Is the relativity principle consistent with classical electrodynamics?

Towards a logico-empiricist reconstruction of a physical theory

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Abstract

It is common in the literature on classical electrodynamics (ED) and relativity theory that the transformation rules for the basic electrodynamical quantities are derived from the hypothesis that the relativity principle (RP) applies to Maxwell's electrodynamics. As it will turn out from our analysis, these derivations raise several problems, and certain steps are logically questionable. This is, however, not our main concern in this paper. Even if these derivations were completely correct, they leave open the following questions: (1) Is the RP a true law of nature for electrodynamical phenomena? (2) Are, at least, the transformation rules of the fundamental electrodynamical quantities, derived from the RP, true? (3) Is the RP consistent with the laws of ED in a single inertial frame of reference? (4) Are, at least, the derived transformation rules consistent with the laws of ED in a single frame of reference? Obviously, (1) and (2) are empirical questions. In this paper, we will investigate problems (3) and (4).

First we will give a general mathematical formulation of the RP. In the second part, we will deal with the operational definitions of the fundamental electrodynamical quantities. As we will see, these semantic issues are not as trivial as one might think. In the third part of the paper, applying what J. S. Bell calls "Lorentzian pedagogy"—according to which the laws of physics in any one reference frame account for all physical phenomena—we will show that the transformation rules of the electrodynamical quantities are identical with the ones obtained by presuming the covariance of the equations of ED, and that the covariance is indeed satisfied.

As to problem (3), the situation is much more complex. As we will see, the RP is actually not a matter of the covariance of the physical equations, but it is a matter of the details of the solutions of the equations, which describe the behavior of moving objects. This raises conceptual problems concerning the meaning of the notion "the same system in a collective motion". In case of ED, there seems no satisfactory solution to this conceptual problem; thus, contrary to the widespread views, the question we asked in the title has no obvious answer.

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Contents

1 Introduction

It is common in the literature on classical electrodynamics (ED) and relativity theory that the transformation rules for the basic electrodynamical quantities are *derived* from the assumption that the relativity principle (RP) applies to Maxwell's electrodynamics. As it will turn out from our analysis (the details are given in Remark 10 and 11), these derivations raise several problems, and certain steps are logically questionable. This is, however, not our main concern in this paper. Even if these derivations were completely correct, they leave open the following questions:

(Q1) Is RP a true law of nature for electrodynamical phenomena?

(Q2) Are, at least, the transformation rules of the fundamental electrodynamical quantities, derived from RP, true?

First of all, one has to clarify what the principle says. The RP is usually formulated as follows: "All the laws of physics take the same form in any inertial frame of reference." This short sentence, however, does not express exactly what the principle actually asserts. For example, consider how Einstein applies the principle in his 1905 paper:

> Let there be given a *stationary* rigid rod; and let its length be l as *measured by a measuring-rod which is also stationary*. We now imagine the axis of the rod lying along the axis of x of the stationary system of co-ordinates, and that a uniform motion of parallel translation with velocity v along the axis of x in the direction of increasing x is then imparted to the rod. We now inquire as to the length of the *moving* rod, and imagine its length to be ascertained by the following *two* operations:
>
> (a) The observer *moves together with the given measuring-rod and the rod to be measured*, and *measures the length* of the rod directly by superposing the measuring-rod, *in just the same way as if all three were at rest*.

(b) By means of *stationary* clocks set up in the *stationary* system and synchronizing in accordance with [the light-signal synchronization], the observer ascertains at what points of the *stationary* system the two ends of the rod to be measured are located at a definite time. The distance between these two points, measured by the measuring-rod already employed, which in this case is *at rest*, is also a length which may be designated "the length of the rod."

In accordance with the principle of relativity the length *to be discovered by the operation (a)*—we will call it "the length of the rod in the moving system"—must be equal to the length l of *the stationary* rod.

The length to be discovered by the operation (b) we will call "the length of the (*moving*) rod in the *stationary* system." This we shall determine on the basis of our two *principles*, and we shall find that it *differs* from l. [all italics added]

From a careful reading of this simple example of Einstein, and also from other usual applications of the RP, for example from the usual derivation of the electromagnetic field of a uniformly moving point charge (Remark 7), one concludes with the following more detailed formulation (Szabó 2004):

(RP) The physical description of the behavior of a system co-moving as a whole with an inertial frame K, expressed in terms of the results of measurements obtainable by means of measuring equipments co-moving with K, take the same form as the description of the similar behavior of the same system when it is co-moving with another inertial frame K', expressed in terms of the measurements with the same equipments when they are co-moving with K'.

(Q1) is a legitime question, in spite of the obvious fact that the RP is a meta-law, that is a law about the laws of nature. For, whether it is true or not is determined by the laws of nature; whether the laws of nature are true or not depends on how the things are in the physical world. So, in spite of the formal differences, the epistemological status of the RP is ultimately the same as that of the ordinary physical laws.

Apparently, to answer question (Q1), that is to verify whether the principle holds for the laws describing electromagnetic phenomena, the following will be needed:

(a) We must be able to tell when two electrodynamical systems are the same except that they are moving, as a whole, relative to each other—one system is co-moving with K, the other is co-moving with K'.

(b) We must have proper descriptions of the behavior of both systems, expressed in terms of two different sets of corresponding variables—one belonging to K the other to K'.

(c) The RP would be completely meaningless if we mix up different physical quantities, because, in terms of different variables, one and the same physical law in one and the same inertial frame of reference can be expressed in different forms. Consequently, we must be able to tell which variable in K corresponds to which variable in K'; that is, how the physical quantities defined in the two different inertial frames are identified. Also, question (Q2) by itself would be meaningless without such an identification. The most obvious idea is that we identify those physical quantities that have identical empirical definitions.

(d) The empirical definition of a physical quantity is based on standard measuring equipments and standard operational procedures. How do the observers in different reference frames share these standard measuring equipments and operational procedures? Do they all base their definitions on the same standard measuring equipments? On the one hand, they must do something like that, otherwise any comparison between their observations would be meaningless. On the other hand, however, it is quite obvious that the principle is understood in a different way—see the above quoted passage of Einstein. That is to say, if the standard measuring equipment defining a physical quantity ξ is, for example, at rest in K and, therefore, moving in K', then the observer in K' does not define the corresponding ξ' as the physical quantity obtainable by means of the original standard equipment—being at rest in K and moving in K'—but rather as the one obtainable by means of the same standard equipment *in another state of motion*, namely when it is at rest in K' and moving in K. Thus, we must be able to tell when two measuring equipments are the same, except that they are moving, as a whole, relative to each other—one is at rest relative to K, the other is at rest relative to K'. Similarly, we must be able to tell when two operational procedures performed by the two observers are the "same"; in spite of the fact that the procedure performed in K' obviously differs from the one performed in K.

(e) Obviously, in order to compare these procedures we must know what the procedures exactly are; that is, we must have precise operational definitions of the quantities in question.

All these issues naturally arise if we want to verify *empirically* whether the RP is a true law of nature for electrodynamical phenomena. For, empirical verification, no doubt, requires the physicist to know which body of observational data indicates that statement (RP) is true or false. Without entering here into the discussion of verificationism in general, we have only two remarks to make.

First, our approach is entirely compatible with confirmation/semantic holism. The position we are advocating here is essentially holistic. We accept it as true that "our statements about the external world face the tribunal of sense experience not individually but only as a corporate body" (Quine 1951). On the one hand this means that a theory, together with its semantics, as a whole is falsified if any single sentence of its deductive closure is empirically falsified; any part of the theory can be reconsidered—the basic deductive system, the applied mathematical tools, and the semantic rules of correspondence included. On the other hand, contrary to what is often claimed, this kind of holism does not imply that the sentences of a physical theory, at least partly, cannot be provided with empirical meaning by reducing them to a sense-datum language.

In our view, on the contrary, what semantic holism implies is that the empirical definition of a physical term must not be regarded in isolation from the empirical definitions of the other terms involved in the definition. For example, as we will see, the empirical definitions of electrodynamical quantities cannot be separated from the notion of mass; in fact, the definitions in the usual ED and mechanics textbooks, together, constitute an incoherent body of definitions with circularities. This is perhaps a forgivable sin in the textbook literature. But, in philosophy of physics, the recognition of these incoherencies should not lead us to jettison the empirical content of an individual statement; on the contrary, we have to reconstruct our theories on the basis of a sufficiently large coherent body of empirical/operational definitions. In our understanding, this is the real holistic approach—a super-holistic, if you like.

Second, in fact, our arguments in this paper will rely on the verificationist theory of meaning in the following very weak sense: In physics, the meaning of a term standing for a *measurable quantity* which is supposed to characterize an objective feature of physical reality is determined by the empirical operations with which the value of the quantity in question can be ascertained. Such a limited verificationism is widely accepted among physicists; almost all ED textbooks start with some descriptions of how the basic quantities like electric charge, electric and magnetic field strengths, etc. are empirically interpreted. Our concern is that these empirical definitions do not satisfy the standard of the above mentioned super-holistic coherence, and the solution of the problem is not entirely trivial.

In any event, in this paper, the demand for precise operational definitions of electrodynamical quantities emerges not from this epistemological context; not from philosophical ideas about the relationship between physical theories, sense-data, and the external reality; not from the context of questions (Q1) and (Q2). The problem of operational definitions is raised as a problem of pure theoretical physics, in the context of the *inner consistency* of our theories. The reason is that instead of the empirical questions (Q1) and (Q2) we will in fact investigate the following two *theoretical* questions:

(Q3) Is the RP consistent with the laws of ED in a single inertial frame of reference?

(Q4) Are, at least, the derived transformation rules consistent with the laws of ED in a single frame of reference?

The basic idea is what J. S. Bell (1987, p. 77) calls "Lorentzian pedagogy", according to which "the laws of physics in any one reference frame account for all physical phenomena, including the observations of moving observers". That is to say, if our physical theories in any one reference frame provide a complete enough account for our world, then all we will say about "operational" definitions and about "empirical" facts—issues (a)–(e) included—must be represented and accounted *within the theory itself*; and the laws of physics—again, in any one reference frame—must determine whether the RP is true or not.

Thus, accordingly, the paper will consists of the following major parts. First of all we will give a general mathematical formulation of the RP and covariance. It will be shown that covariance is not only not sufficient for the RP, but it is not even necessary. In the second part, we will clarify the semantic issues addressed in point (e). In the third part, we will derive the transformation rules of the electrodynamical quantities, from the operational definitions and from the laws of ED in one inertial frame of reference—independently of the RP; by which we will answer our question (Q4). In this way—again, independently of the RP—we will show the covariance of the Maxwell–Lorentz equations.

As we will see, whether the RP holds, as well as whether it implies covariance, hinges on the details of the solutions describing the behavior of moving objects. This raises conceptual problems concerning the meaning of the notion "the same system in a collective motion". As it will be discussed in the last section, in case of ED, there seems no satisfactory solution to this conceptual problem; thus, contrary to the widespread views, the question we asked in the title has no obvious answer.

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Throughout it will be assumed that space and time coordinates are already defined in all inertial frames of reference; that is, in an arbitrary inertial frame K, space tags $\mathbf{r}(A) = (x(A), y(A), z(A)) \in \mathbb{R}^3$ and a time tag $t(A) \in \mathbb{R}$ are assigned to every event A—by means of some empirical operations.[1] We also assume that the assignment is mutually unambiguous, such that there is a one to one correspondence between the space and time tags in arbitrary two inertial frames of reference K and K'; that is, the tags $(x'(A), y'(A), z'(A), t'(A))$ can be expressed by the tags $(x(A), y(A), z(A), t(A))$, and vice versa. The concrete form of this functional relation is an empirical question. In this paper, we will take it for granted that this functional relation is the well-known Lorentz transformation; and the calculations, particularly in section 6, will rest heavily on this assumption. It must be emphasized however that we stipulate the Lorentz transformation of the kinematical quantities as an *empirical* fact, without suggesting that the usual *derivations* of these transformation rules from the RP/constancy of the speed of light are unproblematic. In fact, these derivations raise questions similar to (Q1)–(Q4), concerning the kinematical quantities. In this paper, however, we focus our analysis only on the electrodynamical quantities.

It must be also noted that the transformation of the kinematical quantities, alone, does not determine the transformation of the electrodynamical quantities. As we will see, the latter is determined by the kinematical Lorentz transformation in conjunction with the operational definitions of the electrodynamical quantities and some empirical facts, first of all the relativistic version of the Lorentz equation of motion.

Below we recall the most important formulas we will use. For the sake of simplicity, we will assume the standard situation: the corresponding axises are parallel and K' is moving along the x-axis with velocity $\mathbf{V} = (V, 0, 0)$ relative to K, and the two origins coincide at time 0.[2] Throughout the paper we will use the following notations: $\gamma(\ldots) = \left(1 - \frac{(\ldots)^2}{c^2}\right)^{-\frac{1}{2}}$ and $\gamma = \gamma(V)$.

The connection between the space and time tags of an event A in K and K' is the following:

[1] In fact, to give precise empirical definitions of the basic spatio-temporal quantities in physics is not a trivial problem (Szabó 2009).

[2] All "vectors" are meant to be in $\mathbb{R}^3$; boldface letters $\mathbf{r}, \mathbf{v}, \mathbf{E} \ldots$ simply denote vector matrices.

$$
\begin{aligned}
x'(A) &= \gamma\left(x(A) - Vt(A)\right) & (1)\\
y'(A) &= y(A) & (2)\\
z'(A) &= z(A) & (3)\\
t'(A) &= \gamma\left(t(A) - c^{-2}Vx(A)\right) & (4)
\end{aligned}
$$

Let A be an event on the worldline of a particle. For the velocity of the particle at A we have:

$$
\begin{aligned}
v'_x(A) &= \frac{v_x(A) - V}{1 - c^{-2}v_x(A)V} & (5)\\
v'_y(A) &= \frac{\gamma^{-1}v_y(A)}{1 - c^{-2}v_x(A)V} & (6)\\
v'_z(A) &= \frac{\gamma^{-1}v_z(A)}{1 - c^{-2}v_x(A)V} & (7)
\end{aligned}
$$

We shall use the inverse transformation in the following special case:

$$
\begin{aligned}
\mathbf{v}'(A) = \left(v', 0, 0\right) &\quad\mapsto\quad \mathbf{v}(A) = \left(\frac{v' + V}{1 + c^{-2}v'V}, 0, 0\right) & (8)\\
\mathbf{v}'(A) = \left(0, 0, v'\right) &\quad\mapsto\quad \mathbf{v}(A) = \left(V, 0, \gamma v'\right) & (9)
\end{aligned}
$$

The transformation rule of acceleration is much more complex, but we need it only for $\mathbf{v}'(A) = (0, 0, 0)$:

$$
\begin{aligned}
a'_x(A) &= \gamma^3 a_x(A) & (10)\\
a'_y(A) &= \gamma^2 a_y(A) & (11)\\
a'_z(A) &= \gamma^2 a_z(A) & (12)
\end{aligned}
$$

We will also need the y-component of acceleration in case of $\mathbf{v}'(A) = (0, 0, v')$:

$$
a'_y(A) = \gamma^2 a_y(A) \tag{13}
$$

2 Mathematics of the relativity principle

Let us try to unpack (RP) in a more mathematical way. Consider some variables $\xi_1, \xi_2, \ldots \xi_n$ in K, operationally defined by means of measuring equipments at rest in K. (Depending on the context, variables $\xi_1, \xi_2, \ldots \xi_n$ may contain space-time coordinates and many other variables like the values of field strengths, values of source densities, etc.) Let $\xi'_1, \xi'_2, \ldots \xi'_n$ denote the corresponding variables in K'. "Corresponding" means that $\xi'_1, \xi'_2, \ldots \xi'_n$ are the physical quantities obtainable by means of the *same operations* with the *same equipments*, but *in different state of motion*, namely, in which they are co-moving with K'. Since, for all $i = 1, 2, \ldots n$, both ξ_i and ξ'_i are measured by the same equipment—although in different physical conditions—with the same pointer scale, it is plausible to assume that the possible values of ξ_i and ξ'_i range over the same $\sigma_i \subseteq \mathbb{R}$. We introduce the following notation: $\Sigma = \times_{i=1}^{n}\sigma_i$.

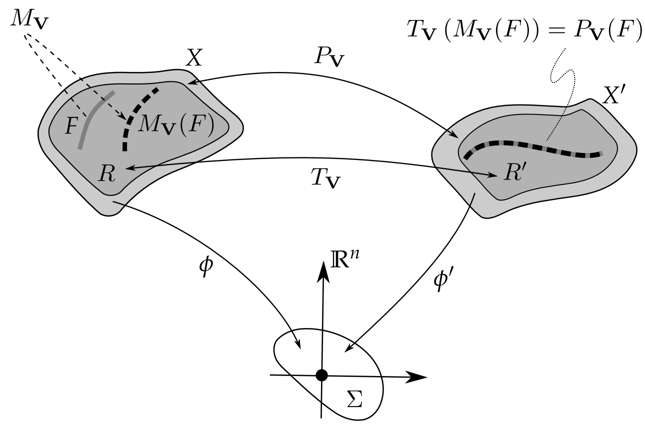

Figure 1: The relativity principle

In spite of the above correspondance, variables $\xi_1, \xi_2, \ldots \xi_n$ and $\xi'_1, \xi'_2, \ldots \xi'_n$ are generally *different* physical quantities—as it will be clearly seen, for example, in ED—due to the fact that the operations by which the quantities are defined are performed under *different* physical conditions; with measuring equipments of *different* states of motion (also see Remark 8). Consequently, $(\xi_1, \xi_2, \ldots \xi_n)$ and $(\xi'_1, \xi'_2, \ldots \xi'_n)$ must not be regarded as elements of the *same* "space"; although the numeric values of the physical quantities, in both cases, can be represented in $\mathbb{R}^n$. In other words, $(5, 12, \ldots 61) \in \mathbb{R}^n$ generally represents different type of *physical* configuration of the world when $\xi_1 = 5, \xi_2 = 12, \ldots \xi_n = 61$ versus $\xi'_1 = 5, \xi'_2 = 12, \ldots \xi'_n = 61$.

Mathematically, this idea can be expressed by considering two different n-dimensional manifolds X and X', each covered by one *global* coordinate system, ϕ and ϕ' respectively (Fig. 1). The coordinate maps ϕ and ϕ' play distinguished roles among the possible coordinate maps of the two manifolds, by carrying physical meaning: $\phi : X \to \Sigma$ assigns to every point of X one of the possible n-tuples of numerical values of physical quantities $\xi_1, \xi_2, \ldots \xi_n$; and $\phi' : X' \to \Sigma$ has similar physical meaning with $\xi'_1, \xi'_2, \ldots \xi'_n$. In this way, a point of the manifold $x \in X$ represents a *type of physical configuration* of the world (a type of state of affairs, a type of situation), namely in which $\xi_1 = \phi_1(x), \xi_2 = \phi_2(x), \ldots \xi_n = \phi_n(x)$.[3] Similarly, a point $x' \in X'$ represents the type of physical configuration of the world in which $\xi'_1 = \phi'_1(x'), \xi'_2 = \phi'_2(x'), \ldots \xi'_n = \phi'_n(x')$. Again, these types of physical configurations are generally different, even if it were the case that $\phi(x) = \phi'(x') \in \mathbb{R}^n$.

In the above sense, the points of X and the points of X' range over all value combinations of physical quantities $\xi_1, \xi_2, \ldots \xi_n$ and $\xi'_1, \xi'_2, \ldots \xi'_n$. It might be the case however that some combinations are impossible in the sense that they never come to existence in the physical world. Let us denote by $R \subseteq X$ and $R' \subseteq X'$ the physically admissible parts of X and X'. Note that $\phi(R) \neq \phi'(R')$, in general.

Now we introduce two maps between X and X', which are of entirely dif-

[3] $\phi_i = \pi_i \circ \phi$, where π_i is the i-th coordinate projection in $\mathbb{R}^n$.

ferent nature. The first one is a bijection $P_{\mathbf{V}} : X \to X'$ which is uniquely determined by the two distinguished coordinate maps ϕ and ϕ':

$$P_{\mathbf{V}} \stackrel{def}{=} \left(\phi'\right)^{-1} \circ \phi \tag{14}$$

In contrast, the second one is determined by contingent physical facts. Assume we *observe* that the types of physical configuration represented by the points of R and R' are not independet from each other; we find a bijection $T_{\mathbf{V}}$ such that whenever the configuration of the world is of type $x \in R$ then is also of type $T_{\mathbf{V}}(x) \in R'$, and vica versa. Thus, we have a bijection

$$T_{\mathbf{V}} : X \supseteq R \to R' \subseteq X' \tag{15}$$

which we call the transformation rules of the physical quantities.

It is of course difficult to give a formal description of a "behavior" of a physical system in general. But we are probably not far from the truth if we assume that a description of a particular behavior of a system in a given situation is a *relation* between the physical quantities. Let F be such a functional relation between the physical quantities $\xi_1, \xi_2, \ldots \xi_n$. In general, it can be given as a subset of R. Consider the following subsets[4] of X', determined by $F \subset R$:

$P_{\mathbf{V}}(F) \subseteq X'$ which formally is the "primed F", that is the "description" of exactly the same "form" as F, but in the primed variables. Note that relation $P_{\mathbf{V}}(F)$ does not necessarily describe a true physical situation, as it can be not realized in nature.

$T_{\mathbf{V}}(F) \subseteq R'$ which is the same description of the same physical situation as F, but *expressed* in the primed variables.

In order to formulate the RP we need one more concept. Let the situation described by F be considered as the one in which the system, as a whole, is co-moving with K. (In principle, arbitrary F allowed by the laws of physics can be considered as describing a situation in which the system is co-moving with K.) Let $M_{\mathbf{V}}(F) \subset R$ be another relation, which is supposed to describe the same system in the same situation, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K, that is, co-moving with reference frame K'. As we will see later on, $M_{\mathbf{V}}$ is a vague concept (see also Szabó 2004). Moreover, one may not assume that every $F \subset R$ describing a situation in which the system is, as a whole, stipulated as co-moving with K, has a counterpart $M_{\mathbf{V}}(F)$ for arbitrary velocity $\mathbf{V}$; because $M_{\mathbf{V}}(F)$ must describe a *real* physical situation, admitted by the relevant physical laws.[5]

Now, applying these concepts, what the RP states is the following:

$$T_{\mathbf{V}}\left(M_{\mathbf{V}}(F)\right) = P_{\mathbf{V}}(F) \tag{16}$$

[4]We denote the map of type $X \to X'$ and its direct image maps of type $2^X \to 2^{X'}$ and $2^{2^X} \to 2^{2^{X'}}$ or their restrictions by the same symbol.

[5]For example, let the system in question be consisting of a single particle, and let F be the description of the particle's behavior when it is moving with constant velocity $\mathbf{w}$ relative to K. And let $M_{\mathbf{V}}(F)$ be understood as the relation describing the motion of a similar particle with a constant velocity $\tilde{\mathbf{w}}$, such that the relative velocity of the two particles is $\tilde{\mathbf{w}} - \mathbf{w} = \mathbf{V}$. (All velocities are relative to K.) Now, $M_{\mathbf{V}}(F)$ represents a possible physical situation only if $|\tilde{\mathbf{w}}| < c$.

or equivalently,

$$P_{\mathbf{V}}(F) \subset R' \text{ and } M_{\mathbf{V}}(F) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{17}$$

for all $F \subset R$ for which there exists a physically admissible $M_{\mathbf{V}}(F)$.

Let us turn to the situation similar to ED, when the physical system in question is described—in K—by a system of equations $\mathcal{E}$; the functional relation $F \subset R$ describing a particular behavior of the system is now given as a solution of $\mathcal{E}$. In general, $\mathcal{E}$ can be a set of algebraic equations, ordinary and partial integro-differential equations, linear and nonlinear, whatever. Without specifying these details, we will identify a system of equations with the set of its solutions; that is, as a set of subsets of R: $\mathcal{E} \subset 2^R$. We only make a physical assumption about $\mathcal{E}$: Let $\mathcal{E}^{\mathbf{V}} \subseteq \mathcal{E}$ denote the subset of those solutions F for which there exists a physically admissible counterpart $M_{\mathbf{V}}(F)$. We assume that $M_{\mathbf{V}}(F) \in \mathcal{E}$ for all $F \in \mathcal{E}^{\mathbf{V}}$; that is to say, the solutions of $\mathcal{E}$ are capable to describe all possible physical situations, in which the system in question is in all physically possible states of motion. Thus, $M_{\mathbf{V}}$ can be regarded as a map $M_{\mathbf{V}} : \mathcal{E} \supseteq \mathcal{E}^{\mathbf{V}} \to \mathcal{E}$.

Thus, in this case, the RP can be formulated as a condition for the solutions of $\mathcal{E}$:

$$T_{\mathbf{V}}\left(M_{\mathbf{V}}(F)\right) = P_{\mathbf{V}}(F) \quad \text{for all } F \in \mathcal{E}^{\mathbf{V}} \tag{18}$$

or, in the more often used equivalent form,

$$P_{\mathbf{V}}(F) \subset R' \text{ and } M_{\mathbf{V}}(F) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \quad \text{for all } F \in \mathcal{E}^{\mathbf{V}} \tag{19}$$

Remark 1. Let us illustrate these concepts with a well-known textbook example of a static versus uniformly moving charged particle. The static field of a charge q being at *rest* at point $\mathbf{r}_0$ in K is the following[6]:

$$F \quad \begin{cases} E_x = \dfrac{q\,(x-x_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_y = \dfrac{q\,(y-y_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ E_z = \dfrac{q\,(z-z_0)}{\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{3/2}} \\ B_x = 0 \\ B_y = 0 \\ B_z = 0 \end{cases} \tag{20}$$

where $F \subset X$ is understood as given in local coordinates, that is, $\phi(F) \subset \mathbb{R}^n$ is determined by the above equations.

The stationary field of a charge q *moving* at constant velocity $\mathbf{V} = (V, 0, 0)$ relative to K can be obtained by solving the equations of ED (in K) with the time-depending source (for example, Jackson 1999, pp. 661–665):

[6] In this example, $\mathbf{E}, \mathbf{B}$ and q denote the usual textbook concepts of field strengths and charge, which are not entirely the same as the ones we will introduce in the next section.

$$M_{\mathbf{V}}(F) \quad \begin{cases} E_x = \dfrac{qX_0}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ E_y = \dfrac{\gamma q\,(y-y_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ E_z = \dfrac{\gamma q\,(z-z_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ B_x = 0 \\ B_y = -c^{-2} V E_z \\ B_z = c^{-2} V E_y \end{cases} \tag{21}$$

where $X_0 = \gamma\,(x - (x_0 + Vt))$.

Now, we form the same expressions as (20)—describing the rest system—but in the *primed* variables of the co-moving reference frame K':

$$P_{\mathbf{V}}\,(F) \quad \begin{cases} E'_x = \dfrac{q'\,(x'-x'_0)}{\left((x'-x'_0)^2 + (y'-y'_0)^2 + (z'-z'_0)^2\right)^{3/2}} \\ E'_y = \dfrac{q'\,(y'-y'_0)}{\left((x'-x'_0)^2 + (y'-y'_0)^2 + (z'-z'_0)^2\right)^{3/2}} \\ E'_z = \dfrac{q'\,(z'-z'_0)}{\left((x'-x'_0)^2 + (y'-y'_0)^2 + (z'-z'_0)^2\right)^{3/2}} \\ B'_x = 0 \\ B'_y = 0 \\ B'_z = 0 \end{cases} \tag{22}$$

By means of the Lorentz transformation rules of the space-time coordinates, the field strengths and the electric charge, one can express (22) in terms of the original variables of K:

$$T_{\mathbf{V}}^{-1}\,(P_{\mathbf{V}}(F)) \quad \begin{cases} E_x = \dfrac{qX_0}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ E_y = \dfrac{\gamma q\,(y-y_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ E_z = \dfrac{\gamma q\,(z-z_0)}{\left(X_0^2 + (y-y_0)^2 + (z-z_0)^2\right)^{3/2}} \\ B_x = 0 \\ B_y = -c^{-2} V E_z \\ B_z = c^{-2} V E_y \end{cases} \tag{23}$$

We find that the result is indeed the same as (21) describing the field of the moving charge: $M_{\mathbf{V}}(F) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right)$. That is to say, we see that RP is true in this particular case.[7]

Reversely, assuming that the RP, that is (17), holds, one can derive the field of the moving charge (21) from the static field (20).

Now we have a strict mathematical formulation of the RP for a physical system described by a system of equations $\mathcal{E}$. Remarkably, however, we still have not encountered the concept of "covariance" of equations $\mathcal{E}$. The reason is that the RP and the covariance of equations $\mathcal{E}$ are not equivalent—in contrast to what many believe. In fact, the logical relationship between the two conditions is much more complex. To see this relationship in more details, we previously need to clarify a few things.

Consider the following two sets: $P_{\mathbf{V}}(\mathcal{E}) = \{P_{\mathbf{V}}(F)|F \in \mathcal{E}\}$ and $T_{\mathbf{V}}(\mathcal{E}) = \{T_{\mathbf{V}}(F)|F \in \mathcal{E}\}$. Since a system of equations can be identified with its set of solutions, $P_{\mathbf{V}}(\mathcal{E}) \subset 2^{X'}$ and $T_{\mathbf{V}}(\mathcal{E}) \subset 2^{R'}$ can be regarded as two systems of equations for functional relations between $\xi'_1, \xi'_2, \ldots \xi'_n$. In the primed variables, $P_{\mathbf{V}}(\mathcal{E})$ has "the same form" as $\mathcal{E}$. Nevertheless, it can be the case that $P_{\mathbf{V}}(\mathcal{E})$ does not express a true physical law, in the sense that its solutions do not necessarily describe true physical situations. In contrast, $T_{\mathbf{V}}(\mathcal{E})$ is nothing but $\mathcal{E}$ expressed in variables $\xi'_1, \xi'_2, \ldots \xi'_n$.

Now, covariance intuitively means that equations $\mathcal{E}$ "preserve their forms against the transformation $T_{\mathbf{V}}$". That is, in terms of the formalism we developed:

$$T_{\mathbf{V}}(\mathcal{E}) = P_{\mathbf{V}}(\mathcal{E}) \tag{24}$$

or, equivalently,

$$P_{\mathbf{V}}(\mathcal{E}) \subset 2^{R'} \text{ and } \mathcal{E} = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\mathcal{E})\right) \tag{25}$$

The first thing we have to make clear is that—even if we know or presume that it holds—covariance (25) is obviously *not sufficient* for the RP (19). For, (25) only guarantees the invariance of the set of solutions, $\mathcal{E}$, against $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$, but it says nothing about which solution of $\mathcal{E}$ corresponds to which solution; while it is the very essence of the RP that the solution $M_{\mathbf{V}}(F)$, describing the system *in motion* relative to K, corresponds to solution $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}(F)$. [8]

What makes the matter more complex is that covariance is not only not sufficient for the RP, but it is *not even necessary* (Fig. 2). The RP only implies that

$$T_{\mathbf{V}}(\mathcal{E}) \supseteq T_{\mathbf{V}}\left(M_{\mathbf{V}}\left(\mathcal{E}^{\mathbf{V}}\right)\right) = P_{\mathbf{V}}\left(\mathcal{E}^{\mathbf{V}}\right) \tag{26}$$

(18) implies (24) only if we have some extra conditions; for example

$$\mathcal{E}^{\mathbf{V}} = \mathcal{E} \tag{27}$$

$$M_{\mathbf{V}}\left(\mathcal{E}\right) = \mathcal{E} \tag{28}$$

[7] It will be clear in section 8 that this is, indeed, a very peculiar case, when the RP is meaningful and true.

[8] The difference between covariance and the RP is obvious from the well-known applications of the RP. For example, what we use in the derivation of electromagnetic field of a uniformly moving point charge (Remark 7) is not the covariance of the equations, but statement (19), that is, what the RP claims about the solutions of the equations in details.

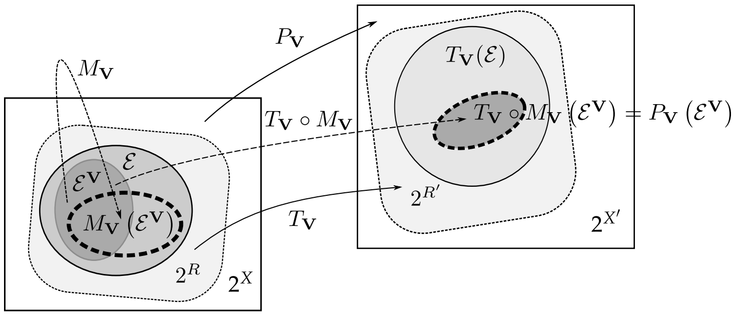

Figure 2: The RP only implies that $T_{\mathbf{V}} \circ M_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}}) = P_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}})$. Covariance of $\mathcal{E}$ would require that $T_{\mathbf{V}}(\mathcal{E}) = P_{\mathbf{V}}(\mathcal{E})$, which is generally not the case

We will return to the problem of how little we can say about $M_{\mathbf{V}}$ in general; what we have to see here is that the RP in itself does not imply the covariance of the physical equations.

What is the situation in ED?

- As we will see later, the very concept of $M_{\mathbf{V}}$ is problematic in ED, and this fact will raise further difficulties. Consequently, there is no guarantee that conditions (27)–(28) are satisfied.

- In any event, we will show the covariance of the Maxwell–Lorentz equations, *independently of the RP*; in the sense that we will determine the transformation of the electrodynamical quantities, independently of the RP—and without presuming the covariance, of course—and will see that the equations are covariant against these transformations.

- The covariance of the Maxwell–Lorentz equations, on the other hand, is not sufficient; whether the RP holds in ED will remain a question we will discuss in section 8.

Let us finally consider the situation, similar to ED, when the solutions of a system of equations $\mathcal{E}$ are specified by (initial and/or boundary value) extra conditions. In our general formalism, an extra condition for $\mathcal{E}$ is a system of equations $\psi \subset 2^X$ such that there exists exactly one solution $[\psi]_{\mathcal{E}}$ satisfying both $\mathcal{E}$ and ψ. That is, $\mathcal{E} \cap \psi = \{[\psi]_{\mathcal{E}}\}$, where $\{[\psi]_{\mathcal{E}}\}$ is a singleton set. Since $\mathcal{E} \subset 2^R$, without loss of generality we may assume that $\psi \subset 2^R$.

Since $P_{\mathbf{V}}$ and $T_{\mathbf{V}}$ are injective, $P_{\mathbf{V}}(\psi)$ and $T_{\mathbf{V}}(\psi)$ are extra conditions for equations $P_{\mathbf{V}}(\mathcal{E})$ and $T_{\mathbf{V}}(\mathcal{E})$ respectively, and we have

$$P_{\mathbf{V}}\left([\psi]_{\mathcal{E}}\right) = \left[P_{\mathbf{V}}(\psi)\right]_{P_{\mathbf{V}}(\mathcal{E})} \tag{29}$$

$$T_{\mathbf{V}}\left([\psi]_{\mathcal{E}}\right) = \left[T_{\mathbf{V}}(\psi)\right]_{T_{\mathbf{V}}(\mathcal{E})} \tag{30}$$

for all extra conditions ψ for $\mathcal{E}$. Similarly, if $P_{\mathbf{V}}(\mathcal{E}), P_{\mathbf{V}}(\psi) \subset 2^{R'}$ then $T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\psi))$ is an extra condition for $T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\mathcal{E}))$, and

$$\left[T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\psi))\right]_{T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\mathcal{E}))} = T_{\mathbf{V}}^{-1}\left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})}\right) \tag{31}$$

Consider now a set of extra conditions $\mathcal{C} \subset 2^{2^R}$. Assume that $\mathcal{C}$ is a *parametrizing set of extra conditions* for $\mathcal{E}$; by which we mean that for all $F \in \mathcal{E}$ there exists exactly one $\psi \in \mathcal{C}$ such that $F = [\psi]_{\mathcal{E}}$; in other words,

$$\mathcal{C} \ni \psi \mapsto [\psi]_{\mathcal{E}} \in \mathcal{E} \tag{32}$$

is a bijection.

Let us introduce the following notation:

$$\mathcal{C}^{\mathbf{V}} \stackrel{def}{=} \left\{ \psi \in \mathcal{C} | \, [\psi]_{\mathcal{E}} \in \mathcal{E}^{\mathbf{V}} \right\} \tag{33}$$

$M_{\mathbf{V}} : \mathcal{E} \supseteq \mathcal{E}^{\mathbf{V}} \to \mathcal{E}$ was introduced as a map between solutions of $\mathcal{E}$. Now, as there is a one-to-one correspondence between the elements of $\mathcal{C}$ and $\mathcal{E}$, it generates a map $M_{\mathbf{V}} : \mathcal{C} \supseteq \mathcal{C}^{\mathbf{V}} \to \mathcal{C}$, such that

$$[M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = M_{\mathbf{V}} \left([\psi]_{\mathcal{E}} \right) \tag{34}$$

Thus, from (29) and (34), the RP, that is (18), has the following form:

$$T_{\mathbf{V}} \left([M_{\mathbf{V}}(\psi)]_{\mathcal{E}} \right) = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \qquad \text{for all } \psi \in \mathcal{C}^{\mathbf{V}} \tag{35}$$

or, equivalently, (19) reads

$$[P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \subset R' \text{ and } [M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = T_{\mathbf{V}}^{-1} \left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \right) \tag{36}$$

We will make use of the following theorem:

Theorem 1. *Assume that the system of equations $\mathcal{E} \subset 2^R$ is covariant, that is, (24) is satisfied. Then,*

(i) for all $\psi \in \mathcal{C}^{\mathbf{V}}$, $T_{\mathbf{V}} \left(M_{\mathbf{V}} \left(\psi \right) \right)$ is an extra condition for the system of equations $P_{\mathbf{V}} \left(\mathcal{E} \right)$, and, (35) is equivalent to the following condition:

$$[T_{\mathbf{V}} \left(M_{\mathbf{V}}(\psi) \right)]_{P_{\mathbf{V}}(\mathcal{E})} = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{37}$$

(ii) for all $\psi \in \mathcal{C}^{\mathbf{V}}$, $P_{\mathbf{V}} \left(\psi \right) \subset 2^{R'}$, $T_{\mathbf{V}}^{-1} \left(P_{\mathbf{V}} \left(\psi \right) \right)$ is an extra condition for the system of equations $\mathcal{E}$ and (36) is equivalent to the following condition:

$$[M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = \left[T_{\mathbf{V}}^{-1} \left(P_{\mathbf{V}} \left(\psi \right) \right) \right]_{\mathcal{E}} \tag{38}$$

Proof. (i) Obviously, $T_{\mathbf{V}} \left(\mathcal{E} \right) \cap T_{\mathbf{V}} \left(M_{\mathbf{V}} \left(\psi \right) \right)$ exists and is a singleton; and, due to (24), it is equal to $P_{\mathbf{V}} \left(\mathcal{E} \right) \cap T_{\mathbf{V}} \left(M_{\mathbf{V}} \left(\psi \right) \right)$; therefore this latter is a singleton, too. Applying (30) and (24), we have

$$T_{\mathbf{V}} \left([M_{\mathbf{V}}(\psi)]_{\mathcal{E}} \right) = [T_{\mathbf{V}} \left(M_{\mathbf{V}} \left(\psi \right) \right)]_{T_{\mathbf{V}}(\mathcal{E})} = [T_{\mathbf{V}} \left(M_{\mathbf{V}} \left(\psi \right) \right)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{39}$$

therefore, (37) implies (36).

(ii) Similarly, due to $P_{\mathbf{V}} \left(\psi \right) \subset 2^{R'}$ and (25), $\mathcal{E} \cap T_{\mathbf{V}}^{-1} \left(P_{\mathbf{V}} \left(\psi \right) \right)$ exists and is a singleton. Applying (31) and (25), we have

$$T_{\mathbf{V}}^{-1} \left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \right) = \left[T_{\mathbf{V}}^{-1} \left(P_{\mathbf{V}} \left(\psi \right) \right) \right]_{T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\mathcal{E}))} = \left[T_{\mathbf{V}}^{-1} \left(P_{\mathbf{V}} \left(\psi \right) \right) \right]_{\mathcal{E}} \tag{40}$$

that is, (38) implies (36). □

Remark 2. As we see, $M_{\mathbf{V}}$ plays a crucial role. Formally, one could say, the RP is relative to a given definition of $M_{\mathbf{V}}$. Therefore, the physical content of the RP depends on how $M_{\mathbf{V}}(F)$ is physically understood. But, what does it mean to say that a physical system is the same and of the same behavior as the one described by F, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K? Without answering this crucial question the RP is meaningless. On the other hand, the answer is not at all obvious. The vagueness of $M_{\mathbf{V}}$ leads to serious problems to which we will return in section 8.

In fact, the same ambiguities are present in the definitions of quantities $\xi'_1, \xi'_2, \ldots \xi'_n$—and, therefore, in the meanings of $T_{\mathbf{V}}$ and $P_{\mathbf{V}}$. For, $\xi'_1, \xi'_2, \ldots \xi'_n$ are not simply arbitrary variables assigned to reference frame K', in one-to-one relations with $\xi_1, \xi_2, \ldots \xi_n$, but the physical quantities obtainable by means of the same operations with the same measuring equipments as in the operational definitions of $\xi_1, \xi_2, \ldots \xi_n$, except that everything is in a collective motion with velocity $\mathbf{V}$. Therefore, we should know what we mean by "the same measuring equipment but in collective motion". From this point of view, it does not matter whether the system in question is the object to be observed or a measuring equipment involved in the observation.

One might claim that $M_{\mathbf{V}}(F)$, describing the moving system, is equal to the "Lorentz boosted solution" *by definition*:

$$M_{\mathbf{V}}(F) \quad \stackrel{def}{=} \quad T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{41}$$

At first sight this suggestion seems to resolve all troubles around $M_{\mathbf{V}}$. But a little reflection shows that it is, in fact, untenable.

(a) In this case, (19) would read

$$T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{42}$$

That is, the RP would become a tautology; a statement which is always true, independently of any contingent fact of nature; independently of the actual behavior of moving physical objects; and independently of the actual empirical meanings of physical quantities $\xi'_1, \xi'_2, \ldots \xi'_n$. But, the RP is supposed to be a fundamental *law of nature*. Note that a tautology is entirely different from a fundamental principle, even if the principle is used as a fundamental hypothesis or fundamental premise of a theory, from which one derives further physical statements. For, a fundamental premise, as expressing a contingent fact of nature, is potentially falsifiable by testing its consequences; a tautology is not.

(b) Even if accepted, (41) can provide physical meaning to $M_{\mathbf{V}}(F)$ only if we know the meanings of $T_{\mathbf{V}}$ and $P_{\mathbf{V}}$, that is, if we know the empirical meanings of the quantities denoted by $\xi'_1, \xi'_2, \ldots \xi'_n$. But, the physical meaning of $\xi'_1, \xi'_2, \ldots \xi'_n$ are obtained from the operational definitions: they are the quantities obtained by "the same measurements with the same equipments when they are, as a whole, co-moving with K' with velocity $\mathbf{V}$ relative to K". Symbolically, we need, priory, the concepts of $M_{\mathbf{V}}(\xi_i\text{-}equipment\ at\ rest)$. And this is a conceptual circularity: in order to have the concept of what it is to be an $M_{\mathbf{V}}(brick\, at\, rest)$ the (size)′ of which we would like to ascertain, we need to have the concept of what it is to be an $M_{\mathbf{V}}(measuring\, rod\, at\, rest)$—which is exactly the same conceptual problem.

(c) One might claim that we do not need to specify the concepts of $M_{\mathbf{V}}(\xi_i\text{-}equipment\, at\, rest)$ in order to know the *values* of quantities $\xi_1', \xi_2', \ldots \xi_n'$ we obtain by the measurements with the moving equipments, given that we can know the transformation rule $T_{\mathbf{V}}$ independently of knowing the operational definitions of $\xi_1', \xi_2', \ldots \xi_n'$. Typically, $T_{\mathbf{V}}$ is thought to be derived from the assumption that the RP (19) holds. If however $M_{\mathbf{V}}$ is, by definition, equal to $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$, then in place of (19) we have the tautology (42), which does not determine $T_{\mathbf{V}}$.

(d) Therefore, unsurprisingly, it is not the RP from which transformation rule $T_{\mathbf{V}}$ is routinely deduced, but the covariance (25). As we have seen, however, covariance is, in general, neither sufficient nor necessary for the RP. Whether (19) implies (25) hinges on physical facts, namely, for example, whether (27)–(28) are statisfied. But, if $M_{\mathbf{V}}$ is taken to be $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$ by definition, the RP becomes true—in the form of tautology (42)—but does not imply covariance $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}(\mathcal{E}) = \mathcal{E}$.

(e) Even if we assume that a "transformation rule" function $\phi' \circ T_{\mathbf{V}} \circ \phi^{-1}$ were derived from some independent premises—from the independent assumption of covariance, for example—how do we know that the $T_{\mathbf{V}}$ we obtained and the quantities of values $\phi' \circ T_{\mathbf{V}} \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ are correct plugins for the RP? How could we verify that $\phi' \circ T_{\mathbf{V}} \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ are indeed the values measured by a moving observer applying the same operations with the same measuring equipments, etc.?—without having an independent concept of $M_{\mathbf{V}}$, at least for the measuring equipments?

(f) One could argue that we do not need such a verification; $\phi' \circ T_{\mathbf{V}} \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n)$ can be regarded *as the empirical definition* of the primed quantities:

$$\left(\xi_1', \xi_2', \ldots \xi_n'\right) \stackrel{def}{=} \phi' \circ T_{\mathbf{V}} \circ \phi^{-1}(\xi_1, \xi_2, \ldots \xi_n) \tag{43}$$

This is of course logically possible. The operational definition of the primed quantities would say: ask the observer at rest in K to measure $\xi_1, \xi_2, \ldots \xi_n$ with the measuring equipments at rest in K, and then perform the mathematical operation (43). In this way, however, even the transformation rules would become tautologies; they would be true, no matter how the things are in the physical world.

Thus, we have to reject the view that $M_{\mathbf{V}}(F)$, describing the moving system, is *by definition* equal to the "Lorentz boosted solution" $T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(F))$. The definition of $M_{\mathbf{V}}(F)$ is a matter of convention, to be sure; but, whether it is equal to $T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(F))$ should be a matter of contingent facts of the world.

Remark 3. Finally, let us note a few important facts which can easily be seen in the formalism we developed:

(a) The covariance of a set of equations $\mathcal{E}$ does *not* imply the covariance of a subset of equations separately. It is because a smaller set of equations corresponds to an $\mathcal{E}^* \subset 2^R$ such that $\mathcal{E} \subset \mathcal{E}^*$; and it does not follow from (24) that $T_{\mathbf{V}}(\mathcal{E}^*) = P_{\mathbf{V}}(\mathcal{E}^*)$.

(b) Similarly, the covariance of a set of equations $\mathcal{E}$ does *not* guarantee the covariance of an arbitrary set of equations which is only satisfactory to $\mathcal{E}$; for example, when the solutions of $\mathcal{E}$ are restricted by some extra conditions. Because from (24) it does not follow that $T_{\mathbf{V}}(\mathcal{E}^*) = P_{\mathbf{V}}(\mathcal{E}^*)$ for an arbitrary $\mathcal{E}^* \subset \mathcal{E}$.

(c) The same holds, of course, for the combination of cases (a) and (b); for example, when we have a smaller set of equations $\mathcal{E}^* \supset \mathcal{E}$ together with some extra conditions ψ. For, (24) does not imply that $T_{\mathbf{V}}(\mathcal{E}^* \cap \psi) = P_{\mathbf{V}}(\mathcal{E}^* \cap \psi)$.

(d) However, covariance is guaranteed if a covariant set of equations is restricted with a *covariant* set of extra conditions; because $T_{\mathbf{V}}(\mathcal{E}) = P_{\mathbf{V}}(\mathcal{E})$ and $T_{\mathbf{V}}(\psi) = P_{\mathbf{V}}(\psi)$ trivially imply that $T_{\mathbf{V}}(\mathcal{E} \cap \psi) = P_{\mathbf{V}}(\mathcal{E} \cap \psi)$.

3 Operational definitions of electrodynamical quantities in K

Now we turn to the operational definitions of the fundamental electrodynamical quantities in a single reference frame K and to the basic observational facts about these quantities.

The operational definition of a physical quantity requires the specification of *etalon* physical objects and standard physical processes by means of which the value of the quantity is ascertained. In case of electrodynamical quantities the only "device" we need is a point-like test particle, and the standard measuring procedures by which the kinematical properties of the test particle are ascertained.

So, assume we have chosen an *etalon* test particle, and let $\mathbf{r}^{etalon}(t)$, $\mathbf{v}^{etalon}(t)$, $\mathbf{a}^{etalon}(t)$ denote its position, velocity and acceleration at time t. It is assumed that we are able to set the *etalon* test particle into motion with arbitrary velocity $\mathbf{v}^{etalon} < c$ at arbitrary location. We will need more "copies" of the *etalon* test particle:

Definition (D0) A particle e is called *test particle* if for all $\mathbf{r}$ and t

$$\mathbf{v}^e(t)\Big|_{\mathbf{r}^e(t)=\mathbf{r}} = \mathbf{v}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{44}$$

implies

$$\mathbf{a}^e(t)\Big|_{\mathbf{r}^e(t)=\mathbf{r}} = \mathbf{a}^{etalon}(t)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}} \tag{45}$$

(The "restriction signs" refer to *physical* situations; for example, $|_{\mathbf{r}^e(t)=\mathbf{r}}$ indicates that the test particle e is at point $\mathbf{r}$ at time t.)

Note, that some of the definitions and statements below require the existence of many test particles; which is, of course, a matter of empirical fact, and will be provided by (E0) below.

First we define the electric and magnetic field strengths. The only measuring device we need is a test particle being at rest relative to K.

Definition (D1) *Electric field strength* at point $\mathbf{r}$ and time t is defined as the acceleration of an arbitrary test particle e, such that $\mathbf{r}^e(t) = \mathbf{r}$ and $\mathbf{v}^e(t) = 0$:

$$\mathbf{E}(\mathbf{r}, t) \stackrel{def}{=} \mathbf{a}^e(t)|_{\mathbf{r}^e(t)=\mathbf{r};\, \mathbf{v}^e(t)=0} \tag{46}$$

Magnetic field strength is defined by means of how the acceleration $\mathbf{a}^e$ of the rest test particle changes with an infinitesimal perturbation of its state of rest, that is, if an infinitesimally small velocity $\mathbf{v}^e$ is imparted to the particle. Of course, we cannot perform various small perturbations simultaneously on one and the same rest test particle, therefore we perform the measurements on many rest test particles with various small perturbations. Let $\delta \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. First we define the following function:

$$\begin{aligned} \mathbf{U}^{\mathbf{r},t} \quad &: \quad \mathbb{R}^3 \supset \delta \to \mathbb{R}^3 \\ & \quad \mathbf{U}^{\mathbf{r},t}(\mathbf{v}) \stackrel{def}{=} \mathbf{a}^e(t)|_{\mathbf{r}^e(t)=\mathbf{r};\, \mathbf{v}^e(t)=\mathbf{v}} \end{aligned} \tag{47}$$

Obviously, $\mathbf{U}^{\mathbf{r},t}(0) = \mathbf{E}(\mathbf{r}, t)$.

Definition (D2) *Magnetic field strength* at point $\mathbf{r}$ and time t is

$$\mathbf{B}(\mathbf{r}, t) \stackrel{def}{=} \left. \begin{pmatrix} \partial_{v_z} U_y^{\mathbf{r},t} \\ \partial_{v_x} U_z^{\mathbf{r},t} \\ \partial_{v_y} U_x^{\mathbf{r},t} \end{pmatrix} \right|_{\mathbf{v}=0} \tag{48}$$

Practically it means that one can determine the value of $\mathbf{B}(\mathbf{r}, t)$, with arbitrary precision, by means of measuring the accelerations of a few test particles of velocity $\mathbf{v}^e \in \delta$.

Next we introduce the concepts of source densities:

Definition (D3)

$$\varrho(\mathbf{r}, t) \stackrel{def}{=} \nabla \cdot \mathbf{E}(\mathbf{r}, t) \tag{49}$$

$$\mathbf{j}(\mathbf{r}, t) \stackrel{def}{=} c^2 \nabla \times \mathbf{B}(\mathbf{r}, t) - \partial_t \mathbf{E}(\mathbf{r}, t) \tag{50}$$

are called *active electric charge density* and *active electric current density*, respectively.

A simple consequence of the *definitions* is that a continuity equation holds for ϱ and $\mathbf{j}$:

Theorem 2.

$$\partial_t \varrho(\mathbf{r},t) + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0 \tag{51}$$

Remark 4. In our construction, the two Maxwell equations (49)–(50), are mere *definitions* of the concepts of active electric charge density and active electric current density. They do not contain information whatsoever about how "matter produces electromagnetic field". And it is not because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are, of course, "unspecified distributions" in these "general laws", but because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ cannot be specified prior to or at least independently of the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. Again, because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are just abbreviations, standing for the expressions on the right hand sides of (49)–(50). In other words, any statement about the "charge distribution" will be a statement about $\nabla \cdot \mathbf{E}$, and any statement about the "current distribution" will be a statement about $c^2\nabla \times \mathbf{B} - \partial_t \mathbf{E}$.

The minimal claim is that this is a possible coherent construction. Though we must add: equations (49)–(50) could be seen as contingent physical laws about the relationship between the charge and current distributions and the electromagnetic field, only if we had an independent empirical definition of charge. However, we do not see how such a definition is possible, without encountering circularities. (Also see Remark 5.)

The operational definitions of the field strengths and the source densities are based on the kinematical properties of the test particles. The following definition describes the concept of a charged point-like particle, in general.

Definition (D4) A particle b is called *charged point-particle* of *specific passive electric charge* π^b and of *active electric charge* α^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \gamma\left(\mathbf{v}^b(t)\right)\mathbf{a}^b(t) &= \pi^b\left\{\mathbf{E}\left(\mathbf{r}^b(t),t\right) + \mathbf{v}^b(t)\times\mathbf{B}\left(\mathbf{r}^b(t),t\right)\right. \\ &\quad \left. -c^{-2}\mathbf{v}^b(t)\left(\mathbf{v}^b(t)\cdot\mathbf{E}\left(\mathbf{r}^b(t),t\right)\right)\right\} \end{aligned} \tag{52}$$

2. If it is the only particle whose worldline intersects a given space-time region Ω, then for all $(\mathbf{r},t)\in\Omega$ the source densities are of the following form:

$$\varrho(\mathbf{r},t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right) \tag{53}$$

$$\mathbf{j}(\mathbf{r},t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right)\mathbf{v}^b(t) \tag{54}$$

where $\mathbf{r}^b(t)$, $\mathbf{v}^b(t)$ and $\mathbf{a}^b(t)$ are the particle's position, velocity and acceleration. The ratio $\mu^b \stackrel{def}{=} \alpha^b/\pi^b$ is called the *electric inertial rest mass* of the particle.

Remark 5. Of course, (52)is equivalent to the standard form of the Lorentz equation:

$$\frac{d}{dt}\left(\gamma(\mathbf{v}(t))\,\mathbf{v}(t)\right) = \pi\left\{\mathbf{E}(\mathbf{r}(t),t) + \mathbf{v}(t)\times\mathbf{B}(\mathbf{r}(t),t)\right\} \tag{55}$$

with $\pi = q/m$ in the usual terminology, where q is the passive electric charge and m is the inertial (rest) mass of the particle—that is why we call π *specific* passive electric charge. Nevertheless, it must be clear that for all charged point-particles we introduced *two independent*, empirically meaningful and experimentally testable quantities: specific passive electric charge π and active electric charge α. There is no universal law-like relationship between these two quantities: the ratio between them varies from particle to p1article. In the traditional sense, this ratio is, however, nothing but the particle's rest mass.

We must emphasize that the concept of mass so obtained, as defined by only means of electrodynamical quantities, is essentially related to ED, that is to say, to electromagnetic interaction. There seems no way to give a consistent and non-circular operational definition of inertial mass in general, independently of the context of a particular type of physical interaction. Without entering here into the detailed discussion of the problem, we only mention that, for example, Weyl's commonly accepted definition (Jammer 2000, pp. 8–10) and all similar definitions based on the conservation of momentum in particle collisions suffer from the following difficulty. There is no "collision" as a purely "mechanical" process. During a collision the particles are moving in a physical field—or fields—of interaction. Therefore: 1) the system of particles, separately, cannot be regarded as a closed system; 2) the inertial properties of the particles, in fact, reveal themselves in the interactions with the field. Thus, the concepts of inertial rest mass belonging to different interactions differ from each other; whether they are equal (proportional) to each other is a matter of contingent fact of nature.

Remark 6. The choice of the *etalon* test particle is, of course, a matter of convention, just as the definitions (D0)–(D4) themselves. It is important to note that all these conventional factors play a constitutive role in the fundamental concepts of ED (Reichenbach 1965). With these choices we not only make semantic conventions determining the meanings of the terms, but also make a decision about the body of concepts by means of which we grasp physical reality. There are a few things, however, that must be pointed out:

(a) This kind of conventionality does not mean that the physical quantities defined in (D0)–(D4) cannot describe *objective* features of physical reality. It only means that we make a decision which objective features of reality we are dealing with. With another body of conventions we have another body of physical concepts/physical quantities and another body of empirical facts.

(b) On the other hand, it does not mean either that our knowledge of the physical world would not be objective but a product of our conventions. If two theories obtained by starting with two different bodies of conventions are complete enough accounts of the physical phenomena, then they describe the same reality, expressed in terms of different physical quantities. Let us spell out an example: Definition (50) is entirely conventional—no objective fact of the world determines the formula on the right hand side. Therefore, we could make another choice, say,

$$\mathbf{j}_\Theta\left(\mathbf{r},t\right) \stackrel{def}{=} \Theta^2 \nabla \times \mathbf{B}\left(\mathbf{r},t\right) - \partial_t \mathbf{E}\left(\mathbf{r},t\right) \tag{56}$$

with some $\Theta \neq c$. At first sight, one might think that this choice will alter the speed of electromagnetic waves. This is however not the case. It will be an empirical fact *about* $\mathbf{j}_\Theta(\mathbf{r},t)$ that if a particle b is the only one whose worldline intersects a given space-time region Ω, then for all $(\mathbf{r},t) \in \Omega$

$$\begin{aligned} \mathbf{j}_\Theta(\mathbf{r},t) &= \alpha^b \delta\left(\mathbf{r} - \mathbf{r}^b(t)\right) \mathbf{v}^b(t) \\ &\quad + \left(\Theta^2 - c^2\right) \nabla \times \mathbf{B}(\mathbf{r},t) \end{aligned} \tag{57}$$

Now, consider a region where there is no particle. Taking into account (57), we have (60)–(61) and

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0 \tag{58}$$

$$\Theta^2 \nabla \times \mathbf{B}(\mathbf{r},t) - \partial_t \mathbf{E}(\mathbf{r},t) = \left(\Theta^2 - c^2\right) \nabla \times \mathbf{B}(\mathbf{r},t) \tag{59}$$

which lead to the usual wave equation with propagation speed c. (Of course, in this particular example, one of the possible choices, namely $\Theta = c$, is distinguished by its simplicity. Note, however, that simplicity is not an epistemologically unproblematic notion.)

4 Empirical facts of electrodynamics

Both "empirical" and "fact" are used in different senses. Statements (E0)–(E4) below are universal generalizations, rather than statements of particular observations. Nevertheless we call them "empirical facts", by which we simply mean that they are truths which can be acquired by *a posteriori* means. Normally, they can be considered as laws obtained by inductive generalization; statements the truths of which can be, in principle, confirmed empirically.

On the other hand, in the context of the consistency questions (Q3) and (Q4), it is not important how these statements are empirically confirmed. (E0)–(E4) can be regarded as axioms of the Maxwell–Lorentz theory in K. What is important for us is that from these *axioms*, in conjunction with the theoretical representations of the measurement operations, there follow assertions about what the moving observer in K' observes. Section 6 will be concerned with these consequences.

(E0) There exist many enough test particles and we can settle them into all required positions and velocities.

Consequently, (D1)–(D4) are sound definitions. From observations about **E**, **B** and the charged point-particles, we have further empirical facts:

(E1) In all situations, the electric and magnetic field strengths satisfy the following two Maxwell equations:

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \tag{60}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) + \partial_t \mathbf{B}(\mathbf{r},t) = 0 \tag{61}$$

(E2) Each particle is a charged point-particle, satisfying (D4) with some specific passive electric charge π and active electric charge α. This is also true for the test particles, with—as follows from the definitions—specific passive electric charge $\pi = 1$.

(E3) If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Ω, then for all $(\mathbf{r}, t) \in \Omega$ the source densities are:

$$\varrho\left(\mathbf{r}, t\right) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \tag{62}$$

$$\mathbf{j}\left(\mathbf{r}, t\right) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \mathbf{v}^{b_i}\left(t\right) \tag{63}$$

Putting facts (E1)–(E3) together, we have the coupled Maxwell–Lorentz equations:

$$\begin{aligned}
\nabla \cdot \mathbf{E}\left(\mathbf{r}, t\right) &= \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) && (64)\\
c^2 \nabla \times \mathbf{B}\left(\mathbf{r}, t\right) - \partial_t \mathbf{E}\left(\mathbf{r}, t\right) &= \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \mathbf{v}^{b_i}\left(t\right) && (65)\\
\nabla \cdot \mathbf{B}\left(\mathbf{r}, t\right) &= 0 && (66)\\
\nabla \times \mathbf{E}\left(\mathbf{r}, t\right) + \partial_t \mathbf{B}\left(\mathbf{r}, t\right) &= 0 && (67)\\
\gamma\left(\mathbf{v}^{b_i}\left(t\right)\right) \mathbf{a}^{b_i}(t) &= \pi^{b_i} \left\{ \mathbf{E}\left(\mathbf{r}^{b_i}\left(t\right), t\right) + \mathbf{v}^{b_i}\left(t\right) \times \mathbf{B}\left(\mathbf{r}^{b_i}\left(t\right), t\right) \right. \\
&\qquad \left. - c^{-2} \mathbf{v}^{b_i}\left(t\right) \left(\mathbf{v}^{b_i}\left(t\right) \cdot \mathbf{E}\left(\mathbf{r}^{b_i}\left(t\right), t\right)\right) \right\} && (68)\\
&\qquad (i = 1, 2, \ldots n)
\end{aligned}$$

These are the fundamental equations of ED, describing an interacting system of n particles and the electromagnetic field.

Remark 7. Without entering into the details of the problem of classical charged particles (Frisch 2005; Rohrlich 2007; Muller 2007), it must be noted that the Maxwell–Lorentz equations (64)–(68), exactly in this form, have *no* solution. The reason is the following. In the Lorentz equation of motion (52), a small but extended particle can be described with a good approximation by one single specific passive electric charge π^b and one single trajectory $\mathbf{r}^b\left(t\right)$. In contrast, however, a similar "idealization" in the source densities (53)–(54) leads to singularities; the field is singular at precisely the points where the coupling happens: on the trajectory of the particle.

The generally accepted answer to this problem is that (53)–(54) should not be taken literally. Due to the inner structure of the particle, the real source densities are some "smoothed out" Dirac deltas. Instead of (53)–(54), therefore, we have some more general equations

$$[\varrho(\mathbf{r}, t)] = \mathcal{R}^b\left[\mathbf{r}^b(t)\right] \tag{69}$$

$$[\mathbf{j}(\mathbf{r}, t)] = \mathcal{J}^b\left[\mathbf{r}^b(t)\right] \tag{70}$$

where $\mathcal{R}^b$ and $\mathcal{J}^b$ are, generally non-linear, operators providing functional relationships between the particle's trajectory $\left[\mathbf{r}^b(t)\right]$ and the source density functions $[\varrho(\mathbf{r},t)]$ and $[\mathbf{j}(\mathbf{r},t)]$. (Notice that (53)–(54) serve as example of such equations.) The concrete forms of equations (69)–(70) are determined by the physical laws of the internal world of the particle—which are, supposedly, outside of the scope of ED. At this level of generality, the only thing we can say is that, for a "point-like" (localized) particle, equations (69)–(70) must be something very close to—but not identical with—equations (53)–(54). With this explanation, for the sake of simplicity we leave the Dirac deltas in the equations. Also, in some of our statements and calculations the Dirac deltas are essentially used; for example, (E3) and, partly, Theorem 8 and 10 would not be true without the exact point-like source densities (53)–(54). But a little reflection shows that the statements in question remain approximately true if the particles are approximately point-like, that is, if equations (69)–(70) are close enough to equations (53)–(54). To be noted that what is actually essential in (53)–(54) is not the point-likeness of the particle, but its stability; no matter how the system moves, it remains a localized object.

5 Operational definitions of electrodynamical quantities in K'

So far we have only considered ED in a single frame of reference K. Now we turn to the question of how a moving observer describes the same phenomena in K'. The observed phenomena are the same, but the measuring equipments by means of which the phenomena are observed are not entirely the same; instead of being at rest in K, they are co-moving with K'.

Accordingly, we will repeat the operational definitions (D0)–(D4) with the following differences:

1. The "rest test particles" will be at rest relative to reference frame K', that is, *in motion with velocity* $\mathbf{V}$ relative to K.
2. The measuring equipments by means of which the kinematical quantities are ascertained—say, the measuring rods and clocks—will be at rest relative to K', that is, *in motion with velocity* $\mathbf{V}$ relative to K. In other words, kinematical quantities $t, \mathbf{r}, \mathbf{v}, \mathbf{a}$ in definitions (D0)–(D4) will be *replaced with*—not expressed in terms of— $t', \mathbf{r}', \mathbf{v}', \mathbf{a}'$.

Definition (D0′) Particle e is called *(test particle)′* if for all $\mathbf{r}'$ and t'

$$\mathbf{v}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{v}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{71}$$

implies

$$\mathbf{a}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{a}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'} \tag{72}$$

A (test particle)′ e moving with velocity $\mathbf{V}$ relative to K is at rest relative to K', that is, $\mathbf{v}'^{e} = 0$. Accordingly:

Definition (D1′) *(Electric field strength)′* at point $\mathbf{r}'$ and time t' is defined as the acceleration of an arbitrary (test particle)′ e, such that $\mathbf{r}'^{e}(t) = \mathbf{r}'$ and $\mathbf{v}'^{e}(t') = 0$:

$$\mathbf{E}'\left(\mathbf{r}', t'\right) \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\,\mathbf{v}'^{e}(t')=0} \tag{73}$$

Similarly, (magnetic field strength)′ is defined by means of how the acceleration $\mathbf{a}'^{e}$ of a rest (test particle)′—rest, of course, relative to K'—changes with a small perturbation of its state of motion, that is, if an infinitesimally small velocity $\mathbf{v}'^{e}$ is imparted to the particle. Just as in (D2), let $\delta' \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. We define the following function:

$$\begin{aligned} \mathbf{U}'^{\mathbf{r}',t'} \quad &: \quad \mathbb{R}^3 \supset \delta' \to \mathbb{R}^3 \\ &\quad \mathbf{U}'^{\mathbf{r}',t'}(\mathbf{v}') \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\,\mathbf{v}'^{e}(t')=\mathbf{v}'} \end{aligned} \tag{74}$$

Definition (D2′) *(Magnetic field strength)′* at point $\mathbf{r}'$ and time t' is

$$\mathbf{B}'(\mathbf{r}', t') \stackrel{def}{=} \left.\left(\begin{array}{c} \partial_{v'_z} U'^{\mathbf{r}',t'}_{y} \\ \partial_{v'_x} U'^{\mathbf{r}',t'}_{z} \\ \partial_{v'_y} U'^{\mathbf{r}',t'}_{x} \end{array}\right)\right|_{\mathbf{v}'=0} \tag{75}$$

Definition (D3′)

$$\varrho'\left(\mathbf{r}', t'\right) \stackrel{def}{=} \nabla \cdot \mathbf{E}'\left(\mathbf{r}', t'\right) \tag{76}$$

$$\mathbf{j}'\left(\mathbf{r}', t'\right) \stackrel{def}{=} c^2 \nabla \times \mathbf{B}'\left(\mathbf{r}', t'\right) - \partial_{t'} \mathbf{E}'\left(\mathbf{r}', t'\right) \tag{77}$$

are called *(active electric charge density)′* and *(active electric current density)′*, respectively.

Of course, we have:

Theorem 3.

$$\partial_{t'} \varrho'\left(\mathbf{r}', t'\right) + \nabla \cdot \mathbf{j}'\left(\mathbf{r}', t'\right) = 0 \tag{78}$$

Definition (D4′) A particle is called *(charged point-particle)′* of *(specific passive electric charge)′* π'^{b} and of *(active electric charge)′* α'^{b} if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \gamma\left(\mathbf{v}'^{b}\left(t'\right)\right) \mathbf{a}'^{b}(t') \;=\; & \pi'^{b}\left\{\mathbf{E}'\left(\mathbf{r}'^{b}\left(t'\right), t'\right) + \mathbf{v}'^{b}\left(t'\right) \times \mathbf{B}'\left(\mathbf{r}'^{b}\left(t'\right), t'\right)\right. \\ & \left. - c^{-2} \mathbf{v}'^{b}\left(t'\right)\left(\mathbf{v}'^{b}\left(t'\right) \cdot \mathbf{E}'\left(\mathbf{r}'^{b}\left(t'\right), t'\right)\right)\right\} \end{aligned} \tag{79}$$

2. If it is the only particle whose worldline intersects a given space-time region Ω', then for all $(\mathbf{r}', t') \in \Omega'$ the (source densities)′ are of the following form:

$$\varrho'\left(\mathbf{r}', t'\right) = \alpha'^{b} \delta\left(\mathbf{r}' - \mathbf{r}'^{b}\left(t'\right)\right) \tag{80}$$

$$\mathbf{j}'\left(\mathbf{r}', t'\right) = \alpha'^{b} \delta\left(\mathbf{r}' - \mathbf{r}'^{b}\left(t'\right)\right) \mathbf{v}'^{b}\left(t'\right) \tag{81}$$

where $\mathbf{r}'^b(t')$, $\mathbf{v}'^b(t')$ and $\mathbf{a}'^b(t')$ is the particle's position, velocity and acceleration in K'. The ratio $\mu'^b \stackrel{def}{=} \alpha'^b/\pi'^b$ is called the *(electric inertial rest mass)'* of the particle.

Remark 8. It is worthwhile to make a few remarks about some epistemological issues:

(a) The physical quantities defined in (D1)–(D4) *differ* from the physical quantities defined in (D1')–(D4'), simply because the physical situation in which a test particle is at rest relative to K differs from the one in which it is co-moving with K' with velocity $\mathbf{V}$ relative to K; and, as we know *from the laws of ED in K*, this difference really matters.

Someone might object that if this is so then any two instances of the same measurement must be regarded as measurements of different physical quantities. For, if the difference in the test particle's velocity is enough reason to say that the two operations determine two different quantities, then, by the same token, two operations must be regarded as different operations—and the corresponding quantities as different physical quantities—if the test particle is at different points of space, or the operations simply happen at different moments of time. And this consequence, the objection goes, seems to be absurd: if it were true, then science would not be possible, because we would not have the power to make law-like assertions at all; therefore we must admit that empiricism fails to explain how natural laws are possible, and, as many argue, science cannot do without metaphysical pre-assumptions.

Our response to such an objections is the following. First, concerning the general epistemological issue, we believe, nothing disastrous follows from admitting that two phenomena observed at different place or at different time *are* distinct. And if they are stated as instances of the same phenomenon, this statement is not a logical or metaphysical necessity—derived from some logical/metaphysical pre-assumptions—but an ordinary scientific hypothesis obtained by induction and confirmed or disconfirmed together with the *whole* scientific theory. In fact, this is precisely the case with respect to the definitions of the fundamental electrodynamical quantities. For example, definition (D1) is in fact a family of definitions each belonging to a particular situation individuated by the space-time locus $(\mathbf{r}, t)$.

Second, in this paper, we must emphasize again, the question of operational definitions of electrodynamical quantities first of all emerges not from an epistemological context, but from the context of the *inner consistency* of our theories, in answering questions (Q3) and (Q4). In the next section, all the results of the measurement operations defined in (D1')–(D4') will be predicted from the laws of ED in K. And, ED itself says that some differences in the conditions are relevant from the point of view of the measured accelerations of the test particles, some others are not; some of the originally distinct quantities are contingently equal, some others not.

(b) From a mathematical point of view, both (D0)–(D4) and (D0′)–(D4′) are definitions. However, while the choice of the *etalon* test particle and definitions (D0)–(D4) are entirely *conventional*, there is no additional conventionality in (D0′)–(D4′). The way in which we define the electrodynamical quantities in inertial frame K' automatically follows from (D0)–(D4) and from the question we would like to answer, namely, whether the RP holds for ED; since the principle is about "quantities obtained by the same operational procedures with the same measuring equipments when they are co-moving with K'".

(c) In fact, one of the constituents of the concepts defined in K' is not determined by the operational definitions in K. Namely, the notion of "the same operational procedures with the same measuring equipments when they are co-moving with K'", that is, the notion of $M_{\mathbf{V}}$ applied for the measuring operation and the measuring equipments. This is however not an additional freedom of conventionality, but a simple vagueness in our physical theories in K. In any event, in our case, the notion of the only moving measuring device, that is, the notion of "a test particle at rest relative to K'" is quite clear.

6 Observations of moving observer

Now we have another collection of operationally defined notions, $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$, the concept of (charged point-particle)′ defined in the primed terms, and its properties π', α' and μ'. Normally, one should investigate these quantities experimentally and collect new empirical facts about both the relationships between the primed quantities and about the relationships between the primed quantities and the ones defined in (D1)–(D4). In contrast, we will continue our analysis in another way; following the "Lorentzian pedagogy", we will determine from the laws of physics in K what an observer co-moving with K' should observe. In fact, with this method, we will answer our question (Q4), whether the textbook transformation rules, derived from the RP, are compatible with the laws of ED in a single frame of reference. We will also see whether the basic equations (64)–(68) are covariant against these transformations.

Throughout the theorems below, it is important that when we compare, for example, $\mathbf{E}(\mathbf{r}, t)$ with $\mathbf{E}'(\mathbf{r}', t')$, we compare the values of the fields *in one and the same event*, that is, we compare $\mathbf{E}(\mathbf{r}(A), t(A))$ with $\mathbf{E}'(\mathbf{r}'(A), t'(A))$. For the sake of brevity, however, we omit the indication of this fact.

The first theorem trivially follows from the fact that the Lorentz transformations of the kinematical quantities are one-to-one:

Theorem 4. *A particle is a (test particle)′ if and only if it is a test particle.*

Consequently, we have many enough (test particles)′ for definitions (D1′)–(D4′); and each is a charged point-particle satisfying the Lorentz equation (52) with specific passive electric charge $\pi = 1$.

Theorem 5.

$$
\begin{aligned}
E'_x &= E_x && (82)\\
E'_y &= \gamma\left(E_y - VB_z\right) && (83)\\
E'_z &= \gamma\left(E_z + VB_y\right) && (84)
\end{aligned}
$$

Proof. When the (test particle)′ is at rest relative to K', it is moving with velocity $\mathbf{v}^e = (V, 0, 0)$ relative to K. From (52) (with $\pi = 1$) we have

$$
\begin{aligned}
a^e_x &= \gamma^{-3} E_x && (85)\\
a^e_y &= \gamma^{-1}\left(E_y - VB_z\right) && (86)\\
a^e_z &= \gamma^{-1}\left(E_z + VB_y\right) && (87)
\end{aligned}
$$

Applying (10)–(12), we can calculate the acceleration $\mathbf{a}'^e$ in K', and, accordingly, we find

$$
\begin{aligned}
E'_x &= a'^e_x = \gamma^3 a^e_x = E_x && (88)\\
E'_y &= a'^e_y = \gamma^2 a^e_y = \gamma\left(E_y - VB_z\right) && (89)\\
E'_z &= a'^e_z = \gamma^2 a^e_z = \gamma\left(E_z + VB_y\right) && (90)
\end{aligned}
$$

□

Theorem 6.

$$
\begin{aligned}
B'_x &= B_x && (91)\\
B'_y &= \gamma\left(B_y + c^{-2} V E_z\right) && (92)\\
B'_z &= \gamma\left(B_z - c^{-2} V E_y\right) && (93)
\end{aligned}
$$

Proof. Consider for instance B'_x. By definition,

$$B'_x = \partial_{v'_z} U'^{\mathbf{r}',t'}_y \Big|_{\mathbf{v}'=0} \tag{94}$$

According to (74), the value of $U'^{\mathbf{r}',t'}_y(\mathbf{v}')$ is equal to

$$a'^e_y \Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=\mathbf{v}'} \tag{95}$$

that is, the y-component of the acceleration of a (test particle)′ e in a situation in which $\mathbf{r}'^e(t') = \mathbf{r}'$ and $\mathbf{v}'^e(t') = \mathbf{v}'$. Accordingly, in order to determine the partial derivative (94) we have to determine

$$\frac{d}{dw}\Big|_{w=0} \left(a'^e_y \Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=(0,0,w)} \right) \tag{96}$$

Now, according to (9), condition $\mathbf{v}'^e = (0, 0, w)$ corresponds to

$$\mathbf{v}^e = \left(V, 0, \gamma^{-1} w\right) \tag{97}$$

Substituting this velocity into (52), we have:

$$a_y^e = \sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}} \left(E_y + w\gamma^{-1}B_x - VB_z\right) \tag{98}$$

Applying (13), one finds:

$$\begin{aligned} a_y'^e &= \gamma^2 a_y^e = \gamma^2 \sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}} \left(E_y + w\gamma^{-1}B_x - VB_z\right) \\ &= \frac{\gamma}{\gamma(w)} \left(E_y + w\gamma^{-1}B_x - VB_z\right) \end{aligned} \tag{99}$$

Differentiating with respect to w at $w = 0$, we obtain

$$B_x' = B_x \tag{100}$$

The other components can be obtained in the same way. □

Theorem 7.

$$\varrho' = \gamma\left(\varrho - c^{-2}Vj_x\right) \tag{101}$$

$$j_x' = \gamma\left(j_x - V\varrho\right) \tag{102}$$

$$j_y' = j_y \tag{103}$$

$$j_z' = j_z \tag{104}$$

Proof. Substituting $\mathbf{E}'$ and $\mathbf{B}'$ with (82)–(84) and (91)–(93), $\mathbf{r}$ and t with the inverse of (1)–(4), then differentiating the composite function and taking into account (49)–(50), we get (101)–(104). □

Theorem 8. *A particle b is charged point-particle of specific passive electric charge π^b and of active electric charge α^b if and only if it is a (charged point-particle)′ of (specific passive electric charge)′ π'^b and of (active electric charge)′ α'^b, such that $\pi'^b = \pi^b$ and $\alpha'^b = \alpha^b$.*

Proof. First we prove (79). For the sake of simplicity, we will verify this in case of $\mathbf{v}'^b = (0, 0, w)$. We can use (98):

$$a_y^b = \pi^b \sqrt{1 - \frac{V^2 + w^2\gamma^{-2}}{c^2}} \left(E_y + w\gamma^{-1}B_x - VB_z\right) \tag{105}$$

From (13), (83), (91), and (93) we have

$$\begin{aligned} a_y'^b &= \pi^b\gamma(w)^{-1}\left(E_y' + wB_x'\right) \\ &= \left[\pi^b\gamma\left(\mathbf{v}'^b\right)^{-1}\left(\mathbf{E}' - c^{-2}v'^b\left(\mathbf{v}'^b\cdot\mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_y \Bigg|_{\mathbf{v}'^b=(0,0,w)} \end{aligned} \tag{106}$$

Similarly,

$$\begin{aligned}
a_x'^b &= \pi^b \gamma(w)^{-1} \left(E_x' - wB_y'\right) \\
&= \left[\pi^b \gamma\left(\mathbf{v}'^b\right)^{-1} \left(\mathbf{E}' - c^{-2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_x \Bigg|_{\mathbf{v}'^b=(0,0,w)} \qquad (107) \\
a_z'^b &= \pi^b \gamma(w)^{-3} E_z' \\
&= \left[\pi^b \gamma\left(\mathbf{v}'^b\right)^{-1} \left(\mathbf{E}' - c^{-2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_z \Bigg|_{\mathbf{v}'^b=(0,0,w)} \qquad (108)
\end{aligned}$$

That is, (79) is satisfied, indeed.

In the second part, we show that (80)–(81) are nothing but (53)–(54) expressed in terms of $\mathbf{r}', t', \varrho'$ and $\mathbf{j}'$, with $\alpha'^b = \alpha^b$.

It will be demonstrated for a particle of trajectory $\mathbf{r}'^b(t') = (wt', 0, 0)$. Applying (8), (53)–(54) have the following forms:

$$\begin{aligned}
\varrho(\mathbf{r}, t) &= \alpha^b \delta(x - \beta t)\, \delta(y)\, \delta(z) \qquad (109) \\
\mathbf{j}(\mathbf{r}, t) &= \alpha^b \delta(x - \beta t)\, \delta(y)\, \delta(z) \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix} \qquad (110)
\end{aligned}$$

where $\beta = \frac{w+V}{1+c^{-2}wV}$. $\mathbf{r}, t, \varrho$ and $\mathbf{j}$ can be expressed with the primed quantities by applying the inverse of (1)–(4) and (101)–(104):

$$\begin{aligned}
\gamma\left(\varrho'(\mathbf{r}', t') + c^{-2}V j'_x(\mathbf{r}', t')\right) &= \alpha^b \delta\left(\gamma\left(x' + Vt' - \beta\left(t' + c^{-2}Vx'\right)\right)\right) \\
&\quad \times \delta(y')\, \delta(z') \qquad (111) \\
\gamma\left(j'_x(\mathbf{r}', t') + V\varrho'(\mathbf{r}', t')\right) &= \alpha^b \delta\left(\gamma\left(x' + Vt' - \beta\left(t' + c^{-2}Vx'\right)\right)\right) \\
&\quad \times \delta(y')\, \delta(z')\, \beta \qquad (112) \\
j'_y(\mathbf{r}', t') &= 0 \qquad (113) \\
j'_z(\mathbf{r}', t') &= 0 \qquad (114)
\end{aligned}$$

One can solve this system of equations for ϱ' and j'_x:

$$\begin{aligned}
\varrho'(\mathbf{r}', t') &= \alpha^b \delta(x' - wt')\, \delta(y')\, \delta(z') \qquad (115) \\
\mathbf{j}'(\mathbf{r}', t') &= \alpha^b \delta(x' - wt')\, \delta(y')\, \delta(z') \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix} \qquad (116)
\end{aligned}$$

□

Theorem 9.

$$\begin{aligned}
\nabla \cdot \mathbf{B}'(\mathbf{r}', t') &= 0 \qquad (117) \\
\nabla \times \mathbf{E}'(\mathbf{r}', t') + \partial_{t'} \mathbf{B}'(\mathbf{r}', t') &= 0 \qquad (118)
\end{aligned}$$

Proof. Expressing (60)–(61) in terms of $\mathbf{r}'$, t', $\mathbf{E}'$ and $\mathbf{B}'$ by means of (1)–(4), (82)–(84) and (91)–(93), we have

$$\begin{aligned}
\nabla\cdot\mathbf{B}' - c^{-2}V\left(\nabla\times\mathbf{E}' + \partial_{t'}\mathbf{B}'\right)_x &= 0 &(119)\\
\left(\nabla\times\mathbf{E}' + \partial_{t'}\mathbf{B}'\right)_x - V\nabla\cdot\mathbf{B}' &= 0 &(120)\\
\left(\nabla\times\mathbf{E}' + \partial_{t'}\mathbf{B}'\right)_y &= 0 &(121)\\
\left(\nabla\times\mathbf{E}' + \partial_{t'}\mathbf{B}'\right)_z &= 0 &(122)
\end{aligned}$$

which is equivalent to (117)–(118), indeed. □

Theorem 10. *If b_1, b_2,..., b_n are the only particles whose worldlines intersect a given space-time region Ω', then for all $(\mathbf{r}', t') \in \Omega'$ the (source densities)′ are:*

$$\begin{aligned}
\varrho'\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right) &(123)\\
\mathbf{j}'\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) &(124)
\end{aligned}$$

Proof. Due to Theorem 8, each (charged point-particle)′ is a charged point-particle with $\alpha'^{b} = \alpha^{b}$. Therefore, we only need to prove that equations (123)–(124) amount to (62)–(63) expressed in the primed variables. On the left hand side of (62)–(63), ϱ and $\mathbf{j}$ can be expressed by means of the inverse of (101)–(104); on the right hand side, we take $\alpha'^{b} = \alpha^{b}$, and apply the inverse of (1)–(4), just as in the derivation of (115)–(116). From the above, we obtain:

$$\begin{aligned}
\varrho'\left(\mathbf{r}', t'\right) + c^{-2}Vj'_x\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)\\
&\quad + c^{-2}V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)v_x'^{b_i}\left(t'\right) &(125)\\
j'_x\left(\mathbf{r}', t'\right) + V\varrho'\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)v_x'^{b_i}\left(t'\right)\\
&\quad + V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right) &(126)\\
j'_y\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)v_y'^{b_i}\left(t'\right) &(127)\\
j'_z\left(\mathbf{r}', t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)v_z'^{b_i}\left(t'\right) &(128)
\end{aligned}$$

Solving these linear equations for ϱ' and $\mathbf{j}'$ we obtain (123)–(124). □

Combining the results we obtained in Theorems 8–10, we have

$$\nabla \cdot \mathbf{E}'\left(\mathbf{r}',t'\right) = \sum_{i=1}^{n} \alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right) \tag{129}$$

$$c^2\nabla \times \mathbf{B}'\left(\mathbf{r}',t'\right) - \partial_{t'}\mathbf{E}'\left(\mathbf{r}',t'\right) = \sum_{i=1}^{n} \alpha^{b_i}\delta\left(\mathbf{r}' - \mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) \tag{130}$$

$$\nabla \cdot \mathbf{B}'\left(\mathbf{r}',t'\right) = 0 \tag{131}$$

$$\nabla \times \mathbf{E}'\left(\mathbf{r}',t'\right) + \partial_{t'}\mathbf{B}'\left(\mathbf{r}',t'\right) = 0 \tag{132}$$

$$\begin{aligned} \gamma\left(\mathbf{v}'^{b_i}\left(t'\right)\right)\mathbf{a}'^{b_i}(t') = {} & \pi'^{b_i}\left\{\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right) + \mathbf{v}'^{b_i}\left(t'\right) \times \mathbf{B}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\right. \\ & \left. -c^{-2}\mathbf{v}'^{b_i}\left(t'\right)\left(\mathbf{v}'^{b_i}\left(t'\right) \cdot \mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\right)\right\} \\ & (i = 1,2,\dots n) \end{aligned} \tag{133}$$

7 Are the textbook transformation rules consistent with the laws of ED in a single frame of reference?

Now, everything is at hand to declare that the textbook transformation rules for electrodynamical quantities, routinely derived from the *presumed* covariance of the Maxwell equations, are in fact true, at least in the sense that they are derivable from the laws of ED in a single frame of reference, including—it must be emphasized—the precise operational definitions of the quantities in question. For, Theorems 5 and 6 show the well-known transformation rules for the field variables. What Theorem 7 asserts is nothing but the well-known transformation rule for charge density and current density. Finally, Theorem 8 shows that a particle's electric specific passive charge, active charge and electric rest mass are invariant scalars.

At this point, having ascertained the transformation rules, we can declare that equations (129)–(133) are nothing but $T_{\mathbf{V}}(\mathcal{E})$ (in coordinates, of course), where $\mathcal{E}$ stands for the equations (64)–(68). At the same time, (129)–(133) are manifestly equal to $P_{\mathbf{V}}(\mathcal{E})$. Therefore, we proved that the Maxwell–Lorentz equations are covariant against the transformations of the kinematical and electrodynamical quantities. In fact, we proved more:

- The Lorentz equation of motion (68) is covariant separately.
- The four Maxwell equations (64)–(67) constitute a covariant set of equations, separately from (68).
- (64)–(65) constitute a covariant set of equations, separately.
- (66)–(67) constitute a covariant set of equations, separately.

As we pointed out in Remark 3, none of these statements follows automatically from the fact that (64)–(68) is a covariant system of equations.

Remark 9. The fact that the proper calculation of the transformation rules for the field strengths and for the source densities leads to the familiar textbook transformation rules hinges on the *relativistic* version of the Lorentz equation, in particular, on the "relativistic mass-formula". Without factor $\gamma\left(\mathbf{v}^b\right)$ in (68), the proper transformation rules were different and the Maxwell equations were not covariant—against the proper transformations.

Remark 10. This is not the place to review the various versions of the textbook derivation of the transformation rules for electrodynamical quantities, nevertheless, a few remarks seem necessary. Among those with which we are acquainted, there are basically two major branches, and both are problematic. The first version follows Einstein's 1905 paper:

(1a) The transformation rules of electric and magnetic field strengths are derived from the presumption of the covariance of the homogeneous (with no sources) Maxwell equations.

(1b) The transformation rules of source densities are derived from the transformations of the field variables.

(1c) From the transformation rules of charge and current densities, it is derived that electric charge is an invariant scalar.

The second version is this:

(2a) The transformation rules of the charge and current densities are derived from some additional assumptions; typically from one of the followings:

  (2a1) the invariance of electric charge (Jackson 1999, pp. 553–558)

  (2a2) the current density is of form $\varrho\mathbf{u}(\mathbf{r},t)$, where $\mathbf{u}(\mathbf{r},t)$ is a velocity field (Tolman 1934, p. 85; Møller 1955, p. 140).

(2b) The transformation of the field strengths are derived from the transformation of ϱ and $\mathbf{j}$ and from the presumption of the covariance of the inhomogeneous Maxwell equations.

Unfortunately, with the only exception of (1b), none of the above steps is completely correct. Without entering into the details, let us mention that (2a1) and (2a2) both involve some further empirical information about the world, which does not follow from the simple assumption of covariance. Even in case of (1a) we must have the tacit assumption that zero charge and current densities go to zero charge and current densities during the transformation—otherwise the covariance of the homogeneous Maxwell equations would not follow from the assumed covariance of the Maxwell equations. (See points (b) and (d) in Remark 3.)

One encounters the next major difficulty in both (1a) and (2b): neither the homogeneous nor the inhomogeneous Maxwell equations determine the transformation rules of the field variables uniquely; $\mathbf{E}'$ and $\mathbf{B}'$ are only determined by $\mathbf{E}$ and $\mathbf{B}$ up to an arbitrary solution of the homogeneous equations.

Finally, let us mention a conceptual confusion that seems to be routinely overlooked in (1c), (2a1) and (2a2). There is no such thing as a simple relation between the scalar invariance of charge and the transformation of charge and current densities, as is usually claimed. For example, it is meaningless to say that

$$Q = \varrho \Delta W = Q' = \varrho' \Delta W' \tag{134}$$

where ΔW denotes a volume element, and

$$\Delta W' = \gamma \Delta W \tag{135}$$

Whose charge is Q, which remains invariant? Whose volume is ΔW and in what sense is that volume Lorentz contracted? In another form, in (2a2), whose velocity is $\mathbf{u}(\mathbf{r}, t)$?

Remark 11. In the previous remark we pointed out typical problems in the derivations of the transformation rules *from the covariance* of the equations. There is however a more fundamental problem: How do we arrive at the covariance itself? Obviously, it would be a completely mistaken idea to regard covariance as a "known/verifiable property of the equations", because we cannot verify that the equations are covariant against the transformations of electrodynamical quantities, *prior to* us knowing the transformations themselves against which the equations must be covariant. Therefore, the usual claim is that the covariance of the equations of ED against the transformations of electrodynamical quantities—whatever these transformations are—is *implied* by the assumption that the RP holds. Now, the problem is that this implication is, as we have seen in section 2, not true. Covariance follows from the RP only if $M_{\mathbf{V}}$ satisfies some extra conditions, for example (27)–(28); which is a questionable assumption, and, as far as we know, it has never been shown. Thus, disregarding the minor flaws mentioned in Remark 10, in the absence of the proof of this implication, one is not entitled to say that either the covariance of the Maxwell–Lorentz equations or the transformation rules of electrodynamical quantities are *derived* from the RP.

In contrast, we have calculated the transformation rules from the proper operational definitions of the basic electrodynamical quantities, and have shown that the Maxwell–Lorentz equations are indeed covariant against these transformations—*independently* of the RP. In fact, the question whether the RP holds for ED has been left open.

8 Is the RP consistent with the laws of ED in a single frame of reference?

One might think, we simply have to verify whether the solutions of equations (64)–(68) satisfy condition (18) in section 2. However, we still have some vagueness in the RP; namely, the vagueness of $M_{\mathbf{V}}(F)$. For, when can we say that a solution describes the same behavior of the same system, except that it is in an additional collective motion at velocity $\mathbf{V}$? While there is unambiguous meaning of $M_{\mathbf{V}}(F)$ in the Galileo covariant classical mechanics, one can show simple situations in relativistic physics, in which a solution of the equations describing the system in question doubtlessly corresponds to the concept of $M_{\mathbf{V}}(F)$ relative to another solution F, but still $M_{\mathbf{V}}(F) \neq T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right)$ (Szabó 2004). Unfortunately, the concept of $M_{\mathbf{V}}(F)$ is especially problematic in case of a coupled particles + electromagnetic field system, as the following considerations will demonstrate.

As is known, a solution of the coupled Maxwell–Lorentz equations is uniquely determined by a set of Cauchy data along a $t = t_0$ Cauchy surface. The Cauchy data are the values of the particles' positions and velocities, and the values of the electric and magnetic field strengths along the Cauchy surface. The corresponding extra conditions are of the following form:

$$\psi \begin{cases} \mathbf{r}^{b_1}(t_0) = \mathbf{r}_0^{b_1} \\ \mathbf{v}^{b_1}(t_0) = \mathbf{v}_0^{b_1} \\ \vdots \\ \mathbf{r}^{b_n}(t_0) = \mathbf{r}_0^{b_n} \\ \mathbf{v}^{b_n}(t_0) = \mathbf{v}_0^{b_n} \\ \mathbf{E}(\mathbf{r}, t_0) = \mathbf{E}_0(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}, t_0) = \mathbf{B}_0(\mathbf{r}) \end{cases} \tag{136}$$

Due to the fact that there is a one-to-one correspondence between the Cauchy data along the $t = t_0$ Cauchy surface and the solutions of the equations, extra conditions of the form (136) constitute a parametrizing set of extra conditions for the Maxwell–Lorentz equations, defined in section 2.

We have proved, independently of the RP, that the Maxwell–Lorentz equations are covariant; therefore, we can apply Theorem 1. That is, the RP for ED is equivalent to

$$[T_{\mathbf{V}}(M_{\mathbf{V}}(\psi))]_{P_{\mathbf{V}}(\mathcal{E})} = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{137}$$

for all $\psi \in \mathcal{C}^{\mathbf{V}}$, where $\mathcal{E}$ stands for the Maxwell–Lorentz equations, $\mathcal{C}$ denotes the parametrizing set of extra conditions of the form (136), $\mathcal{E}^{\mathbf{V}} \subseteq \mathcal{E}$ denotes the set of solutions for which $M_{\mathbf{V}}([\psi]_{\mathcal{E}})$ is physically admissible. So, the question is: what can we say about condition (137) from the laws of ED? In order to answer this question, we should be able to tell what $M_{\mathbf{V}}(\psi)$ *exactly* means in ED. Thus, the basic question we have to answer, in order to answer question (Q3) in the Introduction, is the following:

(Q5) What does it exactly mean that a coupled particles + field system is in such a state at time t_0, that is, the Cauchy date along the $t = t_0$ surface are such, that the corresponding time evolution of the system is the same as the one belonging to ψ, except that the whole system is in an additional *collective motion* with velocity $\mathbf{V}$?

If there were an answer to this question, it would trivially imply the answer to the following more modest question:

(Q6) What does it exactly mean that a coupled particles + field system is in such a state at time t_0 that the corresponding time evolution of the system is the same as the one belonging to ψ, except that the whole system is in an additional collective motion with velocity $\mathbf{V}$, at least *in an infinitesimally small time-window* $(t_0 - \varepsilon, t_0 + \varepsilon)$?

However, as we will see below, even this latter question has no reasonable answer. For, it is perhaps easy to tell when the particles are initiated in this way. For example,

$$M_{\mathbf{V}}(\psi) \quad \begin{cases} \mathbf{r}^{b_1}(t_0) = \mathbf{r}_0^{b_1} \\ \mathbf{v}^{b_1}(t_0) = \mathbf{v}_0^{b_1} + \mathbf{V} \\ \quad\vdots \\ \mathbf{r}^{b_n}(t_0) = \mathbf{r}_0^{b_n} \\ \mathbf{v}^{b_n}(t_0) = \mathbf{v}_0^{b_n} + \mathbf{V} \\ \quad ? \\ \quad ? \end{cases} \tag{138}$$

can be a reasonable definition, if each particle remains in a physically admissible state of motion, that is, $\left|\mathbf{v}^{b_i}(t_0) + \mathbf{V}\right| < c$. But, we also have to tell when the electromagnetic field is initiated with an additional velocity $\mathbf{V}$ relative to K.

It might be thought that it is enough to set into motion the particles, and we do not need to "set into motion" the field; because we can govern only the sources of the field but not the field itself; and because there are supposedly no "wandering waves" in nature, which are traversing across the universe but did not arise originally from moving charges (see Jánossy 1971, p. 171). This can be true from some particular aspect of ED. However:

- We cannot govern the particles better than the field, at least not within the theory we are concerned with, described by the Maxwell–Lorentz equations; any constraint on the motion of the particles would come from outside of the Maxwell–Lorentz theory (see footnote 10).
- In any event, the field configurations $\mathbf{E}(\mathbf{r}, t_0)$ and $\mathbf{B}(\mathbf{r}, t_0)$ are parts of the Cauchy data, therefore one cannot avoid to specify them in order to specify a unique solution of the equations.

Another thought might be that the moving electromagnetic field is the Lorentz boosted one, by definition; that is, $[M_{\mathbf{V}}(\psi)]_{\mathcal{E}} \stackrel{def}{=} \left[T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\psi))\right]_{\mathcal{E}}$. Recall, however, that this idea has been already discussed in Remark 2; and it must be rejected if the RP qualifies as a contingent statement about our physical world, rather than a vacuous tautology.

Thus,

(Q7) What meaning can be attached to the words "the electromagnetic field is *in (an additional) collective motion* with velocity $\mathbf{V}$"?

If this question is meaningful at all, if it is meaningful to talk about an "additional and/or collective motion" of the field, then it must be meaningful to talk about the original and not necessarily collective instantaneous motion of the local parts of the field. That is, we must have a clear answer to the following primary question:

(Q8) What meaning can be attached to the words "the electromagnetic field at point $\mathbf{r}$ and time t *is in motion with some local and instant velocity* $\mathbf{v}(\mathbf{r}, t)$"?

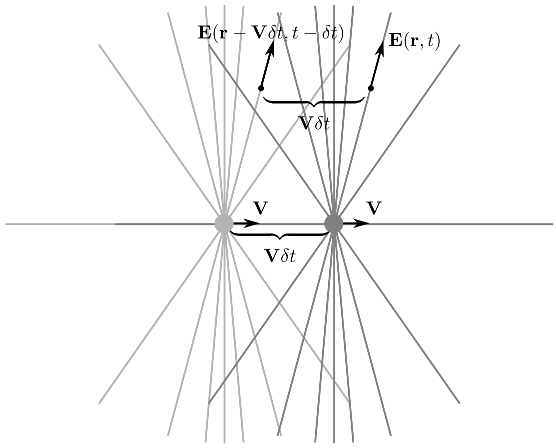

Figure 3: The stationary field of a uniformly moving point charge is in collective motion together with the point charge

To sum up: the RP is meaningful for ED only if we have a clear answer to question (Q5), which implies that we must have an answer to question (Q6), consequently to (Q7) and finally to (Q8). So let us make the first step towards providing meaning to the RP in ED, by trying to answer the most primary question (Q8).

We can rely on what seems to be commonly accepted: Whatever is the answer to question (Q5), according to the *application* of the RP in the derivation of electromagnetic field of a uniformly moving point charge, the system of the moving charged particle + its electromagnetic field *qualifies* as the system of the charged particle + its field in collective motion (Fig. 3). If so, one might think, we can read off the general answer to question (Q7): the electromagnetic field in collective motion with the point charge of velocity $\mathbf{V}$ can be characterized by the following condition:[9]

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r} - \mathbf{V}\delta t, t - \delta t) \tag{139}$$
$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r} - \mathbf{V}\delta t, t - \delta t) \tag{140}$$

[9] It must be pointed out that velocity $\mathbf{V}$ conceptually differs from the speed of light c. Basically, c is a constant of nature in the Maxwell–Lorentz equations, which can emerge in the solutions of the equations; and, in some cases, it can be interpreted as the velocity of propagation of changes in the electromagnetic field. For example, in our case, the stationary field of a uniformly moving point charge, in collective motion with velocity $\mathbf{V}$, can be constructed from the superposition of retarded potentials, in which the retardation is calculated with velocity c; nevertheless, the two velocities are different concepts. To illustrate the difference, consider the fields of a charge at rest (20), and in motion (21). The speed of light c plays the same role in both cases. Both fields can be constructed from the superposition of retarded potentials in which the retardation is calculated with velocity c. Also, in both cases, a small local perturbation in the field configuration would propagate with velocity c. But still, there is a consensus to say that the system described by (20) is at rest while the one described by (21) is moving with velocity $\mathbf{V}$ (together with K', relative to K.) A good analogy would be a Lorentz contracted moving rod: $\mathbf{V}$ is the velocity of the rod, which differs from the speed of sound in the rod.

that is,

$$
\begin{aligned}
-\partial_t \mathbf{E}(\mathbf{r},t) &= \mathrm{D}\mathbf{E}(\mathbf{r},t)\mathbf{V} \qquad (141)\\
-\partial_t \mathbf{B}(\mathbf{r},t) &= \mathrm{D}\mathbf{B}(\mathbf{r},t)\mathbf{V} \qquad (142)
\end{aligned}
$$

where $\mathrm{D}\mathbf{E}(\mathbf{r},t)$ and $\mathrm{D}\mathbf{B}(\mathbf{r},t)$ denote the spatial derivative operators (Jacobians for variables x, y and z); that is, in components:

$$
\begin{aligned}
-\partial_t E_x(\mathbf{r},t) &= V_x\partial_x E_x(\mathbf{r},t) + V_y\partial_y E_x(\mathbf{r},t) + V_z\partial_z E_x(\mathbf{r},t) \qquad (143)\\
-\partial_t E_y(\mathbf{r},t) &= V_x\partial_x E_y(\mathbf{r},t) + V_y\partial_y E_y(\mathbf{r},t) + V_z\partial_z E_y(\mathbf{r},t) \qquad (144)\\
&\vdots\\
-\partial_t B_z(\mathbf{r},t) &= V_x\partial_x B_z(\mathbf{r},t) + V_y\partial_y B_z(\mathbf{r},t) + V_z\partial_z B_z(\mathbf{r},t) \qquad (145)
\end{aligned}
$$

Of course, if conditions (141)–(142) hold for all $(\mathbf{r},t)$ then the general solution of the partial differential equations (141)–(142) has the following form:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0(\mathbf{r}-\mathbf{V}t) \qquad (146)\\
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0(\mathbf{r}-\mathbf{V}t) \qquad (147)
\end{aligned}
$$

with some time-independent $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$. In other words, the field must be a stationary one, that is, a translation of a static field with velocity $\mathbf{V}$. This is correct in the case of a single moving point charge, provided that $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$ are the electric and magnetic parts of the "flattened" Coulomb field (21) at time t_0.[10] But, (146)–(147) is certainly not the case in general; the field is not necessarily stationary.

So, this example does not help to find a general answer to question (Q7), but it may help to find the answer to question (Q8). For, from (139)–(140), it is quite natural to say that the electromagnetic field at point $\mathbf{r}$ and time t is

[10] Here we can observe that we need, indeed, to "set into motion" the electromagnetic field too: if

$$
\psi \quad \begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = 0 \\ \mathbf{E}(\mathbf{r},t_0) = \mathbf{E}_0^C(\mathbf{r}) \\ \mathbf{B}(\mathbf{r},t_0) = 0 \end{cases} \qquad (148)
$$

is the initial state of the rest system, where $\mathbf{E}_0^C(\mathbf{r})$ stands for the Coulomb field (20), then

$$
M_{\mathbf{V}}(\psi) \quad \begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = \mathbf{V} \\ \mathbf{E}(\mathbf{r},t_0) = \mathbf{E}_0^{FC}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r},t_0) = \mathbf{B}_0^{FC}(\mathbf{r}) \end{cases} \qquad (149)
$$

where $\mathbf{E}_0^{FC}(\mathbf{r})$ and $\mathbf{B}_0^{FC}(\mathbf{r})$ stand for the "flattened" fields of the moving charge (that is the electric and magnetic fields (21) at time t_0). Within the framework of the Maxwell–Lorentz theory we cannot describe how the system has been brought into such a state; or we cannot prescribe, by hand, a constraint for the particle to be at rest or to move along a given trajectory—as is the case in many practical applications. The Coulomb field, for example, there appears among the solutions of the Maxwell–Lorentz equations as the one determined by the initial condition (148); and it is a fact about this solution that the particle remains at rest and the field remains the static Coulomb field. ("Solutions of the Maxwell–Lorentz equations", of course, should be understood as explained in Remark 7.)

moving with *local* and *instant* velocity $\mathbf{v}(\mathbf{r},t)$ if and only if

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}\left(\mathbf{r}-\mathbf{v}(\mathbf{r},t)\delta t, t-\delta t\right) &(150)\\
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}\left(\mathbf{r}-\mathbf{v}(\mathbf{r},t)\delta t, t-\delta t\right) &(151)
\end{aligned}$$

are satisfied *locally,* in an *infinitesimally* small space and time region at $(\mathbf{r},t)$, for infinitesimally small δt. In other words, the equations (141)–(142) must be satisfied *locally* at point $(\mathbf{r},t)$ with a local and instant velocity $\mathbf{v}(\mathbf{r},t)$:

$$\begin{aligned}
-\partial_t\mathbf{E}(\mathbf{r},t) &= \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) &(152)\\
-\partial_t\mathbf{B}(\mathbf{r},t) &= \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) &(153)
\end{aligned}$$

Now, if the RP, as it is believed, applies to all physically admissible situations, that is for all solutions from $\mathcal{E}^{\mathbf{V}}$, then it must be meaningful for all solutions in $\mathcal{E}^{\mathbf{V}}$; consequently, the concept of "electromagnetic field moving with velocity $\mathbf{v}(\mathbf{r},t)$ at point $\mathbf{r}$ and time t" must be meaningful, in other words, there must exist a local instant velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying (152)–(153), for all possible solutions of the Maxwell–Lorentz equations, belonging to $\mathcal{E}^{\mathbf{V}}$. That is, substituting an arbitrary solution of (64)–(68), belonging to $\mathcal{E}^{\mathbf{V}}$, into (152)–(153), the overdetermined system of equations must have a solution for $\mathbf{v}(\mathbf{r},t)$.

Since we do not know exactly what $M_{\mathbf{V}}$ is, it is hardly possible to say anything definite about the content of $\mathcal{E}^{\mathbf{V}}$. Nevertheless, it seems quite plausible to assume that $\mathsf{int}\left(\mathcal{E}^{\mathbf{V}}\right) \neq \varnothing$—in the topology induced by the topology on the manifold X of the basic quantities. Otherwise the RP could apply only to some "isolated" solutions of the Maxwell–Lorentz equations; but, it would become inapplicable by an infinitesimally small variation of the solution. In this case, however, one encounters the following difficulty:

Theorem 11. *There exist a solution of the coupled Maxwell–Lorentz equations (64)–(68) which belongs to $\mathcal{E}^{\mathbf{V}}$ but for which there cannot exist a local instant velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying (152)–(153).*

Proof. The proof is almost trivial for a locus $(\mathbf{r},t)$ where there is a charged point particle. However, in order to avoid the eventual difficulties concerning the physical interpretation, we are providing a proof for a point $(\mathbf{r}_*,t_*)$ where there is assumed no source at all.

Consider a solution $\left(\mathbf{r}^{b_1}(t),\ldots\mathbf{r}^{b_n}(t),\mathbf{E}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)\right)$ of the coupled Maxwell–Lorentz equations (64)–(68), which belongs to $\mathsf{int}\left(\mathcal{E}^{\mathbf{V}}\right)$ and which satisfies (152)–(153). At point $(\mathbf{r}_*,t_*)$, the following equations hold:

$$\begin{aligned}
-\partial_t\mathbf{E}(\mathbf{r}_*,t_*) &= \mathsf{D}\mathbf{E}(\mathbf{r}_*,t_*)\mathbf{v}(\mathbf{r}_*,t_*) &(154)\\
-\partial_t\mathbf{B}(\mathbf{r}_*,t_*) &= \mathsf{D}\mathbf{B}(\mathbf{r}_*,t_*)\mathbf{v}(\mathbf{r}_*,t_*) &(155)\\
\partial_t\mathbf{E}(\mathbf{r}_*,t_*) &= c^2\nabla\times\mathbf{B}(\mathbf{r}_*,t_*) &(156)\\
-\partial_t\mathbf{B}(\mathbf{r}_*,t_*) &= \nabla\times\mathbf{E}(\mathbf{r}_*,t_*) &(157)\\
\nabla\cdot\mathbf{E}(\mathbf{r}_*,t_*) &= 0 &(158)\\
\nabla\cdot\mathbf{B}(\mathbf{r}_*,t_*) &= 0 &(159)
\end{aligned}$$

Without loss of generality we can assume—at point $\mathbf{r}_*$ and time t_*—that operators $\mathsf{D}\mathbf{E}(\mathbf{r}_*,t_*)$ and $\mathsf{D}\mathbf{B}(\mathbf{r}_*,t_*)$ are invertible and $v_z(\mathbf{r}_*,t_*) \neq 0$.

Now, consider a 3×3 matrix J such that

$$J = \begin{pmatrix} \partial_x E_x(\mathbf{r}_*, t_*) & J_{xy} & J_{xz} \\ \partial_x E_y(\mathbf{r}_*, t_*) & \partial_y E_y(\mathbf{r}_*, t_*) & \partial_z E_y(\mathbf{r}_*, t_*) \\ \partial_x E_z(\mathbf{r}_*, t_*) & \partial_y E_z(\mathbf{r}_*, t_*) & \partial_z E_z(\mathbf{r}_*, t_*) \end{pmatrix} \tag{160}$$

with

$$J_{xy} = \partial_y E_x(\mathbf{r}_*, t_*) + \lambda \tag{161}$$

$$J_{xz} = \partial_z E_x(\mathbf{r}_*, t_*) - \lambda \frac{v_y(\mathbf{r}_*, t_*)}{v_z(\mathbf{r}_*, t_*)} \tag{162}$$

by virtue of which

$$\begin{aligned} J_{xy} v_y(\mathbf{r}_*, t_*) + J_{xz} v_z(\mathbf{r}_*, t_*) &= v_y(\mathbf{r}_*, t_*) \partial_y E_x(\mathbf{r}_*, t_*) \\ &\quad + v_z(\mathbf{r}_*, t_*) \partial_z E_x(\mathbf{r}_*, t_*) \end{aligned} \tag{163}$$

Therefore, $J\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*)$. There always exists a vector field $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ such that its Jacobian matrix at point $\mathbf{r}_*$ is equal to J. Obviously, from (158) and (160), $\nabla \cdot \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) = 0$. Therefore, there exists a solution of the Maxwell–Lorentz equations, such that the electric and magnetic fields $\mathbf{E}_{\lambda}(\mathbf{r}, t)$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t)$ satisfy the following conditions:[11]

$$\mathbf{E}_{\lambda}(\mathbf{r}, t_*) = \mathbf{E}^{\#}_{\lambda}(\mathbf{r}) \tag{164}$$

$$\mathbf{B}_{\lambda}(\mathbf{r}, t_*) = \mathbf{B}(\mathbf{r}, t_*) \tag{165}$$

At $(\mathbf{r}_*, t_*)$, such a solution obviously satisfies the following equations:

$$\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = c^2 \nabla \times \mathbf{B}(\mathbf{r}_*, t_*) \tag{166}$$

$$-\partial_t \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*) = \nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \tag{167}$$

therefore

$$\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = \partial_t \mathbf{E}(\mathbf{r}_*, t_*) \tag{168}$$

As a little reflection shows, if $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$, that is J, happened to be not invertible, then one can choose a *smaller* λ such that $\mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)$ becomes invertible (due to the fact that $\mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)$ is invertible), and, at the same time,

$$\nabla \times \mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) \neq \nabla \times \mathbf{E}(\mathbf{r}_*, t_*) \tag{169}$$

Consequently, from (168) , (162) and (154) we have

$$-\partial_t \mathbf{E}_{\lambda}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{170}$$

and $\mathbf{v}(\mathbf{r}_*, t_*)$ is uniquely determined by this equation. On the other hand, from (167) and (169) we have

$$-\partial_t \mathbf{B}_{\lambda}(\mathbf{r}_*, t_*) \neq \mathsf{D}\mathbf{B}_{\lambda}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{171}$$

[11] $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t_*)$ can be regarded as the initial configurations at time t_*; we do not need to specify a particular choice of initial values for the sources.

because $\mathrm{D}\mathbf{B}(\mathbf{r}_*, t_*)$ is invertible, too. That is, for $\mathbf{E}_\lambda(\mathbf{r}, t)$ and $\mathbf{B}_\lambda(\mathbf{r}, t)$ there is no local and instant velocity at point $\mathbf{r}_*$ and time t_*. At the same time, λ can be arbitrary small, and

$$\lim_{\lambda \to 0} \mathbf{E}_\lambda(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \tag{172}$$

$$\lim_{\lambda \to 0} \mathbf{B}_\lambda(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t) \tag{173}$$

Therefore solution $\left(\mathbf{r}_\lambda^{b_1}(t), \ldots \mathbf{r}_\lambda^{b_n}(t), \mathbf{E}_\lambda(\mathbf{r}, t), \mathbf{B}_\lambda(\mathbf{r}, t)\right)$ can fall into an arbitrary small neighborhood of $\left(\mathbf{r}^{b_1}(t), \ldots \mathbf{r}^{b_n}(t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)\right)$ in $\operatorname{int}\left(\mathcal{E}^{\mathbf{V}}\right)$, consequently it belongs to $\mathcal{E}^{\mathbf{V}}$. □

Thus, the meaning of the concept of "electromagnetic field moving with velocity $\mathbf{v}(\mathbf{r}, t)$ at point $\mathbf{r}$ and time t", that we obtained by generalizing the example of the stationary field of a uniformly moving charge, is untenable. Perhaps there is no other available rational meaning of this concept. In any event, lacking a better suggestion, we must conclude that the question whether the relativity principle generally holds in classical electrodynamics remains not only unanswered, but even ununderstood.

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