Operational understanding of the covariance of classical electrodynamics

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Abstract

It is common in the literature on classical electrodynamics and relativity theory that the transformation rules for the basic electrodynamical quantities are derived from the pre-assumption that the equations of electrodynamics are covariant against these—unknown—transformation rules. There are several problems to be raised concerning these derivations. This is, however, not our main concern in this paper. Even if these derivations were completely correct, they leave open the following fundamental question: Are the so-obtained transformation rules indeed identical with the true transformation rules of the fundamental electrodynamical quantities? In other words, is it indeed the case that the values calculated from the quantities in one inertial frame by means of the transformation rules we derived are equal to the values of the same quantities obtained by the same operations with the same measuring equipments when they are co-moving with the other inertial frame?

This is of course an empirical question. In this paper, we will investigate the problem in a purely theoretical framework by applying what J. S. Bell calls "Lorentzian pedagogy"—according to which the laws of physics in any one reference frame account for all physical phenomena. We will show that the transformation rules of the electrodynamical quantities are indeed identical with the ones obtained by presuming the covariance of the equations of electrodynamics, and that the covariance is indeed satisfied. Beforehand, however, we need to clarify the operational definitions of the fundamental electrodynamical quantities. As we will see, these semantic issues are not as trivial as one might think.

1 Introduction

Consider two inertial frames of reference K and K'. Let $\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}$ denote the basic electrodynamical quantities, electric and magnetic field strengths and source densities, obtainable by means of measuring equipments co-moving with K; and let $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$ be the same electrodynamical quantities in K', that is, the quantities obtainable by means of the same operations with the same measuring equipments when they are co-moving with K'.

Transformation rule for the electrodynamical quantities is meant to be a one-to-one functional relation expressing the values $\mathbf{E}'(A), \mathbf{B}'(A), \varrho'(A), \mathbf{j}'(A)$ in an arbitrary space-time point A in terms of the values $\mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A)$ in the same space-time point, and vice versa. It is a contingent fact of the physical world whether such a relationship exists and what it looks like.

A system of equations is said to be *covariant* against these transformation rules if expressing variables $\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}$ in the equations by means of $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$ (in conjunction with similar transformations concerning the kinematic quantities) we obtain equations of exactly the same form in the primed variables as the original equations in the original variables.

It is common in the literature on classical electrodynamics (ED) and relativity theory that the transformation rules for the basic electrodynamical quantities are derived from the pre-assumption that the equations of electrodynamics are covariant against these—unknown—transformation rules. Among those with which we are acquainted, there are basically two major versions of these derivations, which are briefly summarized in the Appendix. There are several problems to be raised concerning these derivations, and certain steps are questionable. This is however not our main concern in this paper. No matter whether these derivations are regarded as strictly logical or heuristic ways from the hypothesis that the equations of electrodynamics are covariant to the transformation rules; the following question remains open:

(Q) Are the so-obtained transformation rules indeed identical with the *true* transformation rules of the fundamental electrodynamical quantities?

In other words: is it indeed the case that the values obtained from $\mathbf{E}(A), \mathbf{B}(A), \varrho(A), \mathbf{j}(A)$ by means of the transformation rules we derived are equal to the real $\mathbf{E}'(A), \mathbf{B}'(A), \varrho'(A), \mathbf{j}'(A)$, that is, the quantities obtained by the same operations with the same measuring equipments when they are co-moving with K', in the same space-time point A?

Ultimately this is an empirical question. It is however a natural idea to apply what J. S. Bell (1987, p. 77) calls "Lorentzian pedagogy", according to which "the laws of physics in any one reference frame account for all physical phenomena, including the observations of moving observers". That is to say, the laws of physics that are valid in any one reference frame, say in K, must account for the behaviors of the moving measuring equipments and the results of all measuring operations, therefore must determine the answer to our question (Q).

The answer can be given by the laws of physics only if the question is properly formulated. We must clarify what measuring equipments and etalons are used in the empirical definitions of the electrodynamical quantities; and we must be able to tell when two measuring equipments are the same, except that they are moving, as a whole, relative to each other—one is at rest relative to K, the other is at rest relative to K'. Similarly, we must be able to tell when two operational procedures performed by the two observers are the "same", in spite of the prima facie fact that the procedure performed in K' obviously differs from the one performed in K. In order to compare these procedures, first of all, we must know what the procedures exactly are. All in all, a correct answer to question (Q) can be given only on the bases of *a coherent system of precise operational definitions* of the quantities in question. Interestingly, there is no explicit discussion of these issues in the standard literature on ED and special relativity; although, as we will see, none of these issues are as trivial as one might think.

Thus, accordingly, in the first part of the paper we clarify the operational definitions of the electrodymical quantities and formulate what ED in one inertial frame of reference exactly asserts in terms of the quantities so defined. In the second part, applying the "Lorentzian pedagogy", on the basis of the laws of ED in the "rest" frame, we derive what a moving observer must see in terms of the "rest" frame quantities when repeats the same operational procedures in the "moving" frame. In this way, we obtain the transformation rules of the electrodynamical quantities; that is to say, we derive the transformation rules from the precise operational definitions of the quantities and from the laws of ED in one single inertial frame of reference, *without of the pre-assumption that the equations are covariant* against these transformation rules.

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Throughout it will be assumed that space and time coordinates are already defined in all inertial frames of reference; that is, in an arbitrary inertial frame K, space tags $\mathbf{r}(A) = (x(A), y(A), z(A)) \in \mathbb{R}^3$ and a time tag $t(A) \in \mathbb{R}$ are assigned to every event A—by means of some empirical operations.[1] We also assume that the assignment is mutually unambiguous, such that there is a one to one correspondence between the space and time tags in arbitrary two inertial frames of reference K and K'; that is, the tags $(x'(A), y'(A), z'(A), t'(A))$ can be expressed by the tags $(x(A), y(A), z(A), t(A))$, and vice versa. The concrete form of this functional relation is an empirical question. In this paper, we will take it for granted that this functional relation is the well-known Lorentz transformation; and the calculations, particularly in section 5, will rest heavily on this assumption. It must be emphasized however that we stipulate the Lorentz transformation of the kinematical quantities as an *empirical* fact, without suggesting that the usual *derivations* of these transformation rules from the RP/constancy of the speed of light are unproblematic. In fact, these derivations raise similar questions concerning the kinematical quantities. In this paper, however, we focus our analysis only on the electrodynamical quantities.

It must be also noted that the transformation of the kinematical quantities, alone, does not determine the transformation of the electrodynamical quantities. As we will see, the latter is determined by the kinematical Lorentz transformation in conjunction with the operational definitions of the electrodynamical quantities and some empirical facts, first of all the relativistic version of the Lorentz equation of motion.

Below we recall the most important formulas we will use. For the sake of simplicity, we assume the standard situation: K' is moving along the x-axis with velocity $\mathbf{V} = (V, 0, 0)$ relative to K, the corresponding axises are parallel and the two origins coincide at time 0.[2] Throughout the paper we will use the following notations: $\gamma(\ldots) = \left(1 - \frac{(\ldots)^2}{c^2}\right)^{-\frac{1}{2}}$ and $\gamma = \gamma(V)$.

[1] In fact, to give precise empirical definitions of the basic spatio-temporal quantities in physics is not a trivial problem (Szabó 2009).

[2] All "vectors" are meant to be in $\mathbb{R}^3$; boldface letters $\mathbf{r}, \mathbf{v}, \mathbf{E} \ldots$ simply denote vector matrices.

The connection between the space and time tags of an event A in K and K' is the following:

$$x'(A) = \gamma(x(A) - Vt(A)) \tag{1}$$
$$y'(A) = y(A) \tag{2}$$
$$z'(A) = z(A) \tag{3}$$
$$t'(A) = \gamma\left(t(A) - c^{-2}Vx(A)\right) \tag{4}$$

Let A be an event on the worldline of a particle. For the velocity of the particle at A we have:

$$v'_x(A) = \frac{v_x(A) - V}{1 - c^{-2}v_x(A)V} \tag{5}$$
$$v'_y(A) = \frac{\gamma^{-1}v_y(A)}{1 - c^{-2}v_x(A)V} \tag{6}$$
$$v'_z(A) = \frac{\gamma^{-1}v_z(A)}{1 - c^{-2}v_x(A)V} \tag{7}$$

We shall use the inverse transformation in the following special case:

$$\mathbf{v}'(A) = (v', 0, 0) \quad \mapsto \quad \mathbf{v}(A) = \left(\frac{v' + V}{1 + c^{-2}v'V}, 0, 0\right) \tag{8}$$
$$\mathbf{v}'(A) = (0, 0, v') \quad \mapsto \quad \mathbf{v}(A) = (V, 0, \gamma v') \tag{9}$$

The transformation rule of acceleration is much more complex, but we need it only for $\mathbf{v}'(A) = (0, 0, 0)$:

$$a'_x(A) = \gamma^3 a_x(A) \tag{10}$$
$$a'_y(A) = \gamma^2 a_y(A) \tag{11}$$
$$a'_z(A) = \gamma^2 a_z(A) \tag{12}$$

We will also need the y-component of acceleration in case of $\mathbf{v}'(A) = (0, 0, v')$:

$$a'_y(A) = \gamma^2 a_y(A) \tag{13}$$

2 Operational definitions of electrodynamical quantities in K

Now we turn to the operational definitions of the fundamental electrodynamical quantities in a single reference frame K and to the basic observational facts about these quantities.

The operational definition of a physical quantity requires the specification of *etalon* physical objects and standard physical processes by means of which the value of the quantity is ascertained. In case of electrodynamical quantities the only "device" we need is a point-like test particle, and the standard measuring procedures by which the kinematical properties of the test particle are ascertained.

So, assume we have chosen an *etalon* test particle, and let $\mathbf{r}^{etalon}(t)$, $\mathbf{v}^{etalon}(t)$, $\mathbf{a}^{etalon}(t)$ denote its position, velocity and acceleration at time t. It is assumed that we are able to set the *etalon* test particle into motion with arbitrary velocity $\mathbf{v}^{etalon} < c$ at arbitrary location. We will need more "copies" of the *etalon* test particle:

Definition (D0) A particle e is called *test particle* if for all $\mathbf{r}$ and t

$$\mathbf{v}^{e}(t)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{v}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{14}$$

implies

$$\mathbf{a}^{e}(t)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{a}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{15}$$

(The "restriction signs" refer to *physical* situations; for example, $|_{\mathbf{r}^{e}(t)=\mathbf{r}}$ indicates that the test particle e is at point $\mathbf{r}$ at time t.)

Note, that some of the definitions and statements below require the existence of many test particles; which is, of course, a matter of empirical fact, and will be provided by (E0) below.

First we define the electric and magnetic field strengths. The only measuring device we need is a test particle being at rest relative to K.

Definition (D1) *Electric field strength* at point $\mathbf{r}$ and time t is defined as the acceleration of an arbitrary test particle e, such that $\mathbf{r}^{e}(t) = \mathbf{r}$ and $\mathbf{v}^{e}(t) = 0$:

$$\mathbf{E}(\mathbf{r}, t) \stackrel{def}{=} \mathbf{a}^{e}(t)|_{\mathbf{r}^{e}(t)=\mathbf{r};\, \mathbf{v}^{e}(t)=0} \tag{16}$$

Magnetic field strength is defined by means of how the acceleration $\mathbf{a}^{e}$ of the rest test particle changes with an infinitesimal perturbation of its state of rest, that is, if an infinitesimally small velocity $\mathbf{v}^{e}$ is imparted to the particle. Of course, we cannot perform various small perturbations simultaneously on one and the same rest test particle, therefore we perform the measurements on many rest test particles with various small perturbations. Let $\delta \subset \mathbb{R}^{3}$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^{3}$. First we define the following function:

$$\begin{aligned} \mathbf{U}^{\mathbf{r},t} \quad : \quad & \mathbb{R}^{3} \supset \delta \to \mathbb{R}^{3} \\ & \mathbf{U}^{\mathbf{r},t}(\mathbf{v}) \stackrel{def}{=} \mathbf{a}^{e}(t)|_{\mathbf{r}^{e}(t)=\mathbf{r};\, \mathbf{v}^{e}(t)=\mathbf{v}} \end{aligned} \tag{17}$$

Obviously, $\mathbf{U}^{\mathbf{r},t}(0) = \mathbf{E}(\mathbf{r}, t)$.

Definition (D2) *Magnetic field strength* at point $\mathbf{r}$ and time t is

$$\mathbf{B}(\mathbf{r},t) \stackrel{def}{=} \left. \begin{pmatrix} \partial_{v_z} U_y^{\mathbf{r},t} \\ \partial_{v_x} U_z^{\mathbf{r},t} \\ \partial_{v_y} U_x^{\mathbf{r},t} \end{pmatrix} \right|_{\mathbf{v}=0} \tag{18}$$

Practically it means that one can determine the value of $\mathbf{B}(\mathbf{r},t)$, with arbitrary precision, by means of measuring the accelerations of a few test particles of velocity $\mathbf{v}^e \in \delta$.

Next we introduce the concepts of source densities:

Definition (D3)

$$\varrho(\mathbf{r},t) \stackrel{def}{=} \nabla \cdot \mathbf{E}(\mathbf{r},t) \tag{19}$$

$$\mathbf{j}(\mathbf{r},t) \stackrel{def}{=} c^2 \nabla \times \mathbf{B}(\mathbf{r},t) - \partial_t \mathbf{E}(\mathbf{r},t) \tag{20}$$

are called *active electric charge density* and *active electric current density,* respectively.

A simple consequence of the *definitions* is that a continuity equation holds for ϱ and $\mathbf{j}$:

Theorem 1.

$$\partial_t \varrho(\mathbf{r},t) + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0 \tag{21}$$

Remark 1. In our construction, the two Maxwell equations (19)–(20), are mere *definitions* of the concepts of active electric charge density and active electric current density. They do not contain information whatsoever about how "matter produces electromagnetic field". And it is not because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are, of course, "unspecified distributions" in these "general laws", but because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ cannot be specified prior to or at least independently of the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. Again, because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are just abbreviations, standing for the expressions on the right hand sides of (19)–(20). In other words, any statement about the "charge distribution" will be a statement about $\nabla \cdot \mathbf{E}$, and any statement about the "current distribution" will be a statement about $c^2\nabla \times \mathbf{B} - \partial_t \mathbf{E}$.

The minimal claim is that this is a possible coherent construction. Though we must add: equations (19)–(20) could be seen as contingent physical laws about the relationship between the charge and current distributions and the electromagnetic field, only if we had an independent empirical definition of charge. However, we do not see how such a definition is possible, without encountering circularities. (Also see Remark 2) ⌟

The operational definitions of the field strengths and the source densities are based on the kinematical properties of the test particles. The following definition describes the concept of a charged point-like particle, in general.

Definition (D4) A particle b is called *charged point-particle* of *specific passive electric charge* π^b and of *active electric charge* α^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned}\gamma\left(\mathbf{v}^b(t)\right)\mathbf{a}^b(t) &= \pi^b\left\{\mathbf{E}\left(\mathbf{r}^b(t),t\right)+\mathbf{v}^b(t)\times\mathbf{B}\left(\mathbf{r}^b(t),t\right)\right.\\ &\quad\left.-c^{-2}\mathbf{v}^b(t)\left(\mathbf{v}^b(t)\cdot\mathbf{E}\left(\mathbf{r}^b(t),t\right)\right)\right\}\end{aligned} \tag{22}$$

2. If it is the only particle whose worldline intersects a given space-time region Λ, then for all $(\mathbf{r},t)\in\Lambda$ the source densities are of the following form:

$$\varrho(\mathbf{r},t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right) \tag{23}$$

$$\mathbf{j}(\mathbf{r},t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right)\mathbf{v}^b(t) \tag{24}$$

where $\mathbf{r}^b(t)$, $\mathbf{v}^b(t)$ and $\mathbf{a}^b(t)$ are the particle's position, velocity and acceleration. The ratio $\mu^b \overset{def}{=} \alpha^b/\pi^b$ is called the *electric inertial rest mass* of the particle.

Remark 2. Of course, (22) is equivalent to the standard form of the Lorentz equation:

$$\frac{d}{dt}\left(\gamma(\mathbf{v}(t))\,\mathbf{v}(t)\right)=\pi\left\{\mathbf{E}(\mathbf{r}(t),t)+\mathbf{v}(t)\times\mathbf{B}(\mathbf{r}(t),t)\right\} \tag{25}$$

with $\pi = q/m$ in the usual terminology, where q is the passive electric charge and m is the inertial (rest) mass of the particle—that is why we call π *specific* passive electric charge. Nevertheless, it must be clear that for all charged point-particles we introduced *two independent*, empirically meaningful and experimentally testable quantities: specific passive electric charge π and active electric charge α. There is no universal law-like relationship between these two quantities: the ratio between them varies from particle to particle. In the traditional sense, this ratio is, however, nothing but the particle's rest mass.

We must emphasize that the concept of mass so obtained, as defined by only means of electrodynamical quantities, is essentially related to ED, that is to say, to electromagnetic interaction. There seems no way to give a consistent and non-circular operational definition of inertial mass in general, independently of the context of a particular type of physical interaction. Without entering here into the detailed discussion of the problem, we only mention that, for example, Weyl's commonly accepted definition (Jammer 2000, pp. 8–10) and all similar definitions based on the conservation of momentum in particle collisions suffer from the following difficulty. There is no "collision" as a purely "mechanical" process. During a collision the particles are moving in a physical field—or fields—of interaction. Therefore: 1) the system of particles, separately, cannot be regarded as a closed system; 2) the inertial properties of the particles, in fact, reveal themselves in the interactions with the field. Thus, the concepts of inertial rest mass belonging to different interactions differ from each other; whether they are equal (proportional) to each other is a matter of contingent fact of nature. ⌟

Remark 3. The choice of the *etalon* test particle is, of course, a matter of convention, just as the definitions (D0)–(D4) themselves. It is important to note that all these conventional factors play a constitutive role in the fundamental concepts of ED (Reichenbach 1965). With these choices we not only make semantic conventions determining the meanings of the terms, but also make a decision about the body of concepts by means of which we grasp physical reality. There are a few things, however, that must be pointed out:

(a) This kind of conventionality does not mean that the physical quantities defined in (D0)–(D4) cannot describe *objective* features of physical reality. It only means that we make a decision which objective features of reality we are dealing with. With another body of conventions we have another body of physical concepts/physical quantities and another body of empirical facts.

(b) On the other hand, it does not mean either that our knowledge of the physical world would not be objective but a product of our conventions. If two theories obtained by starting with two different bodies of conventions are complete enough accounts of the physical phenomena, then they describe the same reality, expressed in terms of different physical quantities. Let us spell out an example: Definition (20) is entirely conventional—no objective fact of the world determines the formula on the right hand side. Therefore, we could make another choice, say,

$$\mathbf{j}_\Theta(\mathbf{r},t) \stackrel{def}{=} \Theta^2 \nabla \times \mathbf{B}(\mathbf{r},t) - \partial_t \mathbf{E}(\mathbf{r},t) \tag{26}$$

with some $\Theta \neq c$. At first sight, one might think that this choice will alter the speed of electromagnetic waves. This is however not the case. It will be an empirical fact *about* $\mathbf{j}_\Theta(\mathbf{r},t)$ that if a particle b is the only one whose worldline intersects a given space-time region Λ, then for all $(\mathbf{r},t) \in \Lambda$

$$\begin{aligned} \mathbf{j}_\Theta(\mathbf{r},t) &= \alpha^b \delta\left(\mathbf{r} - \mathbf{r}^b(t)\right) \mathbf{v}^b(t) \\ &\quad + \left(\Theta^2 - c^2\right) \nabla \times \mathbf{B}(\mathbf{r},t) \end{aligned} \tag{27}$$

Now, consider a region where there is no particle. Taking into account (27), we have (30)–(31) and

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0 \tag{28}$$

$$\Theta^2 \nabla \times \mathbf{B}(\mathbf{r},t) - \partial_t \mathbf{E}(\mathbf{r},t) = \left(\Theta^2 - c^2\right) \nabla \times \mathbf{B}(\mathbf{r},t) \tag{29}$$

which lead to the usual wave equation with propagation speed c. (Of course, in this particular example, one of the possible choices, namely $\Theta = c$, is distinguished by its simplicity. Note, however, that simplicity is not an epistemologically interpretable notion.) ⌟

3 Empirical facts of electrodynamics

Both "empirical" and "fact" are used in different senses. Statements (E0)–(E4) below are universal generalizations, rather than statements of particular observations. Nevertheless we call them "empirical facts", by which we simply mean that they are truths which can be acquired by *a posteriori* means. Normally, they can be considered as laws obtained by inductive generalization; statements the truths of which can be, in principle, confirmed empirically.

On the other hand, in our context, it is not important how these statements are empirically confirmed. (E0)–(E4) can be regarded as axioms of the Maxwell–Lorentz theory in K. What is important for us is that from these *axioms*, in conjunction with the theoretical representations of the measurement operations, there follow assertions about what the moving observer in K' observes. Section 5 will be concerned with these consequences.

(E0) There exist many enough test particles and we can settle them into all required positions and velocities.

Consequently, (D1)–(D4) are sound definitions. From observations about $\mathbf{E}$, $\mathbf{B}$ and the charged point-particles, we have further empirical facts:

(E1) In all situations, the electric and magnetic field strengths satisfy the following two Maxwell equations:

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \tag{30}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) + \partial_t \mathbf{B}(\mathbf{r}, t) = 0 \tag{31}$$

(E2) Each particle is a charged point-particle, satisfying (D4) with some specific passive electric charge π and active electric charge α. This is also true for the test particles, with—as follows from the definitions—specific passive electric charge $\pi = 1$.[3]

(E3) If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Λ, then for all $(\mathbf{r}, t) \in \Lambda$ the source densities are:

$$\varrho(\mathbf{r}, t) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}(t)\right) \tag{32}$$

$$\mathbf{j}(\mathbf{r}, t) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}(t)\right) \mathbf{v}^{b_i}(t) \tag{33}$$

Putting facts (E1)–(E3) together, we have the coupled Maxwell–Lorentz

[3] We take it true that the relativistic Lorentz equation is empirically confirmed. (Cf. Huang 1993)

equations:

$$
\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r},t) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}-\mathbf{r}^{b_i}(t)\right) && (34)\\
c^2\nabla\times\mathbf{B}(\mathbf{r},t)-\partial_t\mathbf{E}(\mathbf{r},t) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}-\mathbf{r}^{b_i}(t)\right)\mathbf{v}^{b_i}(t) && (35)\\
\nabla\cdot\mathbf{B}(\mathbf{r},t) &= 0 && (36)\\
\nabla\times\mathbf{E}(\mathbf{r},t)+\partial_t\mathbf{B}(\mathbf{r},t) &= 0 && (37)\\
\gamma\left(\mathbf{v}^{b_i}(t)\right)\mathbf{a}^{b_i}(t) &= \pi^{b_i}\Big\{\mathbf{E}\left(\mathbf{r}^{b_i}(t),t\right)+\mathbf{v}^{b_i}(t)\times\mathbf{B}\left(\mathbf{r}^{b_i}(t),t\right)\\
&\quad -c^{-2}\mathbf{v}^{b_i}(t)\left(\mathbf{v}^{b_i}(t)\cdot\mathbf{E}\left(\mathbf{r}^{b_i}(t),t\right)\right)\Big\} && (38)\\
&\quad (i=1,2,\ldots n)
\end{aligned}
$$

These are the fundamental equations of ED, describing an interacting system of n particles and the electromagnetic field.

Remark 4. Without entering into the details of the problem of classical charged particles (Frisch 2005; Rohrlich 2007; Muller 2007), it must be noted that the Maxwell–Lorentz equations (34)–(38), exactly in this form, have *no* solution. The reason is the following. In the Lorentz equation of motion (22), a small but extended particle can be described with a good approximation by one single specific passive electric charge π^b and one single trajectory $\mathbf{r}^b(t)$. In contrast, however, a similar "idealization" in the source densities (23)–(24) leads to singularities; the field is singular at precisely the points where the coupling happens: on the trajectory of the particle.

The generally accepted answer to this problem is that (23)–(24) should not be taken literally. Due to the inner structure of the particle, the real source densities are some "smoothed out" Dirac deltas. Instead of (23)–(24), therefore, we have some more general equations

$$
\begin{aligned}
[\varrho(\mathbf{r},t)] &= \mathcal{R}^b\left[\mathbf{r}^b(t)\right] && (39)\\
[\mathbf{j}(\mathbf{r},t)] &= \mathcal{J}^b\left[\mathbf{r}^b(t)\right] && (40)
\end{aligned}
$$

where $\mathcal{R}^b$ and $\mathcal{J}^b$ are, generally non-linear, operators providing functional relationships between the particle's trajectory $\left[\mathbf{r}^b(t)\right]$ and the source density functions $[\varrho(\mathbf{r},t)]$ and $[\mathbf{j}(\mathbf{r},t)]$. (Notice that (23)–(24) serve as example of such equations.) The concrete forms of equations (39)–(40) are determined by the physical laws of the internal world of the particle—which are, supposedly, outside of the scope of ED. At this level of generality, the only thing we can say is that, for a "point-like" (localized) particle, equations (39)–(40) must be something very close to—but not identical with—equations (23)–(24). With this explanation, for the sake of simplicity we leave the Dirac deltas in the equations. Also, in some of our statements and calculations the Dirac deltas are essentially used; for example, (E3) and, partly, Theorem 7 and 9 would not be true without the exact point-like source densities (23)–(24). But a little reflection shows that the statements in question remain approximately true if the particles are approximately point-like, that is, if equations (39)–(40) are close enough to equations (23)–(24). To be noted that what is actually essential in (23)–(24) is not the point-likeness of the particle, but its stability; no matter how the system moves, it remains a localized object. ⌟

4 Operational definitions of electrodynamical quantities in K'

So far we have only considered ED in a single frame of reference K. Now we turn to the question of how a moving observer describes the same phenomena in K'. The observed phenomena are the same, but the measuring equipments by means of which the phenomena are observed are not entirely the same; instead of being at rest in K, they are co-moving with K'.

Accordingly, we will repeat the operational definitions (D0)–(D4) with the following differences:

1. The "rest test particles" will be at rest relative to reference frame K', that is, *in motion with velocity* $\mathbf{V}$ relative to K.

2. The measuring equipments by means of which the kinematical quantities are ascertained—say, the measuring rods and clocks—will be at rest relative to K', that is, *in motion with velocity* $\mathbf{V}$ relative to K. In other words, the kinematical quantities $t, \mathbf{r}, \mathbf{v}, \mathbf{a}$ in definitions (D0)–(D4) will be *replaced with*—not expressed in terms of— $t', \mathbf{r}', \mathbf{v}', \mathbf{a}'$.

Definition (D0′) Particle e is called *(test particle)′* if for all $\mathbf{r}'$ and t'

$$\mathbf{v}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{v}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{41}$$

implies

$$\mathbf{a}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{a}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{42}$$

A (test particle)′ e moving with velocity $\mathbf{V}$ relative to K is at rest relative to K', that is, $\mathbf{v}'^{e} = 0$. Accordingly:

Definition (D1′) *(Electric field strength)′* at point $\mathbf{r}'$ and time t' is defined as the acceleration of an arbitrary (test particle)′ e, such that $\mathbf{r}'^{e}(t) = \mathbf{r}'$ and $\mathbf{v}'^{e}(t') = 0$:

$$\mathbf{E}'\left(\mathbf{r}', t'\right) \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=0} \tag{43}$$

Similarly, (magnetic field strength)′ is defined by means of how the acceleration $\mathbf{a}'^{e}$ of a rest (test particle)′—rest, of course, relative to K'—changes with a small perturbation of its state of motion, that is, if an infinitesimally small velocity

$\mathbf{v}'^{e}$ is imparted to the particle. Just as in (D2), let $\delta' \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. We define the following function:

$$\begin{aligned} \mathbf{U}'^{\mathbf{r}',t'} \quad : \quad & \mathbb{R}^3 \supset \delta' \to \mathbb{R}^3 \\ & \mathbf{U}'^{\mathbf{r}',t'}(\mathbf{v}') \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=\mathbf{v}'} \end{aligned} \tag{44}$$

Definition (D2′) *(Magnetic field strength)′* at point $\mathbf{r}'$ and time t' is

$$\mathbf{B}'(\mathbf{r}',t') \stackrel{def}{=} \left. \begin{pmatrix} \partial_{v'_z} U'^{\mathbf{r}',t'}_{y} \\ \partial_{v'_x} U'^{\mathbf{r}',t'}_{z} \\ \partial_{v'_y} U'^{\mathbf{r}',t'}_{x} \end{pmatrix} \right|_{\mathbf{v}'=0} \tag{45}$$

Definition (D3′)

$$\varrho'\left(\mathbf{r}',t'\right) \stackrel{def}{=} \nabla \cdot \mathbf{E}'\left(\mathbf{r}',t'\right) \tag{46}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) \stackrel{def}{=} c^2 \nabla \times \mathbf{B}'\left(\mathbf{r}',t'\right) - \partial_{t'} \mathbf{E}'\left(\mathbf{r}',t'\right) \tag{47}$$

are called *(active electric charge density)′* and *(active electric current density)′*, respectively.

Of course, we have:

Theorem 2.

$$\partial_{t'} \varrho'\left(\mathbf{r}',t'\right) + \nabla \cdot \mathbf{j}'\left(\mathbf{r}',t'\right) = 0 \tag{48}$$

Definition (D4′) A particle is called *(charged point-particle)′* of *(specific passive electric charge)′* π'^{b} and of *(active electric charge)′* α'^{b} if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \gamma\left(\mathbf{v}'^{b}\left(t'\right)\right)\mathbf{a}'^{b}(t') &= \pi'^{b}\Big\{\mathbf{E}'\left(\mathbf{r}'^{b}\left(t'\right),t'\right) + \mathbf{v}'^{b}\left(t'\right) \times \mathbf{B}'\left(\mathbf{r}'^{b}\left(t'\right),t'\right) \\ &\quad - c^{-2}\mathbf{v}'^{b}\left(t'\right)\left(\mathbf{v}'^{b}\left(t'\right) \cdot \mathbf{E}'\left(\mathbf{r}'^{b}\left(t'\right),t'\right)\right)\Big\} \end{aligned} \tag{49}$$

2. If it is the only particle whose worldline intersects a given space-time region Λ', then for all $(\mathbf{r}',t') \in \Lambda'$ the (source densities)′ are of the following form:

$$\varrho'\left(\mathbf{r}',t'\right) = \alpha'^{b}\delta\left(\mathbf{r}' - \mathbf{r}'^{b}\left(t'\right)\right) \tag{50}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) = \alpha'^{b}\delta\left(\mathbf{r}' - \mathbf{r}'^{b}\left(t'\right)\right)\mathbf{v}'^{b}\left(t'\right) \tag{51}$$

where $\mathbf{r}'^{b}\left(t'\right)$, $\mathbf{v}'^{b}\left(t'\right)$ and $\mathbf{a}'^{b}\left(t'\right)$ is the particle's position, velocity and acceleration in K'. The ratio $\mu'^{b} \stackrel{def}{=} \alpha'^{b}/\pi'^{b}$ is called the *(electric inertial rest mass)′* of the particle.

Remark 5. It is worthwhile to make a few remarks about some epistemological issues:

(a) The physical quantities defined in (D1)–(D4) *differ* from the physical quantities defined in (D1')–(D4'), simply because the physical situation in which a test particle is at rest relative to K differs from the one in which it is co-moving with K' with velocity $\mathbf{V}$ relative to K; and, as we know *from the laws of ED in K*, this difference really matters.

Someone might object that if this is so then any two instances of the same measurement must be regarded as measurements of different physical quantities. For, if the difference in the test particle's velocity is enough reason to say that the two operations determine two different quantities, then, by the same token, two operations must be regarded as different operations—and the corresponding quantities as different physical quantities—if the test particle is at different points of space, or the operations simply happen at different moments of time. And this consequence, the objection goes, seems to be absurd: if it were true, then science would not be possible, because we would not have the power to make law-like assertions at all; therefore we must admit that empiricism fails to explain how natural laws are possible, and, as many argue, science cannot do without metaphysical pre-assumptions.

Our response to such an objections is the following. First, concerning the general epistemological issue, we believe, nothing disastrous follows from admitting that two phenomena observed at different place or at different time *are* distinct. And if they are stated as instances of the same phenomenon, this statement is not a logical or metaphysical necessity—derived from some logical/metaphysical pre-assumptions—but an ordinary scientific hypothesis obtained by induction and confirmed or disconfirmed together with the *whole* scientific theory. In fact, this is precisely the case with respect to the definitions of the fundamental electrodynamical quantities. For example, definition (D1) is in fact a family of definitions each belonging to a particular situation individuated by the space-time locus $(\mathbf{r}, t)$.
Second, the question of operational definitions of electrodynamical quantities first of all emerges not from an epistemological context, but from the context of a purely theoretical problem: what do the laws of physics in K say about question (Q)? In the next section, all the results of the measurement operations defined in (D1')–(D4') will be predicted from the laws of ED in K. And, ED itself says that some differences in the conditions are relevant from the point of view of the measured accelerations of the test particles, some others are not; some of the originally distinct quantities are contingently equal, some others not.

(b) From a mathematical point of view, both (D0)–(D4) and (D0')–(D4') are definitions. However, while the choice of the *etalon* test particle and definitions (D0)–(D4) are entirely *conventional*, there is no additional conventionality in (D0')–(D4'). The way in which we define the electrodynamical quantities in inertial frame K' automatically follows from (D0)–(D4) and from the question (Q) we would like to answer; since the question is about the "quantities obtained by the same operational procedures with the same measuring equipments when they are co-moving with K'".

(c) In fact, one of the constituents of the concepts defined in K' is not determined by the operational definitions in K. Namely, the notion of "the same operational procedures with the same measuring equipments when they are co-moving with K'". This is however not an additional freedom of conventionality, but a simple vagueness in our physical theories in K: the vagueness of the general concept of "the same system in the same situation, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K, that is, co-moving with reference frame K'" (Gömöri and Szabó 2011a). In any event, in our case, the notion of the only moving measuring device, that is, the notion of "a test particle at rest relative to K'" is quite clear. ⌟

5 Observations of moving observer

Now we have another collection of operationally defined notions, $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$, the concept of (charged point-particle)′ defined in the primed terms, and its properties π', α' and μ'. Normally, one should investigate these quantities experimentally and collect new empirical facts about both the relationships between the primed quantities and about the relationships between the primed quantities and the ones defined in (D1)–(D4). In contrast, we will continue our analysis in another way; following the "Lorentzian pedagogy", we will determine from the laws of physics in K what an observer co-moving with K' should observe. In fact, with this method, we will answer our question (Q), on the basis of the laws of ED in a single frame of reference. We will also see whether the basic equations (34)–(38) are covariant against these transformations.

Throughout the theorems below, it is important that when we compare, for example, $\mathbf{E}(\mathbf{r}, t)$ with $\mathbf{E}'(\mathbf{r}', t')$, we compare the values of the fields *in one and the same event*, that is, we compare $\mathbf{E}(\mathbf{r}(A), t(A))$ with $\mathbf{E}'(\mathbf{r}'(A), t'(A))$. For the sake of brevity, however, we omit the indication of this fact.

The first theorem trivially follows from the fact that the Lorentz transformations of the kinematical quantities are one-to-one:

Theorem 3. *A particle is a (test particle)′ if and only if it is a test particle.*

Consequently, we have many enough (test particles)′ for definitions (D1′)–(D4′); and each is a charged point-particle satisfying the Lorentz equation (22) with specific passive electric charge $\pi = 1$.

Theorem 4.

$$E'_x = E_x \tag{52}$$
$$E'_y = \gamma\left(E_y - VB_z\right) \tag{53}$$
$$E'_z = \gamma\left(E_z + VB_y\right) \tag{54}$$

Proof. When the (test particle)′ is at rest relative to K', it is moving with velocity $\mathbf{v}^e = (V, 0, 0)$ relative to K. From (22) (with $\pi = 1$) we have

$$\begin{aligned} a_x^e &= \gamma^{-3} E_x && (55) \\ a_y^e &= \gamma^{-1}\left(E_y - VB_z\right) && (56) \\ a_z^e &= \gamma^{-1}\left(E_z + VB_y\right) && (57) \end{aligned}$$

Applying (10)–(12), we can calculate the acceleration $\mathbf{a}'^e$ in K', and, accordingly, we find

$$\begin{aligned} E'_x &= a'^e_x = \gamma^3 a^e_x = E_x && (58) \\ E'_y &= a'^e_y = \gamma^2 a^e_y = \gamma\left(E_y - VB_z\right) && (59) \\ E'_z &= a'^e_z = \gamma^2 a^e_z = \gamma\left(E_z + VB_y\right) && (60) \end{aligned}$$

□

Theorem 5.

$$\begin{aligned} B'_x &= B_x && (61) \\ B'_y &= \gamma\left(B_y + c^{-2} V E_z\right) && (62) \\ B'_z &= \gamma\left(B_z - c^{-2} V E_y\right) && (63) \end{aligned}$$

Proof. Consider for instance B'_x. By definition,

$$B'_x = \partial_{v'_z} U'^{\mathbf{r}',t'}_y \Big|_{\mathbf{v}'=0} \tag{64}$$

According to (44), the value of $U'^{\mathbf{r}',t'}_y(\mathbf{v}')$ is equal to

$$a'^e_y \Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\, \mathbf{v}'^e(t')=\mathbf{v}'} \tag{65}$$

that is, the y-component of the acceleration of a (test particle)′ e in a situation in which $\mathbf{r}'^e(t') = \mathbf{r}'$ and $\mathbf{v}'^e(t') = \mathbf{v}'$. Accordingly, in order to determine the partial derivative (64) we have to determine

$$\frac{d}{dw}\Big|_{w=0} \left(a'^e_y \Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\, \mathbf{v}'^e(t')=(0,0,w)} \right) \tag{66}$$

Now, according to (9), condition $\mathbf{v}'^e = (0, 0, w)$ corresponds to

$$\mathbf{v}^e = \left(V, 0, \gamma^{-1} w\right) \tag{67}$$

Substituting this velocity into (22), we have:

$$a_y^e = \sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}} \left(E_y + w\gamma^{-1} B_x - VB_z\right) \tag{68}$$

Applying (13), one finds:

$$\begin{aligned} a'^e_y &= \gamma^2 a^e_y = \gamma^2 \sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}} \left(E_y + w\gamma^{-1} B_x - VB_z\right) \\ &= \frac{\gamma}{\gamma(w)} \left(E_y + w\gamma^{-1} B_x - VB_z\right) && (69) \end{aligned}$$

Differentiating with respect to w at $w = 0$, we obtain

$$B'_x = B_x \tag{70}$$

The other components can be obtained in the same way. $\square$

Theorem 6.

$$\varrho' = \gamma\left(\varrho - c^{-2}Vj_x\right) \tag{71}$$

$$j'_x = \gamma\left(j_x - V\varrho\right) \tag{72}$$

$$j'_y = j_y \tag{73}$$

$$j'_z = j_z \tag{74}$$

Proof. Substituting $\mathbf{E}'$ and $\mathbf{B}'$ with (52)–(54) and (61)–(63), $\mathbf{r}$ and t with the inverse of (1)–(4), then differentiating the composite function and taking into account (19)–(20), we get (71)–(74). $\square$

Theorem 7. *A particle b is charged point-particle of specific passive electric charge π^b and of active electric charge α^b if and only if it is a (charged point-particle)′ of (specific passive electric charge)′ π'^b and of (active electric charge)′ α'^b, such that $\pi'^b = \pi^b$ and $\alpha'^b = \alpha^b$.*

Proof. First we prove (49). For the sake of simplicity, we will verify this in case of $\mathbf{v}'^b = (0, 0, w)$. We can use (68):

$$a^b_y = \pi^b\sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}}\left(E_y + w\gamma^{-1}B_x - VB_z\right) \tag{75}$$

From (13), (53), (61), and (63) we have

$$\begin{aligned} a'^b_y &= \pi^b\gamma(w)^{-1}\left(E'_y + wB'_x\right) \\ &= \left[\pi^b\gamma\left(\mathbf{v}'^b\right)^{-1}\left(\mathbf{E}' - c^{-2}v'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_y\Bigg|_{\mathbf{v}'^b=(0,0,w)} \end{aligned} \tag{76}$$

Similarly,

$$\begin{aligned} a'^b_x &= \pi^b\gamma(w)^{-1}\left(E'_x - wB'_y\right) \\ &= \left[\pi^b\gamma\left(\mathbf{v}'^b\right)^{-1}\left(\mathbf{E}' - c^{-2}\mathbf{v}'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_x\Bigg|_{\mathbf{v}'^b=(0,0,w)} \end{aligned} \tag{77}$$

$$\begin{aligned} a'^b_z &= \pi^b\gamma(w)^{-3}E'_z \\ &= \left[\pi^b\gamma\left(\mathbf{v}'^b\right)^{-1}\left(\mathbf{E}' - c^{-2}\mathbf{v}'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_z\Bigg|_{\mathbf{v}'^b=(0,0,w)} \end{aligned} \tag{78}$$

That is, (49) is satisfied, indeed.

In the second part, we show that (50)–(51) are nothing but (23)–(24) expressed in terms of $\mathbf{r}', t', \varrho'$ and $\mathbf{j}'$, with $\alpha'^b = \alpha^b$.

It will be demonstrated for a particle of trajectory $\mathbf{r}'^{b}\left(t'\right)=\left(wt',0,0\right)$. Applying (8), (23)–(24) have the following forms:

$$\begin{aligned}\varrho\left(\mathbf{r},t\right) &= \alpha^{b}\delta\left(x-\beta t\right)\delta\left(y\right)\delta\left(z\right) &(79)\\ \mathbf{j}\left(\mathbf{r},t\right) &= \alpha^{b}\delta\left(x-\beta t\right)\delta\left(y\right)\delta\left(z\right)\begin{pmatrix}\beta\\0\\0\end{pmatrix} &(80)\end{aligned}$$

where $\beta=\frac{w+V}{1+c^{-2}wV}$. $\mathbf{r},t,\varrho$ and $\mathbf{j}$ can be expressed with the primed quantities by applying the inverse of (1)–(4) and (71)–(74):

$$\begin{aligned}\gamma\left(\varrho'\left(\mathbf{r}',t'\right)+c^{-2}Vj'_{x}\left(\mathbf{r}',t'\right)\right) &= \alpha^{b}\delta\left(\gamma\left(x'+Vt'-\beta\left(t'+c^{-2}Vx'\right)\right)\right)\\ &\quad\times\delta\left(y'\right)\delta\left(z'\right) &(81)\\ \gamma\left(j'_{x}\left(\mathbf{r}',t'\right)+V\varrho'\left(\mathbf{r}',t'\right)\right) &= \alpha^{b}\delta\left(\gamma\left(x'+Vt'-\beta\left(t'+c^{-2}Vx'\right)\right)\right)\\ &\quad\times\delta\left(y'\right)\delta\left(z'\right)\beta &(82)\\ j'_{y}\left(\mathbf{r}',t'\right) &= 0 &(83)\\ j'_{z}\left(\mathbf{r}',t'\right) &= 0 &(84)\end{aligned}$$

One can solve this system of equations for ϱ' and j'_x:

$$\begin{aligned}\varrho'\left(\mathbf{r}',t'\right) &= \alpha^{b}\delta\left(x'-wt'\right)\delta\left(y'\right)\delta\left(z'\right) &(85)\\ \mathbf{j}'\left(\mathbf{r}',t'\right) &= \alpha^{b}\delta\left(x'-wt'\right)\delta\left(y'\right)\delta\left(z'\right)\begin{pmatrix}w\\0\\0\end{pmatrix} &(86)\end{aligned}$$

□

Theorem 8.

$$\begin{aligned}\nabla\cdot\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 &(87)\\ \nabla\times\mathbf{E}'\left(\mathbf{r}',t'\right)+\partial_{t'}\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 &(88)\end{aligned}$$

Proof. Expressing (30)–(31) in terms of $\mathbf{r}',t',\mathbf{E}'$ and $\mathbf{B}'$ by means of (1)–(4), (52)–(54) and (61)–(63), we have

$$\begin{aligned}\nabla\cdot\mathbf{B}'-c^{-2}V\left(\nabla\times\mathbf{E}'+\partial_{t'}\mathbf{B}'\right)_{x} &= 0 &(89)\\ \left(\nabla\times\mathbf{E}'+\partial_{t'}\mathbf{B}'\right)_{x}-V\nabla\cdot\mathbf{B}' &= 0 &(90)\\ \left(\nabla\times\mathbf{E}'+\partial_{t'}\mathbf{B}'\right)_{y} &= 0 &(91)\\ \left(\nabla\times\mathbf{E}'+\partial_{t'}\mathbf{B}'\right)_{z} &= 0 &(92)\end{aligned}$$

which is equivalent to (87)–(88), indeed. □

Theorem 9. *If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Λ', then for all $\left(\mathbf{r}',t'\right)\in\Lambda'$ the (source densities)′ are:*

$$\begin{aligned}\varrho'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) &(93)\\ \mathbf{j}'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) &(94)\end{aligned}$$

Proof. Due to Theorem 7, each (charged point-particle)′ is a charged point-particle with $\alpha'^{b} = \alpha^{b}$. Therefore, we only need to prove that equations (93)–(94) amount to (32)–(33) expressed in the primed variables. On the left hand side of (32)–(33), ϱ and $\mathbf{j}$ can be expressed by means of the inverse of (71)–(74); on the right hand side, we take $\alpha'^{b} = \alpha^{b}$, and apply the inverse of (1)–(4), just as in the derivation of (85)–(86). From the above, we obtain:

$$\begin{aligned}
\varrho'\left(\mathbf{r}',t'\right)+c^{-2}Vj'_{x}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\\
&\quad +c^{-2}V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v_x'^{b_i}\left(t'\right) \qquad (95)\\
j'_{x}\left(\mathbf{r}',t'\right)+V\varrho'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v_x'^{b_i}\left(t'\right)\\
&\quad +V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) \qquad (96)\\
j'_{y}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v_y'^{b_i}\left(t'\right) \qquad (97)\\
j'_{z}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v_z'^{b_i}\left(t'\right) \qquad (98)
\end{aligned}$$

Solving these linear equations for ϱ' and $\mathbf{j}'$ we obtain (93)–(94). □

Combining the results we obtained in Theorems 7–9, we have

$$\begin{aligned}
\nabla\cdot\mathbf{E}'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha'^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) \qquad (99)\\
c^{2}\nabla\times\mathbf{B}'\left(\mathbf{r}',t'\right)-\partial_{t'}\mathbf{E}'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha'^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) \qquad (100)\\
\nabla\cdot\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 \qquad (101)\\
\nabla\times\mathbf{E}'\left(\mathbf{r}',t'\right)+\partial_{t'}\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 \qquad (102)\\
\gamma\left(\mathbf{v}'^{b_i}\left(t'\right)\right)\mathbf{a}'^{b_i}(t') &= \pi'^{b_i}\Bigg\{\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\\
&\quad +\mathbf{v}'^{b_i}\left(t'\right)\times\mathbf{B}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\\
&\quad -\mathbf{v}'^{b_i}\left(t'\right)\frac{\mathbf{v}'^{b_i}\left(t'\right)\cdot\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)}{c^{2}}\Bigg\} \qquad (103)\\
&\qquad (i=1,2,\ldots n)
\end{aligned}$$

6 Are the textbook transformation rules true?

Our main concern in this paper was: On what grounds can the textbook transformation rules for the electrodynamical quantities—hence the hypothesis of covariance itself, from which the rules are routinely derived—be considered as empirically verified facts of the physical world? Now everything is at hand to declare that the textbook transformation rules are in fact true, at least in the sense that they are derivable from the laws of ED in a single frame of reference—*without the prior assumption of covariance*. For, Theorems 4 and 5 show the well-known transformation rules for the field variables. What Theorem 6 asserts is nothing but the well-known transformation rule for charge density and current density. Finally, Theorem 7 shows that a particle's electric specific passive charge, active charge and electric rest mass are invariant scalars.

At this point, having ascertained the transformation rules, we can declare that equations (99)–(103) are nothing but equations (34)–(38) expressed in the primed variables. At the same time, (99)–(103) are manifestly of the same form as (34)–(38). Therefore, we proved that the Maxwell–Lorentz equations are *indeed* covariant against the *real* transformations of the kinematical and electrodynamical quantities. In fact, we proved more:

- The Lorentz equation of motion (38) is covariant separately.
- The four Maxwell equations (34)–(37) constitute a covariant set of equations, separately from (38).
- (34)–(35) constitute a covariant set of equations, separately.
- (36)–(37) constitute a covariant set of equations, separately.

None of these statements follows automatically from the fact that (34)–(38) is a covariant system of equations (Gömöri and Szabó 2011a).

It is of interest to notice that all these results hinge on the *relativistic* version of the Lorentz equation, in particular, on the "relativistic mass-formula". Without factor $\gamma\left(\mathbf{v}^b\right)$ in (38), the proper transformation rules were different and the Maxwell equations were not covariant—against the proper transformations.

A final remark: what we proved is the *covariance* of the Maxwell–Lorentz equations. But, by no means we proved that the Maxwell–Lorentz electrodynamics satisfies the relativity principle. While it is true that—under quite general conditions—the relativity principle implies covariance; the opposite is not true, covariance is not sufficient for the relativity principle (Bell 1987; Gömöri–Szabó 2011a). Whether and in what sense electrodynamics satisfies the relativity principle is a more complex problem (Gömöri–Szabó 2011c).

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Appendix

There are two major versions of the textbook derivation of the transformation rules for electrodynamical quantities from the hypothesis of covariance. The first version follows Einstein's 1905 paper:

(1a) The transformation rules of electric and magnetic field strengths are derived from the presumption of the covariance of the homogeneous (with no sources) Maxwell equations.

(1b) The transformation rules of source densities are derived from the transformations of the field variables.

(1c) From the transformation rules of charge and current densities, it is derived that electric charge is an invariant scalar.

The second version is this:

(2a) The transformation rules of the charge and current densities are derived from some additional *assumptions*; typically from one of the followings:

(2a1) the invariance of electric charge (Jackson 1999, pp. 553–558)

(2a2) the current density is of form $\varrho \mathbf{u}(\mathbf{r}, t)$, where $\mathbf{u}(\mathbf{r}, t)$ is a velocity field (Tolman 1949, p. 85; Møller 1955, p. 140).

(2b) The transformation of the field strengths are derived from the transformation of ϱ and $\mathbf{j}$ and from the presumption of the covariance of the inhomogeneous Maxwell equations.

Unfortunately, with the only exception of (1b), none of the above steps is completely correct. Without entering into the details, let us mention that (2a1) and (2a2) both involve some further empirical information about the world, which does not follow from the simple assumption of covariance. Even in case of (1a) we must have the tacit assumption that zero charge and current densities go to zero charge and current densities during the transformation—otherwise the covariance of the homogeneous Maxwell equations would not follow from the assumed covariance of the Maxwell equations.

One encounters the next major difficulty in both (1a) and (2b): neither the homogeneous nor the inhomogeneous Maxwell equations determine the transformation rules of the field variables uniquely; $\mathbf{E}'$ and $\mathbf{B}'$ are only determined by $\mathbf{E}$ and $\mathbf{B}$ up to an arbitrary solution of the homogeneous equations.

Finally, let us mention a conceptual confusion that seems to be routinely overlooked in (1c), (2a1) and (2a2). There is no such thing as a simple relation between the scalar invariance of charge and the transformation of charge and current densities, as is usually claimed. For example, it is meaningless to say that

$$Q = \varrho \Delta W = Q' = \varrho' \Delta W' \tag{104}$$

where ΔW denotes a volume element, and

$$\Delta W' = \gamma \Delta W \tag{105}$$

Whose charge is Q, which remains invariant? Whose volume is ΔW and in what sense is that volume Lorentz contracted? In another form, in (2a2), whose velocity is $\mathbf{u}(\mathbf{r}, t)$?

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