Exhaustive Classification of Finite Classical Probability Spaces with Regard to the Notion of Causal Up-to-n-closedness

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16 June 2009

Abstract

Extending the ideas from (Hofer-Szabó and Rédei [2006]), we introduce the notion of causal up-to-n-closedness of probability spaces. A probability space is said to be causally up-to-n-closed with respect to a relation of independence R_{ind} iff for any pair of correlated events belonging to R_{ind} the space provides a common cause or a common cause system of size at most n. We prove that a finite classical probability space is causally up-to-3-closed w.r.t. the relation of logical independence iff its probability measure is constant on the set of atoms of non-0 probability. (The latter condition is a weakening of the notion of measure uniformity.) Other independence relations are also considered.

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1 Introduction

Suppose a probability space contains a correlation between two events we believe to be causally independent. Does the space contain a common cause for the correlation? If not, can the probability space be extended to contain such a cause but ‘preserving’ the old measure? This question has been asked and answered in the positive in (Hofer-Szabó, Rédei, and Szabó [1999]), where the notion of *common cause completability* was introduced: speaking a bit informally, a probability space S is said to be common cause completable with respect to a set $\mathbf{A}$ of pairs of correlated events iff there exists an extension of the space containing common causes of all the correlated pairs in $\mathbf{A}$. Gyenis and Rédei ([2004]) introduced the notion of *common cause closedness*, which (in our slightly different terminology) is equivalent to the following: a probability space S is common cause closed with respect to a relation of independence $R_{ind} \subseteq S^2$ iff it contains common causes for all pairs of correlated events belonging to R_{ind}. The authors have proven therein that a finite classical probability space with no atoms of probability 0 is non-trivially common cause closed w.r.t. the relation of logical independence iff it is the space consisting of a Boolean algebra with 5 atoms and the uniform probability measure.[1] In other words, finite classical probability spaces are in general *not* common cause closed w.r.t. the relation of logical independence, i.e. each contains a correlation between logically independent events for which no common cause in the space exists; the only exceptions to this rule are the spaces which have precisely 5 atoms of probability $\frac{1}{5}$ each and any number of atoms of probability 0.

Still, a common cause is not the only entity which could be used as an explanation for a correlation. Hofer-Szabó and Rédei ([2004]) generalised the idea of common cause, arriving at *common cause systems* (formal details below). Common cause systems (CCSs for short) may have any countable size; the special case of size 2 reduces to the usual notion of common cause. It was proven in (Hofer-Szabó and Rédei [2006]) that there exist CCSs of any finite size, while in (Marczyk, Wroński [unpublished]) an example of a countably infinite CCS is given, as well as a proof that no non-denumerable CCSs exist.

It was natural for corresponding notions of causal closedness to be introduced; a probability space is said to be *causally n-closed* w.r.t. a relation of independence R_{ind} iff it contains a CCS of size n for any correlation between A, B such that $\langle A, B\rangle \in R_{ind}$. It is one of the results of the present paper (see corollaries 18 and 22) that excepting the 5-atom uniform distribution probability space and the related spaces with 0 probability atoms, no finite probability spaces are n-closed, for any $n \geqslant 2$.

We are interested in a slightly different version of causal closedness. If the overarching goal is to find explanations for correlations, why should we expect all explanations to be CCSs of the same size? Perhaps some correlations are explained by common causes and other by CCSs of a bigger size. We propose to explore the idea of *causal up-to-n-closedness* – a probability space is causally up-to-n-closed w.r.t. a relation of independence R_{ind} iff it contains a CCS of size at most n for any correlation between events A, B such that $\langle A, B\rangle \in R_{ind}$.

It turns out that, in the class of finite classical probability spaces with no atoms of probability 0, just as the space with 5 atoms and uniform measure is unique with regard to common cause closedness, the whole class of spaces with the uniform distribution is special with regard to causal up-to-3-closedness – see theorem 9: a finite classical probability space with no atoms of probability

[1]The phrasing of the paper was in fact stronger, omitting the assumption about non-0 probabilities on the atoms (due to a missed special subcase in the proof of case 3 of proposition 4 on p. 1299). The latter is, however, essential; see the counterexample following corollary 20 below.

0 has uniform distribution iff it is causally up-to-3-closed w.r.t. the relation of logical independence. We provide a method of constructing a common cause or a CCS of size 3 for any correlation between logically independent events in any finite classical probability space with uniform distribution.

In the last paragraph we have restricted our attention to spaces containing no atoms of probability 0. However, as will be seen in lemma 19, results obtained in this limited setting can be generalised to arbitrary finite classical probability spaces (theorem 23). This generalisation replaces uniform distributions with distributions constant on the set of atoms of non-0 probability.

Finally, we briefly consider other independence relations.

2 Preliminaries

In the following assume that we are given a classical probability space $\langle S, P\rangle$. The algebra S is to be considered as a field of sets (indeed, a powerset of a finite set in the finite case; by Stone's representation theorem, this causes no loss of generality).

We will now define the relation of logical independence L_{ind}.

Definition 1 *We say that events $A, B \in S$ are logically independent ($\langle A, B\rangle \in L_{ind}$) if all of the following sets are nonempty:*

- $A \cap B$;
- $A \cap B^{\perp}$;
- $A^{\perp} \cap B$;
- $A^{\perp} \cap B^{\perp}$.

Equivalently, two events are logically independent if neither of the events is contained in the other one, their intersection is non-empty and the sum of the two is less than the whole space.

Definition 2 *We say that events $A, B \in S$ are* correlated *if $P(A \cap B) > P(A)P(B)$. They are* negatively correlated *if this inequality is reversed;* uncorrelated *otherwise*[2].

Definition 3 *Let $A, B \in S$. An event C is said to be a* screener-off *for the pair $\{A, B\}$ if $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$. In the case where A and B are correlated we also say that C* screens off *the correlation.*

Definition 4 *Let $A, B \in S$. We say that a family of events $\{C_i\}$ satisfies the* statistical relevance *condition with regard to the pair $\{A, B\}$ if whenever $i \neq j$*

$$(P(A \mid C_i) - P(A \mid C_j))(P(B \mid C_i) - P(B \mid C_j)) > 0$$

The following definition mirrors that of Reichenbach ([1956]).

[2] In the latter case the word 'independent' is normally used; we find it more convenient to reserve this word for logical independence of events.

Definition 5 *Let $A, B \in S$. Then $C \in S - \{A, B\}$ is said to be a* common cause *of these two events if (1) both C and its complement $C^{\perp}$ are screener-offs for the pair $\{A, B\}$ and (2) the pair $\{C, C^{\perp}\}$ satisfies the statistical relevance condition with regard to $\{A, B\}$ with $P(A \mid C) > P(A \mid C^{\perp})$.*

The idea of common cause has been generalized with regard to the number of screener-offs in (Hofer-Szabó and Rédei [2004]). Recall that a partition of unity of S is a family $\{Y_i\}$ of pairwise disjoint non-empty subsets of $\mathbf{1}_S$ such that $\bigcup\{Y_i\} = \mathbf{1}_S$.

Definition 6 *A partition of unity of S is said to be a* common cause system (CCS) *for A and B if it satisfies the statistical relevance condition w.r.t. A and B and all its members are screener-offs for the pair.*

The cardinality of the partition is called the size *of the common cause system.*

We will sometimes say that a probability space *contains* a CCS, which means that the CCS is a partition of unity of the underlying algebra of the space.

Since a common cause C for events A, B may be viewed as a doubleton $\{C, C^{\perp}\}$, it is evident that common causes are just CCSs of size 2, making the first notion a special case of the latter. It was shown in (Hofer-Szabó and Rédei [2004]) that the existence of a common cause system (which was then labelled 'Reichenbachian common cause system') for events $A, B \in S$ entails a correlation between those events, so it can be considered an explanation of the correlation. Should someone prefer it, the following definition could be phrased in terms of CCSs only.

Definition 7 *We say that a classical probability space is* causally up-to-n-closed *w.r.t. to a relation of independence R_{ind} if all pairs of correlated events independent in the sense of R_{ind} possess a common cause or a common cause system of size at most n.*

2.1 Some useful parameters

Before proceeding to the main theorem and its proof, we shall introduce a few particularly useful parameters one may associate with a pair of events A, B in a *finite* classical probability space $\langle S, P \rangle$.

Let n be the number of atoms in the Boolean algebra S. The size of the set of atoms lying below A in the lattice ordering of S will from now on be referred to as a, and likewise for B and b. The analogous parameter associated with the conjunction of events A and B is just the size of the intersection of the relevant sets of atoms and will be called k.

It will soon become apparent that while a and b have some utility in the discussion to follow, the more convenient parameters describe A and B in terms of the number of atoms belonging to one, but not the other. Thus we let $a' = a - k$ and $b' = b - k$. In fact, if we set $z = n - (a' + k + b')$, we obtain a set of four numbers precisely describing the blocks of the partition of the set of atoms of S into the four classes which need to be non-empty for A and B to be logically independent. It is clear that in case of logically independent events a', b', k and z are all non-zero.

2.2 Screening off is enough

It turns out that if we know that events A and B are correlated, all we need to achieve in our search for a common cause is to find a two-element partition of unity of S with both elements screening off the correlation. We do not need additionally to worry about the statistical relevance condition.

Lemma 8 *Suppose A, B are correlated. If both C and $C^\perp$ screen off the correlation, either C or $C^\perp$ is a common cause of it.*

Proof: We need to show that if A and B are correlated, than if both C and $C^\perp$ screen off the correlation, it is either the case that $P(A \mid C) > P(A \mid C^\perp)$ and $P(B \mid C^\perp) > P(B \mid C)$ (and so C is a common cause of the correlation) or $P(A \mid C^\perp) > P(A \mid C)$ and $P(B \mid C^\perp) > P(B \mid C)$ (and so $C^\perp$ is a common cause of the correlation). In other words, we need to prove that

$$((P(A \mid C) - P(A \mid C^\perp))(P(B \mid C) - P(B \mid C^\perp))) > 0.$$

Since A and B are correlated, we know that

$$P(AB) > P(A)P(B).$$

Applying the law of total probability transforms this into

$$P(AB \mid C)P(C) + P(AB \mid C^\perp)P(C^\perp) > \\ (P(A \mid C)P(C) + P(A \mid C^\perp)P(C^\perp))(P(B \mid C)P(C) + P(B \mid C^\perp)P(C^\perp)).$$

By further transforming the last inequality using the screening off conditions on the left side, substituting $1 - P(C)$ for $P(C^\perp)$ and carrying out the multiplication on the right side we eventually arrive at

$$P(C)((P(A \mid C) - P(A \mid C^\perp))(P(B \mid C) - P(B \mid C^\perp))) > \\ P(C)^2((P(A \mid C) - P(A \mid C^\perp))(P(B \mid C) - P(B \mid C^\perp)))$$

which immediately leads to

$$P(C)P(C^\perp)((P(A \mid C) - P(A \mid C^\perp))(P(B \mid C) - P(B \mid C^\perp))) > 0$$

and consequently

$$((P(A \mid C) - P(A \mid C^\perp))(P(B \mid C) - P(B \mid C^\perp))) > 0$$

as desired. ($P(C)$ and $P(C^\perp)$ cannot both be 0, because their sum is 1; and since they are screener-offs for the pair $\{A, B\}$, nor can it be the case that only one of them is 0, since then events A and B would not be correlated.) **Q.E.D.**

The statistical relevance requirement is present in Reichenbach's definition of common cause (reformulated above as definition 5) only to ensure that it is the common cause C, and not its complement $C^\perp$, that is more statistically relevant for both events. But since we are only interested in *finding* common causes for pairs of events which are already known to be correlated, by lemma 8 we never need to concern ourselves with that part of the definition.

3 Results on spaces without 0-probability atoms

Theorem 9 *Let $\langle S, P\rangle$ be a* finite *classical probability space with no atoms of probability* 0*. Suppose S has at least* 4 *atoms.*[3] *The following conditions are equivalent:*

Measure uniformity: *P is the uniform probability measure on S;*

Causal up-to-3-closedness w.r.t. L_{ind}: *$\langle S, P\rangle$ is causally up-to-3-closed w.r.t. the relation of logical independence.*

Before proceeding with the proof we will provide a sketch of the construction and some requisite definitions. Instead of focusing on a particular n-atomic algebra, we will show how the problem presents itself while we 'move' from smaller to bigger algebras. We assume without loss of generality that the set of atoms of an n-atomic Boolean algebra is $\{0, 1, \cdots, n-1\}$ and that each event is a list of atoms. Consider the sequence of all finite classical probability spaces with uniform probability measure, in which the number of atoms of the underlying Boolean algebra of the space increases by 1 at each step, beginning with the algebra with a single atom. We use the shorthand expression "at stage n" to mean "in the probability space with uniform distribution whose underlying Boolean algebra has n atoms". Observe that due to our convention whereby events are identified with lists of atoms, an event present at stage m (one found in the algebra from that stage) is also present at all further stages. In other words, a list of atoms describing an event at stage m can also be interpreted as describing an event at any stage m', with $m' > m$.

Some remarks on the shape of events considered are in order. We will always be talking about pairs of events A, B, with numbers a, a', b, b', k, z and n defined as above (see section 2.1). We assume $a \geqslant b$. Also, since we are dealing with the uniform measure, all relevant characteristics of a pair of events A, B are determined by the numbers a', b', k, and z; therefore, for any combination of these numbers it is sufficient only to consider a single example of a pair displaying them. The rest is just the matter of renaming the atoms. For example, if we are looking for an explanation for the pair $\{(8, 7, 3, 5), (2, 8, 7)\}$ at stage 10, or the pair $\{(1, 3, 5, 6), (1, 6, 4)\}$ at the same stage, we shall search for an explanation for the pair $\{(0, 1, 2, 3), (2, 3, 4)\}$ at stage 10 and then just appropriately 'translate' the result (explicit examples of this follow in section 3.1). In general: the convention we adopt is to have A be a list of consecutive atoms beginning with 0, and B a list of consecutive atoms beginning with $a-k$.

For illustrative purposes we propose to examine the situation at the early stages. The reader interested in the proof proper may jump directly to definition 10 below.

As the reader may verify, there are no correlated pairs of logically independent events at stage 1; similarly for stages 2, 3 and 4. (Remember the measure is uniform and so at stage 4 e.g. the pair $\{(0, 1), (1, 2)\}$, while composed of logically independent events, is not correlated.)

First correlated pairs of logically independent events appear at stage 5. These are of one of the two following types: either $a' = b' = k = 1$, or $a' = b' = 1$

[3] We leave it to the reader to verify that if S has 3 atoms or less, then $\langle S, P\rangle$ contains no correlations between logically independent events.

and $k = 2$. Proposition 3 from (Gyenis and Rédei [2004]) says that all pairs of these types have common causes at stage 5. As noted above, we can without loss of generality consider just two tokens of these types – the pairs $\{(0, 1), (1, 2)\}$ and $\{(0, 1, 2), (1, 2, 3)\}$. In the first case, the events already formed a logically independent pair at stage 4, but were not correlated – we will say that this pair *appears from below at stage* 5 (see definition 10 below). In the second case, stage 5 is the first stage where the events form a logically independent pair, and they are correlated at that stage, too. We will say that the pair $\{(0, 1, 2), (1, 2, 3)\}$ *appears from above at stage* 5. There are no other correlated pairs of logically independent events at stage 5. It will turn out that we can always find common causes for pairs which appear from above or from below at a given stage.

Let us move to stage 6. A new (type of) pair appears from above – $\{(0, 1, 2, 3), (1, 2, 3, 4)\}$. No pairs appear from below, but both pairs which appeared at stage 5 are still correlated and logically independent at stage 6 (as well as at all later stages), so they are again in need of an explanation at this higher stage. It turns out that if a correlated pair of logically independent events at stage n is 'inherited' from the earlier stages, i.e. it appears neither from above nor from below at stage n, we can modify the common cause which we know how to supply for it at the stage where it originally appeared to provide it with an explanation adequate at stage n. This takes the form of a common cause or, in some cases, a common cause system of size 3.

Definition 10 *A pair A, B of events* appears from above at stage n *if it is (1) logically independent at stage n, (2) not logically independent at stage $n-1$ and (3) correlated at stage n.*

A pair A, B of events appears from below at stage n *if it is (1) logically independent at stage n, (2) logically independent at stage $n-1$ and (3) correlated at stage n, but (4) not correlated at stage $n - 1$.*

We will divide common causes into types depending on whether the occurence of a given common cause makes the occurence of at least one of member of the correlation it explains necessary, impossible or possible with probability less then 1.

Definition 11 *A common cause C for a correlated pair of logically independent events A, B is said to be:*

- 1-type *iff $P(A \mid C) = 1$ or $P(B \mid C) = 1$;*
- 0-type *iff $P(A \mid C^{\perp}) = 0$ or $P(B \mid C^{\perp}) = 0$;*
- $\#$-type *iff it is neither 1-type nor 0-type.*

Notice that no common cause C can be both 1-type and 0-type at the same time.

Definition 12 *A common cause system of size n $\{C_i\}_{i \in \{0,\dots,n-1\}}$ is a* 0-type common cause system *(*0-type CCS*) for the correlation iff $P(A \mid C_{n-1}) = 0$ or $P(B \mid C_{n-1}) = 0$.*

We will prove the following:

- if a pair appears from above at stage n, it has a common cause at that stage (lemma 14);
- if a pair appears from below at stage n, it has a common cause at that stage (lemma 15);
- if a pair of logically independent events is correlated at stage n and has a common cause or a 0-type CCS of size 3 at that stage, it has a common cause or a 0-type CCS of size 3 at stage $n+1$ (lemma 16).

It should be straightforward to see that this is enough to prove the theorem in its 'downward' direction. Consider a correlated pair of logically independent events A, B at stage n. If it appears from above, we produce a common cause using the technique described in lemma 14. If it appears from below, we use the method from lemma 15. If it appears neither from above nor from below, it means that it was logically independent at stage $n-1$ and was correlated at that stage, and we repeat the question at the stage $n-1$. This descent terminates at the stage where our pair first appeared, which clearly must have been either from below or from above. This allows us to apply either lemma 14 or lemma 15, as appropriate, followed by lemma 16 to move back up to stage n, where we will now be able to supply the pair with a common cause or a CCS of size 3.

Put $Corr(A,B) := P(AB) - P(A)P(B)$. $Corr(A,B)$ can always be expressed as a fraction with the denumerator being n^2. Of special interest to us will be the numerator of this fraction. Let us call this number $SC_n(A,B)$. (For example, if $A = (0,1,2)$ and $B = (2,3)$, $SC_5(A,B) = -1$.) If $SC_n(A,B) \leqslant 0$, the events are not correlated at stage n. If $SC_n(A,B) > 0$, A and B are correlated at stage n and we need to find either a common cause or a common cause system of size 3 for them. The following lemma will aid us in our endeavour (remember the definitions from section 2.1):

Lemma 13 *Let $\langle S_n, P\rangle$ be a finite classical probability space, S_n being the Boolean algebra with n atoms and P the uniform measure on S_n. Let $A, B \in S_n$. Then $SC_n(A,B) = kz - a'b'$.*

Proof: $Corr(A,B) = P(AB) - P(A)P(B) = \frac{k}{n} - \frac{k+a'}{n}\frac{k+b'}{n} = = \frac{k(n-k-a'-b')-a'b'}{n^2} = \frac{kz-a'b'}{n^2}$. Therefore $SC_n(A,B) = kz - a'b'$. **Q.E.D.**

An immediate consequence of this lemma is that any pair of logically independent events will eventually (at a far enough stage) be correlated – it is just a matter of injecting enough atoms into z. For example, consider events $A = (0,1,2,3,4,5,6)$, $B = (6,7,8,9,10,11)$. At any stage n, $SC_n(A,B)$ is equal to $z - 30$. This means that the pair is correlated at all stages in which $z > 30$; in other words, at stages 43 and up. At some earlier stages (from 13 to 42) the pair is logically independent but not correlated; at stage 12 it is not logically independent; and the events constituting it do not fit in the algebras from stages lower than that.

Notice that since for any A, B: $SC_{n+1}(A,B) = SC_n(A,B) + k$, it follows that at the stage m where the pair first appears (either from above or from below) $SC_m(A,B)$ is positive but lower than k.

We now have all tools we need to prove theorem 9.

Proof: (of theorem 9)

Measure uniformity $\Rightarrow$ Causal up-to-3-closedness w.r.t. L_{ind}

Lemma 14 *Suppose a pair A, B appears from above at stage n. There exists a 1-type common cause for the correlation at that stage.*

Proof: We are at stage n. Since the pair A, B appears from above at this stage, $z = 1$ and so (by lemma 13) $SC_n(A,B) = k - a'b'$. (If z was equal to 0, the events would not be logically independent at stage n; if it was higher than 1, the events would be logically independent at stage $n-1$, too, and so the pair would not appear from above at stage n.) Notice that since A, B are logically independent (and so both a' and b' are non-zero) but correlated at stage n, $0 < SC_n(A,B) = k - a'b' < k$. Let C consist of exactly $SC_n(A,B)$ atoms from the intersection $A \cap B$. Such a C will be a screener-off for the correlation, since $P(AB \mid C) = 1 = P(A \mid C)P(B \mid C)$. What remains is to show that $C^{\perp}$ is a screener-off, too. This follows from the observation that $P(AB \mid C^{\perp}) = \frac{k-(k-a'b')}{n-(k-a'b')} = \frac{a'b'}{n-k+a'b'} = \frac{a'b'(n-k+a'b')}{(n-k+a'b')^2} = \frac{a'b'(1+a'+b'+k)-a'b'k+a'^2b'^2}{(n-k+a'b')^2} = \frac{a'b'+a'b'^2+a'^2b'+a'^2b'^2}{(n-k+a'b')^2} = \frac{a'+a'b'}{n-k+a'b'} \cdot \frac{b'+a'b'}{n-k+a'b'} = \frac{k+a'-(k-a'b')}{n-k+a'b'} \cdot \frac{k+b'-(k-a'b')}{n-k+a'b'} = \frac{k+a'-SC_n(A,B)}{n-k+a'b'} \cdot \frac{k+b'-SC_n(A,B)}{n-k+a'b'} = P(A \mid C^{\perp})P(B \mid C^{\perp})$. **Q.E.D.**

Lemma 15 *Suppose a pair A, B appears from below at stage n. There exists a 1-type common cause or a 0-type common cause for the correlation at that stage.*

Proof:

Case 1: $k > b'$ and $a' > z$.

In this case we will construct a 1-type common cause. Let C consist of $k - b'$ atoms from $A \cap B$ and $a' - z$ atoms from $A \setminus B$. Since $C \subset A$, it screens off the correlation: $P(AB \mid C) = P(B \mid C) = 1 \cdot P(B \mid C) = P(A \mid C)P(B \mid C)$. We need to show that $C^{\perp}$ screens off the correlation as well. This follows from the fact that $P(AB \mid C^{\perp}) = \frac{b'}{n-(k-b')-(a'-z)} = \frac{b'}{2b'+2z} = \frac{2b'^2+2zb'}{(2b'+2z)^2} = \frac{(b'+z)2b'}{(2b'+2z)^2} = \frac{b'+z}{2b'+2z} \cdot \frac{2b'}{2b'+2z} = \frac{b'+z}{n-(k-b')-(a'-z)} \cdot \frac{2b'}{n-(k-b')-(a'-z)} = P(A \mid C^{\perp})P(B \mid C^{\perp})$.

Case 2: $z > b'$ and $a' > k$.

In this case we will construct a 0-type common cause. Let $C^{\perp}$ consist of $a' - k$ atoms from $A \setminus B$ and $z - b'$ atoms from $(A \cup B)^{\perp}$. Since $C^{\perp} \subset B^{\perp}$, it screens off the correlation: $P(AB \mid C^{\perp}) = 0 = P(A \mid C^{\perp}) \cdot 0 = P(A \mid C^{\perp})P(B \mid C^{\perp})$. We need to show that C too screens off the correlation. This follows from the fact that $P(AB \mid C) = \frac{k}{n-(a'-k)-(z-b')} = \frac{k}{2k+2b'} = \frac{2k^2+2kb'}{(2k+2b')^2} = \frac{2k(k+b')}{(2k+2b')^2} = \frac{2k}{2k+2b'} \cdot \frac{k+b'}{2k+2b'} = \frac{2k}{n-(a'-k)-(z-b')} \cdot \frac{k+b'}{n-(a'-k)-(z-b')} = P(A \mid C)P(B \mid C)$.

Case 3a: $z \geqslant a'$, $k \geqslant a'$ and $a' > b'$.

As the reader will verify, in this case $k = z = a'$ and $b' = a' - 1$. We can construct both a 0-type common cause and a 1-type common cause. Suppose we choose to produce the former. An appropriate $C^{\perp}$ would consist of just a single atom from $(A \cup B)^{\perp}$; $C^{\perp}$ screens off the correlation because $P(AB \mid C^{\perp}) = 0 =$

$P(A \mid C^\perp)P(B \mid C^\perp)$. That C is also a screener-off is evidenced by the fact that $P(AB \mid C) - P(A \mid C)P(B \mid C) = \frac{k}{k+a'+b'+z-1} - \frac{k+a'}{k+a'+b'+z-1} \cdot \frac{k+b'}{k+a'+b'+z-1} = \frac{k}{4k-2} - \frac{2k}{2(2k-1)} \cdot \frac{2k-1}{4k-2} = 0$.

To produce a 1-type common cause instead, let C consist of just a single atom from $(A \cap B)$; C screens off the correlation because $P(AB \mid C) = 1 = P(A \mid C)P(B \mid C)$. That $C^\perp$ is also a screener-off follows from the fact that $P(AB \mid C^\perp) = \frac{k-1}{k-1+a'+b'+z} = \frac{b'}{2b'+2a'} = \frac{2b'^2+2a'b'}{(2b'+2a')^2} = \frac{(a'+b')2b'}{(2b'+2a')^2} = \frac{a'+b'}{2b'+2a'} \cdot \frac{2b'}{2b'+2a'} = \frac{k-1+a'}{2b'+2a'} \cdot \frac{k-1+b'}{2b'+2a'} = P(A \mid C^\perp)P(B \mid C^\perp)$.

Case 3b: $z = a' + 1$ and $k = a' = b'$.

In this case we will construct a 0-type common cause. Let $C^\perp$ consist of just a single atom from $(A \cup B)^\perp$; $C^\perp$ screens off the correlation because $P(AB \mid C^\perp) = 0 = P(A \mid C^\perp)P(B \mid C^\perp)$. C screens off the correlation because $P(AB \mid C) = \frac{k}{4k} = \frac{4k^2}{16k^2} = \frac{2k}{4k} \cdot \frac{2k}{4k} = \frac{k+a'}{k+a'+b'+z-1} \cdot \frac{k+b'}{k+a'+b'+z-1} = P(A \mid C)P(B \mid C)$.

Case 3c: $k = a' + 1$ and $z = a' = b'$.

In this case we will construct a 1-type common cause. Let C consist of just a single atom from $(A \cap B)$; as in case 3a, C screens off the correlation. That $C^\perp$ is also a screener-off follows from $P(AB \mid C^\perp) = \frac{a'}{4a'} = \frac{4a'^2}{16a'^2} = \frac{2a'}{4a'} \cdot \frac{2a'}{4a'} = \frac{k-1+a'}{k-1+a'+b'+z} \cdot \frac{k-1+b'}{k-1+a'+b'+z} = P(A \mid C^\perp)P(B \mid C^\perp)$. **Q.E.D.**

Notice that the five cases used in the proof above are exhaustive. For example (due to lemma 13), if $k = a'$, then $z = b' + 1$. (Were $z \leqslant b'$, $SC_n(A, B)$ would not be positive, meaning that the events would not be correlated at stage n; were $z > b' + 1$, it would follow that $SC_n(A, B) > k$, which would mean the pair was already correlated at stage $n - 1$.) Similarly, if $z = a'$, then $k = a' + 1$. Remember than by our convention we always have $a' \geqslant b'$. Finally, notice that if $a' \geqslant k$ and $b' \geqslant z$, then $SC_n(A, B)$ is negative and so there is no correlation; and similarly if $b' \geqslant k$ and $a' \geqslant z$.

Lemma 16 *Suppose A, B form a pair of logically independent events correlated at stage n. Suppose further that they have a common cause or a 0-type CCS of size 3 at that stage. Then they have a common cause or a 0-type CCS of size 3 at stage $n + 1$.*

Proof: (Note that the cases are not exclusive; they are, however, exhaustive.)

Case 1: A, B have a 0-type common cause at stage n.

Let C be a 0-type common cause for the correlation. When moving from stage n to $n + 1$, a new atom $(n + 1)$ is added. Let $C'^\perp = C^\perp \cup \{n + 1\}$. Notice that C and $C'^\perp$ form a partition of unity of the algebra at stage $n + 1$. C contains exclusively atoms from the algebra at stage n and so continues to be a screener off. Notice that since C was a 0-type common cause at stage n, at that stage $P(A \mid C^\perp) = 0$ or $P(B \mid C^\perp) = 0$. Since the atom $n + 1$ lies outside the events A and B, at stage $n + 1$ $P(A \mid C'^\perp) = 0$ or $P(B \mid C'^\perp) = 0$, and so $C'^\perp$ is a screener-off, too. Thus C and $C'^\perp$ form a partition of unity composed of screener-offs at stage $n + 1$. By lemma 8, this is enough to conclude that A, B have a 0-type common cause at stage $n + 1$.

Case 2: A, B have a common cause which is not a 0-type common cause at stage n.

Let C be a non-0-type common cause for the correlation at stage n. Notice that both $P(AB \mid C)$ and $P(AB \mid C^{\perp})$ are non-zero. In this case the 'new' atom cannot be added to any element of the common cause without breaking the screening-off condition. But, since—as we remarked in the previous case—the atom $n+1$ lies outside the events A and B, and so is trivially a screener-off for the pair. Therefore our explanation of the correlation at stage $n+1$ will be a 0-type CCS of size 3: $C' = \{C, C^{\perp}, \{n+1\}\}$.[4]

Case 3: A, B have a 0-type CCS of size 3 at stage n.

Let the partition $C = \{C_i\}_{i \in \{0,1,2\}}$ be a 0-type CCS of size 3 at stage n for the correlation, with C_2 being the zero element (that is $P(A \mid C_2) = 0$ or $P(B \mid C_2) = 0$ (or possibly both), with the conditional probabilities involving C_0 and C_1 being positive). Let $C' = \{C_0, C_1, C_2 \cup \{n+1\}\}$. Since $n+1 \notin A \cup B$, $C_2 \cup \{n+1\}$ screens off the correlation at stage $n+1$ and C' is a 0-type CCS of size 3 at stage $n+1$ for the correlation. **Q.E.D.**

As mentioned above, lemmas 14–16 complete the proof of this direction of the theorem since a method is given for obtaining a common cause or a CCS of size 3 for any correlation between logically independent events in any finite probability space with the uniform distribution.

We proceed with the proof of the 'upward' direction of theorem 9.

Causal up-to-3-closedness w.r.t. $L_{ind} \Rightarrow$ Measure uniformity

In fact, we will prove the contrapositive: if in a finite probability space with no 0-probability atoms the measure is not uniform, then there exist logically independent, correlated events A, B possessing neither a common cause nor a CCS of size 3. In the remainder of the proof we extend the reasoning from case 2 of proposition 4 from (Gyenis and Rédei [2004]), which only covers common causes.

Consider the space with n atoms; arrange the atoms in the order of decreasing probability and label them as numbers $0, 1, \ldots, n-1$. Let $A = (0, n-1)$ and $B = (0, n-2)$. Gyenis and Rédei ([2004]) prove that A, B are correlated and do not have a common cause. We will now show that they do not have a CCS of size 3 either.

Suppose $C = \{C_i\}_{i \in \{0,1,2\}}$ is a CCS of size 3 for the pair A, B. If for some $i \in \{0,1,2\}$ $A \subseteq C_i$, C violates the statistical relevance condition, since for the remaining $j, k \in \{0,1,2\}, j \neq k, i \neq j, i \neq k$, $P(A \mid C_j) = 0 = P(A \mid C_k)$. Similarly if B is substituted for A in the above reasoning. It follows that none of the elements of C can contain the whole event A or B. Notice also that no C_i can contain the atoms $n-1$ and $n-2$, but not the atom 0, as then it would not be a screener-off, because in such a case $P(AB \mid C_i) = 0$ despite $P(A \mid C_i) \neq 0$ and $P(B \mid C_i) \neq 0$. But since C is a partition of unity of the space, each of the three atoms forming $A \cup B$ has to belong to an element of C, and so each C_i contains exactly one atom from $A \cup B$. Therefore for some $j, k \in \{0,1,2\}$ $P(A \mid C_j) > P(A \mid C_k)$ but $P(B \mid C_j) < P(B \mid C_k)$, which means that C violates the statistical relevance condition. All options exhausted, we conclude

[4] The fact that a correlation has a CCS of size 3 does not necessarily mean it has no common causes.

that the pair A, B does not have a CCS of size 3. And so the probability space is not causally up-to-3-closed. **Q.E.D.**

In fact, the reasoning from the 'upward' direction of the theorem can be extended to show that if a probability space with no 0-probability atoms has a non-uniform probability measure, it is not causally up-to-n-closed for any $n \geqslant 2$. The sum of the two events A and B described above only contains 3 atoms; it follows that the pair cannot have a CCS of size greater than 3, since it would have to violate the statistical relevance condition (two or more of its elements would, when conditionalised upon, give probability 0 to event A or B). This, together with proposition 3 of (Gyenis and Rédei [2004]) justifies the following claims:

Theorem 17 *No finite probability space with a non-uniform measure and without 0-probability atoms is causally up-to-n-closed for any $n \geqslant 2$.*

Corollary 18 *No finite probability space with a non-uniform measure and without 0-probability atoms is causally n-closed for any $n \geqslant 2$.*

3.1 Examples

We will now present a few examples of how our method of finding explanations for correlations works in practice, analysing a few cases of correlated logically independent events in probability spaces of various sizes.

Example 1. $n = 7$, $A = (0, 2, 3, 5, 6)$, $B = (1, 2, 5, 6)$.

We see that $a' = 2$, $b' = 1$ and $k = 3$, so we should analyse the pair $A_1 = (0, 1, 2, 3, 4)$, $B_1 = (2, 3, 4, 5)$. We now check whether A_1,B_1 were independent at stage 6, and since at that stage $A_1^\perp \cap B_1^\perp = \emptyset$ we conclude that they were not. Therefore the pair A_1, B_1 appears from above at stage 7. Notice that $SC_7(A_1, B_1) = 1$. By construction from lemma 14 we know that an event consisting of just a single atom from the intersection of the two events satisfies the requirements for being a common cause of the correlation. Therefore $C = (2)$ is a common cause of the correlation between A and B at stage 7.

Example 2. $n = 10$, $A = (2, 3, 8)$, $B = (2, 8, 9)$.

We see that $a' = 1$, $b' = 1$ and $k = 2$, so we should analyse the pair $A_1 = (0, 1, 2)$, $B_1 = (1, 2, 3)$. Since $SC_{10}(A_1, B_1) = 11$ (as remarked above, SC changes by k from stage to stage), we conclude that the lowest stage at which the pair is correlated is 5. A_1 and B_1 are logically independent at that stage, but not at stage 4, which means that the pair appears from above at stage 5. We employ the same method as in the previous example to come up with a 1-type common cause of the correlation at that stage – let it be the event (1). Now the reasoning from case 2 of lemma 16 is used to 'translate' the explanation to stage 6, where it becomes the following 0-type CCS: $\{(1), (0, 2, 3, 4), (5)\}$. Case 3 of the same lemma allows us to arrive at the CCS for A_1,B_1 at stage 10: $\{(1), (0, 2, 3, 4), (5, 6, 7, 8, 9)\}$. Its structure is as follows: one element contains a single atom from the intersection of the two events, another the remainder of $A_1 \cup B_1$ as well as one atom not belonging to any of the two events, while the third element of the CCS contains the rest of the atoms of the algebra at stage 10. We can therefore produce a 0-type CCS for A,B at stage 10: $\{(2),(0,3,8,9),(1,4,5,6,7)\}$.

Example 3. $n = 12$, $A = (2,4,6,8,9,10,11)$, $B = (1,3,6,10,11)$.

We see that $a' = 4$, $b' = 2$ and $k = 3$, so we should analyse the pair $A_1 = (0,1,2,3,4,5,6)$, $B_1 = (4,5,6,7,8)$. We also see that A_1 and B_1 were logically independent at stage 11, but were not correlated at that stage. Therefore the pair A_1, B_1 appears from below at stage 12. Notice that $z = 3$. Therefore we see that $z > b'$ and $a' > k$, which means we can use the method from case 2 of lemma 15 to construct a 0-type common cause. The complement of it consists of 1 atom from $A_1 \setminus B_1$ and 1 atom from $(A_1 \cup B_1)^{\perp}$. Going back to A and B, we see that the complement of our common cause can be put e.g. as $C^{\perp} = (0,2)$. Therefore $C = (1,3,4,5,6,7,8,9,10,11)$ is a 0-type common cause of the correlation between A and B at stage 12.[5]

4 Results on arbitrary spaces

The results presented so far only concern probability spaces without 0-probability atoms. If we admit spaces containing such atoms, the 'upward' direction of theorem 9 breaks down, because a finite classical probability space such that all its non-zero probability atoms have the same probability is causally up-to-n-closed w.r.t. L_{ind} precisely if the space obtained by restricting the probability measure to the algebra containing all and only these atoms is. This stands in contradiction with proposition 4 from (Gyenis, Redei [2004]). The reason is that the aforementioned proposition, to hold in its stated form, requires the unstated assumption of no 0-probability atoms. Case 3 of the proof given in (Gyenis, Redei [2004]) is incomplete. It ends with the following sentence:

> "Since (S_{n-k}, p_{n-k}) contains no atom with non-zero probability, if it is not equal with (S_5, p_u), then Case 3 is reduced to Case 1 or to Case 2 and the proof is complete" (p. 1299),

but the case when (S_{n-k}, p_{n-k}) *is* equal to (S_5, p_u) is not considered. Corollary 20 below makes it clear how to construct counterexamples to Gyenis and Redei's proposition 4, which is only true with respect to spaces with no 0-probability atoms.

The following lemma expresses the simple fact that to check whether a finite classical probability space $\langle S, P\rangle$ is causally up-to-n-closed w.r.t. L_{ind} it is enough to consider the space $\langle S^+, P^+\rangle$, where S^+ is the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ is the restriction of P to S^+. The 0-probability atoms of S are irrelevant to the issue of causal up-to-n-closedness of $\langle S, P\rangle$ w.r.t. L_{ind}.

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[5]Incidentally, at stage 12 a 1-type common cause for A,B also exists: just put $C = (2,11)$, in which case $P(A \mid C) = 1$. But such behaviour is not universal and there are cases in which only 0-type common causes (or only 1-type common causes) are possible. For a concrete example, take the pair $\{(0,1,2,3,4),(4,5)\}$, which appears from below at stage 11 and, as the reader may verify, only has 0-type common causes at that stage.

Lemma 19 *Let $\langle S, P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has 4 atoms or more. Then*

$\langle S, P\rangle$ is causally up-to-n-closed w.r.t. L_{ind}	*iff*	*$\langle S^+, P^+\rangle$ is causally up-to-n-closed w.r.t. L_{ind}.*

Proof: Let $A \in S$. As before, we can think of A as a list of atoms of S. Let A^+ be the set of non-zero probability atoms in A:

$$A^+ := A \setminus \{a \mid a \text{ is an atom of } S \text{ and } P(a) = 0\}.$$

Notice that

$$P(A) = \sum_{a\in A} P(a) = \sum_{a\in A^+} P(a) = P^+(A^+) \tag{1}$$

and also that

$$P(A) = P(A^+) = P^+(A^+). \tag{2}$$

Suppose $A, B, C \in S$. From (1) it follows that if A, B are correlated in $\langle S, P\rangle$, A^+, B^+ are correlated in $\langle S^+, P^+\rangle$. Similarly, for any $D \in S$, $P(D \mid C) = P^+(D^+ \mid C^+)$. So, if C screens off the correlated events A, B in $\langle S, P\rangle$, then C^+ screens off the correlated events A^+, B^+ in $\langle S^+, P^+\rangle$. Also, if a family $\mathbf{C} = \{C_i\}_{i\in I}$ satisfies the statistical relevance condition w.r.t. A, B in $\langle S, P\rangle$, then the family $\mathbf{C}^+ = \{C_i^+\}_{i\in I}$ satisfies the statistical relevance condition w.r.t. A^+, B^+ in $\langle S^+, P^+\rangle$. It follows that if $\mathbf{C} = \{C_i\}_{i\in\{0,\ldots,n-1\}}$ is a CCS of size n for the correlation between events A, B in $\langle S, P\rangle$, $\mathbf{C}^+ = \{C_i^+\}_{i\in\{0,\ldots,n-1\}}$ is a CCS of size n for the correlation between events A^+, B^+ in $\langle S^+, P^+\rangle$.

1. If $\langle S^+, P^+\rangle$ is causally up-to-n-closed w.r.t. L_{ind}, then $\langle S, P\rangle$ is causally up-to-n-closed w.r.t. L_{ind}. Suppose $\langle S^+, P^+\rangle$ is causally up-to-n-closed w.r.t. L_{ind}. Let A, B be logically independent and correlated events from $\langle S, P\rangle$. Then A^+, B^+ are logically independent and correlated in $\langle S^+, P^+\rangle$. Since $\langle S^+, P^+\rangle$ is causally up-to-n-closed w.r.t. L_{ind}, there exists in $\langle S^+, P^+\rangle$ a CCS of size n $\mathbf{C}^+ = \{C_i\}_{i\in\{0,\ldots,n-1\}}$ for the correlation. Then $\mathbf{C} := \{C_0 \cup \{a \mid a \text{ is an atom of } S \text{ and } P(a) = 0\}, C_1, \ldots, C_{n-1}\}$ is a CCS of size n for the correlation between A and B in $\langle S, P\rangle$. Since the choice of A and B was arbitrary, it follows that $\langle S, P\rangle$ is causally up-to-n-closed w.r.t. L_{ind}.

2. If $\langle S^+, P^+\rangle$ is not causally up-to-n-closed w.r.t. L_{ind}, then $\langle S, P\rangle$ is not causally up-to-n-closed w.r.t. L_{ind}. Suppose $\langle S^+, P^+\rangle$ is not causally up-to-n-closed w.r.t. L_{ind}. Then there exist logically independent, correlated events A^+, B^+ which do not have a CCS of size at most n in $\langle S^+, P^+\rangle$. The two events are also logically independent and correlated in $\langle S, P\rangle$. We will show that $\langle S, P\rangle$ also contains no CCS of size at most n for them. For suppose that for some $m \leqslant n$, $\mathbf{C} = \{C_i\}_{i\in\{0,\ldots,m-1\}}$ was a CCS of size m for the correlation between A^+ and B^+ in $\langle S, P\rangle$. Then it follows that $\mathbf{C}^+ := \{C_i^+\}_{i\in\{0,\ldots,m-1\}}$ would be a CCS of size m for the correlation between A^+ and B^+ in $\langle S^+, P^+\rangle$, but by our assumption no such CCSs exist. We infer that the correlated events A, B have no CCS of size up to n in $\langle S, P\rangle$, so the space $\langle S, P\rangle$ is not causally up-to-n-closed w.r.t. L_{ind}. **Q.E.D.**

From proposition 3 and case 2 of proposition 4 from (Gyenis, Redei [2004]) it follows that, if we restrict our attention to spaces with no 0-probability atoms, a finite classical probability space $\langle S,P\rangle$ is common cause closed (which is equivalent to being causally 2-closed) w.r.t. L_{ind} if and only if S is the Boolean algebra with 5 atoms and P is the uniform measure on S. This, together with the reasoning used in the proof of lemma 19, allows us to infer the following corollary:

Corollary 20 *Let $\langle S,P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has at least 4 atoms. Then*

$$\begin{array}{c}\langle S,P\rangle \textit{ is common cause}\\ \textit{closed (causally 2-closed)}\\ \textit{w.r.t. } L_{ind}\end{array} \quad \textit{iff} \quad \begin{array}{c}S^+ \textit{ has 5 atoms}\\ \textit{and } P^+ \textit{ is uniform.}\end{array}$$

Observe for example that the space $\langle S_6, P_{u5}\rangle$, where S_6 is the Boolean algebra with 6 atoms labelled $a_0,\ldots,a_5$ and

$$P_{u5}(a_i) := \begin{cases} \frac{1}{5} & for\ i \in \{0,\ldots 4\} \\ 0 & for\ i = 5 \end{cases}$$

is causally 2-closed w.r.t. L_{ind} (every two logically independent, correlated events in it have a common cause) even though its measure is not uniform; as such, it is a counterexample to proposition 4 of Gyenis, Redei ([2004]).

We leave it to the reader to justify the following generalisations of theorem 17 and lemma 18:

Theorem 21 *Let $\langle S,P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has at least 4 atoms. Then if P^+ is not uniform, $\langle S,P\rangle$ is not is causally up-to-n-closed w.r.t. L_{ind} for any $n \geqslant 2$.*

Corollary 22 *Let $\langle S,P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has at least 4 atoms. Then if P^+ is not uniform, $\langle S,P\rangle$ is not is causally n-closed w.r.t. L_{ind} for any $n \geqslant 2$.*

The final theorem of this section—which provides the main motivation for the paper's title—ties theorem 9, which only concerned probability spaces with no 0-probability atoms, with lemma 19, which covers arbitrary spaces.

Theorem 23 *Let $\langle S,P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has at least 4 atoms. Then*

$$\langle S,P\rangle \textit{ is causally up-to-3-closed w.r.t. } L_{ind} \quad \textit{iff} \quad P^+ \textit{ is uniform.}$$

Proof: Immediate from theorem 9 and lemma 19. **Q.E.D.**

5 Other independence relations

So far, the relation of independence—determining which correlations between two events require explanation—was the relation of logical independence. Let us consider using a 'broader' relation $R_{ind} \supset L_{ind}$, which apart from all pairs of logically independent events would also include some pairs of logically dependent events. (Assume uniformity of the probability measure.) Will we have more correlations to explain? If so, will they have common causes?

In the case that events A, B are correlated but one of them (say B) equals $\mathbf{1}_S$, there can be no common cause of the correlation, because for any C for which the appropriate conditional probabilities are defined $1 = P(B \mid C) = P(B \mid C^{\perp})$ which violates the statistical relevance condition. In the sequel assume that neither A nor B equals $\mathbf{1}_S$.

First, note that if $A \cap B = \emptyset$, then $P(AB) = 0$ and no (positive) correlation arises.

Second, if $A^{\perp} \cap B^{\perp} = \emptyset$, there is again no positive correlation. This is because in such a case $A \cup B = \mathbf{1}_S$ and thus $SC_n(A, B) = k(k+a'+b') - (k+a')(k+b') = -a'b' < 0$.

Consider the last possible configuration in which the events A, B are logically dependent: namely, that one is included in the other. Suppose $A \subseteq B$. The events will be correlated, since $SC_n(A, B) > 0$. The reader may check that when $A \subseteq B$ but $B \neq \mathbf{1}_S$, any C which screens off the correlation and has a non-empty intersection with A has to be a subset of B. And so if C is a common cause, it is necessary that $C^{\perp} \cap A = \emptyset$. In the other direction, it is evident that if $A \subseteq C \subseteq B$, both C and $C^{\perp}$ screen off the correlation and the statistical relevance condition is satisfied. The only pitfall is that the definition of a common cause requires it be distinct from both A and B, and so none exist when $b' = 1$.

To summarise, the only correlated pairs of logically dependent events A, B are these in which one of the events is included in the other. Assume $A \subseteq B$. Then:

- if $B = \mathbf{1}_S$ or $b' = 1$, there is no common cause of the correlation;
- otherwise the common causes of the correlation are precisely all the events C such that $A \subset C \subset B$.

Lastly, notice that in a space $\langle S_n, P \rangle$ (S_n being the Boolean algebra with n atoms and P being the uniform measure) we could proceed in the opposite direction and restrict rather than broaden the relation L_{ind}. If we take the independence relation R_{ind} to be the relation of logical independence restricted to the pairs which appear from above or below at stage n, then our probability space is common cause closed w.r.t. R_{ind}.

6 Conclusions and problems

The main result is that in finite classical probability spaces with the uniform probability measure (and so no atoms with probability 0) all correlations between logically independent events have an explanation by means of a common cause or a common cause system of size 3. A few remarks are in order.

First, notice that the only CCSs employed in our method described in section 3 are 0-type CCSs, and that they are required only when 'translating' the explanation from a smaller space to a bigger one. Sometimes (if the common cause we found in the smaller space is 0-type; see example 3 above) such a translation can succeed without invoking the notion of CCS at all.

Second, #-type common causes, which some would view as 'genuinely indeterministic', are never *required* to explain a correlation – that is, a correlation can always be explained by means of a 0-type CCS, a 0-type common cause, or a 1-type common cause[6]. Therefore the 'right-to-left' direction of the equivalence in theorem 23 can be strengthened:

Theorem 24 *Let $\langle S, P\rangle$ be a finite classical probability space. Let S^+ be the subalgebra of S containing all and only the non-zero probability atoms of S and P^+ be the restriction of P to S^+. Suppose S^+ has at least 4 atoms.*

If P^+ is the uniform probability measure on S^+, then any pair of correlated and logically independent events in $\langle S, P\rangle$ has a 1-type common cause, a 0-type common cause or a 0-type common cause system of size 3 in $\langle S, P\rangle$.

A natural question to ask is to what extent could the results of this paper be extended to finite non-distributive orthomodular lattices and non-classical probability measures.

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[6]But #-type common causes do exist: e.g. in the space with 12 atoms and the uniform measure the pair of events A, B, where $A = (0, 1, 2, 3, 4, 5, 6)$, $B = (4, 5, 6, 7, 8)$ (the same we dealt with in example 3) has, apart from both 0- and 1-type common causes, a #-type common cause of shape $C = (1, 2, 4, 5, 7, 9)$, $C^\perp = (0, 3, 6, 8, 10, 11)$; $P(A \mid C) = \frac{2}{3}$, $P(B \mid C) = \frac{1}{2}$, $P(A \mid C^\perp) = \frac{1}{2}$, $P(B \mid C^\perp) = \frac{1}{3}$.