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Part III Essay

Bell Correlations in Quantum Field Theory

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Preface

There is a sense in which the notion of entanglement captures much of the essence of quantum theory. In the first instance, it exposes the inherent nonlocality of the theory, which so repulsed Einstein and led to his allegations of the incompleteness of the theory. Secondly, entanglement is not merely an accidental feature of quantum theory, but is *generic* and even *intrinsic* in a number of ways, which we shall attempt to make precise in this essay.

The existence of quantum mechanical states of composite systems which maintain correlations through arbitrarily large spatial separations reveals an unsettling tension between quantum theory and special relativity. Einstein, among others, held his doctrine of *Trennungsprinzip* in the highest regard, and produced the famous EPR-paradox in defiance of quantum theory's version of "reality". The search for local hidden variable models underlying quantum mechanics met a strong arbitrator in J. S. Bell's Theorem, with impressive experiments by A. Aspect *et al.* [2] ruling largely against local realism. Quantum no-signaling theorems serve to ease the tension somewhat, but the question of whether quantum theory and special relativity are formally compatible remains an open one [27]. One might expect a resolution in *relativistic* quantum field theories, but as we shall see, spacelike Bell-correlations become pervasive, and force a closer scrutiny of the status of relativistic causality.

We will primarily be concerned with studying spacelike correlations in quantum field theory, with occasional visits to non-relativistic quantum mechanics when the need arises. Because the issues we hope to tackle are so subtle, we will find it useful to work within the rigorous framework of algebraic quantum field theory (AQFT). The first chapter of this essay will be a minimal summary of the requisite mathematical tools, along with a brief outline of AQFT. In the second chapter, we review a number of results (mainly due to Summers, Werner, Clifton and Halvorson) demonstrating that the violation of Bell inequalities in AQFT is "generic", vis-à-vis the choices of spacetime regions, observable quantities, and states. We will then explore the suprising consequences of two mathematical features of AQFT — the Reeh–Schlieder theorem, and the so-called Type III_1 factors — which illustrate just how deeply entrenched entanglement is.

The third chapter will address the issue of coexistence between AQFT and special relativity. We shall see that there are several inequivalent ways of formulating relativistic causality in AQFT, and consequently, several different ways (due to Clifton, Halvorson, Landsman and Butterfield) to address this question of coexistence. In these approaches, the advantage of the algebraic framework becomes clear, in that the notions of events, causal influences and probabilities can be made precise and so studied fruitfully.

This essay aims to be both expository and mathematically precise — two styles which are usually incompatible. The compromise will be an omission of the detailed proofs of most theorems, which can often take up the content of an entire paper while delivering a message that can be succinctly presented. Finally, the author should mention that the overall presentation of this essay was inspired by a *Philosophy of Physics* seminar delivered by Dr J. Butterfield in February 2011.

Contents

Chapter 1

Operator algebras in AQFT

1.1 Mathematical preliminaries

There are a number of good references for the theory of operator algebras [1, 13, 15, 17], and we merely summarize the important aspects required in AQFT and our discussion of Bell correlations. There is a considerable amount of preliminary mathematics to be developed, but the reward is a clear and rigorous framework of quantum field theory in which we can analyse the surprising nature of entanglement.

1.1.1 Operators on Hilbert spaces

A *Hilbert space* is a complete complex inner product space. In any Hilbert space $\mathcal{H}$, one can find orthonormal bases, which are sets of vectors $\{e_\alpha\}_{\alpha\in A}$ with the following properties:

$$\langle e_\alpha|e_\beta\rangle = \delta_{\alpha\beta} \qquad \text{(orthonormality)}$$

$$\left\{v \in \mathcal{H} : v = \sum_{n=1}^{N} c_{\alpha_n} e_{\alpha_n},\ c_{\alpha_n} \in \mathbb{C}\right\} \text{ is dense in } \mathcal{H} \qquad \text{(completeness).}$$

With respect to an orthonormal basis, one can write any vector $v \in \mathcal{H}$ as $v = \sum_\alpha \langle e_\alpha|v\rangle\, e_\alpha$. The cardinality of an orthonormal basis for $\mathcal{H}$ does not depend on the choice of basis, and is called the *dimension* of $\mathcal{H}$. Two Hilbert spaces are isometrically isomorphic iff they have the same dimension. A *separable* Hilbert space is one with dimension of at most $\aleph_0$. Unless otherwise stated, the Hilbert spaces that we shall be considering will be separable.

A closed (in the topological sense) linear subset of $\mathcal{H}$ is called a *subspace*. Given a subspace $\mathcal{K}$, $\mathcal{K}^\perp$ is called the *orthogonal complement* of $\mathcal{K}$, and we can always uniquely decompose any vector $v \in \mathcal{H}$ into $v = v_\parallel + v_\perp, v_\parallel \in \mathcal{K},\ v_\perp \in \mathcal{K}^\perp$. One can then define the map $P_{\mathcal{K}} : v \mapsto v_{\|}$, which is a bounded linear operator that is *idempotent* ($P_{\mathcal{K}} = P_{\mathcal{K}}^2$), and *self-adjoint* ($P_{\mathcal{K}}^* = P_{\mathcal{K}}$). In fact, any linear operator P satisfying $P = P^2 = P^*$ is an *orthogonal projection operator* onto a subspace corresponding to its range. Furthermore, the operator $\mathbf{1} - P$ is the projection operator onto the corresponding orthogonal subspace.

1.1.2 Algebras of operators

We begin our discussion of operator algebras by considering the set $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space. Here, the boundedness of $A \in \mathfrak{B}(\mathcal{H})$ means that its operator norm,

$$\|A\| = \sup_{v \in \mathcal{H} \setminus \{0\}} \frac{\|Av\|}{\|v\|}, \tag{1.1}$$

is finite. Of interest are subalgebras of $\mathfrak{B}(\mathcal{H})$, i.e., subsets of $\mathfrak{B}(\mathcal{H})$ that are closed under linear combination and taking products. If in addition, a subalgebra is closed under taking adjoints, we call it a *-subalgebra.

There are several topologies on $\mathfrak{B}(\mathcal{H})$ that we will consider in this essay. These can all be built from a family of seminorms:

Norm or uniform topology This is the topology associated with the operator norm, and is the strongest of the topologies listed here. We can also view it as the topology of uniform convergence.

Strong operator topology This is defined by the family of seminorms $\|Av\|$ for all $v \in \mathcal{H}$. A sequence of operators $\{A_n\}$ converges strongly to A if the sequence of vectors $\{A_n v\}$ converges to Av for all $v \in \mathcal{H}$. Thus we can view this topology as that of pointwise convergence.

Weak operator topology The family of seminorms is given by $|\langle u| A |v\rangle|$ for all choices of $u, v \in \mathcal{H}$. This is the topology of matrix element convergence, and is the weakest topology in this list.

Weak *-topology The family of seminorms is given by $|\mathrm{tr}\{\varrho A\}|$, where ϱ is a trace-class operator on $\mathcal{H}$. This is the topology induced by the set of *normal states* (see Section 1.1.3).

We can now define two types of *-subalgebras:

Definition 1.1.1. A (concrete) *C^*-algebra* is a uniformly closed *-subalgebra of $\mathfrak{B}(\mathcal{H})$.

Definition 1.1.2. A *von Neumann algebra* is a weakly closed *-subalgebra of $\mathfrak{B}(\mathcal{H})$ which contains the identity operator $\mathbf{1}$.

Note that a von Neumann algebra is automatically a C^*-algebra. The closures in the strong operator, weak operator, and weak *-topologies coincide for *-subalgebras of $\mathfrak{B}(\mathcal{H})$, so that we shall only be concerned with the two types of *-subalgebras of $\mathfrak{B}(\mathcal{H})$ defined above.

A famous theorem by von Neumann himself characterises a von Neumann algebra as a self-adjoint subset of $\mathfrak{B}(\mathcal{H})$ which is its own *double commutant*. Here, the *commutant* of a subset $S \subset \mathfrak{B}(\mathcal{H})$ is the set of all bounded linear operators which commute with every element of S, and is denoted by S'. Commutants are closely linked to von Neumann algebras, as the following theorem elucidates:

Theorem 1.1.3. (pp. 114 of [13]) *For any self-adjoint subset $S \subset \mathfrak{B}(\mathcal{H})$ (i.e. closed under taking adjoints),*

1. *S' is a von Neumann algebra,*
2. *S'' is the smallest von Neumann algebra containing S (bicommutant theorem),*
3. *$S''' = S'$.*

One should notice how this theorem relates a *topological* property (closure) with an *algebraic* one (commutant). For any pair of von Neumann algebras M_1, M_2, we denote the largest (smallest) von Neumann algebra contained in (containing) both M_1 and M_2 by $M_1 \wedge M_2$ ($M_1 \vee M_2$). Then, we have

$$M_1 \vee M_2 = (M_1 \cup M_2)'', \qquad \text{by (2) of Thm. 1.1.3,} \tag{1.2}$$

$$M_1' \cap M_2' = M_1' \wedge M_2' = (M_1 \vee M_2)'. \tag{1.3}$$

The classification of von Neumann algebras was worked out by Francis J. Murray and von Neumann. In general, a von Neumann algebra may be decomposed as a direct integral of "irreducible" factors, which we will now define.

Definition 1.1.4. A *factor* is a von Neumann algebra M whose centre is trivial, i.e., $M \cap M' = \{\lambda \mathbf{1} : \lambda \in \mathbb{C}\}$, or equivalently, $M \vee M' = \mathfrak{B}(\mathcal{H})$. The last expression is called a *factorization* of $\mathfrak{B}(\mathcal{H})$.

An equivalence relation can be defined on the set of projection operators of a factor M, with the projections P and Q being equivalent if there exists $U \in M$ such that $P = U^\dagger U$ and $Q = UU^\dagger$. We can introduce an ordering on the equivalence classes $\{[P]\}$ by imposing $[P] \leq [Q]$ whenever $P \leq Q$. This is in fact a total order, which may be represented faithfully via a non-negative *dimension function* $d(P)$ (with $d(P) = d(Q)$ if

$[P] = [Q]$ implicit) satisfying the property,

$$P \perp Q \Rightarrow d(P+Q) = d(P) + d(Q).$$

A factor is classified into one of the following types, depending on the nature of its dimension function, which is uniquely determined up to a constant.

1. Type I_n: $d(P) \in \{0, 1, \ldots, n\}, n \in \mathbb{N} \cup \{\infty\}$
2. Type II_1: $d(P) \in [0, 1]$
3. Type II_∞: $d(P) \in [0, \infty]$
4. Type III: $d(P) \in \{0, \infty\}$, with $d(P) = \infty$ if $P \neq 0$.

A factor is called *finite* if the projection $\mathbf{1}$ is finite, and (properly) infinite otherwise. This classification into Type I_n, Type II_1 etc., and finite/infinite, can be generalized to von Neumann algebras which are not themselves factors.

1.1.3 States

It is slightly unfortunate that the word "state" takes on different meanings in different contexts. In the most general sense, a state ω on a C^*-algebra $\mathcal{A}$ is a normalized, positive linear functional on $\mathcal{A}$. Here, positivity of ω means that $\omega(A^*A) \geq$ for all $A \in \mathcal{A}$, and normalization entails $\|\omega\| \equiv \sup\{|\omega(A)| : A \in \mathcal{A}, \|A\| \leq 1\} = 1$.

The same definition applies for a von Neumann algebra M, where positivity of ω corresponds to positive operators (in the usual sense) having a non-negative expectation value in the state ω, and normalization simplifies to $\omega(\mathbf{1}) = 1$. We will have occasion to refer to the finer notion of *normal* states[1], which roughly speaking, correspond to *completely additive* measures (or countably additive, for the case of separable Hilbert spaces). More formally, we have the following definition:

Definition 1.1.5. A state ω on a von Neumann algebra M is *normal* if $\omega(\sup A_\nu) = \sup \omega(A_\nu)$ for all bounded increasing nets of positive operators $A_\nu \in M$.

Given a normal state ω on M, there is a density operator $\rho \in \mathcal{T}(\mathcal{H})$ such that $\omega(A) = \mathrm{tr}\{\rho A\}$ for all $A \in M$; ($\mathcal{T}(\mathcal{H})$ refers to the trace-class operators on $\mathcal{H}$). In particular, when $M = \mathfrak{B}(\mathcal{H})$, the completely additive measures and the normal states are in one-to-one correspondence, and can each be represented by a density operator on $\mathcal{H}$.

[1]Chapter 2.1 of [1] provides a physical motivation for these definitions, as well as some technical remarks.

Every unit vector $v \in \mathcal{H}$ defines a state ω_v on $M \subseteq \mathfrak{B}(\mathcal{H})$, given by

$$\omega_v(A) = \langle v|Av\rangle \qquad \forall A \in M. \tag{1.4}$$

We call ω_v the induced vector state, and ω_v is a normal state on M.

The notions of *pure* and *mixed* states are defined in the sense of convex geometry: a state ω is pure if it cannot be expressed as a non-trivial convex combination of other states, and is mixed otherwise.

1.1.4 GNS construction

A C^*-algebra may also be defined abstractly, as a Banach (i.e. norm-complete) *-algebra $\mathcal{A}$ (where $A \mapsto A^*$ is an involution), whose norm satisfies the C^*-condition $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. The Gelfand–Naimark lemma [12] allows a concrete realization of such an abstract C^*-algebra $\mathcal{A}$ via a faithful representation as a norm-closed *-subalgebra of $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, making contact with our earlier definition.

There is a broad dichotomy of C^*-algebras into commutative and non-commutative algebras — a distinction which hints suggestively at a classical versus quantum divide (more on this in Section 3.1.2). Furthermore, the (commutative) Gelfand–Naimark lemma states that any commutative C^*-algebra is isomorphic to $C_0(X)$, i.e., the space of complex-valued continuous functions on a locally compact Hausdorff space X which vanish at infinity. The C^*-norm is given by the supremum norm $\|f\|_\infty \equiv \sup_{x\in X}\{|f(x)|\}$, while the involution is simply pointwise complex conjugation.

A state on a C^*-algebra plays an additional role in defining a distinguished representation of $\mathcal{A}$ via the Gelfand–Naimark–Segal (GNS) construction, which we state as the following theorem:

Theorem 1.1.6. (pp. 34 of [1]) *For each state ω on a C^*-algebra $\mathcal{A}$, there exists a triple — a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$, and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ — such that:*

1. $\omega(A) = \langle \Omega_\omega|\pi_\omega(A)\Omega_\omega\rangle \quad \forall A \in \mathcal{A}$,

2. *Ω_ω is cyclic for the representation π_ω, that is, $\pi_\omega(\mathcal{A}) \equiv \{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.*

The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is unique up to unitary equivalence, i.e., if $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ is another triple satisfying the above two conditions, then there is a unitary map $U : \mathcal{H}_\omega \to \mathcal{H}'_\omega$

satisfying

$$U\pi_\omega(A) = \pi'_\omega(A)U \quad \forall A \in \mathcal{A},$$
$$U\Omega_\omega = \Omega'_\omega.$$

1.2 Algebraic quantum field theory

Already in ordinary quantum mechanics, one quickly realizes that a study of Bell's theorem and its consequences would at least require a clear mathematical setting from the outset. We will find the Haag–Kastler axiomatic framework for quantum field theory fruitful for our investigations of spacelike correlations in relativistic quantum theory. Its distinctive feature is the emphasis on algebraic relations among observables, which take on fundamental roles in the description of a physical system. This approach, also called algebraic quantum field theory (AQFT), will be discussed briefly in this section. A more thorough treament can be found in the textbooks of Haag [13] and Araki [1], and an excellent review can be found in [20].

A motivating idea behind AQFT is that measurements of a physical quantity can be thought of as being performed in some specific limited spacetime domain. We denote the C^*-algebra generated by *all* the observables by $\mathcal{A}$. The basic mathematical object in AQFT is then an assignment to each bounded open set $\mathcal{O}$ of Minkowski spacetime[2], of a C^*-subalgebra $\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}$, which is interpreted as the C^*-algebra generated by the observables measurable in $\mathcal{O}$. The association $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ satisfies a number of axioms:

1. *Isotony*: If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_\in)$.

2. *Locality*: If $\mathcal{O}_1 \subseteq \mathcal{O}'_2$, then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)'$.

3. *Covariance*: $\alpha_g\mathcal{A}(\mathcal{O}) = \mathcal{A}(g\mathcal{O}) \;\; \forall g \in \mathscr{P}^\uparrow_+$.

4. *Generating property*: $\bigcup_\mathcal{O} \mathcal{A}(\mathcal{O})$ is dense in $\mathcal{A}$ (with respect to the norm topology).

A few explanatory remarks about these axioms are necessary. Isotony is hopefully a natural assumption, given our interpretation of the algebras $\mathcal{A}(\mathcal{O})$. The locality axiom is the statement of spacelike commutativity; (note the double-duty of the '$\,'\,$' — $\mathcal{O}'$ refers to the causal complement of $\mathcal{O}$, i.e., the set of all points spacelike to all of $\mathcal{O}$, while $\mathcal{A}(\mathcal{O})'$ refers to the commutant of $\mathcal{A}(\mathcal{O})$). Relativistic symmetry is given by the (restricted)

[2] One may apply the formalism of AQFT on more general spacetimes (e.g., Chap. 4.5 of [36]), but for most of this Essay, we will restrict ourselves to the basic case of Minkowski spacetime.

Poincaré group $\mathscr{P}_+^\uparrow$, with each symmetry $g \in \mathscr{P}_+^\uparrow$ represented by an automorphism α_g of $\mathcal{A}$. The third axiom is then the statement that the automorphism α_g brings the algebra of observables associated to a spacetime region $\mathcal{O}$, to the algebra associated with the transformed spacetime region $g\mathcal{O}$. Finally, the generating property is a reflection of our emphasis on "local" observables, which axiom 4 declares to suffice to generate the entire observable algebra. Correspondingly, we shall refer to $\mathcal{A}$ as a *local C^*-system*.

Concrete realizations of $\mathcal{A}$ as operators on Hilbert spaces can be recovered, for example, via the GNS-construction. The class of physically relevant representations of $\mathcal{A}$ is decided by certain desiderata, for instance, the existence of a $\mathscr{P}_+^\uparrow$-invariant (or translation-invariant) vacuum vector, a strongly continuous unitary representation of $\mathscr{P}_+^\uparrow$, and irreducibility[3]. We will often find it useful to take each local algebra $\mathcal{A}(\mathcal{O})$ to be a von Neumann algebra. This can be obtained, for instance, by considering an appropriate representation π of the local C^*-system, and taking $M(\mathcal{O}) = \pi(\mathcal{A}(\mathcal{O}))''$.

[3] For a more precise treatment on vacuum representations, see, for example, Chapter 4 of [1].

Chapter 2

Generic Bell Correlations between Local Algebras

If we regard the violation of a Bell-type inequality as a manifestation of non-local correlations, then non-locality can be said to be *ubiquitous* in algebraic quantum field theory. A precise meaning of this statement was given by Halvorson and Clifton [14], and we shall present their argument in Section 2.1. A different sense in which Bell-corrrelations can be considered to be generic is also presented in Section 2.1.2, following Landau's paper [19].

2.1 Bell correlations are commonplace in AQFT

Let us first clarify what we mean by "Bell correlations" between von Neumann algebras. Consider then, two commuting von Neumann algebras M_1, M_2 acting on a Hilbert space $\mathcal{H}$. We define the set of *Bell operators* $\mathscr{T}_{12}$ for $M_1 \vee M_2 =: M_{12}$ to be the following combinations of self-adjoint contractions:

$$\mathscr{T}_{12} := \left\{ \frac{1}{2}[A_1(B_1 + B_2) + A_2(B_1 - B_2)] \,:\, A_i = A_i^\dagger \in M_1, B_i = B_i^\dagger \in M_2, -\mathbf{1} \leq A_i, B_i \leq \mathbf{1} \right\}. \tag{2.1}$$

This definition has, of course, been formulated with the celebrated CHSH-inequality [7] in mind. To each state ω of M_{12}, we associate a number, the *maximal Bell correlation* of ω, defined by

$$\beta(\omega) := \sup \{ |\omega(T)| \,:\, T \in \mathscr{T}_{12} \}, \tag{2.2}$$

which is a continuous map from the state space of M_{12} into the interval $[1, \sqrt{2}]$ (Lemma 2.1 of [33]). Then Bell's theorem says essentially that a local hidden variable model, which accounts for the correlations between the observables in M_1 and M_2 in the state ω, exists only if $\beta(\omega) = 1$. A state ω is said to violate Bell's inequality (or is a *Bell-correlated state for* M_{12}), if $\beta(\omega) > 1$, and maximally violates Bell's inequality if $\beta(\omega) = \sqrt{2}$.

We shall also assume that M_1, M_2 satisfy the *Schlieder property* — if $A \in M_1, B \in M_2$ and $AB = 0$, then either $A = 0$ or $B = 0$. If M_1, M_2 are of infinite type (which is typically the case in quantum field theories, see the remark at the end of Sec. III.2.1 of [13]), then the following proposition holds:

Proposition 2.1.1. [14] *Let* $M_1, M_2 \subseteq \mathfrak{B}(\mathcal{H})$ *be commuting infinite-type non-Abelian von Neumann algebras satisfying the Schlieder property. Then there is an open dense subset of vector states which are Bell-correlated for* M_{12}.

Here, we are referring to subsets of $\mathcal{S}$ where $\mathcal{S}$ is defined as the set of unit vectors in $\mathcal{H}$, and by "vector states", we mean the states on $\mathfrak{B}(\mathcal{H})$ induced by these unit vectors. Also, to be precise, we should really speak about the restriction of these vector states to the algebra M_{12}. Note that this proposition also holds in any faithful representation of M_{12}. For example, in the universal normal representation (pp. 458 of [17]), normal states and vector states coincide. It follows that the set of Bell-correlated states for M_{12} is (norm) dense in the normal state space of M_{12}.

We should also clarify the link between Bell-correlations and entanglement. In the quantum information literature, a *separable*, or *classically correlated* (mixed) state ρ on $\mathfrak{B}(\mathbb{C}^n) \otimes \mathfrak{B}(\mathbb{C}^n)$ is defined as a density operator which can be approximated in the trace norm by convex combinations of product density operators $\rho_1 \otimes \rho_2$. An *entangled* state is then a *non-separable* state. Equivalently, one can take the set of separable states to be the norm-closed convex hull of the product states on $\mathfrak{B}(\mathbb{C}^n) \otimes \mathfrak{B}(\mathbb{C}^n)$. This definition is due surely to Werner [38], who also demonstrated that an entangled state does not necessarily violate a Bell inequality, but nevertheless exhibits some form of non-locality. When we pass to the case of commuting infinite-dimensional von Neumann algebras M_1, M_2, the choice of topology becomes important. Specifically, we will *define* the separable states of M_{12} to be the normal states in the weak *-closed convex hull of the normal product states. Here, a normal product state refers to a normal state ω on M_{12}, such that $\omega(AB) = \omega_1(A)\omega_2(B)$ for all $A \in M_1, B \in M_2$, where ω_1, ω_2 are normal states on M_1 and M_2 respectively. There are physical grounds for this choice of topology[1]; but for our purposes, it is pertinent that non-separability is necessary for Bell correlation under our definitions, i.e., $\beta(\omega) > 1 \Rightarrow \omega$ is entangled.

Given a von Neumann algebra M on $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is said to be *cyclic* if $M\xi := \{A\xi \ : \ A \in M\}$ is dense in $\mathcal{H}$. A *separating* vector for M is a vector $\xi \in \mathcal{H}$ such

[1] This is related to the limited accuracy of experimental measurements and the finite number of times one can perform an experiment. See pp. 125 of [13], or the discussion of "physical topology" on pp. 13 of [1], for instance.

that for a pair of operators $A, B \in M$, the equation $A\xi = B\xi$ implies that $A = B$. Note that "separating" and "separable" are two distinct notions. One can show from these definitions that if ξ is cyclic for M, then it is separating for M', and likewise, if ξ is separating for M, then ξ is cyclic for M'. Furthermore, if M has at least one cyclic vector in $\mathcal{S}$, it even has a set of cyclic vectors which is dense in $\mathcal{S}$ [10]. The next proposition relates cyclic vectors to non-separability:

Proposition 2.1.2. [14] *Let M_1, M_2 be two commuting, non-Abelian von Neumann algebras acting on $\mathcal{H}$. If $\xi \in \mathcal{S}$ is cyclic for M_1, then the vector state induced by ξ is non-separable across M_{12}.*

Consequently, provided that M_1 has a cyclic vector, the set of vectors inducing non-separable states across M_{12} is dense in $\mathcal{S}$. Note, however, that we cannot make a similar conclusion as we did after Proposition 2.1.1 for the normal state space of M_{12}. For the existence of a cyclic vector required by Proposition 2.1.2 is not retained under isomorphisms of M_{12}; so that, for instance, the trick of utilizing the universal normal representation no longer works (see remarks after Prop. 2 of [14]). The simplest illustration of this failure is seen by considering $M_1 = \mathfrak{B}(\mathbb{C}^2) \otimes \mathbf{1}_2, M_2 = \mathbf{1}_2 \otimes \mathfrak{B}(\mathbb{C}^2)$, i.e., M_1, M_2 considered as the algebras associated to each of the two subsystems of a two-qubit system. In this example, *any* entangled (pure) unit vector in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ is cyclic for M_1, and indeed the entangled pure states are dense in the set of vector states. To see this, one may use the Schmidt decomposition to write any $|\psi\rangle = \sum_{i=1,2} a_i \, |i\rangle_1 \otimes |i\rangle_2$, where $\{|i\rangle_1\}$ and $\{|i\rangle_2\}$ are orthonormal bases for the respective copy of $\mathbb{C}^2$ in $\mathcal{H}$. However, the non-separable normal states (density matrices in this case) are *not* dense in the normal state space (see Sec. 2.1.1 below and [39]).

For quantum field theories, it is often the case that the M_{12} under consideration has a separating vector (see the discussion on the Reeh–Schlieder theorem in Sec. 2.2.1). All normal states of M_{12} are then vector states (Theorem 7.2.3 in [17]), i.e., they are the restriction to M_{12} of the states on $\mathfrak{B}(\mathcal{H})$ induced by some $\xi \in \mathcal{S}$. Under such circumstances, Proposition 2.1.2 tells us that the non-separable states are norm dense in the normal state space of M_{12}.

We now link these results to algebraic quantum field theory proper. We shall first consider Minkowski spacetime, together with its local C^*-system. Then a third proposition by Halvorson and Clifton says that Bell-correlated states are generic in the following sense[2]:

[2]Halvorson and Clifton use weaker assumptions (in place of axioms 3 and 4 in Chap. 1.2) on the local C^*-system — *translational* covariance: $\alpha_{\mathbf{x}}\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O} + \mathbf{x})$ (i.e., a faithful representation $\mathbf{x} \to \alpha_{\mathbf{x}}$ of the translation subgroup of $\mathscr{P}_+^\uparrow$ in the automorphism group of $\mathcal{A}$), and weak additivity: for any $\mathcal{O}$, $\mathcal{A}$ is the smallest C^*-algebra containing $\bigcup_{\mathbf{x}} \mathcal{A}(\mathcal{O} + \mathbf{x})$.

Proposition 2.1.3. [14] *Let π be a physically relevant irreducible vacuum representation*[3] *of the local C^*-system $\{\mathcal{A}(\mathcal{O})\}$ (or any other representation in the same local quasiequivalence class*[4]*) over Minkowski space, with the Hilbert space $\mathcal{H}_\pi$. Let $\mathcal{S}_\pi$ be the set of unit vectors in $\mathcal{H}_\pi$, and $M_\pi(\mathcal{O}) := \pi(\mathcal{A}(\mathcal{O}))''$ be the local von Neumann algebra associated with $\mathcal{A}(\mathcal{O})$ on this representation space. Then for any two spacelike-separated open sets $\mathcal{O}_1, \mathcal{O}_2$ of Minkowski space, the set of vectors inducing Bell-correlated states between $M_\pi(\mathcal{O}_1)$ and $M_\pi(\mathcal{O}_2)$ is an open, dense subset of $\mathcal{S}_\pi$.*

The above result essentially follows from the fact that local von Neumann algebras in an irreducible vacuum representation are of the infinite type (see Sec. 2.2.2), and that the Schlieder property holds for the local algebras of appropriate subsets of $\mathcal{O}_1$ and $\mathcal{O}_2$. That is, Proposition 2.1.1 applies to the local von Neumann algebras associated to any pair of spacelike-separated open regions. In fact, Proposition 2.1.3 also holds for any globally hyperbolic spacetime, for the local C^*-system associated with the free Klein–Gordon field, realized on the GNS representation of some quasifree Hadamard state. For further details, refer to Proposition 4 of [14].

2.1.1 The volume of separable states in quantum mechanics

It is interesting to also consider a similar question in quantum mechanics — are there "more" separable states or entangled states? This question was pursued by Życzowski *et al.* in [39], where they considered states on composite systems in ordinary quantum mechanics, i.e. (unit trace) density operators on tensor products of finite-dimensional Hilbert spaces. To compare volumes in this state space, it is necessary to first define a natural measure on the space of density operators. The authors used a product measure $\nu \times \mathcal{L}$ based on natural quantities appearing in the spectral decomposition of a density operator. Roughly speaking, ν corresponds to the Haar measure on the unitary group associated with the orthonormal eigenbasis of a density operator, while $\mathcal{L}$ corresponds to the Lebesgue measure on the convex hull of the possible sets of eigenvalues of the density operator.

They proved that the volume of the set of separable states is *non-zero*, regardless of the (finite) dimension of the Hilbert space and the number of subsystems involved.

[3] With the group of translation automorphisms unitarily represented, and satisfying the spectrum condition — see also Chapter 3.1.

[4] Two representations π_1, π_2 of a C^*-algebra $\mathcal{A}$, on $\mathcal{H}_1, \mathcal{H}_2$ respectively, are *unitarily equivalent* if there exists a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(A) = \pi_2(A)U$ for all $A \in \mathcal{A}$. Let $\hat{\pi}_1, \hat{\pi}_2$ be representations of $\mathcal{A}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, defined by $\hat{\pi}_1(A) = \pi_1(A) \otimes \mathbf{1}, \hat{\pi}_2(A) = \mathbf{1} \otimes \pi_2(A)$. We say that π_1 and π_2 are *quasiequivalent* if $\hat{\pi}_1$ and $\hat{\pi}_2$ are unitarily equivalent — this is equivalent to the existence of an isomorphism of von Neumann algebras generated by the representations, $\pi_1(\mathcal{A})'' \cong \pi_2(\mathcal{A})''$ (pp. 212 of [1]).

That is, some significant semblance of classicality, as captured by (non-)entanglement, always persists in the finite-dimensional case. On the other hand, one expects that this finite volume, however small, approaches zero as the Hilbert space dimension tends towards infinity, so as to make contact with our earlier remark that in AQFT, entangled states are norm-dense in the normal state space of M_{12}. While no analytic proof of this conjecture was given by Życzowski *et al.*, they provided strong numerical evidence that the volume of the set of separable states decreases exponentially with the dimension of the composite quantum system.

Another related study was carried out by Ferraro *et al.* [11], where the idea of *quantum discord* was studied. While we will not need a precise definition of discord, it is pertinent that zero discord is a necessary condition for purely classical correlations; (in a slightly different sense from that which is used the definition of separable states). The authors prove the the set of zero-discord states has measure zero, and is nowhere dense in the set of quantum states, *independently* of the Hilbert space dimension. The point is that even in the finite-dimensional case, there is already a sense in which states with classical-only correlations are negligible.

2.1.2 Other aspects in which entanglement is generic

One might argue that not all choices of observables are "interesting", and that maybe certain relevant local observables, a set of "yes–no" questions perhaps, admit a classical description in terms of a joint distribution for all states. This turns out to be a false hope, as Landau demonstated in [19]. Consider, then, the local algebras $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ associated to a pair of spacelike separated regions. We also assume that the Schlieder property holds for $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$. Let $P_1, P_2 \in \mathcal{A}(\mathcal{O}_1), Q_1, Q_2 \in \mathcal{A}(\mathcal{O}_2)$ be projection operators, with $[P_1, P_2] \neq 0 \neq [Q_1, Q_2]$. We form the usual combination

$$C := \frac{1}{2}\left[P_1(Q_1 + Q_2) + P_2(Q_1 - Q_2)\right], \tag{2.3}$$

and study its expectation value in some state ω. We say that Bell's inequality is violated[5] if $|\omega(C)| > 1$. Landau's result (Prop. 3 of [19]) says that there is some state ω satisfying $|\omega(C)| = \beta$, where

$$\beta \equiv \sqrt{1 + 4\|[P_1, P_2]\|\|[Q_1, Q_2]\|} > 1. \tag{2.4}$$

Therefore, we can be as picky as we wish about the "yes-no" questions that we ask in either region (excepting the trivial case where P_1 commutes with P_2, or Q_1 commutes with Q_2), and there will still be some state which violates the Bell inequality.

[5] This is a slight adaptation of our earlier definition in Sec. 2.1; nevertheless, "violation" retains the same consequence of not admitting a local hidden variable description.

Finally, we also mention the technical results obtained by Summers and Werner (reviewed in [32]), which describe various situations in which Bell-correlation is not only generic, but "maximal", in the sense that Bell's inequalities are maximally violated in all normal states of certain pairs of local algebras. A particularly strong result (Thm. 6.9 of [32]) states that in a number of AQFTs, the pair $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ is maximally correlated in the above sense for *any* pair $\mathcal{O}_1, \mathcal{O}_2$ of tangent (i.e. intersecting closures) "diamonds". We will define a "diamond" region and return to this line of thought when we consider the so-called "Type *III* factors" in Section 2.2.2.

2.2 Bell correlations are entrenched in AQFT

Already, the preceding Section suggests a radical shift in the way one *describes* physics with respect to the cherished concepts of locality, causality, and "closed systems" embodied by classical field theories. Indeed, Streater, Wightman (pp. 139 of [29]) and Haag (pp. 298 of [13], 2nd edn) have all made claims that quantum field systems are unavoidably and intrinsically open to entanglement. Clifton and Halvorson [8] pursue this even further, revealing as a matter of principle, fundamental limitations on one's ability to isolate field systems from entanglement.

The situation is much less grim in non-relativistic quantum mechanics, where there is a general way in which entanglement can be "brought under control" by the experimentalist. Consider an entangled state between two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Then the measurement on subsystem A of any non-degenerate observable with a discrete spectrum serves precisely to turn any initial (pure or mixed) state of $\mathcal{H}_A \otimes \mathcal{H}_B$ into a convex combination of product states, i.e., an unentangled state in Section 2.1's sense (pp. 4 of [8]). This example highlights the conceptual departure from AQFT, where entanglement between spacelike-separated systems is "robust" against local operations — a point which we will make more precise later.

To endow our experimentalist with as much power as possible, we consider the most general transformation of a state of a quantum system with Hilbert space $\mathcal{H}$ [37]. These are described via positive, weak *-continuous, linear maps[6] $T : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ such that $0 \leq T(\mathbf{1}) \leq \mathbf{1}$. Such a map T induces a transformation $\omega \to \omega^T$ from the state space of

[6] In quantum information theory, a related notion is that of *quantum operations*, which are captured by *completely positive* maps $T : \mathcal{D}(\mathcal{H}_A) \to \mathcal{D}(\mathcal{H}_B)$, where $\mathcal{D}(\mathcal{H}_i)$ is the set of density operators on the Hilbert space $\mathcal{H}_i$. Stinespring's dilation theorem [28] allows one to represent a completely positive map T as a (Kraus) operator-sum $T(\rho) = \sum_i K_i \rho K_i^\dagger$, where $K_i : \mathcal{H}_A \to \mathcal{H}_B$ and $\sum_i K_i^\dagger K_i \leq \mathbf{1}$ (see Sec. 8.2.4 of [23]). Interestingly, the class of *positive but not completely positive* (PnCP) maps provides a separability criterion for distinguishing entangled states from separable states — a state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable iff $\mathbf{1}_A \otimes T_B \geq 0$ for all PnCP maps T_B [16].

$\mathfrak{B}(\mathcal{H})$ to itself (or 0), with the action of ω^T on $A \in \mathcal{A}$ given by

$$\omega^T(A) = \begin{cases} \frac{\omega(T(A))}{\omega(T(\mathbf{1}))} & \text{if } \omega(T(\mathbf{1})) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We call T a *selective* operation if $T(\mathbf{1}) < \mathbf{1}$ and a *non-selective* operation if $T(\mathbf{1}) = \mathbf{1}$. By the Kraus representation theorem [18], any operation $T : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ can be represented by a sequence $\{K_i\} \subseteq \mathfrak{B}(\mathcal{H})$ of operators, as

$$T(A) = \sum_i K_i^\dagger A K_i \qquad \forall A \in \mathfrak{B}(\mathcal{H}), \tag{2.6}$$

with the Kraus operators $\{K_i\}$ satisfying $0 \leq \sum_i K_i^\dagger K_i \leq \mathbf{1}$, and with convergence of infinite sums understood to be in the weak *-sense. A *pure* operation is one whose induced map takes a normal pure state to another normal pure state, and is represented by a single Kraus operator. The Kraus representation theorem thus allows us to understand a general operation T as the result of mixing the effects of pure operations on various subensembles in accordance with the Kraus operators appearing in the Kraus representation of T. We also define a *local operation* on the local algebra represented by a von Neumann algebra M, to be one which satisfies $T(B) = T(\mathbf{1})B$ for all $B \in M'$. These will have a representation by Kraus operators belonging to the local algebra M (pp. 13-14 of [8]).

2.2.1 Implications of the Reeh–Schlieder theorem

In AQFT, the Reeh–Schlieder theorem, in its various guises, makes the surprising assertion that the vacuum vector Ω is cyclic and separating for any local algebra $\mathcal{A}(\mathcal{O})$ (with $\mathcal{O}' \neq \emptyset$). More generally, any state with bounded energy[7] has these properties. An immediate observation is that the local algebra associated to a bounded open region cannot contain a number operator N, for we would have $N\Omega = 0 \Rightarrow N = 0$ since Ω is separating. In fact, there can be no non-zero, localized observable that annihilates the vacuum (or indeed, any vector state with bounded energy).

Now, recall that in non-relativistic quantum mechanics, a tell-tale sign of entanglement between two subsystems is the mixed nature of the local restriction of a state. More precisely, a pure vector state in $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff its reduced density matrix (on either subsystem) is mixed[8]. Likewise, as a consequence of the Reeh–Schlieder theorem,

[7] Let E be the spectral measure for the global Hamiltonian of the field. A pure global state with representative vector x is said to have *bounded energy* if $E([r, s])x = x$ for some $-\infty < r < s < \infty$, i.e., the field's energy in the state x is confined to some bounded interval $[r, s]$ with probability 1.

[8] In fact, the von Neumann entropy of the reduced state provides a *measure* of entanglement, so that a highly mixed local state indicates a strong degree of entanglement.

Clifton and Halvorson (Sec. 3 of [8]) have shown that the local restriction of a bounded energy vector state $\omega_x|_{\mathcal{A}(\mathcal{O})}$ is highly mixed, in the sense that $\omega_x|_{\mathcal{A}(\mathcal{O})}$ has a norm-dense set of components in the (normal) state space of $\mathcal{A}(\mathcal{O})$. Furthermore, bounded energy states are typical — they form a norm-dense subset of the pure state space of $\mathfrak{B}(\mathcal{H})$ (pp. 17 of [8]).

Recall the connection between cyclicity and entanglement given in Proposition 2.1.2, which holds generally for commuting non-Abelian von Neumann algebras. Then, in particular, we can apply the Reeh–Schlieder theorem, together with this proposition, to the local algebras of any spacelike-separated regions $\mathcal{O}_1$ and $\mathcal{O}_2$. It follows that the dense set of bounded-energy vector states are all entangled across $\mathcal{O}_1$ and $\mathcal{O}_2$. There is, on the other hand, the general statement that for any non-trivial $A \in \mathcal{A}(\mathcal{O})$ and any bounded-energy vector state ω_x, any state of the field induced by a vector of the form Ax does not have bounded energy (Cor. 7 of [4]), and is consequently outside the purview of the Reeh–Schlieder theorem. Thus, it appears that an experimentalist could simply perform a pure operation within $\mathcal{O}_1$ to circumvent the Reeh–Schlieder theorem, and the resulting state will not necessarily be entangled across $\mathcal{O}_1, \mathcal{O}_2$. However, almost all pure operations that the experimentalist at $\mathcal{O}$ may perform, *do* preserve cyclicity and thus entanglement (pp. 20 of [8]).[9]

Furthermore, applying the observations after Proposition 2.1.2 to $\mathcal{O}_1$ and $\mathcal{O}_2$, we obtain Clifton and Halvorson's "Generic Result", that the generic state of $\mathcal{A}(\mathcal{O}_{12}) \equiv \mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ will be entangled across $\mathcal{O}_1$ and $\mathcal{O}_2$; (the Reeh–Schlieder theorem provides the requisite cyclic and separating vectors for this argument, provided $(\mathcal{O}_1 \cup \mathcal{O}_2)' \neq \emptyset$). The experimentalist at $\mathcal{O}_1$, having painstakingly worked out a disentangling scheme, still has the mammoth task of ascertaining that he is *not* in possession of one of the typical entangled states of $\mathcal{A}(\mathcal{O}_{12})$. Thus, even if we allow him to perform *any* (pure or mixed) local operation of his choosing, there remains the inherent practical impossibility of distinguishing the resulting state of his operation from the overwhelming majority of states on $\mathcal{A}(\mathcal{O}_{12})$ which are entangled.

2.2.2 Type III factors, intrinsic entanglement, and intrinsically mixed states

As we noted in passing before Prop. 2.1.1, the local algebras associated with bounded open regions of Minkowski space in AQFT models are of the infinite type; more precisely, they are Type *III* factors. These have some remarkable properties which have profound

[9] As remarked by the authors (pp. 12-13 of [8]), one can consider more general mixing operations in $\mathcal{O}$, but there is an ambiguity in interpreting the status of an apparently unentangled mixture of entangled states.

consequences. A useful fact about a Type III algebra (acting on a separable Hilbert space) is that it possesses a cyclic and separating vector (Cor. 2.9.28 of [26], Ex. 14.4.12 of [17]). Consequently, it has a dense set of cyclic vectors in $\mathcal{S}$, and all its normal states are (the restrictions of) vector states. Despite this, Type III algebras have *no* pure normal states at all.[10] This strange property means that there is no way to understand a state as a mixture of pure states, as is usual in the ignorance interpretation of mixtures in ordinary quantum mechanics; so we shall call them *intrinsically mixed* (for a detailed "appreciation", see [25]).

Furthermore, a Type III factor M is *non-abelian*, so that its commutant M' in $\mathfrak{B}(\mathcal{H})$ is also a Type III non-abelian factor, because taking the commutant preserves type (Thm. 9.1.3 in [17]). In this case, a pure state on $\mathfrak{B}(\mathcal{H}) = M \vee M'$ restricts to an intrinsically mixed state on each subalgebra M and M'. This suggests the presence of entanglement between the commuting algebras M and M', which is similarly "intrinsic". This is in fact correct, as there are no normal product states on $M \vee M'$ (pp. 213 of [32]). There are no unentangled states between M and M' to even speak of!

Pursuing this line of thought even further, one finds that the local algebras in known models of AQFT are actually (hyperfinite) Type III_1. The technical details of the subclasses of Type III factors are not important here; what is essential is that III_1 factors have further remarkable properties and implications for our study of Bell correlations. Connes and Størmer have provided a useful characterization of Type III_1 factors M as follows [9]: for any two normal states ρ, ω of $\mathfrak{B}(\mathcal{H})$ and any $\epsilon < 0$, there exist unitary operators $U \in M, U' \in M'$ such that $\|\rho - \omega^{UU'}\| < \epsilon$. In quantum information theory, one usually demands that a measure of entanglement remains invariant under local unitary operations, and is norm-continuous [34]. Under these requirements, the Connes–Størmer characterization essentially prevents us from defining a non-trivial measure of entanglement across M and M'.

In a series of remarkable papers, Summers and Werner showed that for a Type III_1 factor M on a separable Hilbert space $\mathcal{H}$, *every* normal state on $M \vee M' = \mathfrak{B}(\mathcal{H})$ is maximally entangled ([30, 31, 33], Thm. 3.19 in [15])[11]. There are a large number of technical results in these papers, which state various conditions under which this phenomenon would occur. They go on to demonstrate that such conditions are typically satisfied in a variety of AQFTs, so that entanglement there is not only endemic, but

[10] This in in stark contrast with Type I factors, which arise as $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. There, pure states are aplenty — they are simply the projectors onto one-dimensional subspaces of $\mathcal{H}$.

[11] The authors appeared to have proved the more general statement that for a strongly stable von Neumann algebra acting on a separable Hilbert space equipped with a cyclic and separating vector, every normal state on $\mathfrak{B}(\mathcal{H})$ is maximally entangled across M and M'. The corresponding statement for Type III_1 algebras acting on a separable Hilbert space hold as well, since Cor. 2.9.28 of [26] guarantees the existence of the required cyclic and separating vector.

maximally so.

Although we have spoken of Type III_1 factors, we have not yet described the nature of the region from which the factor arises as a local algebra. Following Haag (pp. 111 of [13]), we now consider the *diamonds* as our archetypal causally complete region.[12]. These are defined, for a given pair of spacetime points p, q with $p - q$ positive timelike, by

$$V_q^p \equiv \{x : \ p - x \in V_+, x - q \in V_+\}\,, \tag{2.7}$$

where $V_+ \equiv \{x : \|x\| > 0, x^0 > 0\}$ is the set of positive timelike vectors. More intuitively, V_q^p is the intersection of the interior of the future light cone of q with the interior of the past light cone of p. It turns out that many models of AQFT in Minkowski space satisfy the *duality relation*, $\mathcal{A}((V_p^q)') = \mathcal{A}(V_q^p)'$ for all diamond regions V_q^p (also called *Haag duality*, see pp. 145 of [13]). Consequently, we have intrinsic entanglement of every global state of the field across V_q^p and $(V_q^p)'$. Thus one is forced to conclude that the field system in a diamond region is entangled with its spacelike complement.

Clifton and Halvorson [8] suggest the more modest goal of disentangling a state across a pair of *strictly* spacelike-separated regions $\mathcal{O}_1, \mathcal{O}_2$ (i.e., the two regions remain spacelike when either region undergoes a small translation). In this case, there do exist normal product states across $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ (see pp. 26 of [8] and pp. 239-240 of [32] for a technical discussion), which our experimentalist in $\mathcal{O}_1$ could, in principle, hope to prepare via local operations. However, as Clifton and Halvorson proceed to demonstrate, a norm-dense set of entangled states of $\mathcal{A}(\mathcal{O}_{12})$ may *not* be disentangled via pure local operations carried out in $\mathcal{O}_1$.[13]

We therefore see that in AQFT, there is a deep entrenchment of entanglement between a spacetime region and its causal complement, or indeed, between a spacetime region and any strictly spacelike-separated region. On the other hand, Clifton and Halvorson offer a clever respite — they demonstrated that our experimentalist, if allowed to perform *approximately* local operations, *can* disentangle his field system from other strictly spacelike-separated field systems, and can even prepare any local normal state ρ that he desires (pp. 28-29 of [8]). More specifically, we allow operations which are local to a "super-region" $\tilde{\mathcal{O}}_1$, whose interior contains the closure of $\mathcal{O}_1$. Choosing $\tilde{\mathcal{O}}_1$ to approximate $\mathcal{O}_1$ sufficiently closely so that $\mathcal{O}_2 \subseteq (\tilde{\mathcal{O}}_1)'$ (this is possible when $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly spacelike-separated), the experimentalist can perform a non-selective operation

[12] The *causal completion* of a set of points S in Minkowski space is S'' We say that S is *causally complete* if $S'' = S$.

[13] Halvorson and Clifton showed further that the same conclusion follows if we allow mixed projective operations; but as remarked in footnote 9, ambiguity over the meaning of "disentangling" arises if we consider mixing operations.

T, local to $\tilde{\mathcal{O}}_1$, to turn any normal state ω on $\mathcal{O}_{12}$ into the product state,

$$\omega^T(AB) = \rho(A)\omega(B) \qquad \forall A \in \mathcal{A}(\mathcal{O}_1), B \in \mathcal{A}(\mathcal{O}_2). \tag{2.8}$$

In this manner, they suggest, the indelible nature of entanglement that we had previously attested to does not really pose a practical problem to the experimentalist.

Chapter 3

Coexistence between AQFT and Special Relativity

Relativity's adherence to the principle of local action and quantum theory's non-local nature appear to be at odds, and this tension is made explicit by Einstein's own derisive "spooky action-at-a-distance" remark about quantum mechanics. In light of the genericity and indestructibility of entanglement described in the previous chapter, it is even "...ironic, considering Einstein's point of view, that such limits should be forced upon us once we make the transition to a fully *relativistic* formulation of quantum theory..." (pp. 5 of [8]). Yet, there are subtle ways in which special relativity and AQFT can and do coexist "peacefully". In this chapter, we merely highlight a few interesting points of view in the literature.

3.1 Relativistic causality in AQFT

In plain language, relativistic causality is the requirement that causal influences travel subluminally. A precise definition, in quantum theories, is a far more tricky issue. Following [5], we describe three notions of relativistic causality that arise in AQFT (in Minkowski spacetime). The first is the notion of *primitive causality* (Axiom G, Sec. II.1.2 of [13]), or introduced as an axiom for AQFT, the *Diamond axiom*: $\mathcal{A}(D(\mathcal{O})) = \mathcal{A}(\mathcal{O})$. Here, $D(\mathcal{O})$ refers to the *domain of dependence*[1] of the spacetime region $\mathcal{O}$. This is motivated by the hyperbolic equations of motion encountered in classical field theories (e.g. Maxwell's electromagnetism) where the initial conditions on a spacelike patch determine the conditions

[1]For a region $\mathcal{O}$, the *future domain of dependence* $D^+(\mathcal{O})$ is the set of spacetime points p such that every past-inextendible causal curve through p intersects $\mathcal{O}$. The *past domain of dependence* $D^-(\mathcal{O})$ is similarly defined, and the total domain of dependence is $D(\mathcal{O}) = D^+(\mathcal{O}) \cup D^-(\mathcal{O})$. See also Chapter 8.3 of [35].

on its domain of dependence. Thus for instance, for a region $\mathcal{O}_1 \subset D^+(\mathcal{O})$ and disjoint from $\mathcal{O}$, an observable $A \in \mathcal{A}(\mathcal{O}_1)$ could just as well be measured in $\mathcal{O}$, although of course by a different procedure from what one would have adopted in $\mathcal{O}_1$ so as to measure A.

The second notion of relativistic causality is spacelike commutativity, which we have already encountered in the locality axiom of AQFT in Section 1.2. Explicitly, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike-separated, then $[A_1, A_2] = 0$ for all $A_1 \in \mathcal{A}(\mathcal{O}_1), A_2 \in \mathcal{A}(\mathcal{O}_2)$. From a physical viewpoint, this is the idea that observables at spacelike distances should be co-measurable and therefore commute. Later on, we shall explore this notion in the light of the Reeh–Schlieder theorem.

The final notion is the *spectrum condition* which we briefly alluded to in footnote 3 of Chapter 2 (see Sec. II.1.2 and pp. 110 of [13], and Axiom 4, pp. 104 of [1]). This states that the energy-momentum operators (generators of the translation group unitarily represented in an irreducible vacuum representation) have their spectra contained in the (closed) forward light-cone,

$$\overline{V}_+ \equiv \left\{p : p^2 \geq 0, p^0 \geq 0\right\}. \tag{3.1}$$

It is important to note that these notions are logically independent [5]; the basic lesson is that relativistic causality, carried over to the quantum case, is a subtle issue.

3.1.1 Reeh–Schlieder theorem revisited

A bizarre consequence of the cyclic nature of the vacuum vector Ω, is that one can apply pure operations confined within $\mathcal{O}$ on Ω, to prepare essentially any global state of the field. This suggests a type of "action-at-a-distance" by procedures localized in an arbitrarily small region, which appears to violate relativistic causality. This is reminiscent of the analogous situation of "remote steering" in non-relativistic quantum mechanics. There, the usual response to the bizarre ("spooky" in Einstein's words) nature of remote steering is to emphasize:

(a) the *selective* nature of the operation concerned (cf. Sec. 2.2); and

(b) a *non-selective* Lüders rule measurement of an observable A does not affect the measurement probabilities of another observable B which commutes with A — this is a statement of the no-signalling theorem.

Similarly in AQFT, the locality axiom entails *spacelike commutativity*, so that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike-separated, then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$. And Clifton and Halvorson demonstrate that the general $A \in \mathcal{A}(\mathcal{O})$ represents a *selective* operation, which "... involves performing a physical operation on an ensemble followed by a *purely conceptual* operation in which one makes a selection of a subensemble based on the outcome of the physical operation (assigning "state" 0 to the remainder)..."(Sec. 2 in [8]). From this point of view, the remote steering of the field system at a spacelike separation is effected not by a physical operation alone, but a conceptual one.

3.1.2 Landsman's take on the Bohr–Einstein debate

One major advantage of the algebraic approach to quantum theory is that it provides a common mathematical framework in which we can analyse *both* the classical and quantum cases (covering both non-relativistic quantum mechanics and relativistic AQFT). This feature is cleverly exploited by Landsman [21], who in one fell swoop, managed a remarkable reconciliation between Bohr and Einstein. The following is an outline of Landsman's treatment.

We will first make a return to the more general setting of C^*-algebras. Recall Section 1.1.4, which tells us that any C^*-algebra is isomorphic to a norm-closed self-adjoint subalgebra of the bounded operators on some Hilbert space $\mathcal{H}$. Consider two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, which we interpret as the algebras of observables of two physical systems. A *product* state on $\mathcal{A}\hat{\otimes}\mathcal{B}$[2] is a state ω which satisfies, for some states ρ of $\mathcal{A}$ and σ of $\mathcal{B}$, the equation $\omega(A \otimes B) = \rho(A)\sigma(B)$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. As in the case of von Neumann algebras, we say that a state ω on $\mathcal{A}\hat{\otimes}\mathcal{B}$ is *separable* if it is in the weak *-closure of the convex hull of the product states on $\mathcal{A}\hat{\otimes}\mathcal{B}$, and is *entangled* otherwise. Also, in analogy to Eq. 2.1 and Eq. 2.2, we define the *Bell operators* $\mathscr{T}_{12}$ for $\mathcal{A}\hat{\otimes}\mathcal{B}$ to be the combination

$$\mathscr{T}_{12} := \left\{ \frac{1}{2}[A_1(B_1 + B_2) + A_2(B_1 - B_2)] \,:\, A_i = A_i^* \in \mathcal{A}, B_i = B_i^* \in \mathcal{B}, \|A_i\| \leq 1, \|B_i\| \leq 1 \right\}, \tag{3.2}$$

and the *maximal Bell correlation* of a state ω on $\mathcal{A}\hat{\otimes}\mathcal{B}$, by

$$\beta(\omega) := \sup\left\{|\omega(T)| \,:\, T \in \mathscr{T}_{12}\right\}. \tag{3.3}$$

We say that ω violates Bell's inequality if $\beta(\omega) > 1$ and satisfies it if $\beta(\omega) \leq 1$. With these definitions, we can now state Raggio's theorem [24].

Theorem 3.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. The following are equivalent:*

1. *Each state on $\mathcal{A}\hat{\otimes}\mathcal{B}$ is separable;*

[2] A 'hat' has appeared subtly, because the tensor product of two C^*-algebras is not unique in general, so we need to take the *projective* tensor product $\mathcal{A}\hat{\otimes}\mathcal{B}$, which is defined to be the completion in the maximal C^*-cross-norm of the *algebraic* tensor product $\mathcal{A} \otimes \mathcal{B}$. The technical reasons for choosing the projective tensor product is explained in Landsman's paper [21]. For our purposes, this choice ensures that product states $\rho \otimes \sigma$ and mixtures $\sum_i p_i\rho_i \otimes \sigma_i$ are well-defined (by linearity and continuity) on the projective tensor product.

2. $\mathcal{A}$ or $\mathcal{B}$ is commutative;

3. Each state on $\mathcal{A}\hat{\otimes}\mathcal{B}$ satisfies Bell's inequality.

Now, let $\mathcal{A}$ and $\mathcal{B}$ be the algebra of observables of a quantum system and a measuring instrument respectively. Recall Einstein's *Trennungsprinzip*, which we will interpret as the statement that each pure state of $\mathcal{A}\hat{\otimes}\mathcal{B}$ restricts to a pure state on both $\mathcal{A}$ and $\mathcal{B}$. Adhering to this doctrine, Condition 1 is satisfied, and with $\mathcal{A}$ non-commutative (since it describes a quantum system), we conclude that $\mathcal{B}$ is commutative *à la* Bohr. In summary: Einstein's position really implies Bohr's! Conversely, based on Bohr's doctrine of classical concepts, we assume that $\mathcal{B}$ is commutative, and then Condition 1 is implied. This in particular, means that a pure state of the joint system restricts to a pure state on either subsystem. Each subsystem has its own "real state", which is precisely Einstein's demand. So again, in the sense of Theorem 3.1.1: Bohr's position implies Einstein's!

3.1.3 Stochastic Einstein Locality and AQFT

There is another formulation of relativistic causality, which was introduced to address concerns arising from EPR-Bell correlations, and is suited for the case of Minkowski spacetime. This conception of relativistic causality is called "Stochastic Einstein Locality" (SEL). For an event E occurring in a region $\mathcal{O}$, it is natural to expect that the probability at a time t (preceding $\mathcal{O}$) that E does occur should be determined by the events that occurred within that part of the past light cone of $\mathcal{O}$ that precedes t. To formulate this intuition precisely, we follow the guidance of Butterfield in [6]. As well shall see, there are a number of distinct ways to define SEL.

For a subset $\mathcal{O}$ of Minkowski space, we denote its past light cone[3] by $C^-(\mathcal{O})$. For convenience, we will use also E to denote the spacetime region in which event E occurs. Since Minkowski space is globally hyperbolic, it can be foliated by a family of Cauchy surfaces ("global instantaneous slices"), and one can define a global time function f on Minkowski space so that each constant-f surface is a Cauchy surface (Chapter 8.3 of [35]). Actually, we shall only be using hypersurfaces, labelled by t, which are spacelike at least within $C^-(E)$, and which divide $C^-(E)$ into two disjoint parts, $C^+(t) \cap C^-(E)$ (the "summit") and $C^-(t) \cap C^-(E)$ (the "base"). It then makes sense to speak of "time-dependent" probabilities $\mathrm{Pr}_t(E)$.

We also imagine a set of "possible worlds" $\mathcal{W}$, where a "possible world" w is a dynamically possible total history of the system under consideration. Because we are assuming

[3] As pointed out in [6], there is a slight distinction between *chronological* and *causal* pasts, which thankfully, will not affect our discussion. Also, Butterfield defined his SEL for more general spacetimes that exhibit *stable causality.* As we are mostly concerned with AQFT on Minkowski spacetime, we only need the assurance that Minkowski space is stably causal.

a fixed background Minkowski spaectime, we may identify times t between worlds in $\mathcal{W}$ in the following manner. For two worlds which "match" in their respective histories up to two hypersurfaces each in its world, we identify the hypersurfaces by labelling them with t. By "matching", we mean that all properties and relations intrinsic to the regions concerned are isomorphic (i.e., there is a bijection of the regions that is an isomorphism of the fields etc.). Consequently, we may speak of $\Pr_{t,w}(E)$ as the probability, at time t in world w, that event E occurs.

The first version of SEL is called SELS, with the last 'S' standing, with the benefit of hindsight, for 'satisfied'. It is stated as follows:

Definition 3.1.2 (SELS)**.** Suppose two worlds $w, w' \in \mathcal{W}$ match in their history in $C^-(E) \cap C^-(t)$. Then,

$$\Pr_{t,w}(E) = \Pr_{t,w'}(E). \tag{3.4}$$

Thus, SELS says that the probability in either world at time t that E occurs is the same.

We might equally well have formulated SEL with a statement along the lines of "the probability of E occuring is independent of an event F that is far away". It is not immediately clear which events we want E to be stochastically independent of, since we are dealing with probability functions $\Pr_{t,w}(E)$ that are both time-dependent and world-dependent. We therefore define a different probability function, which is determined not by all of a world's history up to t, but by the world's history lying both in the past of t and within $C^-(E)$. We denote this truncated history by (H, w), and the corresponding probability function by $\Pr_{H,w}(\cdot)$. Then, we have the second formulation of SEL, called SELD2[4]:

Definition 3.1.3 (SELD2)**.** For any world $w \in \mathcal{W}$, any hypersurface t, and any event $F \subset C^-(t) - C^-(E)$ (i.e., before t *and* outside the past light cone of E),

$$\Pr_{H,w}(E \text{ and } F) = \Pr_{H,w}(E) \cdot \Pr_{H,w}(F), \tag{3.5}$$

where (H, w) is the history of w in the intersection $C^-(E) \cup C^-(t)$.

Our burden is now to transfer these notions of SEL to the arena of AQFT. Instead of the "worlds" that we had considered previously, we now have models of a AQFT on Minkowski spacetime, each given by the pair $(\mathcal{A}, \omega)$, where $\mathcal{A}$ encodes the local algebra assignment $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, and ω is a state on $\mathcal{A}$. A (localized) event E now becomes a projector (also called E) in a local algebra $\mathcal{A}(\mathcal{O})$, with the probability of E's occurence replaced by $\omega(E)$. The natural transcription of SELS is

[4]There is also a SELD1 [6]. The 'D' stands for 'denied', in hindsight.

Proposition 3.1.4 (SELS for AQFT, Sec. 4.1 of [6])**.** *Consider any two models* $(\mathcal{A}_1, \omega_1)$ *and* $(\mathcal{A}_2, \omega_2)$*, any bounded open region* $\mathcal{O}$*, and any projection* $E \in \mathcal{A}_1(\mathcal{O}) \cap \mathcal{A}_2(\mathcal{O})$*. Let* t *be any hypersurface preceding* $\mathcal{O}$*. Suppose the two models match throughout the region* $C^-(t) \cap C^-(\mathcal{O})$[5]*. Then,*

$$\omega_1(E) = \omega_2(E). \tag{3.6}$$

It is then a trivial matter to demonstrate that SELS is satisfied in AQFT. The diamond axiom tells us that ω_1 and ω_2 match on $\mathcal{A}(D(\mathcal{R})) = \mathcal{A}(\mathcal{R})$ for each bounded open subset $\mathcal{R} \subset C^-(t) \cap C^-(\mathcal{O})$. The region $\mathcal{O}$ lies inside the future domain of dependence of some suitably chosen $\tilde{\mathcal{R}}$, i.e., $\mathcal{O} \subset D^+(\tilde{\mathcal{R}})$, so by isotony, $\mathcal{A}_i(\mathcal{O}) \subset \mathcal{A}(D(\tilde{\mathcal{R}}))$. Therefore, ω_1 and ω_2 match on $\mathcal{A}_i(\mathcal{O}) \ni E$.

As for SELD2, we have the transcription,

Proposition 3.1.5 (SELD2 for AQFT, Sec. 4.1 of [6])**.** *Consider any model* $(\mathcal{A}, \omega)$*, any bounded open region* $\mathcal{O}_1$*, and any projection* $E \in \mathcal{A}(\mathcal{O}_1)$*. Let* t *be any hypersurface preceding* $\mathcal{O}_1$*,* $\mathcal{O}_2$ *be any bounded open subset of* $C^-(t) - C^-(\mathcal{O}_1)$*, and* F *be any projection in* $\mathcal{A}(\mathcal{O}_2)$*. Then,*

$$\omega(EF) = \omega(E) \cdot \omega(F), \tag{3.7}$$

or equivalently,

$$\omega(E) = \omega(E/F). \tag{3.8}$$

In Section 2.1, we showed how endemic Bell-correlated states (across two spacelike separated regions) are in a AQFT on Minkowski space. In view of this, SELD2 in AQFT is emphatically denied!

[5]This means that there is an isomorphism of the local algebras $\mathcal{A}_1(\mathcal{R}) \cong \mathcal{A}_2(\mathcal{R})$ associated to each bounded open subset $\mathcal{R}$ of $C^-(t) \cap C^-(\mathcal{O})$, which we denote simply by $\mathcal{A}(\mathcal{R})$, and that the expectation values of ω_1 and ω_2 are equal on the elements of $\mathcal{A}(\mathcal{R})$.

Chapter 4

Conclusion

Broadly speaking, we have covered two issues in this essay. The first concerns the surprising nature of Bell-correlations in algebraic quantum field theory — we have found that they are generic, maximal, and even indestructible. What then, are the implications of these superlatives for the status of special relativity in quantum field theory? The former insists on 'no action-at-a-distance', but the latter is so flagrantly non-local that it is difficult to see how the two can be consistently combined.

To a large extent, the answer depends on the notion of relativistic causality that one wishes to maintain, and we have seen a few examples of the ways in which quantum nonlocality can coexist 'peacefully' with special relativity. Admittedly, there is a case for the statement that AQFT is *constructed* precisely to preclude super-luminal signalling, a point perhaps best illustrated by "micro-causality": $\mathcal{A}(\mathcal{O}') \subseteq \mathcal{A}(\mathcal{O})'$. There is therefore a real danger of postulating what we had set out to prove. However, as we have seen with the example of SEL, not all natural notions of relativistic causality are satisfied by AQFT. If nothing else, we have at least learnt to appreciate the subtlety of the issue.

In honesty, we are still far from a completely satisfactory resolution, and it would be appropriate to close this essay with a telling quote from J. S. Bell [3].

> "...we have an apparent incompatibility, at the deepest level, between the two fundamental pillars of contemporary theory...It may be that a real synthesis of quantum and relativity theories requires not just technical developments but radical conceptual renewal."

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