indit}(\varphi)$.

Proof: The proof uses induction over the complexity of the formulas [where complexity is defined in the standard way in propositional logic (Fitting 1969)].

If φ is one of the constants 0 or 1, then the proposition holds since: $0^d = \widehat{1} = \text{indit}(0)$ and $1^d = \widehat{0} = \text{indit}(1)$. If $\varphi = \alpha$ is atomic, then it is true by the definition: $\sigma^d = \text{indit}(\sigma)$. If φ is a compound formula then the main connective in φ is one of the four primitive partition operations and the main connective in φ^d is one of the four primitive equivalence relation operations. Consider the case: $\varphi = \pi \wedge \sigma$ so that $\varphi^d = \pi^d \vee \sigma^d$. By the induction hypothesis, $\pi^d = \text{indit}(\pi)$ and $\sigma^d = \text{indit}(\sigma)$, and by the definition of the equivalence relation join: $\varphi^d = \pi^d \vee \sigma^d = \text{indit}(\pi) \vee \text{indit}(\sigma) = \overline{\{\text{indit}(\pi) \cup \text{indit}(\sigma)\}} = \text{indit}(\varphi)$. The other three cases proceed in a similar manner. ■

Corollary 1 *The map* $\varphi \longmapsto \text{indit}(\varphi)$ *is a dual-isomorphism:* $\Pi(U) \rightarrow \Pi(U)^{op}$ *between the partition algebra and the dual equivalence relation algebra.*

Proof: Clearly the mapping is a set isomorphism since each partition φ on U is uniquely determined by its dit set $\text{dit}(\varphi)$, and thus by its complement $\text{indit}(\varphi)$. By "dual-isomorphism," we mean that each operation in the partition algebra is mapped to the dual operation in the equivalence relation algebra. Suppose $\varphi = \sigma \Rightarrow \pi$ so that $\varphi^d = \pi^d - \sigma^d$. By the proposition, this means that $\text{indit}(\varphi) = \text{indit}(\pi) - \text{indit}(\sigma)$ (where we must be careful to note that "$-$" is the difference operation on equivalence relations which is the closure of the set-difference operation $\text{indit}(\pi) \cap \text{indit}(\sigma)^c$ on subsets of $U \times U$) so that $\varphi \longmapsto \text{indit}(\varphi)$ maps the partition operation of implication to the equivalence relation operation of difference. The other operations are treated in a similar manner. ■

The previous result $\text{int}[A \cap B] = \text{int}[\text{int}(A) \cap \text{int}(B)]$ for $A, B \subseteq U \times U$ could also be expressed using the closure operation as $\overline{[A \cup B]} = \overline{[\overline{A} \cup \overline{B}]}$ and thus the conjunctive normal form treatment of the 15 binary operations on partitions in terms of the operations of $\vee$, $\wedge$, $\Rightarrow$, and $|$ dualizes to the disjunctive normal form treatment of the 15 (dual) binary operations on equivalence relations in terms of the dual operations $\wedge$, $\vee$, $-$, and $\overline{\vee}$, which are the primitive operations in the algebra of equivalence relations $\Pi(U)^{op}$.

The previous two tables giving the CNF treatment of the 15 partition operations dualize to give two similar tables for the DNF treatment of the 15 non-zero operations on equivalence relations. In the following table, let $S' = \text{indit}(\sigma)$ and $T' = \text{indit}(\tau)$ where $()^c$ is complementation in $U \times U$. We have also taken the liberty of writing the "converse non-implication" operation as the difference operation on both equivalence relations and partitions: $\tau^d - \sigma^d = \sigma^d \not\Leftarrow \tau^d$ and $\tau - \sigma = \sigma \not\Leftarrow \tau$.

15 regions Disjunctive Normal Form	Bin. op. on eq. rel.	Dual to
$S'^c \cap T'^c$	$\sigma^d \overline{\vee} \tau^d$	$\sigma \mid \tau$
$S' \cap T'^c$	$\sigma^d - \tau^d$	$\tau \Rightarrow \sigma$
$(S' \cap T'^c) \cup (S'^c \cap T'^c)$	$\neg \tau^d$	$\neg \tau$
$S'^c \cap T'$	$\tau^d - \sigma^d$	$\sigma \Rightarrow \tau$
$(S'^c \cap T') \cup (S'^c \cap T'^c)$	$\neg \sigma^d$	$\neg \sigma$
$(S'^c \cap T') \cup (S' \cap T'^c)$	$\sigma^d \not\equiv \tau^d$	$\sigma \equiv \tau$
$(S'^c \cap T') \cup (S'^c \cap T'^c) \cup (S' \cap T'^c)$	$\sigma^d \mid \tau^d$	$\sigma \overline{\vee} \tau$
$S' \cap T'$	$\sigma^d \wedge \tau^d$	$\sigma \vee \tau$
$(S' \cap T') \cup (S'^c \cap T'^c)$	$\sigma^d \equiv \tau^d$	$\sigma \not\equiv \tau$
$(S' \cap T') \cup (S' \cap T'^c)$	σ^d	σ
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T'^c)$	$\tau^d \Rightarrow \sigma^d$	$\sigma - \tau$
$(S' \cap T') \cup (S'^c \cap T')$	τ^d	τ
$(S'^c \cap T') \cup (S'^c \cap T'^c) \cup (S' \cap T')$	$\sigma^d \Rightarrow \tau^d$	$\tau - \sigma$
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T')$	$\sigma^d \vee \tau^d$	$\sigma \wedge \tau$
$(S' \cap T') \cup (S' \cap T'^c) \cup (S'^c \cap T') \cup (S'^c \cap T'^c)$	$\widehat{1}$	0

Closure of column 1 gives equivalence relation binary operation in column 2

For instance, the CNF expression for the partition inequivalence or symmetric difference is: $\sigma \not\equiv \tau = (\sigma \vee \tau) \wedge (\sigma \mid \tau)$ so that:

$$\begin{aligned} \operatorname{dit}(\sigma \not\equiv \tau) &= \operatorname{int}\left[\operatorname{int}(\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)) \cap \operatorname{int}(\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\tau)^c)\right] \\ &= \operatorname{int}\left[(\operatorname{dit}(\sigma) \cup \operatorname{dit}(\tau)) \cap (\operatorname{dit}(\sigma)^c \cup \operatorname{dit}(\tau)^c)\right]. \end{aligned}$$

Taking complements yields:

$$\begin{aligned} \operatorname{indit}(\sigma \not\equiv \tau) &= \overline{\left[(\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau)) \cup (\operatorname{indit}(\sigma)^c \cap \operatorname{indit}(\tau)^c)\right]} \\ &= \overline{\left[\overline{(\operatorname{indit}(\sigma) \cap \operatorname{indit}(\tau))} \cup \overline{(\operatorname{indit}(\sigma)^c \cap \operatorname{indit}(\tau)^c)}\right]} \\ &= \overline{\left[(\sigma^d \wedge \tau^d) \cup (\sigma^d \overline{\vee} \tau^d)\right]} \\ &= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right) \\ &= \sigma^d \equiv \tau^d. \end{aligned}$$

Thus the equivalence $\sigma^d \equiv \tau^d$ of equivalence relations has the disjunctive normal form: $\sigma^d \equiv \tau^d = \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$ in the "dual" logic of equivalence relations. The disjunctive normal forms for the 15 operations on equivalence relations is given in the following table.

Binary operation	Equivalence relation DNF for 15 binary operations
$\sigma^d \overline{\vee} \tau^d$	$= \sigma^d \overline{\vee} \tau^d$
$\sigma^d - \tau^d$	$= \sigma^d - \tau^d$
$\neg \tau^d$	$= \left(\sigma^d - \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$
$\tau^d - \sigma^d$	$= \tau^d - \sigma^d$
$\neg \sigma^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$
$\sigma^d \not\equiv \tau^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\sigma^d \mid \tau^d$	$= \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\sigma^d \wedge \tau^d$	$= \sigma^d \wedge \tau^d$
$\sigma^d \equiv \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$
σ^d	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right)$
$\tau^d \Rightarrow \sigma^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$
τ^d	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\tau^d - \sigma^d\right)$
$\sigma^d \Rightarrow \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$
$\sigma^d \vee \tau^d$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\tau^d - \sigma^d\right)$
$\widehat{1}$	$= \left(\sigma^d \wedge \tau^d\right) \vee \left(\sigma^d - \tau^d\right) \vee \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$

Distributing closure across unions gives equivalence relation DNF

The table gives the expression of the non-primitive binary operations on equivalence relations, e.g., $\equiv$, $\not\equiv$, $|$,and $\Rightarrow$, in terms of the primitive operations. The unary operation of negation on equivalence relations, of course, has the simpler definition $\neg\sigma^d = \sigma^d \overline{\vee} \sigma^d$ in addition to the above DNF equation $\neg\sigma^d = \left(\tau^d - \sigma^d\right) \vee \left(\sigma^d \overline{\vee} \tau^d\right)$ which treats it as a binary operation.

In referring to the dual logic of equivalence relations, we must keep distinct different notions of duality. Partition logic is dual to subset logic in the sense of the duality between monomorphisms and epimorphisms (or between subsets and quotient sets). But equivalence relation logic is only dual to partition logic in the sense of complementation–analogous to the duality between Heyting algebras and co-Heyting algebras, or between open subsets and closed subsets of a topological space. Since the complement of an open set is a closed set that is not necessarily open, complementation-duality for partition logic and intuitionistic propositional logic is a duality between two types of algebras (partition algebras and equivalence relation algebras in the one case and Heyting and co-Heyting algebras in the other case). But the complement of a general subset is another subset so complementation-duality for subset logic is a duality within a Boolean algebra.

1.13 Subset and partition tautologies

For present purposes, we may take the formulas of classical propositional logic (i.e., subset logic) as using the binary operations of $\vee$, $\wedge$, $\Rightarrow$, and $|$ along with the constants 0 and 1 so that we have exactly the same well-formed formulas in subset logic and partition logic. A *classical tautology* or *subset tautology* is a formula that always evaluates to 1 (the universe set U) in the Boolean algebra $\mathcal{P}(U)$ regardless of the subsets assigned to the atomic variables. A *partition* *tautology* is a formula that always evaluates to 1 (the discrete partition) in the partition algebra $\Pi(U)$ regardless of the partitions assigned to the atomic variables.[20] It is also useful to define a *weak partition tautology* as a formula that never evaluates to 0 (the indiscrete partition) regardless of the partitions assigned to the atomic variables. Of course, any partition tautology is a weak partition tautology. Moreover, it is easily seen that:

Proposition 3 *φ is a weak partition tautology iff $\neg\neg\varphi$ is a partition tautology.*

An immediate question is the relationship of partition tautologies and weak partition tautologies to the classical subset tautologies as well as to the valid formulas of intuitionistic propositional logic (where formulas are assumed to be written in the same language).

There is a sense in which results in partition logic can be trivially seen as a generalization of results in ordinary subset logic. This *reduction principle* (only treated informally here) is based on the observation that any partition logic result holding for all U will hold when restricted to any two element universe $|U| = 2$. There is an isomorphism between the partition algebra $\Pi(2)$ on the two-element set and the Boolean algebra $\mathcal{P}(1)$ on the one-element set. There are only two partitions, the bottom 0 and top 1 on U where $|U| = 2$. Moreover, the partition operations of join, meet, implication, and nand in this special case satisfy the truth tables for the corresponding Boolean operations on subsets (using 0 and 1 in the usual manner in the truth tables). For instance, in $\Pi(U)$ where $|U| = 2$, we can only substitute 0 or 1 for the atomic variables in $\sigma \Rightarrow \tau$. The result is 0 in the case where $\sigma = 1$ and $\tau = 0$, and the result in 1 in the other three cases. But that is just the truth table for the Boolean implication operation in $\mathcal{P}(1)$. Similarly for the other operations so there is an isomorphism: $\Pi(2) \cong \mathcal{P}(1)$. Hence if a partition logic result holds for all U, then it holds for a two-element U where the partition operations on on the partitions 0 and 1 are isomorphic to the Boolean operations on the subsets 0 and 1 (where 0 and 1 in the Boolean case stand for the null subset and the universe set of a one-element universe). But if a result in subset logic holds on the one-element universe, i.e., in $\mathcal{P}(1)$, then it holds in subset logic. This might be summarized in the slogan:

Partition logic restricted to a two-element universe is Boolean logic:
$\Pi(2) \cong \mathcal{P}(1)$.
Reduction Principle

For instance, if φ is a weak partition tautology, e.g., $\varphi = \sigma \vee \neg\sigma$, then it will never evaluate to 0 in any $\Pi(U)$ where it is always assumed $|U| \geq 2$. For $|U| = 2$, there are only two partitions 0 and 1, so never evaluating to 0 means always evaluating to the partition 1. By the reduction principle, the Boolean operations in $\mathcal{P}(1)$ would always evaluate to the subset 1. This proves the following proposition.

[20] Needless to say, the constants 0 and 1 are always assigned the bottom and top, respectively, in any evaluation or interpretation of a formula in either $\mathcal{P}(U)$ or $\Pi(U)$.

Proposition 4 *All weak partition tautologies are classical subset tautologies.* ■

Corollary 2 *All partition tautologies are classical subset tautologies.* ■

The converse is not true with Peirce's law, $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma$, accumulation, $\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))$, and distributivity, $((\pi \vee \sigma) \wedge (\pi \vee \tau)) \Rightarrow (\pi \vee (\sigma \wedge \tau))$, being examples of classical tautologies that are not partition tautologies.

There is no inclusion either way between partition tautologies and the valid formulas of intuitionistic propositional logic. In view of the complex nature of the partition meet, it is not surprising that a formula such as the accumulation formula, $\sigma \Rightarrow (\pi \Rightarrow (\pi \wedge \sigma))$, is valid in both classical and intuitionistic logic but not in partition logic. The ("non-weak") law of excluded middle, $\sigma \vee \neg\sigma$, is a weak partition tautology, and the weak law of excluded middle, $\neg\sigma \vee \neg\neg\sigma$, is a ("non-weak") partition tautology that is not intuitionistically valid.

In the dual algebra $\Pi(U)^{op}$ of equivalence relations, the bottom is the smallest equivalence relation $\widehat{0} = \Delta = \operatorname{indit}(1)$ containing only the diagonal pairs (u, u). Dual to the notion of a partition tautology is the notion of an *equivalence relation contradiction* which is a formula (with the atomic variables written with the "d" superscript) that always evaluates to the bottom $\widehat{0} = \Delta = 1^d$ of $\Pi(U)^{op}$ regardless of the equivalence relations substituted for the atomic variables. Similarly, a formula (with the atomic variables written with the "d" superscript) is a *weak equivalence relation contradiction* if it never evaluates to the top $\widehat{1} = U \times U = 0^d$ of $\Pi(U)^{op}$. We then have the following duality theorem.

Proposition 5 (Principle of duality for partition logic) *Given a formula φ, φ is a (weak) partition tautology iff φ^d is a (resp. weak) equivalence relation contradiction.*

Proof: Using the complementation anti-isomorphism $\Pi(U) \to \Pi(U)^{op}$, a partition formula φ evaluates to the top of $\Pi(U)$, i.e., $\operatorname{dit}(\varphi) = \operatorname{dit}(1) = U \times U - \Delta$ when any partitions are substituted for the atomic variables of φ iff φ^d evaluates to the bottom of $\Pi(U)^{op}$, i.e., $1^d = \operatorname{dit}(1)^c = \operatorname{indit}(1) = \widehat{0} = \Delta$, when any equivalence relations are substituted for the atomic variables of φ^d. Similarly for the weak notions. ■

Using the reduction principle, restricting the above proposition and its related concepts to $|U| = 2$ would yield the usual Boolean duality principle (Church 1956, p. 107) that φ is a tautology iff φ^d is a contradiction (where the weak or "non-weak" notions coincide in the Boolean case and where $\Pi(2) \cong \mathcal{P}(1) \cong \Pi(2)^{op}$). The reduction principle can be used to show how our earlier results, such as the conjunctive normal form in the logic of partitions and disjunctive normal form in the logic of equivalence relations (for two-variable formulas), would reduce to the corresponding classical results by taking $|U| = 2$.

In the Boolean case, if a formula φ is not a subset tautology, then there is a non-empty universe set U and an assignment of subsets of U to the atomic variables of φ so that φ does not evaluate to 1 (the universe set U). Such a model showing that φ is not a tautology is called a *countermodel* for φ. In the Boolean case, it suffices to restrict the universe set U to a one-element set. If φ has a countermodel, then it has a countermodel using the subsets of a one-element set.

Analogous questions can be posed in partition logic. Is there a finite number n so that if φ always evaluates to 1 for any partitions on U with $|U| \leq n$, then φ is a partition tautology? For instance, if φ is not a partition tautology and is also not a Boolean tautology, then it suffices to take $n = 2$ since $\Pi(2) \cong \mathcal{P}(1)$ so a Boolean countermodel in $\mathcal{P}(1)$ also provides a partition countermodel in $\Pi(2)$. Hence the question is only open for formulas φ which are classical tautologies but not partition tautologies. A standard device answers this question in the negative.

Proposition 6 *There is no fixed n such that if any φ has no partition countermodel on any universe U with $|U| \leq n$, then φ has no partition countermodel, i.e., is a partition tautology.*

Proof: Consider any fixed $n \geq 2$. We use the standard device of a "universal disjunction of equations" (Grätzer 2003, p. 316) to construct a formula ω_n that evaluates to 1 for any substitutions of partitions on U with $|U| \leq n$ and yet the formula is not a partition tautology. Let B_n be the Bell number, the number of partitions on a set U with $|U| = n$. Take the atomic variables to be π_i for $i = 0, 1, ..., B_n$ so that there are $B_n + 1$ atomic variables. Let ω_n be the join of all the equivalences between distinct atomic variables:

$$\omega_n = \bigvee \{\pi_i \equiv \pi_j : 0 \leq i < j \leq B_n\}.$$

Then for any substitution of partitions on U where $|U| \leq n$ for the atomic variables, there is, by the pigeonhole principle, some "disjunct" $\pi_i \equiv \pi_j = (\pi_i \Rightarrow \pi_j) \wedge (\pi_j \Rightarrow \pi_i)$ which has the same partition substituted for the two variables so the disjunct evaluates to 1 and thus the join ω_n evaluates to 1. Thus ω_n evaluates to 1 for any substitutions of partitions on any U where $|U| \leq n$. To see that ω_n is not a partition tautology, take $U = \{0, 1, ..., B_n\}$ and let π_i be the atomic partition which has i as a singleton and all the other elements of U as a block, i.e., $\pi_i = \{\{0, 1, ..., i-1, i+1, ..., B_n\}, \{i\}\}$. Then $\pi_i \Rightarrow \pi_j = \pi_j$ and $\pi_j \wedge \pi_i = 0$ so that $\omega_n = 0$ for that substitution and thus ω_n is not even a weak partition tautology. ■

For $n = 2$, $B_2 = 2$ so that $\omega_2 = (\pi_0 \equiv \pi_1) \vee (\pi_0 \equiv \pi_2) \vee (\pi_1 \equiv \pi_2)$. Thus ω_2 is a Boolean tautology and hence so is any larger join ω_n for $n > 2$.

There is no upper bound n so that if any formula has a countermodel, then it has a countermodel with $|U| \leq n$. However, it seems likely to the author that if a partition formula has a countermodel, then it has a finite countermodel (i.e., the finite model property) but that question remains open.

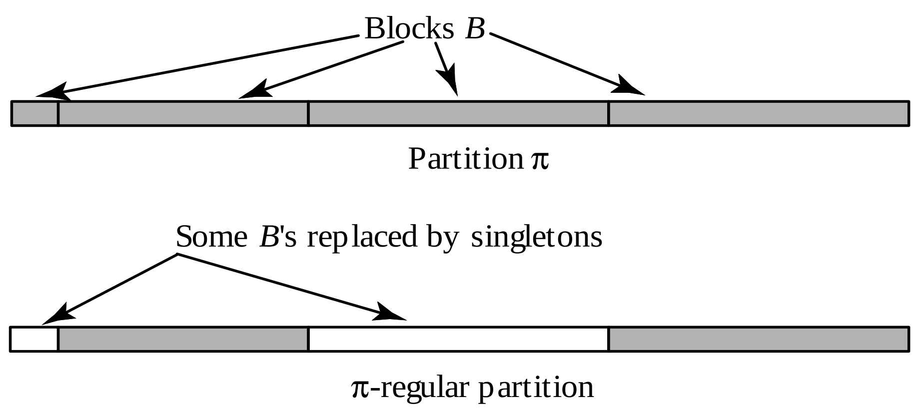
1.14 Boolean subalgebras $\mathcal{B}_\pi$ of $\Pi(U)$ for any partition π

In any Heyting algebra, the elements of the form $\neg\sigma = \sigma \Rightarrow 0$ for some σ are the *regular* elements. They form a Boolean algebra but it is not a subalgebra since the join of two regular elements is not necessarily regular (so one must take the double negation of the join to have the Boolean algebra join). In the topological interpretation, the regular elements of the Heyting algebra of open subsets are the regular open sets (the regular open sets are obtained as the interior of the closure of a subset) and the union of two regular open subsets is open but not necessarily regular open.

Following the analogy, we define a partition as being π*-regular* if it can be obtained as the implication $\sigma \Rightarrow \pi$ for some partitions σ and π. Intuitively, a π-regular partition is like π except that some blocks may have been discretized. Let

$$\mathcal{B}_\pi = \{\sigma \Rightarrow \pi : \text{ for some } \sigma \in \Pi(U)\}$$

be the subset of π-regular partitions with the induced partial ordering of refinement. The top is still 1 but the bottom is $\pi = 1 \Rightarrow \pi$ itself. The implication partition $\sigma \Rightarrow \pi$ can be interpreted as a Boolean probe for containment between blocks. If $B \subseteq C$ for some $C \in \sigma$, then the probe finds containment and this is indicated by setting the π-block B locally equal to 1, i.e., by discretizing B, and otherwise B stays locally like 0, i.e., stays as a whole block (or "mini-blob") B. Whenever the refinement relation $\sigma \preceq \pi$ holds, then all the non-singleton blocks $B \in \pi$ are discretized in $\sigma \Rightarrow \pi$ (and the singleton blocks are already discrete) so that $\sigma \Rightarrow \pi = 1$ (and vice-versa).



B-slots in π-regular partition

The partition operations of meet and join operate on the blocks of π-regular partitions in a completely Boolean manner. Since every π-regular partition is like π except that some blocks may be set locally to 1 while the others remain locally like 0, the meet of two π-regular partitions, say $\sigma \Rightarrow \pi$ and $\tau \Rightarrow \pi$, will have no interaction between distinct π-blocks. Each block of the meet will be "truth-functionally" determined by whatever is in the B-slot of the two constituents. If either of the B's remains locally equal to 0, then the whole block B fills the B-slot of the meet, i.e., B is locally equal to 0 in the meet $(\sigma \Rightarrow \pi) \wedge (\tau \Rightarrow \pi)$. But if both B's were discretized in the constituents, i.e., both are set locally to 1, then the blocks in that B-slot of the meet are the singletons from B, i.e., the discretized B or B set locally to 1. That local pattern of 0's and 1's is precisely the truth table for the Boolean meet.

If π_{ns} is the set of non-singleton blocks of the partition π, then the π-regular partitions are in one-to-one correspondence with the subsets of π_{ns}, each of which can be represented by its characteristic function $\chi : \pi_{ns} \to 2 = \{0,1\}$ which takes each non-singleton block of π to its local assignment. Thus for a π-regular partition with the form $\sigma \Rightarrow \pi$, $\chi(\sigma \Rightarrow \pi) : \pi_{ns} \to 2$ takes a non-singleton block $B \in \pi$ to 1 if B is discretized in $\sigma \Rightarrow \pi$ and otherwise to 0. The argument just given shows that the characteristic function for the meet of two π-regular partitions is obtained by the component-wise Boolean meets of "conjuncts":

$$\chi((\sigma \Rightarrow \pi) \wedge (\tau \Rightarrow \pi)) = \chi(\sigma \Rightarrow \pi) \wedge \chi(\tau \Rightarrow \pi).$$

In a similar manner, the blocks in the join of two π-regular partitions, $\sigma \Rightarrow \pi$ and $\tau \Rightarrow \pi$, would be the intersections of what is in the B-slots. If B was discretized (set locally to 1) in either of the constituents, then B would be discretized in the join $(\tau \Rightarrow \pi) \vee (\sigma \Rightarrow \pi) = \overset{\pi}{\neg}\tau \vee \overset{\pi}{\neg}\sigma$ (since the intersection of a discretized B with a whole B is still the discretized B). But if both B's were still whole (set locally to 0) then their intersection would still be the whole block B. This pattern of 0's and 1's is precisely the truth table for the Boolean join or disjunction. In terms of the characteristic functions of local assignments:

$$\chi((\tau \Rightarrow \pi) \vee (\sigma \Rightarrow \pi)) = \chi(\tau \Rightarrow \pi) \vee \chi(\sigma \Rightarrow \pi).$$

For the implication $(\sigma \Rightarrow \pi) \Rightarrow (\tau \Rightarrow \pi)$ between two π-regular partitions, the result would have B remaining whole, i.e., being set to 0, only in the case where B was whole in the consequent partition $\tau \Rightarrow \pi$ but discretized in the antecedent partition $\sigma \Rightarrow \pi$; otherwise B is discretized, i.e., set to 1. This pattern of 0's and 1's is precisely the truth table for the ordinary Boolean implication. In terms of the characteristic functions:

$$\chi((\sigma \Rightarrow \pi) \Rightarrow (\tau \Rightarrow \pi)) = \chi(\sigma \Rightarrow \pi) \Rightarrow \chi(\tau \Rightarrow \pi).$$

To show that $\mathcal{B}_\pi$ is a Boolean algebra, we must define negation inside of $\mathcal{B}_\pi$. The negation of a π-regular element $\sigma \Rightarrow \pi$ would be its implication to the bottom element which in $\mathcal{B}_\pi$ is π itself. Thus the negation of $\sigma \Rightarrow \pi = \overset{\pi}{\neg}\sigma$ is just the iterated implication: $(\sigma \Rightarrow \pi) \Rightarrow \pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$, the double π-negation. It is easily seen that this just "flips" the B-slots to the opposite state. The B's set (locally) to 1 in $\sigma \Rightarrow \pi$ are flipped back to (locally) 0 in $(\sigma \Rightarrow \pi) \Rightarrow \pi$, and the B's left whole in $\sigma \Rightarrow \pi$ are flipped to 1 or discretized in $(\sigma \Rightarrow \pi) \Rightarrow \pi$. This pattern of 0's and 1's is just the truth table for the Boolean negation. In terms of the characteristic functions,

$$\chi((\sigma \Rightarrow \pi) \Rightarrow \pi) = \neg\chi(\sigma \Rightarrow \pi).$$

Thus it is easily seen that the set of π-regular elements $\mathcal{B}_\pi$ is a Boolean algebra, called the *Boolean core* of the upper interval $[\pi, 1] = \{\sigma \in \Pi(U) : \pi \preceq \sigma \preceq 1\}$, since it is isomorphic to the powerset Boolean algebra $\mathcal{P}(\pi_{ns})$ of the set π_{ns} (when the subsets are represented by their characteristic functions).

Proposition 7 $\mathcal{B}_\pi \cong \mathcal{P}(\pi_{ns})$. ■

We previously saw that the partition lattice $\Pi(U)$ could be represented by the lattice of open subsets $\operatorname{dit}(\pi)$ of the product $U \times U$ (when taken as a closure space). The representation of the partition lattice by the open subsets of the closure space $U \times U$ continues to hold when the lattice is enriched with the implication and nand operations.

$$\boxed{\Pi(U) \cong O(U \times U)}$$

Representation of algebra of partitions $\Pi(U)$
as the algebra of open subsets $O(U \times U)$

Now we can see the dual representation of the Boolean algebra of subsets $\mathcal{P}(U)$ by a certain Boolean algebra of partitions. Start with the dual constructions of subsets $\operatorname{indit}(\pi)$ of the product $U \times U$ and the partitions $\Delta(S)$ on the coproduct $U \uplus U$. For the dual representation of $\mathcal{P}(U)$ we consider the partition algebra $\Pi(U \uplus U)$ on the coproduct and the Boolean core $\mathcal{B}_\Delta$, or $\mathcal{B}_\Delta(U \uplus U)$ to make the underlying universe explicit, associated with the diagonal partition $\Delta(U) = \Delta(\emptyset^c)$ consisting of all the pairs $\{u, u^*\}$ for $u \in U$. Just as we previous took the complement of $\operatorname{indit}(\pi)$ to arrive at the partition relations $\operatorname{dit}(\pi) = \operatorname{indit}(\pi)^c$, and we now consider the Δ-complements $\overset{\Delta}{\neg}\Delta(S) = \Delta(S^c)$ which are the *subset corelations*. The Δ-regular partitions of $\Pi(U \uplus U)$ are precisely the subset corelations $\Delta(S^c)$. The subset corelation $\Delta(S^c)$ locally assigns $\{u, u^*\} \in \Delta$ to 1 (i.e., discretizes it) if $u \in S$ and locally assigns $\{u, u^*\} \in \Delta$ to 0 (i.e., leaves it whole) if $u \in S^c$. Rather than associate each partition π with the partition relation $\operatorname{dit}(\pi)$ on the product $U \times U$, we now associate each subset $S \in \mathcal{P}(U)$ with the subset corelation $\Delta(S^c)$ on the coproduct $U \uplus U$ to get the dual representation:

$$\boxed{\mathcal{P}(U) \cong \mathcal{B}_\Delta(U \uplus U)}$$

Representation of the Boolean algebra of subsets $\mathcal{P}(U)$
as the BA of subset corelations $\mathcal{B}_\Delta(U \uplus U)$.

The universe sets U are assumed to have two or more elements to avoid the degenerate case of a singleton universe where $0 = 1$, i.e., the indiscrete and discrete partitions are the same. But in partitions π, singleton blocks cannot be avoided and the same problem emerges locally. For a singleton block B, being locally like 0 (i.e., remaining whole) and being locally like 1 (being discretized) are the same. Hence they play no role in the Boolean algebras $\mathcal{B}_\pi$.

We previously saw another Boolean algebra $\mathcal{B}(\pi)$ associated with every partition π on a set U, and the singletons will play a role in connecting the two BAs.

For each partition π on U, $\mathcal{B}(\pi) \subseteq \mathcal{P}(U)$ is the complete subalgebra generated by the blocks of π as the atoms so that all the elements of $\mathcal{B}(\pi)$ are formed as the arbitrary unions and intersections of blocks of π. Conversely, given any complete subalgebra $\mathcal{B}$ of $\mathcal{P}(U)$, the intersection of all elements of $\mathcal{B}$ containing an element $u \in U$ will provide the atoms of $\mathcal{B}$ which are the blocks in a partition π on U so that $\mathcal{B} = \mathcal{B}(\pi)$. Since each element of $\mathcal{B}(\pi)$ is the union of a set of blocks of π, it is isomorphic to the powerset BA of the set of blocks that make up π, i.e., $\mathcal{B}(\pi) \cong \mathcal{P}(\pi)$. Since $\mathcal{B}_\pi \cong \mathcal{P}(\pi_{ns})$ is isomorphic to the powerset BA of the set of non-singleton blocks of π, and since the introduction of each singleton $\{u\}$ will have the effect of doubling the elements of $\mathcal{P}(\pi_{ns})$ (with or without the singleton), we can reach $\mathcal{P}(\pi)$ from $\mathcal{P}(\pi_{ns})$ by taking the direct product with the two element BA 2 for each singleton in π. Thus we have the following result which relates the two BAs associated with each partition π.

Proposition 8 $\mathcal{B}(\pi) \cong \mathcal{B}_\pi \times \prod_{\{u\} \in \pi} 2$. ■

1.15 Transforming subset tautologies into partition tautologies

Unlike the case of the Boolean algebra of regular elements in a Heyting algebra, the Boolean core $\mathcal{B}_\pi$ is a *sub*algebra of the partition algebra $\Pi(U)$ for the "Boolean" operations of join, meet, and implication (but not nand), i.e., the Boolean operations in $\mathcal{B}_\pi$ are the partition operations from the partition algebra $\Pi(U)$. The BA $\mathcal{B}_\pi$ even has the same top 1 as the partition algebra; only the bottoms are different, i.e., π in $\mathcal{B}_\pi$ and 0 in $\Pi(U)$.

Since the Boolean core $\mathcal{B}_\pi$ of the interval $[\pi, 1]$ and the whole partition algebra $\Pi(U)$ have the same top 1 and the same operations of join, meet, and implication, we immediately have a way to transform any classical tautology into a partition tautology. But we must be careful about the connectives used in the classical tautology. The partition operations of the join, meet, and implication are the same as the Boolean operations in the Boolean core $\mathcal{B}_\pi$. But the negation in that BA is not the partition negation $\neg$ but the π-negation $\overset{\pi}{\neg}$. Similarly, the nand operation in the Boolean algebra $\mathcal{B}_\pi$ is not the partition nand $|$ but the π-*nand* defined by the ternary partition operation: $\operatorname{dit}(\sigma \mid_\pi \tau) = \operatorname{int}[\operatorname{indit}(\sigma) \cup \operatorname{indit}(\tau) \cup \operatorname{dit}(\pi)]$ which agrees with the usual nand when $\pi = 0$. But the nand operation in the BA $\mathcal{B}_\pi$ can be defined in terms of the other BA operations so we may assume that the classical tautology is written without a nand operation $|$. Similarly we may assume that negations $\neg\sigma$ are written as $\sigma \Rightarrow 0$ so that no negation signs $\neg$ occur in the partition tautology.

Given any propositional formula using the connectives of $\vee$, $\wedge$, $\Rightarrow$ and the constants of 0 and 1, its *single π-negation transform* is obtained by replacing each atomic variable σ by its single π-negation $\overset{\pi}{\neg}\sigma = \sigma \Rightarrow \pi$ and by replacing the constant 0 by π. The binary operations $\vee$, $\wedge$, and $\Rightarrow$ as well as the constant 1 all remain the same. For instance, the single π-negation transform of the excluded middle formula $\sigma \vee \neg\sigma = \sigma \vee (\sigma \Rightarrow 0)$ is the weak excluded middle formula for π-negation:

$$(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi) = \overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma.$$

A formula that is a classical tautology will always evaluate to 1 in a Boolean algebra regardless of what elements of the Boolean algebra are assigned to the atomic variables. The single π-negation transformation maps any formula into a formula for an element of the Boolean core $\mathcal{B}_\pi$. If the original formula with the atomic variables σ, τ,... was a classical tautology, then the single π-negation transform of the formula will evaluate to 1 in $\mathcal{B}_\pi$ for any partitions (π-regular or not) assigned to the original atomic variables σ, τ, ... with π fixed. But this is true for any π so the single π-negation transform of any classical tautology will evaluate to 1 for any partitions assigned to the atomic variables π, σ, τ,.... Thus it is a partition tautology.

Proposition 9 *The single π-negation transform of any classical tautology is a partition tautology.* ■

For example, since the law of excluded middle, $\sigma \vee \neg\sigma$, is a classical tautology, its single π-negation transform, $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$, is a partition tautology. This particular example is also intuitively obvious since the blocks B that were not discretized in $\overset{\pi}{\neg}\sigma$ are discretized in the double π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so all the non-singleton blocks are discretized in $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ (and the singleton blocks were already "discretized") so it is a partition tautology. This formula is also an example of a partition tautology that is not a valid formula of intuitionistic logic (either for $\pi = 0$ or in general).

We can similarly define the *double π-negation transform* of a formula as the formula where each atomic variable σ is replaced by its double π-negation $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ and by replacing the constant 0 by π. By the same argument, the double π-negation transform of any classical tautology is a partition tautology so there are at least two ways to transform any classical subset tautology into a partition tautology.

Proposition 10 *The double π-negation transform of any classical tautology is a partition tautology.* ■

The double π-negation transform of excluded middle, $\sigma \vee \neg\sigma$, is the partition tautology $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. Since the π-negation has the effect of flipping the π-blocks B back and forth being locally equal to 0 or 1 (i.e., from being whole to being discretized), it is clear that $\overset{\pi}{\neg}\sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so the formula $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ is equivalent to $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\sigma$.

There is also a partition analogue of the Gödel transform (Gödel 1933) that produces an intuitionistic validity from each classical tautology. For any classical formula φ in the language of $\vee$, $\wedge$, and $\Rightarrow$ as well as 0 and 1, we define the *Gödel π-transform* φ_π^g of the formula as follows:

- If φ is atomic, then $\varphi^g_\pi = \varphi \vee \pi$;
- If $\varphi = 0$, then $\varphi^g_\pi = \pi$, and if $\varphi = 1$, then $\varphi^g_\pi = 1$;
- If $\varphi = \sigma \vee \tau$, then $\varphi^g_\pi = \sigma^g_\pi \vee \tau^g_\pi$;
- If $\varphi = \sigma \Rightarrow \tau$, then $\varphi^g_\pi = \sigma^g_\pi \Rightarrow \tau^g_\pi$; and
- if $\varphi = \sigma \wedge \tau$, then $\varphi^g_\pi = \overset{\pi\,\pi}{\neg\neg}\sigma^g_\pi \wedge \overset{\pi\,\pi}{\neg\neg}\tau^g_\pi$.

When $\pi = 0$, then we write $\varphi^g_0 = \varphi^g$.

Lemma 10 *φ is a classical tautology iff φ^g is a weak partition tautology iff $\neg\neg\varphi^g$ is a partition tautology.*

Proof: The idea of the proof is that the partition operations on the Gödel 0-transform φ^g mimic the Boolean $0, 1$-operations on φ if we associate the partition interpretation $\sigma^g = 0$ with the Boolean $\sigma = 0$ and $\sigma^g \neq 0$ with the Boolean $\sigma = 1$. We proceed by induction over the complexity of the formula φ where the induction hypothesis is that: $\varphi = 1$ in the Boolean case iff $\varphi^g \neq 0$ in the partition case, which could also be stated as: $\varphi = 0$ in the Boolean case iff $\varphi^g = 0$ in the partition case. If φ is atomic, then $\varphi^g = \varphi \vee 0 = \varphi$. The Boolean assignment $\varphi = 0$ (the Boolean truth value 0) is associated with the partition assignment of $\varphi = 0$ (the indiscrete partition) and for atomic φ, $\varphi = \varphi \vee 0 = \varphi^g$ so the hypothesis holds in the base case.

For the join in the Boolean case, $\varphi = \sigma \vee \tau = 1$ iff $\sigma = 1$ or $\tau = 1$. In the partition case, $\varphi^g = \sigma^g \vee \tau^g \neq 0$ iff $\sigma^g \neq 0$ or $\tau^g \neq 0$, so by the induction hypothesis, $\varphi = \sigma \vee \tau = 1$ iff $\sigma = 1$ or $\tau = 1$ iff $\sigma^g \neq 0$ or $\tau^g \neq 0$ iff $\varphi^g = \sigma^g \vee \tau^g \neq 0$.

For the implication in the Boolean case, $\varphi = \sigma \Rightarrow \tau = 0$ iff $\sigma = 1$ and $\tau = 0$. In the partition case, $\varphi^g = \sigma^g \Rightarrow \tau^g = 0$ iff $\sigma^g \neq 0$ and $\tau^g = 0$. Hence using the induction hypothesis, $\varphi = \sigma \Rightarrow \tau = 1$ iff $\sigma = 0$ or $\tau = 1$ iff $\sigma^g = 0$ or $\tau^g \neq 0$ iff $\varphi^g = \sigma^g \Rightarrow \tau^g \neq 0$.

For the meet in the Boolean case, $\varphi = \sigma \wedge \tau = 1$ iff $\sigma = 1 = \tau$. In the partition case, $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = 1$ iff $\neg\neg\sigma^g = 1 = \neg\neg\tau^g$ iff $\sigma^g \neq 0 \neq \tau^g$. By the induction hypothesis, $\varphi = \sigma \wedge \tau = 1$ iff $\sigma = 1 = \tau$ iff $\sigma^g \neq 0 \neq \tau^g$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = 1$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g \neq 0$.

Thus φ is a classical tautology iff under any Boolean interpretation, $\varphi = 1$ iff for any partition interpretation, $\varphi^g \neq 0$ iff φ^g is a weak partition tautology iff $\neg\neg\varphi^g$ is a partition tautology. ■

In this case of $\pi = 0$, the negation $\neg\sigma = \sigma \Rightarrow 0$ is unchanged and, for atomic variables φ, $\varphi \vee 0 = \varphi$ so atomic variables are left unchanged in the Gödel 0-transform. Hence any classical formula φ expressed in the language of $\neg$, $\vee$, and $\Rightarrow$ (excluding the meet $\wedge$) would be unchanged by the Gödel 0-transform.

Corollary 3 *For any formula φ in the language of $\neg$, $\vee$, and $\Rightarrow$ along with 0 and 1, φ is a classical tautology iff φ is a weak partition tautology iff $\neg\neg\varphi$ is a partition tautology.*

For instance, the Gödel 0-transform of excluded middle $\sigma \vee \neg\sigma$ is the same formula, $\sigma \vee \neg\sigma$, which is a weak partition tautology, and $\neg\neg(\sigma \vee \neg\sigma)$ is a partition tautology.

The lemma generalizes to any π in the following form.

Proposition 11 *φ is a classical tautology iff $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$ is a partition tautology.*

Proof: For any fixed partition π on a universe set U, the interpretation of the Gödel π-transform φ^g_π is in the upper interval $[\pi, 1] \subseteq \Pi(U)$. The key to the generalization is the standard result that the upper interval $[\pi, 1]$ can be represented as the product of the sets $\Pi(B)$ where B is a non-singleton block of π:

$$[\pi, 1] \cong \prod \{\Pi(B) : B \in \pi,\ B \text{ non-singleton}\}.$$[21]

Once we establish that the Gödel π-transform φ^g_π can be obtained, using the isomorphism, by computing the Gödel 0-transform φ^g "component-wise" in $\Pi(B)$, then we can apply the lemma component-wise to obtain the result.

Given a partition π on U, any interpretation of an atomic φ as a partition on U can be cut down to each non-singleton block $B \in \pi$ to yield a partition on B. Then $\varphi^g_\pi = \varphi \vee \pi$ has a block $B \in \pi$ iff $\varphi^g_0 = \varphi^g$ is equal to the zero 0_B of $\Pi(B)$. Proceeding by induction over the complexity of φ, if $\varphi = \sigma \vee \tau$, then a block of $\varphi^g_\pi = \sigma^g_\pi \vee \tau^g_\pi$ is B iff B is a block of both σ^g_π and τ^g_π iff $\sigma^g = 0_B = \tau^g$ in $\Pi(B)$ iff $\varphi^g = \sigma^g \vee \tau^g = 0_B$ in $\Pi(B)$. If $\varphi = \sigma \Rightarrow \tau$, then $\varphi^g_\pi = \sigma^g_\pi \Rightarrow \tau^g_\pi$ has a block $B \in \pi$ iff σ^g_π does not have the block B and τ^g_π has the block B iff σ^g is not equal to 0_B and τ^g is equal to 0_B in $\Pi(B)$ iff $\varphi^g = \sigma^g \Rightarrow \tau^g = 0_B$ in $\Pi(B)$. If $\varphi = \sigma \wedge \tau$, then $\varphi^g_\pi = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma^g_\pi \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\tau^g_\pi$ has a block $B \in \pi$ iff both σ^g_π and τ^g_π have a block B iff $\sigma^g = 0_B = \tau^g$ in $\Pi(B)$ iff $\varphi^g = \neg\neg\sigma^g \wedge \neg\neg\tau^g = 0_B$ in $\Pi(B)$.

Hence applying the lemma component-wise, φ is a classical tautology iff φ^g never evaluates to 0_B in $\Pi(B)$ iff B is never a block of φ^g_π iff every block $B \in \pi$ is discretized in $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$, i.e., $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi^g_\pi$ is a partition tautology. ■

Thus the Gödel π-transform of excluded middle $\varphi = \sigma \vee (\sigma \Rightarrow 0)$ is $\varphi^g_\pi = (\sigma \vee \pi) \vee (\sigma \Rightarrow \pi)$ and $\overset{\pi}{\neg}\overset{\pi}{\neg}[(\sigma \vee \pi) \vee (\sigma \Rightarrow \pi)]$ is a partition tautology. Note that the single π-negation transform, the double π-negation transform, and the Gödel π-transform all gave different formulas starting with the classical excluded middle tautology.

1.16 Some partition results

Before turning to the proof theory of partition logic, we might mention a few interesting results. For many purposes, the structure of the partition algebras $\Pi(U)$ is best analyzed by analyzing the upper intervals $[\pi, 1]$ for any partition π. Every partition $\varphi \in [\pi, 1]$ (i.e., every partition that refines π) has a unique "Booleanization" $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi \in \mathcal{B}_\pi$ in the Boolean core of the interval. Since φ refines

[21] Since the partition lattice is conventionally written upside down, the usual result is stated in terms of the interval below π (Grätzer 2003, p. 252).

π, each block $B \in \pi$ is either the same in φ or is strictly refined in φ, and $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi$ is essentially a π-*characteristic partition* for those two cases. If B remains whole in φ, then B is whole in $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi$ (i.e., set locally to 0), and if B is strictly refined in φ, then B is discretized in $\overset{\pi}{\neg}\overset{\pi}{\neg}\varphi$ (i.e., set locally to 1). For each π-regular partition $\overset{\pi}{\neg}\sigma$, all the partitions in $[\pi, 1]$ that Booleanize to $\overset{\pi}{\neg}\sigma$, i.e., for which $\overset{\pi}{\neg}\sigma$ is the π-characteristic partition, form the *shadow* of $\overset{\pi}{\neg}\sigma$. The shadows of the π-regular partitions partition the interval $[\pi, 1]$ so the interval is classified by its π-characteristic partitions in the Boolean core $\mathcal{B}_\pi$.

Partition lattices are the "standard" examples of non-distributive lattices, but one can do much better than simply say a partition lattice is non-distributive. The Boolean core of each interval $[\pi, 1]$ is, of course, distributive since it is a Boolean algebra using the meet and join operations of the partition lattice. Moreover, each partition in the interval distributes across the Boolean core. To see this, note that one of these distributivity results is essentially due to Oystein Ore. Ore (1942) did much of the path-breaking work on partitions. He defined two partitions as being *associable* if each block in their meet is a block in one (or both) of the partitions.[22] Although Ore did not consider π-regular partitions, any two π-regular partitions are associable. Ore showed that any partition joined with the meet of two associable partitions will distribute across the meet. Hence we have the following result for any partitions φ, σ, τ, and π.

Lemma 11 (Ore's associability theorem) $\varphi \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) = \left(\varphi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\varphi \vee \overset{\pi}{\neg}\tau\right)$.

Ore's theorem does not assume that φ is in the interval $[\pi, 1]$ but we can interchange join and meet if we restrict φ to the interval.

Lemma 12 ("Dual" to Ore's theorem) *If* $\varphi \in [\pi, 1]$, *then* $\varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right) = \left(\varphi \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\varphi \wedge \overset{\pi}{\neg}\tau\right)$.

Proposition 12 (Distributivity over the Boolean core) *If* $\pi \preceq \varphi$,

$$\begin{aligned} \varphi \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\tau\right) &= \left(\varphi \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\varphi \vee \overset{\pi}{\neg}\tau\right) \\ \varphi \wedge \left(\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\tau\right) &= \left(\varphi \wedge \overset{\pi}{\neg}\sigma\right) \vee \left(\varphi \wedge \overset{\pi}{\neg}\tau\right). \end{aligned}$$

Lawvere, (1986) and (1991), has explored two interesting formulas in the context of co-Heyting algebras (e.g., the closed subsets of a topological space) but both formulas are also true in the partition algebras $\Pi(U)$. Since Lawvere was working in a co-Heyting algebra, his suggestive terminology would be more

[22] Ore actually dealt with the join of equivalence relations but we are using the opposite presentation.

fitting in the algebra of equivalence relations (represented by the closed subsets in the non-topological closure space $U \times U$). For instance, Lawvere uses the negation that is "difference from 1" (e.g., $(\neg\sigma)^d = (\sigma \Rightarrow 0)^d = \sigma^d \not\Leftarrow 0^d = 0^d - \sigma^d$ in the algebra of equivalence relations where $0^d = \widehat{1}$ is the top or "one" of that algebra) which is dual to the "implication to 0," i.e., $\neg\sigma = \sigma \Rightarrow 0$, in the partition algebra. Moreover, we will relativize the negation using an arbitrary π in place of 0.

Lawvere defines the "boundary" of an element as its meet with its negation, so dualizing and using π-negation, we define the π-*coboundary* of a partition as the partition obtained from the excluded middle formula using π-negation:

$$\partial^\pi \sigma = \sigma \vee \overset{\pi}{\neg}\sigma$$

π-coboundary of a partition σ

Lawvere's boundary was "nowhere dense" in the sense that its double negation was the zero element. In the dual, the π-coboundary is π-*dense* in the sense that its double π-negation is 1. That is, the double π-negation of the excluded middle formula using π-negation is a partition tautology.

$$\overset{\pi}{\neg}\overset{\pi}{\neg}\partial^\pi \sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\left(\sigma \vee \overset{\pi}{\neg}\sigma\right) = 1.$$

Lawvere defined the "core" of an element as its double negation but we could just extend the notion of the double π-negation as the π-*characteristic* of a partition σ to arbitrary σ rather than just $\sigma \in [\pi, 1]$. Lawvere then shows that each element is equal to its boundary joined with its core. In the opposite presentation, this result is: $(\sigma \vee (\sigma \Rightarrow 0)) \wedge ((\sigma \Rightarrow 0) \Rightarrow 0) = \sigma$. Generalizing from 0 to any π then gives the following result true in any $\Pi(U)$.

Proposition 13 (Lawvere's boundary + core law for partitions) $\partial^\pi \sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \sigma \vee \pi$.

Proof: This is easily proved from Ore's associability theorem using some basic identities such as: $\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \pi$ and $\sigma \preceq \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ so that $\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma = \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. Then using Ore's theorem:

$$\sigma \vee \pi = \sigma \vee \left(\overset{\pi}{\neg}\sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \left(\sigma \vee \overset{\pi}{\neg}\sigma\right) \wedge \left(\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma\right) = \partial^\pi \sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma. \blacksquare$$

Restricting to $\sigma \in [\pi, 1]$, any σ that refines π can be reconstructed from its π-characteristic partition $\overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$ by taking the meet with its π-coboundary $\partial^\pi \sigma$.

Corollary 4 *If* $\sigma \in [\pi, 1]$, *then* $\sigma = \partial^\pi \sigma \wedge \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$. $\blacksquare$

Lawvere also shows that the Leibniz rule for taking the derivative of the product of functions, i.e., $(fg)' = f(g') + (f')g$, applies in, say, the co-Heyting algebra of closed subsets of a topological space using the notion of boundary in place of the derivative. The Leibniz rule holds in the dual algebra of equivalence relations using the dual notion of π-boundary, and the dual of the Leibniz rule holds in the partition algebras $\Pi(U)$ using the notion of π-coboundary.

Proposition 14 (co-Leibniz rule for partitions) $\partial^\pi(\sigma \vee \tau) = (\partial^\pi\sigma \vee \tau) \wedge (\sigma \vee \partial^\pi\tau)$.

2 Correctness and completeness for partition logic

2.1 Beth-style tableaus for partition logic

2.1.1 Classical, intuitionistic, and partition "forcing" models

It is a familiar fact from classical and intuitionistic logic that logics might be syntactically presented in a number of ways: Hilbert-style axiom systems, Beth-style tableaus, natural deduction systems, or Gentzen-style sequent systems. For partition logic, it seems that the Beth-style tableaus provide the easiest and most transparent approach so they will be exclusively used here.

Beth-style tableaus are often called "semantic" since the rules, in effect, try to construct a model for a formula at the syntactic level. For each of the connectives, it will be useful to consider the corresponding classical and intuitionistic tableaus for purposes of comparison. This requires presenting an appropriate form of the classical and intuitionistic tableaus adapted to the subset interpretation. As remarked before, classical and intuitionistic logic are to be interpreted as being about subsets (open subsets in the intuitionistic case). The rules for the connectives govern when the subsets contain or don't contain a generic element u. Then the partition case is motivated by elements-distinctions analogy with the generic element u replaced by a generic pair (u, u') of distinct elements. The conditions governing when subsets contain elements are replaced by the conditions governing when partitions make distinctions.

Tableaus with signed formulas $T\sigma$ or $F\sigma$ will be used (Smullyan 1968). But each signed formula must be accompanied by a generic element or generic pair as in "$u : T\sigma$" or "$(u, u') : T\sigma$." In the classical or intuitionistic case, $u : T\sigma$ would be interpreted as saying that the subsets represented by σ contains the element u while $u : F\sigma$ would mean that σ (i.e., the subset it represents) does not contain u. In the more common propositional interpretation, σ would represent a proposition and u would be a "possible world" where σ would hold or not hold. Similarly, $(u, u') : T\sigma$ means that the partition represented by σ makes the distinction (u, u'), i.e., u and u' are in distinct blocks of σ, and $(u, u') : F\sigma$ would mean that u and u' are in the same block of σ.

For classical "propositional" logic, the subsets in the intended interpretation are the subsets of any non-empty universe set U. For intuitionistic "propositional" logic, the intended interpretation is known as a *Kripke structure* or *intuitionistic forcing model* (Fitting 1969). The universe U is endowed with a partial ordering $\leq$ and the relevant subsets are the up-closed subsets where $S \subseteq U$ is *up-closed* if $u \in S$ and $u \leq u'$ implies that $u' \in S$. These subsets satisfy the conditions for being the open sets of a topology on U. Ordinarily one has a forcing relation ($\models$) between the points of U and the unsigned formulas. However, signed formulas will be used here to facilitate the connection to the tableaus using signed formulas:

$u \models \varphi$ will be written $u : T\varphi$ and $u \not\models \varphi$ is written $u : F\varphi$.

A Kripke structure satisfies the structural rule for any T-formula φ, $\forall u' \geq u$, if $u : T\varphi$ then $u' : T\varphi$ so that all T*-sets* $T_\varphi = \{u|\ u : T\varphi\}$ are up-closed (i.e., open). The T-conditions for the connectives are given below while the F-conditions are obtained by contraposition.

$u : T(\pi \vee \sigma)$ iff $u : T\pi$ or $u : T\sigma$;
$u : T(\pi \wedge \sigma)$ iff $u : T\pi$ and $u : T\sigma$;
$u : T(\sigma \Rightarrow \pi)$ iff $\forall u' \geq u$, $u' : F\sigma$ or $u' : T\pi$; and
$u : T(\sigma \mid \pi)$ iff $\forall u' \geq u$, $u' : F\sigma$ or $u' : F\pi$.

Ordinarily, Kripke structures are defined using negation as a primitive connective but we can define $\neg\sigma = \sigma \mid \sigma$ so that setting $\pi = \sigma$ in the condition for $T(\sigma \mid \pi)$ gives the derived condition for the negation:

$u : T(\neg\sigma)$ iff $\forall u' \geq u$, $u' : F\sigma$.

Kripke structures make explicit certain features which are left implicit in classical logic but which must be explicit in partition logic so they are useful as an expository bridge. It was emphasized from the outset that classical "propositional" logic should be seen as being about the subsets of a universe set U and that the "truth table" rules for the connectives are really the subset membership conditions for a generic element u. Since the classical operations on subsets do not require ever "leaving" the base point u, say, to some other point u', all explicit reference to u is dropped. The definitions can all be interpreted as being about the subsets 0 and 1 of a one point set $\{u\}$ which, in turn, can be interpreted as falsity and truth for propositions. But once we have the notion of Kripke structures, then we can see that classical propositional logic arises, as it were, when the partial ordering on U is discrete which gives the discrete topology where all subsets are open subsets.

In a Kripke structure, the atomic variables are, in effect, interpreted as T-sets (open subsets) and the conditions for the Kripke structure just give the membership conditions for the T-sets of compound formulas since: $u \in T_\varphi$ iff $u : T\varphi$. Thus a *classical model* for classical "propositional" logic would be a discrete Kripke structure, i.e., a non-empty universe set U together with the "forcing" or membership conditions:

$u : T(\pi \vee \sigma)$ iff $u : T\pi$ or $u : T\sigma$;
$u : T(\pi \wedge \sigma)$ iff $u : T\pi$ and $u : T\sigma$;
$u : T(\sigma \Rightarrow \pi)$ iff $u : F\sigma$ or $u : T\pi$; and
$u : T(\sigma \mid \pi)$ iff $u : F\sigma$ or $u : F\pi$.

These conditions for a classical model of propositional logic are just disguised versions of the usual truth tables but they make explicit the subset interpretation of the logic. Each formula φ would be interpreted in a model by a subset $T_\varphi = \{u|\ u : T\varphi\}$, and the rules could be restated as membership conditions for a generic element, e.g., $u \in T_{\pi\vee\sigma}$ iff $u \in T_\pi$ or $u \in T_\sigma$, and so forth.

In the usual treatment of Kripke structures, a formula φ is *intuitionistically valid* if it is forced at every point in any Kripke structure. But this is equivalent to saying that for any interpretation of the atomic variables of φ as open subsets of the model, the whole formula evaluates to the universe set U. In the discrete or classical case, it means that a formula is a classical or subset tautology if regardless of the subsets of U assigned to the atomic variables of the formula, the formula evaluates to the universe set U (for any non-empty U).

We now have sufficient motivation to define the analogous *partition forcing models*. We start with a universe set U with two or more elements. The points in the classical and Kripke structures are replaced by the pairs (u,u') of distinct points from U. Instead of using an explicit forcing relation between pairs and formulas, we will again use signed formulas so that:

$(u,u') \models \varphi$ is written as $(u,u') : T\varphi$, and $(u,u') \not\models \varphi$ is written as $(u,u') : F\varphi$.

Unlike the points u or u', the pairs (u,u') have an internal structure; a pair (u,u') can be reversed to (u',u) and pairs can be connected in triangles as in (u,u'), (u,a), and (a,u') or in longer chains. Hence a partition forcing model has two structural conditions reflecting the symmetry and anti-transitivity of partition relations:

if $(u,u') : T\varphi$, then $(u',u) : T\varphi$;

if $(u,u') : T\varphi$, then for any other a, $(u,a) : T\varphi$ or $(a,u') : T\varphi$.

No rule is needed to enforce the anti-reflexivity of partition relations since the notation always assumes that (u,u') is a pair of distinct elements.

The unstructured universe set U still determines the complete undirected graph $K(U)$ on U which has a link (u,u') between any two distinct points. A u,u'*-chain* is a finite sequence of links, (u_1,u_2),(u_2,u_3),...,(u_{n-1},u_n), with $u = u_1$ and $u' = u_n$ as the endpoints. In particular, the *base pair* (u,u') is a one-link u,u'-chain, and any third element a gives the two-link u,u'-chain (u,a) and (a,u'). Recall that the *Boolean condition* for any signed compound formula $\pi * \sigma$ is the disjunction or conjunction of the pair of signed formulas that hold in a classical model for the constituents π and σ where $*$ is any binary operation.

Now the "forcing conditions" for a partition forcing model can be stated for the T-signed formulas (with the F-rules obtained by contraposition).

$(u,u') : T(\pi \vee \sigma)$ iff $(u,u') : T\pi$ or $(u,u') : T\sigma$ (i.e., the Boolean condition holds at the base pair);

$(u,u') : T(\sigma \Rightarrow \pi)$ iff for any 1- or 2-link u,u'-chain, the Boolean condition (i.e., $F\sigma$ or $T\pi$) holds on some chain link;

$(u,u') : T(\pi \wedge \sigma)$ iff for any u,u'-chain, the Boolean condition (i.e., $T\pi$ and $T\sigma$) holds on some chain link; and

$(u,u') : T(\sigma \mid \pi)$ iff for any u,u'-chain, the Boolean condition (i.e., $F\sigma$ or $F\pi$) holds on some chain link.

The T-sets are $T_\varphi = \{(u,u') \mid \ (u,u') : T\varphi\}$, and a partition validity would be a formula whose T-set consisted of all pairs (u,u') of distinct elements in all partition forcing models.

These partition forcing models have been defined so that one can see the analogies between Kripke structures (and classical structures as the discrete special case). But we have met the partition forcing models before; they are just a different presentation of the dit-set representation of the partition algebras $\Pi(U)$:

$$\text{Partition forcing model} = \text{dit-set representation of } \Pi(U).$$

The T-sets are the dit sets since $(u, u') : T\varphi$ is the same as $(u, u') \in \operatorname{dit}(\varphi)$ so that $T_\varphi = \operatorname{dit}(\varphi)$.

The presentation of the dit-set representation as a "partition forcing model" nevertheless brings out a number of analogies between the distinguishing-cut and falsifying-chain theorems in partition and related results in intuitionistic and classical logic. In a Kripke structure, the order structural condition is that if $T\varphi$ holds at a point u, then it holds at any higher point $u' \geq u$. In a partition forcing model, the anti-transitivity structure condition is that if $T\varphi$ holds at any pair (u, u'), then it holds at some link on any u, u'-chain. Moreover, the conditions for the connectives provide a stronger version of the analogy. Let $*$ be any operation such as $\vee$, $\Rightarrow$, $\wedge$, or $|$.

Partition forcing model
$\sigma * \pi$ distinguishes (u, u'), i.e., $(u, u') : T(\sigma * \pi)$ iff $\forall$ u, u'-chains, the Boolean conditions for $T(\sigma * \pi)$ hold at some link on the chain.
$\sigma * \pi$ identifies (u, u'), i.e., $(u, u') : F(\sigma * \pi)$ iff $\exists$ u, u'-chain, with the Boolean conditions for $F(\sigma * \pi)$ holding at every link.

Intuitionistic forcing model
$\sigma * \pi$ contains u, i.e., $u : T(\sigma * \pi)$ iff $\forall$ $u' \geq u$, the Boolean conditions for $T(\sigma * \pi)$ hold at u'.
$\sigma * \pi$ does not contain u, i.e., $u : F(\sigma * \pi)$ iff $\exists$ $u' \geq u$, such that the Boolean conditions for $F(\sigma * \pi)$ hold at u'.[23]

Classical forcing model
$\sigma * \pi$ contains u, i.e., $u : T(\sigma * \pi)$ iff the Boolean conditions for $T(\sigma * \pi)$ hold at u.
$\sigma * \pi$ does not contain u, i.e., $u : F(\sigma * \pi)$ iff the Boolean conditions for $F(\sigma * \pi)$ hold at u.

Some pains have been taken to emphasize the analogies between the Kripke structure model and the classical and partition "forcing" models. But the classical and partition models are just a fancy way to describe, respectively, the membership conditions for subsets of a set U and the distinction conditions for partitions on a set U. Moreover, ordinary subset logic and partition logic are at the same mathematical level in the sense that both start with an unstructured set U. The subsets of a set and the partitions on a set can both be described without assuming any additional structure. In the intuitionistic case, either a

[23] The chain-cut results in partition logic have an even closer analogy in the intuitionistic case if one uses the Beth semantics of paths and bars in partially ordered sets, see (van Dalen 2001, p. 237) or (Restall 2000, p. 276).

topology or a partial order (which induces the topology of up-closed subsets as the open subsets) is assumed on the universe set U.[24]

2.1.2 Tableau structural rules

In general, the intuitionistic and partition F-rules will have a similar form. For any connective $*$, the intuitionistic rule is that $u : F(\pi * \sigma)$ iff $\exists u' \geq u$ such that the Boolean condition for $F(\pi * \sigma)$ holds at u', while the partition rule is that $(u, u') : F(\pi * \sigma)$ iff $\exists u, u'$-chain such that the Boolean condition for $F(\pi * \sigma)$ holds at every link on the chain.

By the same token, we could formulate the intuitionistic and partition T-rules as contrapositives. For the intuitionistic $T*$ rule, $u : T(\pi * \sigma)$ holds iff $\forall u' \geq u$, the Boolean condition for $T(\pi * \sigma)$ holds at u', and the partition rule is that $(u, u') : T(\pi * \sigma)$ iff $\forall u, u'$-chains, there is a link on the chain where the Boolean condition for $T(\pi * \sigma)$ holds on that link.

But the T-rules are written in a simplified way where the Boolean condition for $T(\pi * \sigma)$ holds at the base, and then is transmitted to a new base with that Boolean condition also holding there. For instance, the intuitionistic T-rule for $\pi * \sigma$ will be given in the simplified form as $u : T(\pi * \sigma)$ implies the Boolean condition for $T(\pi * \sigma)$ holds at u, together with a T-transmitting rule so any T's are transmitted to higher points in the ordering. Similarly in the partition case, we have used the simplified rule where $(u, u') : T(\pi * \sigma)$ implies the Boolean condition for $T(\pi * \sigma)$ also holds at (u, u') and then the following T-anti-transitivity rule transmits any T's to some link in any u, u'-chain.

The two T-transmitting structural rules for the intuitionistic and partition cases are as follows. The T-anti-transitivity rule splits into two alternatives given by the vertical line |. Context should suffice to avoid confusion between the vertical line | separating branches in the tableau tree and the Sheffer stroke | of the nand operation.

$\dfrac{u : T\varphi}{\forall a > u,\ a : T\varphi}$	$\dfrac{(u, u') : T\varphi}{\forall a,\ (u, a) : T\varphi \mid (a, u') : T\varphi}$
Intuitionistic T-transmitting rule	Partition T-anti-transitivity rule

An easy corollary implies that a $T\varphi$ holding at (u, u') is transmitted to some link in any u, u'-chain.

The T-transmitting rules can also be contraposited to derive "F-reflecting" rules.

$\dfrac{\exists a > u,\ a : F\varphi}{u : F\varphi}$	$\dfrac{\exists a,\ (u, a) : F\varphi \text{ and } (a, u') : F\varphi}{(u, u') : F\varphi}$
Intuitionistic F-reflecting rule	Partition F-transitivity rule

[24] Starting with Kripke structures as models for intuitionistic and modal logics, there has recently been a vast proliferation of logics modeled by sets with orderings or closure operations along with a variety of compatibility and accessibility relations on the sets; see (Restall 2000) for a survey. In contrast to this profusion of logics, partition logic, like classical subset logic, is modeled using only unstructured sets U.

Thus if $F\varphi$ holds at each link on any u,u'-chain, then $(u,u'):F\varphi$ follows.

Partition relations and their complementary equivalence relations are symmetric. Since we are using the ordered pairs (u,u') rather than the unordered pairs $\{u,u'\}$, we need rules to enforce that symmetry for the ordered pairs.

$\frac{(u,u'):T\varphi}{(u',u):T\varphi}$	$\frac{(u,u'):F\varphi}{(u',u):F\varphi}$
Partition T symmetric rule	Partition F symmetric rule

Equivalence relations are reflexive and partition relations are anti-reflexive but we don't need rules to enforce that since we have stipulated that the ordered pairs (u,u') in the rules are always of distinct elements.

2.1.3 Tableaus for the partition join

The tableau rules are given, for comparison purposes, for the three logics: subset, intuitionistic, and partition. The terms $u,u',a,b,...,c$ are now elements in the syntactic machinery of the tableau rules with the intended interpretations that have been already given; $u:T\varphi$ would be interpreted as u is a member of the set that interprets φ in the classical and intuitionistic rules while $(u,u'):T\varphi$ would be interpreted as (u,u') is a distinction of the partition that interprets φ, and similarly for the F-formulas. The four operations of $\vee$, $\wedge$, $\Rightarrow$, and $|$ will be taken as primitive in all the logics with the constant 1 defined as $\sigma\Rightarrow\sigma$ for any σ and 0 defined as $1\mid 1$.

In general, the syntactic eliminative T rules give the left-to-right implication in the "forcing" models described above, and the F rules are obtained by contrapositing the implication in the other direction. To compare the tableaus for these three logics, we start with the join where the eliminative tableaus are the most alike.

$\frac{u:F(\pi\vee\sigma)}{u:F\pi,F\sigma}$	$\frac{u:F(\pi\vee\sigma)}{u:F\pi,F\sigma}$	$\frac{(u,u'):F(\pi\vee\sigma)}{(u,u'):F\pi,F\sigma}$
Classical $F\vee$ rule	Intuitionistic $F\vee$ rule	Partition $F\vee$ rule

The $T\vee$ rules use the notion of a splitting of alternatives which is indicated by a vertical line.

$\frac{u:T(\pi\vee\sigma)}{u:T\pi\mid u:T\sigma}$	$\frac{u:T(\pi\vee\sigma)}{u:T\pi\mid u:T\sigma}$	$\frac{(u,u'):T(\pi\vee\sigma)}{(u,u'):T\pi\mid(u,u'):T\sigma}$
Classical $T\vee$ rule	Intuitionistic $T\vee$ rule	Partition $T\vee$ rule

The close analogies between the classical and intuitionistic rules on the one hand and the partition rules on the other hand are all by virtue of putting the lattice of partitions right side up. The penchant to write the lattice of partitions upside down seems to be one of the reasons why it has taken so long to develop the logic of partitions.

2.1.4 Tableaus for the partition implication

The complications arise in the F rules so we begin with the T rules.

$\dfrac{u : T\,(\sigma \Rightarrow \pi)}{u : F\sigma \mid u : T\pi}$	$\dfrac{u : T\,(\sigma \Rightarrow \pi)}{u : F\sigma \mid u : T\pi}$	$\dfrac{(u, u') : T\,(\sigma \Rightarrow \pi)}{(u, u') : F\sigma \mid (u, u') : T\pi}$
Classical $T \Rightarrow$ rule	Intuitionistic $T \Rightarrow$ rule	Partition $T \Rightarrow$ rule

The classical rules never leave the base point u so u is usually left implicit. In the intuitionistic $F \Rightarrow$ rule, a new element a may be introduced. Since the Beth-style tableau rules, in effect, try to construct a model of a formula using syntactic machinery, the ordering between the points in a Kripke structure must already be introduced as an ordering between elements. In particular, in the intuitionistic $F \Rightarrow$ rule, the new element a introduced in the consequence of the rule is higher in the ordering of elements than the base point used in the premise of the rule. In other treatments of the intuitionistic rule $F \Rightarrow$ as in Fitting (1969), the elements such as u and a are also left implicit but the rules that require leaving the base point to move higher in the ordering (i.e., the $F \Rightarrow$ and $F\neg$ rules) are indicated by dropping any other F-formulas in the premise and keeping only the T-formulas since only the T-formulas are transmitted to points higher in the ordering. We will not fully develop our version of the intuitionistic tableaus but we are presenting them to bring out the analogies with the partition tableaus.

In the partition $F \Rightarrow$ we may introduce a new element a but there is no ordering on the elements. There is always the notion of a chain of pairs of elements, and the partition $F \Rightarrow$ rule says that the Boolean condition for $F\,(\sigma \Rightarrow \pi)$ holds on each link of the chain $(u, a), (a, u')$.

$\dfrac{u : F\,(\sigma \Rightarrow \pi)}{u : T\sigma, F\pi}$	$\dfrac{u : F\,(\sigma \Rightarrow \pi)}{\exists a \geq u,\ a : T\sigma, F\pi}$
Classical $F \Rightarrow$ rule	Intuitionistic $F \Rightarrow$ rule

$\dfrac{(u, u') : F\,(\sigma \Rightarrow \pi)}{\exists u, u'\text{-chain (1 or 2 links) with } T\sigma, F\pi \text{ on each link}}$
Partition $F \Rightarrow$ rule

Since this is the first partition tableau rule that might introduce a new element, we have to be more explicit about how the tableau rules will be used here. We are given some partition formula φ and we begin a tableau for φ with the statement $(u_0, u_1) : F\varphi$. Since a tableau can branch like an upside-down tree, this initial statement $(u_0, u_1) : F\varphi$ is the root of the tree. The application of the tableau rules attempts to construct a partition on some model set U containing u_0 and u_1 where $(u_0, u_1) : F\varphi$ holds, i.e., to construct a countermodel for φ. The universe set starts at $U_0 = \{u_0, u_1\}$, and each application of a rule introducing one or more new elements will take the developing model from some U_n to U_{n+1} which is U_n plus the new elements.[25] Each U_n might be called a *stage* of the developing model.

[25] It may be useful to keep in mind the analogies with the development of models in classical first-order logic using tableaus (Smullyan 1968). We are from the outset seeing the new

New elements should be introduced only as a last resort. Since new element might be introduced only by F-rules, before introducing new elements to make a falsifying chain, we need to first check that a falsifying chain could not be formed using the existing elements. Given the premise $(u,u') : F(\sigma \Rightarrow \pi)$, the "base pair" (u,u') might be a one-link falsifying chain if $(u,u') : T\sigma, F\pi$, or the falsifying chain might be constructed using an element $a \in U_n$ of the evolving universe set at that stage. For instance, we might have already derived $(u,u') : F\sigma, F\pi$ so the base pair was not a one-link falsifying chain, but there might be an element $a \in U_n$ in the evolving universe set at that stage and on that branch of the tableau where, say, $(u,a) : F\pi$ held. Then the F-transitivity rule given below would yield $(a,u') : F\pi$. If we then initiated a new branch with the assumption $(u,a) : T\sigma$ then the T-anti-transitivity rule given below would imply $(a,u') : T\sigma$ and we would have a falsifying chain for $(u,u') : F(\sigma \Rightarrow \pi)$ without introducing any new elements. Such a falsifying chain using existing elements might be called a *back-chain*. Thus there are a finite number of options to establish a falsifying back-chain before taking the "last option" of introducing a new element and thus a new stage in the developing countermodel. Each of these options creates a branch in the tree. Since each U_n is a finite set, there are only a finite number of possible back-chains (including the one-link back-chain of the base pair) so only a finite number of branches might be created by applying the rule.

We know from the previous falsifying-chain theorem that when the atomic variables of some formula $\sigma * \pi$ are interpreted as partitions on some universe set U, then $F(\sigma * \pi)$ will hold at some pair (u,u') iff there is a falsifying chain with the Boolean conditions for $F(\sigma * \pi)$ holding at each link on the chain. The $F \Rightarrow$ rule gives us the syntactic version of that semantic theorem in the following sense. Each branch resulting from applying the rule to $(u,u') : F(\sigma \Rightarrow \pi)$ will have the statements for a falsifying chain either at the base pair $(u,u') : T\sigma, F\pi$ or on a two-link falsifying chain $(u,a),(a,u') : T\sigma, F\pi$ which might be a back-chain if $a \in U_n$ or a new chain if a is a new element that yields $U_{n+1} = U_n \cup \{a\}$. For the operations of the meet $\wedge$ and nand $|$, the falsifying chains could have more than two links and thus involve two or more elements other than the base pair (u,u'). In that case, the possibility arises of *mixed chains* using some existing elements in U_n and some new elements.

We also know from the distinguishing cut theorem that when the atomic variables of some formula $\sigma * \pi$ are interpreted as partitions on some universe set U, then $T(\sigma * \pi)$ will hold at some pair (u,u') iff for every u,u'-chain, there is a link (a,b) where the Boolean conditions for $T(\sigma * \pi)$ hold. The T-rules together with the T-anti-transitivity rule ensure that the corresponding formulas are derived in the developing branch of a tableau. For instance, in the present case of the implication $(u,u') : T(\sigma \Rightarrow \tau)$, the T-anti-transitivity rule implies that for any u,u'-chain using the elements of U_n, there is a link (a,b) where $T(\sigma \Rightarrow \pi)$ holds and then the $T \Rightarrow$ rule implies that either $(a,b) : F\sigma$

"constants" being introduced as elements in a potential model (in a manner reminiscent of the Löwenheim-Skolem theorem in classical first-order logic).

or $(a,b) : T\pi$ holds–which are the Boolean conditions for $T(\sigma \Rightarrow \pi)$ holding at (a,b).

Similar remarks apply to all the $T*$ and $F*$ rules where $*$ is $\vee$, $\Rightarrow$, $\wedge$, or $|$.

2.1.5 Tableaus for the partition meet

All three of the $T\wedge$ rules are rather standard.

$\frac{u : T(\pi \wedge \sigma)}{u : T\pi, T\sigma}$	$\frac{u : T(\pi \wedge \sigma)}{u : T\pi, T\sigma}$	$\frac{(u,u') : T(\pi \wedge \sigma)}{(u,u') : T\pi, T\sigma}$
Classical $T\wedge$ rule	Intuitionistic $T\wedge$ rule	Partition $T\wedge$ rule

The classical and intuitionistic rules for $F\wedge$ are standard while the partition $F\wedge$ is complicated since it involves a chain of elements with the Boolean condition, $F\pi$ or $F\sigma$, holding on each link. In the eliminative rule for the universal quantifier in classical first-order logic, we go from a premise $u : (\forall x)\varphi(x)$ to a conclusion of either $u : \varphi(a)$ where a is a constant in the developing model or $u : \varphi(x')$ where x' is a variable that can latter be replaced by a constant. In the partition $F\wedge$ rule, we have a similar situation when there is no falsifying back-chain so we need to introduce new elements to be strung together to make a falsifying chain. How many new elements should be introduced? In each branch of a tableau, we may eventually arrive at a contradiction in the form $(a,b) : T\sigma, F\sigma$ at some pair in which case the branch would *close*. Along that branch, no countermodel can be constructed so the branch is terminated. But a branch might be "falsely" terminated if we don't introduce enough new links in the falsifying chain of the $F\wedge$ rule. For instance, suppose we also had $(u,u') : T\phi_1, T\phi_2, T\phi_3$ in the branch and any two of these formulas holding at the same pair would give rise to a contradiction. Then if we had only introduced one new element to give the two-link falsifying chain $(u,a) : F\sigma$ and $(a,u') : F\pi$, then the T-anti-transitivity rule would have to "transmit" two of the three formulas $T\phi_1, T\phi_2, T\phi_3$ to one of the links in the chain and we would seem to have a closure of the branch. But we could just as well have introduced two new elements a and b so we had a falsifying chain of the three links $(u,a), (a,b), (b,u')$ and then each of the three formulas could be transmitted to a different link avoiding the contradiction. A crude upper bound on the number of necessary links is the number of subformulas of the formula φ in the root of the tree.

Hence when a branch closes, we must be sure that it would still close regardless of the length of the falsifying chain introduced in the $F\wedge$ rule. This can be done by ensuring that any falsifying chain from the $F\wedge$ rule in a closed branch could have been treated as a "variable" or generic chain so that whenever some $T\phi_i$ holding at (u,u') is transmitted to the chain, then it must have its "own" link and must not be forced to unnecessarily share a link with some other $T\phi_j$. If a branch does not close, then we need to construct a countermodel from the elements introduced in that branch (see the Satisfaction Theorem below) so we need to have introduced specific elements in an open branch.

In the $F\wedge$ rule, the elements $a, b, ..., c$ form a u,u'-chain, $(u,a), (a,b), ..., (c,u')$.

$\dfrac{u : F\,(\sigma \wedge \pi)}{u : F\sigma \mid u : F\pi}$	$\dfrac{u : F\,(\sigma \wedge \pi)}{u : F\sigma \mid u : F\pi}$
Classical $F\wedge$ rule	Intuitionistic $F\wedge$ rule

$\dfrac{(u,u') : F\,(\sigma \wedge \pi)}{\exists a, b, ...c \text{ so the } u,u'\text{-chain has } F\sigma \text{ or } F\pi \text{ on each link}}$
Partition $F\wedge$ rule

By the F-transitivity rule, two consecutive $F\sigma$ links could be shorted to one $F\sigma$ link so we may assume that the links of the falsifying chain are alternating. As in the case of the $F \Rightarrow$ rule, the falsifying chain might be a back-chain established using the elements of the current stage U_n without introducing new elements or a mixed chain with some old and some new elements. When new assumptions are made to have a falsifying back-chain, that creates a new branch. When new elements are introduced and T-formulas are transmitted to the links, then each way this could be done is a new branch. The possibilities quickly multiply but they are always finite at each stage.

2.1.6 Tableaus for the partition nand

All three of the $T \mid$ rules are rather standard.

$\dfrac{u : T\,(\pi \mid \sigma)}{u : F\pi \mid u : F\sigma}$	$\dfrac{u : T\,(\pi \mid \sigma)}{u : F\pi \mid u : F\sigma}$	$\dfrac{(u,u') : T\,(\pi \mid \sigma)}{(u,u') : F\pi \mid (u,u') : F\sigma}$
Classical $T \mid$ rule	Intuitionistic $T \mid$ rule	Partition $T \mid$ rule

In the "intuitionistic" $F \mid$ rule (which we have invented since the nand operation is not ordinarily used in intuitionistic logic), a new element a is introduced so that $a \geq u$ in the partial ordering of elements so that the Boolean condition for $F\,(\pi \mid \sigma)$, i.e., $T\pi, T\sigma$, holds at that point. In the $F \mid$ rule for partitions we already know that four links suffice in an falsifying chain so we only need to introduce at most three new elements a, b, c to form the falsifying u, u'-chain where the same Boolean conditions hold at each link.

$\dfrac{u : F\,(\pi \mid \sigma)}{u : T\pi, T\sigma}$	$\dfrac{u : F\,(\pi \mid \sigma)}{\exists a \geq u,\ a : T\pi, T\sigma}$
Classical $F \mid$ rule	Intuitionistic $F \mid$ rule

$\dfrac{(u,u') : F\,(\pi \mid \sigma)}{\exists\ u,u'\text{-chain (at most four links) with } T\pi, T\sigma \text{ on each link}}$
Partition $F \mid$ rule

As before, the falsifying chain could be a back-chain. For the option where new elements are introduced, at most three elements need to be introduced since four links suffice in any falsifying chain for the nand $\pi \mid \sigma$.

2.1.7 Examples of proofs and countermodels using the $F\wedge$ rule

Starting with the assumption that a "root" formula φ does not distinguish a generic pair (u_0, u_1), i.e., $(u_0, u_1) : F\varphi$, the tableau rules for the connectives (as opposed to the structural rules) eliminate the main connective of a formula at each step. If all branches terminate with a contradiction such as $T\sigma, F\sigma$ at some pair, then the tableau constitutes a proof of the formula φ, i.e., φ is a *theorem* of the tableau system. If a branch arrives at atomic signed formulas without any contradiction but where all the possible rules have been applied, then the open tableau branch will give a model of $(u_0, u_1) : F\varphi$, i.e., a countermodel to φ being a partition tautology.

The $F\wedge$ rule will be illustrated by developing tableaus for two related formulas, $\varphi_1 = (\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow (\sigma \wedge \pi)$ and $\varphi_2 = \sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$, where both formulas are classical tautologies but only the first is a partition tautology. To save space, we have ignored the base pair and back-chain branches for $F \Rightarrow$ and $F\wedge$ since we show that the branches with new multiple-link falsifying chains close. Hence the base pair and back-chain branches would, a fortiori, close since they allow even fewer possibilities to avoid contradictions. When a formula appears on a branch with both signs, e.g., $(u_0, b) : F\sigma, T\sigma$, then the branch closes as indicated with an X.

1	$(u_0, u_1) : F[(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow (\sigma \wedge \pi)]$	Rules used
2	$\exists a, (u_0, a), (a, u_1) : T(\sigma \wedge (\sigma \Rightarrow \pi)), F(\sigma \wedge \pi)$	$F \Rightarrow$
	Continuing the analysis at (u_0, a)	
3	$\exists b, c,\ (b, c) : F\sigma, T(\sigma \wedge (\sigma \Rightarrow \pi))$ | cont.	$F\wedge$ and T-a-t
	$(b, c) : F\pi, T(\sigma \wedge (\sigma \Rightarrow \pi))$	$F\wedge$ and T-a-t
4	$(b, c) : T\sigma, T(\sigma \Rightarrow \pi)$ X | $(b, c) : T\sigma, T(\sigma \Rightarrow \pi)$	$T\wedge$ both branches
5	X | $(b, c) : F\sigma$X || $(b, c) : T\pi$ X	$T \Rightarrow$

Closed tableau for: $(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow (\sigma \wedge \pi)$

In the second line, there was only one T-formula $T(\sigma \wedge (\sigma \Rightarrow \pi))$ to transmit to a link in the falsifying chain for $F(\sigma \wedge \pi)$ so a two-link chain would suffice to give $T(\sigma \wedge (\sigma \Rightarrow \pi))$ the alternatives of going to a $F\sigma$ link (the left-hand alternative) or to a $F\pi$ link (the right-hand alternative). But we use the example to illustrate a generic u_0, a-chain with a link (b, c) in the chain. No matter how long the chain is, there are only two alternatives created since $T(\sigma \wedge (\sigma \Rightarrow \pi))$ is either transmitted to an $F\sigma$ link (the left branch) or to a $F\pi$ link (the right branch). In the last line, two vertical lines || were used to indicate a second branching in the right-hand branch. The use of multiple vertical lines helps one to keep track of the level of branching in the tree.

In the second row of the above tableau, the same Boolean conditions would hold on (a, u_1) as hold on (u_0, a). They are related by an "and" and are not alternatives. Hence if contradictions can be obtained on all branches resulting from analyzing (u_0, a)–as indeed happened–then one does not need any more analysis on (a, u_1).

When a tableau has an open branch, a branch where the formulas have been "atomized" with no contradictions appearing and all the rules have been exhausted, then we will see that a countermodel can be constructed using the branch. If the formula is not a classical tautology, then one can stick entirely to the original base pair since there is a countermodel with $|U| = 2$. But if the formula is a classical tautology but not a partition tautology, then a multiple-link falsifying chain is required at some point. For the formula $\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$, which is a classical but not partition tautology, there is an open branch where a multiple-link falsifying chain was only used for the $F\wedge$ rule.

1	$(u_0,u_1): F[\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))]$	Rules used
2	$(u_0,u_1): T\sigma, F((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$	$F\Rightarrow$ (base pair)
3	$(u_0,u_1): T\sigma, T(\sigma \Rightarrow \pi), F(\sigma \wedge \pi)$	$F\Rightarrow$ (base pair)
4	$(u_0,u_1): T\pi \mid (u_0,u_1): F\sigma$ X	$T\Rightarrow$
5	$\exists a$, $(u_0,a): F\sigma, T(\sigma \Rightarrow \pi)$ and $(a,u_1): F\pi, T\sigma$ $\|$ ...$\mid$ X	$F\wedge$ and T-a-t
6	$(u_0,a): T\pi$ $\|\|$ $(u_0,a): F\sigma$ $\|$ ...$\mid$ X	$T\Rightarrow$

Simple tableau for $\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$ with an open branch.

Taking the left branches at the three splittings, which terminates with $(u_0,a): T\pi$, we can use the atomic signed formulas on each branch to construct a "countermodel", namely a model where $(u_0,u_1): F[\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))]$ holds so that the formula cannot be a partition tautology.

But to construct the model, the branch needs to be "completed" by applying the eliminative rules to any signed compound formulas in the branch until signed atomic formulas are reached, and by assigning signed atomic variables to any remaining branches in a manner consistent with T-anti-transitivity and F-transitivity (symmetry is assumed as a matter of course).

How does one know if this is always possible? If an assignment of signed atomic variables to the other pairs was not possible given the signed formulas that already have to hold at the pairs, then either there is some contradiction that could be derived using the rules so the branch was not really open, or the rules are incomplete (so that one has a partition tautology where the rules were unable to close all the branches)–the latter possibility being ruled out by the satisfaction theorem below. In the case at hand, there is already a consistent assignment of signed atomic variables to all the links, i.e., $(u_0,u_1): T\sigma, T\pi$, $(u_0,a): F\sigma, T\pi$, and $(a,u_1): T\sigma, F\pi$. This immediately generates the partitions $\sigma = \{\{u_0,a\},\{u_1\}\}$ and $\pi = \{\{u_0\},\{u_1,a\}\}$. Then $\sigma \wedge \pi = 0$, $\sigma \Rightarrow \pi = \pi$, $(\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi) = 0$, and the whole formula $\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$ then also evaluates to 0 which gives a model for $(u_0,u_1): F[\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))]$ and thus a countermodel to that formula being a partition tautology.

Thus the two similar classical tautologies, $(\sigma \wedge (\sigma \Rightarrow \pi)) \Rightarrow (\sigma \wedge \pi)$ and $\sigma \Rightarrow ((\sigma \Rightarrow \pi) \Rightarrow (\sigma \wedge \pi))$, give rather different results for partitions since only the first formula is a partition tautology. The difference in the two cases was that for the first formula, we had $T(\sigma \wedge (\sigma \Rightarrow \pi))$ being transmitted to some link in the falsifying chain for $F(\sigma \wedge \pi)$, where a contradiction would then arise. But in the second formula, it was the pair of T-formulas, $T\sigma, T(\sigma \Rightarrow \pi)$, which were being transmitted so there was no necessity that they be transmitted to the same link in the falsifying chain. By spreading them out with $T\sigma$ going to a $F\pi$ link and $T(\sigma \Rightarrow \pi)$ going to a $F\sigma$ link, no contradiction arose and in fact a countermodel could be constructed.

2.1.8 Tableaus for partition negation

It may be useful to also have tableau rules for negation which can be derived from the other rules. Since we are only taking the four operations $\vee$, $\wedge$, $\Rightarrow$, and $|$ as primitive, we could define the constant 1 as $\sigma \Rightarrow \sigma$ for any atomic variable σ and we could define 0 as $1 \mid 1$. Then we could define negation (as in intuitionistic logic) as $\neg\sigma = \sigma \Rightarrow 0$. But since we have the nand operation, it is far simpler to equivalently define negation as: $\neg\sigma = \sigma \mid \sigma$. Then the tableau rules for negation are just a special case of the rules for the nand.

$\dfrac{u : T(\neg\sigma)}{u : F\sigma}$ Classical $T\neg$ rule	$\dfrac{u : T(\neg\sigma)}{u : F\sigma}$ Intuitionistic $T\neg$ rule	$\dfrac{(u,u') : T(\neg\sigma)}{(u,u') : F\sigma}$ Partition $T\neg$ rule

We know for the nand that four links suffice in any falsifying chain for $F(\pi \mid \sigma)$, and it can easily be shown the only two links suffice if $\sigma = \pi$. The same holds if we had defined the negation as the implication to 0.

$\dfrac{u : F(\neg\sigma)}{u : T\sigma}$ Classical $F\neg$ rule	$\dfrac{u : F(\neg\sigma)}{\exists a \geq u,\, a : T\sigma}$ Intuitionistic $F\neg$ rule

$\dfrac{(u,u') : F(\neg\sigma)}{\exists\ u,u'\text{-chain (one or two links) with } T\sigma \text{ on each link.}}$ Partition $F\neg$ rule

The $T(\neg\sigma)$ rule is an example of a T-formula implying an F-formula. In any such case, the F-formula has to hold everywhere. If we consider any other $a \in U_n$, then by the T-anti-transitivity rule, $(u,u') : T(\neg\sigma)$ implies either $(u,a) : T(\neg\sigma)$ or $(a,u') : T(\neg\sigma)$. Whichever one holds, it implies that $F\sigma$ holds on the link which together with $(u,u') : F\sigma$ implies that $F\sigma$ holds on the other link by F-transitivity. Similarly for any other $b \in U_n$, and then $(a,b) : F\sigma$ follows from $(u,a),(u,b) : F\sigma$ by F-transitivity where (a,b) is any link in the complete graph $K(U_n)$.

2.1.9 Possibility of infinite open branches: the Devil's tableau

In the usual treatment of intuitionistic tableaus (Fitting 1969), the elements of the developing potential countermodel are left implicit and another device is used to construct a countermodel when a tableau does not close. However, in the partition tableaus we have treated the pairs (u,u') quite explicitly. But then we need the T-anti-transitivity and F-transitivity rules which do not reduce the complexity of formulas. The cost is that we do not have the usual proof of the finiteness of tableaus based on the fact that each of the non-structural rules for the connectives reduces the complexity of formulas so each branch must terminate after a finite number of steps in either a contradiction or in an open branch. That argument is unavailable due to the two complexity-preserving rules.

Moreover, the T-anti-transitivity rule leads to the possibility of cycles that can introduce an infinite sequence of stages: $U_0 \subseteq ... \subseteq U_n \subseteq U_{n+1} \subseteq ....$ The F-transitivity rule never forces the introduction of new elements. If we had a chain $(u,a),(a,b),...,(c,u')$ of elements in U_n with $F\sigma$ holding at each link with falsifying chains in U_n, then we can simply hook the chains together to give a falsifying chain for $(u,u'):F\sigma$, the conclusion of the F-transitivity rule. Hence the F-transitivity rule would never force new elements to be added to U_n.

But we have seen that $T(\neg\varphi)$ implies $F\varphi$, and for an appropriate φ, the $F\varphi$ might imply new elements yield a falsifying chain. And then the cycle repeats itself. The formula $\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))$ is a classical tautology that is not a partition tautology. But is it a weak partition tautology so that its double negation would be a partition tautology? That tableau would have the following infinite branch.

$(u_0,u_1):F(\neg\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))))$	Rules used
$(u_0,u_1):T(\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))))$	$F\neg$ (base pair)
$(u_0,u_1):F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$	$T\neg$
$\exists u_2,(u_0,u_2),(u_2,u_1):T\sigma,F(\pi \Rightarrow (\sigma \wedge \pi))$	$F\Rightarrow$
$(u_0,u_2):T\pi,F(\sigma \wedge \pi)$	$F\Rightarrow$ (base pair)
$(u_0,u_1):T\pi$	T-anti-trans.
$(u_0,u_1):F\sigma$ and $(u_2,u_1):F\pi$	$F\wedge$ back-chain u_0,u_1,u_2
$(u_0,u_2):T(\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))))$	T-anti-trans.
$(u_0,u_2):F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$	$T\neg$
$\exists u_3,(u_0,u_3),(u_3,u_2):T\sigma,F(\pi \Rightarrow (\sigma \wedge \pi))$	$F\Rightarrow$
$(u_0,u_3):T(\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))))$	T-anti-trans.
Cycle repeats...	

Infinite open branch in tableau

This tableau adds a single new element at each stage: $U_0 = \{u_0,u_1\} \subseteq U_1 = \{u_0,u_1,u_2\} \subseteq ...$ so the universe set associated with the infinite branch is the union $U = \bigcup U_n$. For the branch to be *finished*, then at each stage, each rule needs to be applied wherever possible. For instance, at the end of stage 1 (the double line in the table), the F-transitivity rule could be applied to derive $(u_2,u_1):F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$ and $(u_0,u_1):F(\pi \Rightarrow (\sigma \wedge \pi))$ but the status of σ on (u_0,u_1) is undetermined.

To construct the countermodel, the partitions are defined on U by using all the atomic F-statements so that a and b are in the same block of the α if $(a,b):F\alpha$ occurred at some finite stage. If for two elements $u,u' \in U$, the formula $(u,u'):F\alpha$ never occurs at any stage, then those two elements would be in separate blocks of α. Otherwise, there would have been a finite u,u'-chain where $(a,b):F\alpha$ holds at each link (a,b) in the chain. But then at some finite stage, all the links and the statements $(a,b):F\alpha$ would be present so $(u,u'):F\alpha$ would be implied by the F-transitivity rule at that stage.

By the satisfaction theorem proven below, this will provide a countermodel for $\neg\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$. But the tableau construction of an infinite model does not show the absence of any finite models. Indeed, the above tableau could have been stopped at the double line, the end of the first stage. For instance, the formula $(u_0, u_2) : F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$ was used to introduce a new element u_3 and to move to another stage. But that formula is already satisfied at its base pair (u_0, u_2) so the introduction of a new element was unnecessary. If we stop at the double line (after applying some more rules to "finish" that stage), the model on $U_1 = \{u_0, u_1, u_2\}$ given by the atomic F-statements is: $\sigma = \{\{u_0, u_1\}, \{u_2\}\}$ and $\pi = \{\{u_0\}, \{u_1, u_2\}\}$ which, in this case, provides a countermodel for $\neg\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$. This shows, incidentally, that $\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))$ is not even a weak partition tautology.

It is easy to see why this sort of an infinite branch generated by a simple cycle was unnecessary. On the links of the chain introduced by the new element a, the Boolean conditions for $F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$ had to hold. But when $T(\neg(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi))))$ was sent to one of the links and $F(\sigma \Rightarrow (\pi \Rightarrow (\sigma \wedge \pi)))$ again derived, then its Boolean conditions would hold at that link so it was unnecessary to introduce a new element.

There is a much more devilish pattern that can generate an infinite branch, a pattern we might call the "Devil's tableau." The idea is to take two formulas with complementary Boolean conditions, such as $F(\sigma \wedge \pi)$ and $F(\sigma \mid \pi)$, where one or both might introduce new elements. Thus one or the other of the formulas would not have their Boolean conditions satisfied at the base pair. In the following Devil's tableau, we develop an infinite open branch taking the set of elements being introduced as the natural numbers $\mathbb{N}$.

$(0,1) : F(\neg\neg[(\sigma \wedge \tau) \vee (\sigma \mid \tau)])$	Rules used
$(0,1) : T(\neg[(\sigma \wedge \tau) \vee (\sigma \mid \tau)])$	$F\neg$
$(0,1) : F[(\sigma \wedge \tau) \vee (\sigma \mid \tau)]$	$T\neg$
$(0,1) : F(\sigma \wedge \tau), F(\sigma \mid \tau)$	$F\vee$
$(0,1) : T\sigma, T\tau$	$F \mid$ (base pair)
$\exists 2, (0,2) : F\sigma$ and $(1,2) : F\tau$	$F\wedge$
$(0,2) : F(\sigma \wedge \tau), F(\sigma \mid \tau)$	T-a-t etc.
$\exists 3, (0,3), (2,3) : T\sigma, T\tau$	$F \mid$
$(0,3) : F(\sigma \wedge \tau), F(\sigma \mid \tau)$	T-a-t etc.
$\exists 4, (0,4) : F\sigma$ and $(3,4) : F\tau$	$F\wedge$
$(0,4) : F(\sigma \wedge \tau), F(\sigma \mid \tau)$	T-a-t etc.
$\exists 5, (0,5), (4,5) : T\sigma, T\tau$	$F \mid$
...	

Infinite open branch of a Devil's tableau

The even stages $U_0 = \{0, 1\}$, $U_2 = \{0, 1, 2, 3\}$, ... use $F\wedge$ to introduce a new element and the odd stages use $F \mid$ to introduce a new element. This generates the pattern

$$(0, even) : F\sigma \text{ and } (even - 1, even) : F\tau \text{ and } (0, even - 1) : T\sigma, T\tau$$

$$(0, odd), (odd - 1, odd) : T\sigma, T\tau \text{ and } (0, odd - 1) = (0, even) : F\sigma.$$

The union of the stages $U_n = \{0, 1, ..., n+1\}$ is the natural numbers $\mathbb{N}$ and the partitions defined by the atomic F-statements are:

$$\sigma = \{\{0, 2, 4, 6, ...\}, \{1\}, \{3\}, \{5\}, ...\}$$
$$\tau = \{\{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, ...\}.$$

This is indeed a model since $\sigma \wedge \tau = 0 = \sigma \mid \tau$. The fact that $\sigma \wedge \tau = 0$ is easily seen since σ identifies all the even numbers and τ identifies each odd number with its successor even number. To see that $\sigma \mid \tau = 0$, consider its graph which will have links $n \sim m$ whenever n and m are distinguished by both partitions. Thus in that graph $even \sim even + 1$ ($= odd$) and $odd \sim odd + 2$ so there is a finite chain connecting any $n, m \in \mathbb{N}$.

By alternating between the two potentially element-introducing F-formulas, $F(\sigma \wedge \tau)$ and $F(\sigma \mid \tau)$, the Devil's tableau avoids having both formulas satisfied at the base pair at the same time. But there is still the possibility that both formulas could be satisfied by back-chains at the same time–so that there would be no need to introduce any new constants and the branch could be terminated there. Indeed, that is the case with this Devil's tableau. If we stop the tableau at the double line where the stage is $U_3 = \{0, 1, 2, 3, 4\}$, then the partitions are: $\sigma = \{\{0, 2, 4\}, \{1\}, \{3\}\}$ and $\tau = \{\{0\}, \{1, 2\}, \{3, 4\}\}$ and $\sigma \wedge \tau = 0 = \sigma \mid \tau$ as well. Hence in this case, the consideration of back-chains gives a finite tableau that provides a finite countermodel, but the question of whether there is always a finite countermodel is left open along with the related question of the decidability of the set of partition tautologies. The necessity of considering back-chains in order to have a finite open branch of the Devil's tableau shows why back-chains are included in the "$\exists u, u'$-chain" clause in the conclusions of the element-introducing F rules.

2.1.10 More proofs and countermodels using tableaus

A few more examples may be helpful. The partition tautology $\pi \Rightarrow (\sigma \Rightarrow \pi)$ provides a simple example. But even for this example, tableau trees expand rapidly without shortcuts and symmetry arguments.

$(u_0, u_1) : F[\pi \Rightarrow (\sigma \Rightarrow \pi)]$	Rules used:
$(u_0, u_1) : T\pi, F(\sigma \Rightarrow \pi) \mid$	$F \Rightarrow$
$\exists a\ (u_0, a), (a, u_1) : T\pi, F(\sigma \Rightarrow \pi)$ and $(u_0, u_1) : F\pi$	$F \Rightarrow$, F-t
(continuing with the right branch)	
...$\mid$ $(u_0, a) : T\sigma, F\pi$X $\mid\mid$ $\exists b, (u_0, b), (b, a) : T\sigma, F\pi$ and $(u_0, a) : F\pi$X	$F \Rightarrow$, F-t.
(right branch closed so picking up the left branch)	
$(u_0, u_1) : T\pi, F(\sigma \Rightarrow \pi) \mid$ X $\mid\mid$ X	
$(u_0, u_1) : T\sigma, F\pi$X $\mid\mid$ cont.	$F \Rightarrow$
$\exists a, (u_0, a), (a, u_1) : T\sigma, F\pi$ and $(u_0, u_1) : F\pi$ X $\mid$ X $\mid\mid$ X	$F \Rightarrow$, F-t

Closed tableau for: $\pi \Rightarrow (\sigma \Rightarrow \pi)$

Taking the leftmost branches, we stay at the base pair (u_0, u_1) and have essentially the classical closing tableau since this formula is a classical tautology. Since the other element-introducing branches also close, the formula is a partition tautology (assuming the correctness theorem proved below).

Peirce's law, $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma$, is a good example of non-closing tableau which must generate a model where the formula does not distinguish some pair.

$(u_0, u_1) : F[((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma]$	Rules used
$(u_0, u_1) : T[((\sigma \Rightarrow \pi) \Rightarrow \sigma)], F\sigma$	$F \Rightarrow$ (base)
$(u_0, u_1) : F(\sigma \Rightarrow \pi) \mid (u_0, u_1) : T\sigma$ X	$T \Rightarrow$
$\exists a, (u_0, a), (a, u_1) : T\sigma, F\pi$ and $(u_0, u_1) : F\pi\mid$ X	$F \Rightarrow$, F-t
$(u_0, a) : T[((\sigma \Rightarrow \pi) \Rightarrow \sigma)]\mid$ X	T-a-t
$(u_0, a) : F(\sigma \Rightarrow \pi) \mid\mid (u_0, a) : T\sigma \mid$ X	$T \Rightarrow$

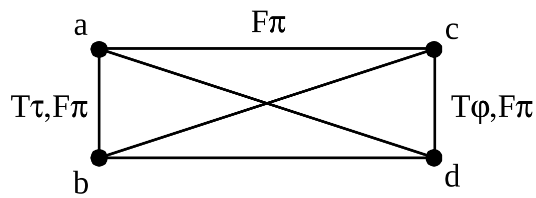
Non-closed tableau for: $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma$

The branch terminating with $(u_0, a) : T\sigma$ in the last row is an open branch (atomic formulas with no contradiction) so it may be used to generate of model of $F[((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma]$, i.e., a countermodel to Peirce's law being a partition tautology. To generate the model, we need to fill out the atomic signed formulas on all the links but that is already done on the indicated branch. The universe set is the three elements used in the tableau: $U = \{u_0, u_1, a\}$. The partition σ has $(u_0, u_1) : F\sigma$ while $T\sigma$ holds at (u_0, a) and (a, u_1). Thus $\sigma = \{\{u_0, u_1\}, \{a\}\}$. The partition π has $F\pi$ on all links so π is the blob: $\pi = \{\{u_0, u_1, a\}\} = 0$. The compound partitions are then: $\sigma \Rightarrow \pi = \pi = 0$ (since no non-singleton block of π is contained in a block of σ), $(\sigma \Rightarrow \pi) \Rightarrow \sigma = 1$ (since all blocks of σ are contained in the blob), and finally $((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma = \sigma$ (since $1 \Rightarrow \sigma = \sigma$) so that $(u_0, u_1) : F[((\sigma \Rightarrow \pi) \Rightarrow \sigma) \Rightarrow \sigma]$ holds and Peirce's law is not a partition tautology.

Essentially the same argument as in the common-dits theorem yields a powerful result that can be used to close branches of a tableau. It gives conditions under which a contradiction has to exist on some link without forcing one to work through all the possibilities on sub-branches to show they close.

Lemma 13 (Branch-closing lemma) *Suppose $(a, b) : T\tau, F\pi$ and $(c, d) : T\varphi, F\pi$ where there is a chain connecting the two links that has $F\pi$ holding at each link on the chain. Then there exists a link where $T\tau, T\varphi, F\pi$ all hold on the link.*

Proof: The $F\pi$-chain needs to connect a or b with c or d. If it connects, say, a and c, then by F-transitivity, $(a, c) : F\pi$. Then we have the following situation regarding those four points..



By F-transitivity again, $F\pi$ has to hold at all the links connecting the four points. Consider the triangle formed by a, b, and c. By T-anti-transitivity, $T\tau$ has to hold on either (a, c) or (b, c). Case 1. If $T\tau$ holds on (a, c), then by considering the triangle formed by a, c, and d, then $T\tau$ has to hold at (a, d) or (c, d). If it holds at (c, d), then we are finished so suppose it holds on (a, d). But since $T\varphi$ holds on (c, d), by T-anti-transitivity again, $T\varphi$ has to hold at either (a, c) or (a, d) and we are finished in either case. Case 2. If $T\tau$ holds on (b, c), then we repeat the same argument but for the triangle formed by b, c, and d. ■

The principal use of the branch-closing lemma is when, on a branch of a tableau, we have signed formulas $T\sigma, F\pi$ on one link, $T(\sigma \Rightarrow \pi), F\pi$ on another link, with a chain connecting the two links with $F\pi$ holding on each link of the chain. Then by the branch-closing lemma, there exists a link where $T\sigma, T(\sigma \Rightarrow \pi), F\pi$ all hold and thus the branch closes since there is a contradiction on that link regardless of whether $T(\sigma \Rightarrow \pi)$ is developed as $F\sigma$ or $T\pi$ by the $T \Rightarrow$ rule.

The single π-negation transform, $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$, of the law of excluded middle, $\sigma \vee \neg\sigma$, is an example of a partition tautology that is not an intuitionistic validity. It is the π-negation version of $\neg\sigma \vee \neg\neg\sigma$, the weak law of excluded middle. The tableau proof of the formula is also an example of using the branch-closing lemma.

$(u_0, u_1) : F[(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi)]$	Rules used
$(u_0, u_1) : F(\sigma \Rightarrow \pi), F[(\sigma \Rightarrow \pi) \Rightarrow \pi]$	$F\vee$
$(u_0, u_1) : T\sigma, F\pi \mid \exists a,\ (u_0, a), (a, u_1) : T\sigma, F\pi$ and $(u_0, u_1) : F\pi$	$F \Rightarrow$, F-t
... $\mid \exists b,\ (u_0, b), (b, u_1) : T(\sigma \Rightarrow \pi), F\pi$ and $(u_0, u_1) : F\pi$	$F \Rightarrow$, F-t
...$\mid$ $F\pi$ holds on chain u_0, u_1, a, b so $(u_0, a) : T\sigma$ (cont.)	
and $(u_0, b) : T(\sigma \Rightarrow \pi)$ collide with $F\pi$. X	B-C lemma

Closed tableau for: $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$

The branch with both falsifying chains allows maximal freedom from contradiction but it still closes (by the branch-closing lemma) so the left branch starting with a base-pair application of $F \Rightarrow$ would, a fortiori, close.

To see why this formula is not intuitionistically valid, we could develop its intuitionistic tableau. In the partition case, we have used repeatedly the fact that when $F(\sigma \Rightarrow \pi)$ is satisfied by a falsifying chain, then F-transitivity implies that $F\pi$ has to hold at the base pair. A similar result holds in the intuitionistic case. By the $F \Rightarrow$ rule, $u : F(\sigma \Rightarrow \pi)$ implies that the Boolean condition $T\sigma, F\pi$ has to hold at some higher point $a \geq u$. But if $u : T\pi$ held, then the structural rule would imply that $T\pi$ had to hold at all higher points (contradicting $a : F\pi$), so $u : F\pi$ must hold.

$u : F[(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi)]$	Rules used
$u : F(\sigma \Rightarrow \pi), F[(\sigma \Rightarrow \pi) \Rightarrow \pi]$	$F\vee$
$\exists a \geq u,\ a : T\sigma, F\pi$ and $\exists b \geq u,\ b : T(\sigma \Rightarrow \pi), F\pi$	$F \Rightarrow$ twice
$b : F\sigma \mid b : T\pi$ X	$T \Rightarrow$

Open intuitionistic tableau for: $\overset{\pi}{\neg}\sigma \vee \overset{\pi}{\neg}\overset{\pi}{\neg}\sigma$

As with partition tableaus, a model can be constructed from an open branch of an intuitionistic tableau. There are three points in $U = \{u, a, b\}$ and the partial ordering is given by $u \leq a$ and $u \leq b$. Then $F\pi$ holds at all points so π is modeled by the empty set $\emptyset$. $T\sigma$ holds at a but cannot hold at b and thus cannot hold at u. Hence σ is modeled by the up-closed set $\{a\}$. The sets formed by the connectives are then: $\sigma \Rightarrow \pi = \operatorname{int}(\sigma^c \cup \pi) = \{b\}$ and $((\sigma \Rightarrow \pi) \Rightarrow \pi) = \operatorname{int}(\{b\}^c \cup \pi) = \{a\}$ so that: $(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi) = \{b\} \cup \{a\} = \{a, b\} \neq U$ and we have a model for $u : F[(\sigma \Rightarrow \pi) \vee ((\sigma \Rightarrow \pi) \Rightarrow \pi)]$.

The reason why the intuitionistic tableau does not close is that once u branches to the two points a and b, those branches in the ordered set U do not need to interact so the "conflict" between the two branches in the ordering never gives a contradiction to close the tableau. However in partition logic, for any two links, there is always a direct connection so the conflict becomes a contradiction. For instance, in the partition tableau for this formula, the potential conflict at the two separate links $(u_0, a) : T\sigma, F\pi$ and $(b, u_1) : T(\sigma \Rightarrow \pi), F\pi$ is connected by the link $(u_0, u_1) : F\pi$ so the branch-closing lemma brings out the contradiction.

2.2 Correctness theorem for partition tableaus

A tableau for φ, i.e., a tableau with the root $(u_0, u_1) : F\varphi$, *closes* if all the possible branches terminate with a contradiction $(a, b) : T\pi, F\pi$ at some pair (a, b) for some subformula π. But this definition requires special attention to the $F\wedge$ rule. If a branch does not close with a contradiction, then the branch should generate a countermodel which requires any element-introducing use of the $F\wedge$ rule to introduce specific elements in the falsifying chain. But if a branch is to close with a contradiction for each alternative, then it is not enough to have it close from some finite set of specific falsifying chains since there is an infinite set of possible finite falsifying chains (and the $F\wedge$ rule would not have the finite-branching property). This is why some special attention is required in a tableau that uses the $F\wedge$ and that closes. The $F\wedge$ rule is interpreted as only introducing a generic finite chain of finite length, and the links in the chain only become specific when the T-anti-transitivity rule transmits a T-formula to some link. By taking it to be the shortest falsifying chain we could ensure that the links are alternating. Thus if $(u, u') : F(\sigma \wedge \pi)$, then the links would alternate between $F\sigma, T\pi$ and $T\sigma, F\pi$. If there were, say, three other T-formulas, $T\phi_1$, $T\phi_2$, and $T\phi_3$, holding at (u, u'), then each $T\phi_i$ could be transmitted to either a $F\sigma$ link or a $F\pi$ link (and always to different $F\sigma$ or $F\pi$ links fromt the other $T\phi_j$ formulas).[26] Hence there are only $2^3 = 8$ branches generated by the $F\wedge$ that would ultimately need to close for the tableau to close.

[26] If a branch would close when the $T\phi_i$ formulas were spread out on different links, then it would, a fortiori, close when some of the formulas were bunched together on the same type of link of the falsifying chain, so those alternatives may be ignored.

The correctness theorem for tableaus asserts that if the tableau for $F\varphi$ closes, then φ is a partition tautology, and the completeness theorem proves the reverse. The strategy of the proof of the correctness theorem is to show that if there is an interpretation in $\Pi(U)$ of the premise of a tableau rule, then there is an interpretation of the conclusion. Hence if the tableau closes, then since there can be no interpretation of the conclusions that close a tableau, there can be no interpretation of the beginning of the tableau, $(u_0, u_1) : F\varphi$ and thus φ is a partition tautology.

An *interpretation* or *model* of the formulas has a universe set U with two or more elements, interprets the atomic variables as partitions on U, and interprets the operation symbols $\vee$, $\wedge$, $\Rightarrow$, and $|$ as those operations in the partition algebra $\Pi(U)$. When convenient, we use the dit-set representation of $\Pi(U)$ so the variables and formula would refer to dit sets or partition relations rather than set-of-blocks partitions. Statements like $B \in \pi$ are interpreted in the obvious manner without pedantically saying that B is a block in the partition interpreting the symbol "π" and so forth. We are also already accustomed to using statements like "$T\varphi$ holds at (u, u')" as saying that (u, u') is a distinction of the partition interpreting φ, and similarly for F statements.

Theorem 14 (Correctness of partition tableaus) *If the tableau for $F\varphi$ closes, then φ is a partition tautology.*

Proof: We assume we have an interpretation of the formulas in a universe set U where the premises of the rules hold, and then we show that one of the possible conclusions holds.

All the T rules can be handled in a uniform way. Where $*$ is $\vee$, $\wedge$, $\Rightarrow$, or $|$, if $T(\sigma * \pi)$ holds at (u, u'), then the Boolean conditions for $T(\sigma * \pi)$ must hold at some link on any u, u'-chain which means they must hold at the one-link chain (u, u') which are the conclusions in the four T rules.

All the F rules have the general form that the premise $(u, u') : F(\sigma * \pi)$ implies the existence of a u, u'-chain where the Boolean conditions for $F(\sigma * \pi)$ hold at every link of the chain. The assumption is that at a certain stage where the set of elements or "constants" is U_n, then the elements u and u' of U_n are interpreted in U and there are partitions on U interpreting the atomic variables so that $F(\sigma * \pi)$ holds at $u, u' \in U_n$. Then by the falsifying-chain theorem, there is a finite u, u'-chain of elements of U where the Boolean conditions for $F(\sigma * \pi)$ hold at each link. In terms of U_n, that falsifying chain could be a back-chain, a mixed chain, or a chain of new elements linking u and u'. Thus by adding a finite number of new elements of U to U_n if necessary to have U_{n+1}, one of the alternatives of the $F*$ rule is the set of assignments to the links of that chain that hold in the model on U.

The structural rules are also correct by similar reasoning. In any interpretation, $(u, u') : T\varphi$ means that $(u, u') \in \text{dit}(\varphi)$ which is a partition relation and thus anti-transitive so the conclusion of the T anti-transitivity rule holds. If $(u, a), (a, u') : F\varphi$, then $(u, a), (a, u') \in \text{indit}(\varphi)$ which is an equivalence relation so its transitivity gives the conclusion of the F transitivity rule. In any interpretation, both dit sets and indit sets are symmetric so if the premise holds, then the conclusion holds in each of the symmetry rules.

Hence if the premise in any of the rules has an interpretation, then so does one of the alternatives in the conclusion. Since the conclusions of the closed branches have no interpretation, a closed tableau for $F\varphi$ implies there is no interpretation for the premise of $(u_0, u_1) : F\varphi$ so that $(u_0, u_1) : T\varphi$ holds for any pair in any interpretation and thus φ is a partition tautology. ■

2.3 Completeness theorem for partition tableaus

2.3.1 Completing a tableau

The correctness theorem shows that if all branches in the tableau with the root $(u_0, u_1) : F\varphi$ close, then φ is a partition tautology. The goal now is to prove the converse: if φ is a partition tautology, then there is a tableau for $F\varphi$ where all branches close, i.e., φ is a theorem of the tableau system. It would be equivalent to prove the contrapositive that if there was an open branch (i.e., a branch that could not be closed), then the branch would provide a countermodel to φ, i.e., a model for $(u_0, u_1) : F\varphi$.

A branch of a tableau is *closed* if for some pair (a, b) and some formula π, both $(a, b) : T\pi$ and $(a, b) : F\pi$ occur on the branch. In terms of stages, a closed branch must close at some finite stage and the branch terminates at that stage. If a tableau with the root $(u_0, u_1) : F\varphi$ is *closed* in the sense that all branches are closed, then, since all rules are finitely-branching (using the generic falsifying chain in the $F\wedge$ rule which only branches for the finite number of possible ways that T-formulas $T\phi_i$ could be transmitted to the chain), a closed tableau is finite and thus constitutes a tableau proof of φ.

A branch of a tableau is *complete at stage n* with the universe set U_n if all applications of the rules that can be made have been made. There are two types of rules, the connective rules for the four connectives and the structural rules (T-anti-transitivity, F-transitivity, and the symmetry rules). When a connective rule with a premise $(u, u') : F\phi$ or $(u, u') : T\phi$ has been used then it can be checked (✓) once. But the same premise could also be used in the premise for many structural rules so a premise would get a second check mark when all the structural rules have been applied at that stage. In the order of applying rules systematically, the structural rules and the non-element-introducing rules should be used first and should involve only elements from the universe set U_n at that stage. Then the potentially element-introducing rules ($F \Rightarrow$, $F\wedge$, and $F\,|$) are used. Then any applicable rules may need to be applied again if any new formulas $(a, b) : F\phi$ or $(a, b) : T\phi$ were introduced for old elements $a, b \in U_n$. This cycling over the rules at each stage terminates after a finite number of steps since there are a finite number of elements in each U_n and we are not yet considering any new elements introduced into the next stage. No infinite regress (like in the Devil's tableau) is possible since we are only considering formulas at pairs of elements of the given finite stage.

Being "complete" is defined stage by stage since when new elements are introduced, there is a new stage and new applications of the structural rules of T-anti-transitivity and F-transitivity to premises of former stages may occur. Thus at each new stage, the second check mark on the old formulas is erased until all the new applications using pairs involving new elements have been used. For instance, the premise $(u, u') : T\phi$ might be checked a second time when applied to all $a \in U_n$ to yield $(u, a) : T\phi$ or $(a, u') : T\phi$ but then could be applied again using new $b \in U_{n+1}$.

When a stage is complete but new elements were introduced, then the same process continues at the next stage. If a stage is completed with no new elements introduced, then the branch is complete (with no further stages). A branch of the tableau is *complete* when it is complete at each of its stages. A tableau is *completed* if every branch is either complete or closed.

2.3.2 Satisfaction and completeness theorems

A completed tableau that is not closed must have at least one open complete branch.

Theorem 15 (Satisfaction theorem) *An open complete branch of a partition tableau with the root formula $(u_0, u_1) : F\varphi$ gives a model where the root formula is satisfied, i.e., $(u_0, u_1) \in \operatorname{indit}(\varphi)$ in the model.*

Proof: An open complete branch of a tableau will be used to define a model on a set U. If the complete open branch terminated at the stage U_n, then take $U = U_n$. Otherwise, there is an infinite sequence of stages $U_0 \subseteq U_1 \subseteq ...$ and $U = \bigcup_n U_n$. The partitions on U are defined by the formulas $(a, b) : F\alpha$ occurring in the branch for the atomic variables α occurring in the root formula φ. Using the graph machinery, these atomic F-formulas occurring in the branch define the links of a graph on the node set U, and the blocks of the partition α are the sets of nodes in the connected components of the graph. This defines the partitions interpreting the atomic variables of φ and then the partition operations of $\Pi(U)$ will give an interpretation of φ using partitions on U. We need to show that $(u_0, u_1) \in \operatorname{indit}(\varphi)$ under that interpretation.

The proof is by induction over the complexity of the subformulas of φ. The basis step is that every signed atomic formula which occurs in the branch is true in the model. If $(u, u') : F\alpha$ occurs in the branch then it is true by definition in the model, i.e., $(u, u') \in \operatorname{indit}(\alpha)$. If $(u, u') : T\alpha$ occurs in the branch but does not hold in the model, i.e., $(u, u') \in \operatorname{indit}(\alpha)$, then using the graph constructed for α and using the falsifying-chain theorem, there is a finite u, u'-chain with $(u_i, u_{i+1}) : F\alpha$ holding at each link. Moreover, there is a finite stage U_n where all these formulas would have occurred. But completeness at that stage would then imply, by using the F -transitivity rule, that the formula $(u, u') : F\alpha$ held at that stage which would contradict $(u, u') : T\alpha$ holding at some stage on the complete open branch. Hence if $(u, u') : T\alpha$ did occur in the open branch, then $(u, u') \in \operatorname{dit}(\alpha)$ in the model.

The induction steps can be efficiently treated using the graph machinery. Suppose $(u, u') : T(\sigma * \pi)$ occurs in the complete open branch. In order for $(u, u') \in \operatorname{dit}(\sigma * \pi)$ in the model on U, then for every finite u, u'-chain in U, the Boolean conditions for $T(\sigma * \pi)$ must hold at some link in the chain. Suppose not, so there is a u, u'-chain where the complementary Boolean conditions for $F(\sigma * \pi)$ hold at each link. There is a finite stage U_n of the branch in which all the elements of that chain have appeared and where $(u, u') : T(\sigma * \pi)$ also occurs. But then by the completeness of applying the T-anti-transitivity at that stage, there is a link (u_i, u_{i+1}) in the chain where $(u_i, u_{i+1}) : T(\sigma * \pi)$ holds. Then by completeness and the connective rule for $T(\sigma * \pi)$, the formulas for the Boolean conditions for $T(\sigma * \pi)$ holding at (u_i, u_{i+1}) would be in the branch at that stage as well. But they are formulas of lower complexity than $\sigma * \pi$, so by the induction hypothesis, those formulas must hold in the model which contradicts the complementary Boolean conditions holding at all links of that chain in the model. Hence $(u, u') \in \operatorname{dit}(\sigma * \pi)$ holds in the model.

Suppose $(u, u') : F(\sigma * \pi)$ occurs at some stage U_n in the open branch. Then by completeness and the connective rule for $F(\sigma * \pi)$, there is a finite u, u'-chain in U_{n+1} (or in U_n if no new elements were introduced) where the formulas for the Boolean conditions for $F(\sigma * \pi)$ occur at each link in the chain. But all those formulas are of lower complexity than $\sigma * \pi$ so by the induction hypothesis, they are true in the model on U, which in turn implies that $(u, u') \in \operatorname{indit}(\sigma * \pi)$ in that model.

Since the formula $(u_0, u_1) : F\varphi$ occurs in every branch, the open complete branch supplies a model where $(u_0, u_1) \in \operatorname{indit}(\varphi)$. ■

Theorem 16 (Completeness theorem for partition tableaus) *If φ is a partition tautology, then any completed tableau beginning with $(u_0, u_1) : F\varphi$ must close, and thus every partition tautology is provable by the tableau method.*

Proof: If a completed tableau beginning with $(u_0, u_1) : F\varphi$ had a complete open branch, then by the satisfaction theorem there would be an interpretation where $(u_0, u_1) \in \operatorname{indit}(\varphi)$ and thus φ is not a partition tautology. Hence if φ is a partition tautology, then any completed tableau beginning with $(u_0, u_1) : F\varphi$ must close so that φ is a theorem by the tableau method. ■

3 Concluding remarks

Classical "propositional" logic should be interpreted as having its variables refer to subsets of an unstructured universe set U, with the propositional interpretation being isomorphic to the subsets 0 and 1 of a one element universe. Intuitionistic logic adds structure to the universe set U to define a topology (e.g., the up-closed subsets from a partial ordering on the universe set) so that the relevant subsets for the interpretation are the open subsets. Classical "propositional" logic can be seen as the special case with the discrete topology on U so that all subsets are open and the intuitionistic operations reduce to the classical ones.

Partition "propositional" logic, like classical logic, starts with an unstructured universe set U (two or more elements). The subsets of the powerset Boolean algebra $\mathcal{P}(U)$ and the partitions of the partition algebra $\Pi(U)$ are both defined simply on the basis of the set U. Thus subset logic and partition logic are at the same mathematical level, and are based on the dual concepts of subsets and partitions.

Partition logic provides a dual semantics for propositional formulas, a semantics based on the distinctions of partitions rather than the elements of subsets. One can go further with the elements-distinctions duality. Probability theory conceptually starts with the finite case where the probability is the ratio of the number of elements in a subset ("event") to the size of the finite universe U ("sample space"). This conceptual continuation from subset logic to finite probability theory was there from the beginning in Boole. Quoting Poisson, Boole defined "the measure of the probability of an event [as] the ratio of the number of cases favourable to that event, to the total number of cases favourable and unfavourable, and all equally possible." (Boole 1854, p. 253) Replacing elements and subsets with distinctions and partitions yields a *logical information theory* where the *logical entropy* of a partition is defined as the ratio of the number of distinctions of the partition to the size of the finite closure space $U \times U$. The resulting logical information theory provides a conceptual foundation for Shannon's information theory (Ellerman 2009).

Finally, we might speculate about why it has taken so long for partition logic to be developed. The subset interpretation dates back to Boole and DeMorgan, and the subset-partition duality is at least as old as category theory. There seems to be a cluster of reasons.

From the side of logic, most non-category-theoretic treatments of logic give only the propositional interpretation of Boolean logic. Moreover, the progression from "propositional" logic to "quantification theory" is usually based entirely on analyzing propositions as quantified formulas. Tarski's semantics developed as model theory has been very successful in applications. Model theory interprets open formulas and atomic relations as subsets of an n-fold product U^n of some underlying universe set, and then closed formulas are propositions which are either true or false. But Lawvere's development of categorical logic brings out the general setting in the category of sets. Given a set map $f : V \to U$ between two universe sets, the two quantifiers will map subsets of V to subsets of U.[27] In the special case of classical quantification theory, quantifying over a variable in effect takes the set map as the projection $U^n \to U^{n-1}$ that leaves out the variable so that the subset quantifiers carry subsets of U^n to subsets of U^{n-1}. When $n = 1$, quantifying over the single variable is usually interpreted as turning an open single-variable formula into a closed formula or proposition which is true or false, but the interpretation in categorical logic is mapping subsets of U^1 to subsets of $U^0 = 1$ where the subsets of 1 behave like the usual propositional

[27] The technical details are not relevant to our point here since this paper does not deal with quantifiers for partition logic. The categorical logic treatment of the subset quantifiers is covered in Mac Lane (1971), Lawvere and Rosebrugh (2003), Awodey (2006), or Mac Lane and Moerdijk (1992).

truth values. This propositional special case has been so important that the general case of subset logic and subset quantifiers has been rather eclipsed and neglected. The part has been taken as the whole. The point is that since propositions do not have a dual notion of partitions, the idea of a dual logic of partitions does not arise in the conventional treatment of "propositional" logic.

From the partition side, one reason was simply that the "lattice of partitions" was traditionally defined "upside down" as (isomorphic to) the lattice of equivalence relations rather than its opposite. But the element-distinction duality makes it clear that the lattice of partitions should use the partial ordering given by the set of distinctions (dit set) of a partition rather than its set of indistinctions (just as the lattice of subsets uses the partial ordering given by the set of elements of a subset rather than its set of non-elements). This is what allowed the direct comparison of formulas in classical, intuitionistic, and partition logic as well as the proof-theoretic parallels between the tableaus for the three logics.

Another reason is that (at least to our knowledge) the implication, nand and other new binary operations on partitions (aside from the join and meet) have not been previously studied. In a recent paper in a commemorative volume for Gian-Carlo Rota, the three authors remark that in spite of the importance of equivalence relations, only the operations of join and meet have been studied.

> Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence. Much is still unknown about equivalence relations. Were this situation remedied, the theory of equivalence relations could initiate a chain reaction generating new insights and discoveries in many fields dependent upon it.
>
> This paper springs from a simple acknowledgement: the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations. (Britz, Mainetti, and Pezzoli 2001, p. 445)

Yet the new operations, particularly the implication, are crucial to the whole development. The only partition tautologies with only lattice operations are trivialities such as 1 and $1 \vee \pi$. Without the non-lattice operations, one can always study identities in the partition lattice such as $\pi \preceq \pi \vee \sigma$ (which corresponds to the tautology $\pi \Rightarrow \pi \vee \sigma$). But it has been shown (Whitman 1946) that partition lattices are so versatile that any formula in the language of lattices (i.e., without the implication or other non-lattice operations) that is an identity in all partition lattices (or lattices of equivalence relations) is actually a general lattice-theoretic identity. Hence the logic taking models in all partition algebras $\Pi(U)$ only became interesting by moving beyond the lattice operations on partitions.

Throughout his career, Gian-Carlo Rota emphasized the analogies between the Boolean lattice of subsets of a set and the lattice of equivalence relations on a set. Partition logic, with the heavy emphasis on the analogies with subset logic, should be seen as a continuation of that Rota program. The closest earlier work in the vein of partition logic was indeed by Rota and colleagues [(Finberg, Mainetti, and Rota 1996),(Haiman 1985)], but it used the lattice of equivalence relations and did not define the partition implication (which would be the difference operation on equivalence relations) or other non-lattice operations. It was restricted to the important class of commuting equivalence relations (Dubreil and Dubreil-Jacotin 1939) where identities hold which are not general lattice-theoretic identities.

In sum, the subset interpretation of ordinary logic (so the subset-partition duality would come into play), the turning of the lattice of partitions right side up, and the introduction of the non-lattice operations (particularly the implication) were all important in the development of partition logic.

References

Awodey, S. (2006). *Category Theory.* Oxford: Clarendon Press.

Birkhoff, G. (1948). *Lattice Theory.* New York: American Mathematical Society.

Boole, G. (1854). *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities.* Cambridge: Macmillan and Co.

Britz, T., M. Mainetti, and L. Pezzoli (2001). Some operations on the family of equivalence relations. In H. Crapo and D. Senato (Eds.), *Algebraic Combinatorics and Computer Science: A Tribute to Gian-Carlo Rota*, pp. 445–59. Milano: Springer.

Church, A. (1956). *Introduction to Mathematical Logic.* Princeton: Princeton University Press.

Dubreil, P. and M.-L. Dubreil-Jacotin (1939). Théorie algébrique des relations d'équivalence. *J. de Mathématique 18*, 63–95.

Ellerman, D. (2006). A theory of adjoint functors with some thoughts on their philosophical significance. In G. Sica (Ed.), *What is Category Theory?*, pp. 127–183. Milan: Polimetrica.

Ellerman, D. (2009). Counting distinctions: On the conceptual foundations of Shannon's information theory. *Synthese 168*(1 (May)), 119–149.

Finberg, D., M. Mainetti, and G.-C. Rota (1996). The logic of commuting equivalence relations. In A. Ursini and P. Agliano (Eds.), *Logic and Algebra*, pp. 69–96. New York: Marcel Dekker.

Fitting, M. C. (1969). *Intuitionistic Logic, Model Theory, and Forcing.* Amsterdam: North-Holland.

Grätzer, G. (2003). *General Lattice Theory (2nd ed.).* Boston: Birkhäuser Verlag.

Gödel, K. (1933). Zur intuitionistischen arithmetik und zahlentheorie. *Ergebnisse eines mathematischen Kolloquiums 4*, 34–38.

Haiman, M. (1985). Proof theory for linear lattices. *Advances in Mathematics 58*, 209–242.

Kung, J. P., G.-C. Rota, and C. H. Yan (2009). *Combinatorics: The Rota Way*. New York: Cambridge University Press.

Kuratowski, C. (1922). Sur l'operation Clos(A) de l'analysis situs. *Fundamenta Mathematicae 3*, 182–199.

Lawvere, F. W. (1986). Introduction. In F. W. Lawvere and S. Schanuel (Eds.), *Categories in Continuum Physics (Buffalo 1982) LNM 1174*, pp. 1–16. Springer-Verlag.

Lawvere, F. W. (1991). Intrinsic co-Heyting boundaries and the Leibniz rule in certain toposes. In A. Carboni, M. Pedicchio, and G. Rosolini (Eds.), *Category Theory (Como 1990) LNM 1488*, pp. 279–297. Springer-Verlag.

Lawvere, F. W. and R. Rosebrugh (2003). *Sets for Mathematics*. Cambridge: Cambridge University Press.

Lawvere, F. W. and S. Schanuel (1997). *Conceptual Mathematics: A first introduction to categories*. New York: Cambridge University Press.

Mac Lane, S. (1971). *Categories for the Working Mathematician*. New York: Verlag.

Mac Lane, S. and I. Moerdijk (1992). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. New York: Springer.

Ore, O. (1942). Theory of equivalence relations. *Duke Mathematical Journal 9*, 573–627.

Restall, G. (2000). *An Introduction to Substructural Logics*. London: Routledge.

Rockafellar, R. (1984). *Network Flows and Monotropic Optimization*. New York: John Wiley.

Smullyan, R. M. (1995 (1968)). *First-Order Logic*. New York: Dover.

Stanley, R. P. (1997). *Enumerative Combinatorics Vol. 1*. New York: Cambridge University Press.

van Dalen, D. (2001). Intuitionistic logic. In L. Goble (Ed.), *The Blackwell Guide to Philosophical Logic*, pp. 224–257. Oxford: Blackwell.

Whitman, P. (1946). Lattices, equivalence relations, and subgroups. *Bull. Am. Math. Soc. 52*, 507–522.

Wilson, R. J. (1972). *Introduction to Graph Theory*. London: Longman.