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Discerning Elementary Particles

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Abstract. We extend the quantum-mechanical results of Muller & Saunders (2008) establishing the *weak discernibility* of an arbitrary number of similar fermions in finite-dimensional Hilbert-spaces in two ways: (a) from fermions to bosons for all finite-dimensional Hilbert-spaces; and (b) from finite-dimensional to infinite-dimensional Hilbert-spaces for all elementary particles. In both cases this is performed using operators whose physical significance is beyond doubt. This confutes the currently dominant view that (A) the quantum-mechanical description of similar particles conflicts with Leibniz's Principle of the Identity of Indiscernibles (PII); and that (B) the only way to save PII is by adopting some pre-Kantian metaphysical notion such as Scotusian haecceittas or Adamsian primitive thisness. We take sides with Muller & Saunders (2008) against this currently dominant view, which has been expounded and defended by, among others, Schrödinger, Margenau, Cortes, Dalla Chiara, Di Francia, Redhead, French, Teller, Butterfield, Mittelstaedt, Giuntini, Castellani, Krause and Huggett.

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1 Introduction

According to the founding father of wave mechanics Erwin Schrödinger (1996: 121–122), one of the ontological lessons that quantum mechanics (QM) has taught us is, as he told an audience in Dublin, February 1950, that the elementary building blocks of the physical world are entirely *indiscernible*:

> I beg to emphasize this and I beg you to believe it: it is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of the 'sameness', of identity, really and truly has no meaning.

Similar elementary particles have no 'identity', there is nothing that discerns one particle from another, neither properties nor relations can tell them apart, they are not individuals. Thus Schrödinger famously compared the elementary particles to "the shillings and pennies in your bank account", in contrast to the coins in your piggy-bank. In 1928, Hermann Weyl (1950: 241) had preceded Schrödinger when he wrote that "even in principle one cannot demand an alibi from an electron".

Over the past decades, several philosophers have scrutinised this *Indiscernibility Thesis* (IT) by providing various rigorous arguments in favour of it: *similar* elementary particles (same mass, charge, spin, etc.), when forming a composite physical system, are indiscernible by quantum-mechanical means. Leibniz's metaphysical Principle of the Identity of Indiscernibles (PII) is thus refuted by physics (QM) — and perhaps is therefore not so metaphysical after all. This does not rule out conclusively that particles *really* are discernible, but *if* they are, they have to be discerned by means that go above and beyond physics (QM), such as by ascribing Scotusian *haecceitas* to the particles, or ascribing sibling attributes to them from scholastic and neo-scholastic metaphysics. Nevertheless few philosophers have considered this move to save the discernibility of the elementary particles to be attractive — if this move is mentioned, then usually as a possibility and rarely as a plausibility. Mild naturalistic inclinations seem sufficient to accommodate IT in our general metaphysical view of the world. We ought to let well-established scientific knowledge inform our metaphysical view of the world whenever possible and appropriate, and this is exactly what Schrödinger begged us to do. The only respectable metaphysics is naturalised metaphysics; see further Ladyman & Ross (2007: 1–38). Prominent defenders of IT include: Margenau (1944); Cortes (1976), who brandished PII "a false principle"; Barnette (1978), French & Redhead (1988); Giuntini & Mittelstaedt (1989), who argued that although demonstrably valid in classical logic, in quantum logic the validity of PII cannot be established; French (1989a), who assured us in the title that PII "is not contingently true either"; French (1989a; 1989b; 1998; 2006), Redhead & Teller (1992), Butterfield (1993), Castellani & Mittelstaedt (2000), Massimi (2001), Teller (1998), French & Rickles (2003), Huggett (2003), French & Krause (2006: Ch. 4).

There have however been dissenters. B.C. van Fraassen (1991) is one of them; see Muller & Saunders (2006: 517–518) for an analysis of his arguments. We follow the other dissenters: S.W. Saunders and one of us (Saunders (2006), Muller & Saunders (2008)). They neither claim that fermions are individuals nor do they rely on a particular interpretation of QM. On the basis of standard mathematics (standard set-theory and classical predicate logic) and only uncontroversial postulates of QM (notably leaving out the projection postulate, the strong property postulate and the quantum-mechanical probabilities), they demonstrate that similar *fermions* are *weakly discernible*, i.e. they are discerned by relations that are irreflexive and symmetric, in every admissible state of the composite system. So according to Muller & Saunders, the elementary building blocks of matter (fermions) are *not* indiscernibles after all, contra IT. They prove this, however, only for *finite-dimensional* Hilbert-spaces (their Theorem 1), which is a rather serious restriction because most applications of QM to physical systems employ complex wave-functions and these live in the *infinite-dimensional* Hilbert-space $L^2(\mathbb{R}^{3N})$; nonetheless they confidently conjecture that their result will hold good for infinite-dimensional Hilbert-spaces as well (their Conjecture 1). Furthermore, Muller & Saunders (2008: 534–535) need to *assume* for their proof there is a maximal self-adjoint operator acting on finite-dimensional Hilbert-spaces that is physically significant. In the case of dimension 2 of a single-fermion Hilbert-space, Pauli's spin-1/2 operator qualifies as such a maximal self-adjoint operator, but for higher dimensions, spin is degenerate. What this maximal operator in those cases corresponds to they gloss over.

When it comes to the elementary quanta of interaction (bosons), Muller & Saunders (2008: 537–540) claim that bosons are also weak discernibles, but of a *probabilistic* kind (the discerning relation involves quantum-mechanical probabilities and therefore their proof needs the Probability Postulate of QM), whereas the discerning relation of the fermions is of a *categorical* kind (no probabilities involved). More precisely, the categorical discernibility of bosons turns out to be a contingent matter: in some states they are categorically discernible, in others, e.g. direct-product states, they are not; this prevents one to conclude that bosons are categorical discernibles *simpliciter*. But the boson's probabilistic discernibility is a quantum-mechanical necessity; Theorem 3 (*ibid.*) establishes it for two bosons, with no restrictions on the dimensionality of Hilbert-space but conditional on whether a particular sort of operator can be found (again, a maximal self-adjoint one of physical significance). The fermions are also probabilistic discernibles; their Theorem 2 states it for finite-dimensional Hilbert-spaces only and is therefore equally restrictive as their Theorem 1.

The central aim of the current paper is the completion of the project initiated and developed in Muller & Saunders (2008), by demonstrating that *all* restrictions in *their* discernibility theorems can be removed by proving more general theorems and proving them differently than they have done, employing only quantum mechanical operators that have obvious physical significance. We shall then be in a position to conclude in utter generality that *all* kinds of similar particles in *all* their physical states, pure and mixed, in *all* infinite-dimensional or finite-dimensional Hilbert-spaces can be *categorically* discerned on the basis of quantum-mechanical postulates. This result, then, should be the death-knell for IT, and, by implication, establishes the universal reign of Leibniz's PII in QM.

In Sections 3 and 4, we prove the theorems that establish the general result. First we introduce some terminology, state explicitly what we need of QM, and address the issue of what has the license to discern elementary particles (Section 2).

2 Preliminaries

For the motivation and further elaboration of the terminology we are about to introduce, we refer to Muller & Saunders (2008: 503–505) because we follow them closely (readers of that paper can jump to the next Section of the current paper). Here we only mention what is necessary in order to keep the current paper comparatively self-contained.

We call physical objects in a set *absolutely discernible*, or *individuals*, iff for every object there is some physical property that it has but all others lack; and *relationally discernible* iff for every object there is some physical relation that discerns it from all others (see below). An object is *indiscernible* iff it is both absolutely and relationally indiscernible, and hence *discernible* iff it is discernible either way or both ways. Objects that are not individuals but are relationally discernible from all other objects we call *relationals*; then *indiscernibles* are objects that are neither individuals nor relationals. Quine (1981: 129–133) was the first to inquire into different kinds of discernibility; he discovered there are *only two* independent logical categories of relational discernibility (by means of a binary relation): either the relation is irreflexive and asymmetric, in which case we speak of *relative discernibility*; or the relation is irreflexive and symmetric, in which case we speak of *weak discernibility.* We call attention to the logical fact that if relation R discerns particles $\boldsymbol{1}$ and $\boldsymbol{2}$ *relatively*, then its *complement relation*, defined as $\neg R$, is also asymmetric but reflexive; and if R discerns particles $\boldsymbol{1}$ and $\boldsymbol{2}$ *weakly*, then its complement relation $\neg R$ is reflexive and symmetric but does not hold for $\boldsymbol{a} \neq \boldsymbol{b}$ whenever

R holds for $\boldsymbol{a} \neq \boldsymbol{b}$.

Leibniz's *Principle of the Identity of Indiscernibles* (PII) for physical objects states that *no two physical objects are absolutely and relationally indiscernible; or synonymously, two physical objects are numerically discernible only if they are qualitatively discernible.* One can further distinguish principles for absolute and for relational indiscernibles and then inquire into the logical relations between these and PII; see Muller & Saunders (2008: 504–505). Similarly one can distinguish three indiscernibility theses as the corresponding negations of the Leibnizian principles. We restrict ourselves to the *Indiscernibility Thesis* (IT): *there are composite systems of similar physical objects that consist of absolutely and relationally indiscernible physical objects.* Then either it is a theorem of logic that PII holds and IT fails, or conversely:

$$\vdash \text{ PII } \longleftrightarrow \neg\text{IT} . \tag{1}$$

Next we rehearse the postulates of QM that we shall use in our Discernibility Theorems.

The *State Postulate* (StateP) associates some super-selected sector Hilbert-space $\mathcal{H}$ to every given physical system S and represents every *physical state* of S by a statistical operator $W \in \mathcal{S}(\mathcal{H})$; the pure states lie on the boundary of this convex set $\mathcal{S}(\mathcal{H})$ of all statistical operators and the mixed states lie inside. If S consists of N similar elementary particles, then the associated Hilbert-space is a direct-product Hilbert-space $\mathcal{H}^N = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ of N identical single-particles Hilbert-spaces.

The *Weak Magnitude Postulate* (WkMP) says that every physical magnitude is represented by an operator that acts on $\mathcal{H}$. Stronger magnitude postulates are not needed, because they all imply the logically weaker WkMP, which is sufficient for our purposes.

In order to state the Symmetrisation Postulate, we need to define first the orthogonal projectors $\Pi_N^{\pm}$ of the lattice $\mathcal{P}(\mathcal{H}^N)$ of all projectors, defined as

$$\Pi_N^+ \equiv \frac{1}{N!} \sum_{\pi \in \mathbb{P}_N}^{N!} U_\pi \quad \text{and} \quad \Pi_N^- \equiv \frac{1}{N!} \sum_{\pi \in \mathbb{P}_N}^{N!} \text{sign}(\pi) U_\pi \ , \tag{2}$$

where $\text{sign}(\pi) \in \{\pm 1\}$ is the *sign* of the permutation $\pi \in \mathbb{P}_N$ on $\{1, 2, \ldots, N\}$ (+1 if it is even, -1 if odd), and where U_π is a unitary operator acting on $\mathcal{H}^N$ corresponding to permutation π (these U_π form a unitary representation on $\mathcal{H}^N$ of the permutation group $\mathbb{P}_N$). The projectors (2) lead to the following permutation-invariant orthogonal subspaces:

$$\mathcal{H}_+^N \equiv \Pi_N^+ \big[\mathcal{H}^N\big] \quad \text{and} \quad \mathcal{H}_-^N \equiv \Pi_N^- \big[\mathcal{H}^N\big] \ , \tag{3}$$

which are called the BE-*symmetric* (Bose-Einstein) and the FD-*symmetric* (Fermi-Dirac) subspaces of $\mathcal{H}^N$, respectively. These subspaces can, alternatively, be seen as generated by the symmetrised and anti-symmetrised versions of the products of basis-vectors in $\mathcal{H}^N$. Only for $N = 2$, we have that $\mathcal{H}^N_- \oplus \mathcal{H}^N_+ = \mathcal{H}^N$.

The *Symmetrisation Postulate* (SymP) states for a composite system of $N \geqslant 2$ similar particles with N-fold direct-product Hilbert-space $\mathcal{H}^N$ the following: (i) the projectors $\Pi^{\pm}_N$ (2) are super-selection operators; (ii) integer and half-integer spin particles are confined to the BE- and the FD-symmetric subspaces (3), respectively; and (iii) all composite systems of similar particles consist of particles that have all either integer spin or half-integer spin (Dichotomy).

We represent a *quantitative physical property* associated with physical magnitude $\mathcal{A}$ mathematically by ordered pair $\langle A, a\rangle$, where A is the operator representing $\mathcal{A}$ and $a \in \mathbb{C}$. The *Weak Property Postulate* (WkPP) says that if the physical state of physical system S is an eigenstate of A having eigenvalue a, then it has property $\langle A, a\rangle$; the *Strong Property Postulate* (StrPP) adds the converse conditional to WkPP. (We mention that eigenstates can be *mixed*, so that physical systems in *mixed states* can posses properties too (by WkPP). see Muller & Saunders (2008: 513) for details.) WkPP implies that every physical system S always has the same quantitative properties associated with all super-selected physical magnitudes because S always is in the same common eigenstate of the super-selected operators. We call these possessed quantitative physical properties *super-selected* and we call physical systems, e.g. particles, that have the same super-selected quantitative physical properties *similar* (this is the precise definition of 'similar', a word that we have been using loosely until now, following Dirac).

We also adopt the following *Semantic Condition* (SemC). When talking of a physical system at a given time, we ascribe to it at most *one* quantitative physical property associated with physical magnitude $\mathcal{A}$:

$$\text{(SemC)}\quad \textit{If physical system } S \textit{ possesses } \langle A, a\rangle \textit{ and } \langle A, a'\rangle,\textit{ then } \; a = a' \,. \tag{4}$$

For example, particles cannot possess two different masses at the same time and (4) is the generalisation of this in the language of QM. In other words: if S possesses quantitative physical property $\langle A, a\rangle$, then S does *not* posses property $\langle A, a'\rangle$ for every $a' \neq a$. Statement (4) is neither a tautology nor a theorem of logic, but we agree with Muller & Saunders (2008: 515) in that "it seems absurd to deny it all the same".

Notice there is neither mention of measurements nor of probabilities in the postulates mentioned above, let alone interpretational glosses such as dispositions.

For an outline of *the elementary language of* QM, we refer again to Muller & Saunders (2008: 520–521). In this language, the proper formulation of PII is that *physically indiscernible physical systems are identical* (Muller & Saunders 2008: 521–523):

$$\mathsf{PhysInd}(\boldsymbol{a},\ \boldsymbol{b}) \quad\longrightarrow\quad \boldsymbol{a} = \boldsymbol{b}\ , \tag{5}$$

where '$\boldsymbol{a}$' and '$\boldsymbol{b}$' are physical-system variables, ranging over all physical systems, and where $\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})$ comprises everything that is in principle permitted to discern the particles: roughly, all physical relations and all physical properties. The properties and the relations may involve, in their definition, probabilities, in which case we call them *probabilistic*; otherwise, in the absence of probabilities, we call them *categorical*. So the three logical kinds of discernibility — (a) absolute and (r) relational, which further branches in (r.w) weak and (r.r) relative discernibility — come in a probabilistic and a categorical variety. In their analysis of the traditional arguments in favour of IT, Muller & Saunders (2008: 524–526) make the case that, setting conditional probabilities aside, (r) relational discernibility has been largely overlooked by the tradition. (Parenthetically, Leibniz also included relations in his PII *because* he held that all relations reduce to properties and thus could make do with an explicit formulation of PII that only mentions properties; give up his reducibility thesis of relations to properties and one can no longer make do with his formulation; see Muller & Saunders (2008: 504–505).) What in particular has been overlooked, and is employed by Muller & Saunders, are properties of wholes that are relations between their constitutive parts: the distance between the Sun and the Earth is a property of the solar system; the Coulomb-interaction between the electron and the proton is a property of the Hydrogen atom; etc.

Muller & Saunders (2008: 524–528) argue at length that only those properties and relations are permitted to occur in $\mathsf{PhysInd}$ (5) that meet the following two requirements.

(Req1) *Physical meaning.* All properties and relations, as they occur WkPP, should be transparently defined in terms of physical states and operators that correspond to physical magnitudes in order for the properties and relations to be physically meaningful.

(Req2) *Permutation invariance.* Any property of one particle is a property of any other; relations should be permutation-invariant, so *binary* relations should be symmetric *and* either reflexive or irreflexive.

All proponents of the Indiscernibility Thesis (IT) have considered quantum-mechanical means of discerning similar particles that obey these two Requirements (see the references listed in the Introduction) — and have found them all to fail. They were correct in this. They were not correct in *not* considering categorical relations.

To close this Section, we want address another distinction from the recent flourishing literature on indiscernibility and inquire briefly whether this motivates a *third* requirement, call it Req3. One easily shows that absolute discernibles are relational discernibles by *defining* a relation (expressed by dyadic predicate R_M) in terms of the discerning properties (expressed by monadic predicate M); see Muller & Saunders (2008: 529). One could submit that this is not a case of 'genuine' but of 'fake' relational discernibility, because there is nothing inherently relational about the way this relational discernibility is achieved: R_M is completely reducible to property M, which already discerns the particles absolutely. Similarly, one may also object that a case of absolute discernibility implied by relational discernibility by means of a monadic predicate M_R that is defined in terms of the discerning relation R is not a case of 'genuine' but of 'fake' absolute discernibility (the terminology of 'genuine' and 'fake' is not Ladyman's (2007: 36), who calls 'fake' and 'genuine' more neutrally "contextual" and "intrinsic", respectively). Definitions: physical systems $\boldsymbol{a}$ and $\boldsymbol{b}$ are *genuine* relationals, or *genuine* (weak, relative) relational discernibles, iff they are discerned by some dyadic predicate *that is not reducible to monadic predicates of which some discern* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *absolutely*; $\boldsymbol{a}$ and $\boldsymbol{b}$ are *genuine* individuals, or *genuine* absolute discernibles, iff they are discerned by some monadic predicate *that is not reducible to dyadic predicates of which some discern* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *relationally*; discernibles are *fake* iff they are not genuine. Hence there is a *prima facie* case for adding a third Requirement that excludes fake discernibility:

> (Req3) *Authenticity.* Predicates expressing discerning relations and properties must be genuine.

In turn, a *fake* property or relation can be defined rigorously as its *undefinability in terms of the predicates in the language of* QM *that meet* Req1 *and* Req2. In order to inquire logically into genuineness and fakeness, thus defined, at an appreciable level of rigour, the entire formal language must be spelled out and all axioms of QM must be spelled out in that formal language. Such a logical inquiry is however far beyond the scope of the current paper. Nonetheless we shall see that our discerning relations plausibly *are* genuine.

But besides formalise-fobia, there is a respectable reason for not adding Req3 to our list. To see why, consider the following two cases: (a) indiscernibles and (b) discernibles.

(a) Suppose particles turn out to be indiscernibles in that they are indiscernible by all genuine relations and all genuine properties. Then they are also indiscernible by *all* properties and *all* relations that are defined in terms of these, which one can presumably prove by induction over the complexity of the defined predicates. So indiscernibles remain indiscernibles, whether we require the candidate properties and relations to be genuine or not.

(b) Suppose next that the particles turn out to be discernibles. ($b.i$) If they are discerned by a relation that turns out to be definable in terms of genuine properties one of which discerns the particles absolutely, then the relationals become individuals — good news for admirers of PII. But the important point to notice is that discernibles remain discernibles. (*b.ii*) If the particles are discerned by a property that turns turns out to be definable in terms of genuine relations one of which discerns the particles relationally, then the individuals loose their individuality and become relationals. They had a fake-identity and are now exposed as metaphysical imposters. But again, the important point to notice is that discernibles remain discernibles.

To conclude, adding Req3 will not have any consequences for crossing the border between discernibles and indiscernibles. This seems a respectable reason not to add Req3 to our list of two Requirements.

3 To Discern in Infinite-Dimensional Hilbert-Spaces

We first prove a Lemma, from which our categorical discernibility theorems then immediately follow.

Lemma 1 (StateP, WkMP, WkPP, SemC) *Given a composite physical system of $N \geqslant 2$ similar particles and its associated direct-product Hilbert-space $\mathcal{H}^N$. If there are two single-particle operators, A and B, acting in single-particle Hilbert-space $\mathcal{H}$, and they correspond to physical magnitudes $\mathcal{A}$ and $\mathcal{B}$, respectively, and there is a non-zero number $c \in \mathbb{C}$ such that in every pure state $|\phi\rangle \in \mathcal{H}$ in the domain of their commutator the following holds:*

$$[A,\, B]|\phi\rangle \;=\; c|\phi\rangle\;, \tag{6}$$

then all particles are categorically weakly discernible.

Proof. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{j}$ be particle-variables, ranging over the set $\{\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{N}\}$ of N particles. We proceed Step-wise, as follows.

[S1] Case for $N = 2$, pure states.
[S2] Case for $N = 2$, mixed states.
[S3] Case for $N > 2$, pure states.
[S4] Case for $N > 2$, mixed states.

[S1]. *Case for $N = 2$, pure states.* Assume the antecedent. Define the following operators on $\mathcal{H}^2 = \mathcal{H} \otimes \mathcal{H}$:

$$A_{\boldsymbol{1}} \;\equiv\; A \otimes \mathit{1} \quad \text{and} \quad A_{\boldsymbol{2}} \;\equiv\; \mathit{1} \otimes A\;, \tag{7}$$

where the operator *1* is the identity-operator on $\mathcal{H}$; and *mutatis mutandis* for B. Define next the following *commutator-relation*:

$$\mathsf{C}(\boldsymbol{a},\boldsymbol{b}) \quad \text{iff} \quad \forall\,|\Psi\rangle \in \mathcal{D}:\ \big[A_{\boldsymbol{a}},\,B_{\boldsymbol{b}}\big]|\Psi\rangle \;=\; c|\Psi\rangle\ , \tag{8}$$

where $\mathcal{D} \subseteq \mathcal{H}\otimes\mathcal{H}$ is the domain of the commutator. An arbitrary vector $|\Psi\rangle$ can be expanded:

$$|\Psi\rangle = \sum_{j,k=1}^{d} \gamma_{jk}|\phi_j\rangle \otimes |\phi_k\rangle\ , \qquad \sum_{j,k=1}^{d} |\gamma_{jk}|^2 = 1, \tag{9}$$

where d is a positive integer or ∞, and $\{|\phi_1, |\phi_2\rangle, \ldots\}$ is a basis for $\mathcal{H}$ that lies in the domain of the commutator $[A,B]$. Then using expansion (9) and eq. (6) one quickly shows that ($\boldsymbol{a}=\boldsymbol{b}$):

$$\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{a}}\big]|\Psi\rangle \;=\; c|\Psi\rangle\ , \tag{10}$$

and that for $\boldsymbol{a}\neq\boldsymbol{b}$:

$$\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{b}}\big]|\Psi\rangle \;=\; 0|\Psi\rangle \;\neq\; c|\Psi\rangle\ , \tag{11}$$

because by assumption $c \neq 0$. By WkPP, the composite system then possesses the following *four* quantitative physical properties (when substituting ***1*** or ***2*** for $\boldsymbol{a}$ in the first, and ***1*** for $\boldsymbol{a}$ and ***2*** for $\boldsymbol{b}$, or conversely, in the second):

$$\big\langle\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{a}}\big],\,c\big\rangle \quad \text{and} \quad \big\langle\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{b}}\big],\,0\big\rangle \quad (\boldsymbol{a}\neq\boldsymbol{b})\ . \tag{12}$$

In virtue of the Semantic Conditional (4), the composite system then does not possess the following *four* quantitative physical properties (recall that $c \neq 0$):

$$\big\langle\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{a}}\big],\,0\big\rangle \quad \text{and} \quad \big\langle\big[A_{\boldsymbol{a}},\,B_{\boldsymbol{b}}\big],\,c\big\rangle \quad (\boldsymbol{a}\neq\boldsymbol{b})\ . \tag{13}$$

The composite system possesses the property (12) that *is* a relation between its constituent parts, namely C (8), which is reflexive: $\mathsf{C}(\boldsymbol{a},\boldsymbol{a})$ for every $\boldsymbol{a}$ due to (10). Similarly, but now using SemC (4), the composite system does *not* possess the property (13) that *is* a relation between its constituent parts, namely C. Therefore ***1*** is not related to ***2***, and ***2*** is not related to ***1*** either, because $\neg\mathsf{C}(\boldsymbol{a},\boldsymbol{b})$ and $\neg\mathsf{C}(\boldsymbol{b},\boldsymbol{a})$ ($\boldsymbol{a}\neq\boldsymbol{b}$); and then, due to the following theorem of logic:

$$\vdash\ \big(\neg\mathsf{C}(\boldsymbol{a},\boldsymbol{b}) \wedge \neg\mathsf{C}(\boldsymbol{b},\boldsymbol{a})\big) \ \longrightarrow\ \big(\mathsf{C}(\boldsymbol{a},\boldsymbol{b}) \longleftrightarrow \mathsf{C}(\boldsymbol{b},\boldsymbol{a})\big)\ , \tag{14}$$

we conclude that C is symmetric (Req2). Since by assumption A and B correspond to physical magnitudes, relation C (8) is physically meaningful (Req1) and hence is admissible, because it meets Req1 and Req2.

Further, it was just shown that the relation C (8) is reflexive and symmetric but fails for $\boldsymbol{a} \neq \boldsymbol{b}$ due to (11), which means that C discerns the two particles *weakly* in every pure state $|\Psi\rangle$ of the composite system. Since probabilities do not occur in C (8), the particles are discerned *categorically.*

[S2]. *Case for $N = 2$, mixed states.* The equations in (8) can also be written as an equation for 1-dimensional projectors that project onto the ray that contains $|\Psi\rangle$:

$$\big[A_{\boldsymbol{a}},\, B_{\boldsymbol{b}}\big]|\Psi\rangle\langle\Psi| \;=\; c|\Psi\rangle\langle\Psi|\,. \tag{15}$$

Due to the linearity of the operators, this equation remains valid for arbitrary linear combinations of projectors. This includes all convex combinations of projectors, which exhausts the set $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ of *all* mixed states. The commutator-relation C (8) is easily extended to mixed states $W \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ and the ensuing relation also discerns the particles categorically and weakly.

[S3], [S4]. *Case for $N > 2$, pure and mixed states.* Cases [S1] and [S2] are immediately extended to the N-particle cases, by considering the following N-factor operators:

$$A_{\boldsymbol{j}} \;\equiv\; \mathit{1} \otimes \;\cdots\; \otimes \mathit{1} \otimes A \otimes \mathit{1} \otimes \;\cdots\; \otimes \mathit{1}\;, \tag{16}$$

where A is the j-th factor and $\boldsymbol{j}$ a particle-variable running over the N labeled particles, and similarly for $B^{(j)}$. The extension to the mixed states then proceeds as in [S2]. **Q.e.d.**

Theorem 1 (StateP, WkMP, WkPP, SemC) *In a composite physical system of a finite number of similar particles, all particles are categorically weakly discernible in every physical state, pure and mixed, for every infinite-dimensional Hilbert-space.*

Proof. In Lemma 1, choose for A the linear momentum operator $\widehat{P}$, for B the Cartesian position-operator $\widehat{Q}$, and for c the value $-\mathrm{i}\hbar$. The physical significance of these operators and their commutator, which is the celebrated canonical commutator

$$\big[\widehat{P},\, \widehat{Q}\,\big] = -\mathrm{i}\hbar\mathit{1}\;, \tag{17}$$

is beyond doubt and so is the ensuing commutator relation C (8), which we baptise *the Heisenberg-relation.* The operators $\widehat{P}$ and $\widehat{Q}$ act on the infinite-dimensional Hilbert-space of the complex wave-functions $L^2(\mathbb{R}^3)$, which is isomorphic to *every infinite-dimensional* Hilbert-space. **Q.e.d.**

But is Theorem 1 not only applicable to particles having spin-0 and have we forgotten to mention this? Yes and No. Yes, we have deliberately forgotten to mention this. No, it is a corollary of Theorem 1 that it holds for all spin-magnitudes, which is the content of the next theorem.

Corollary 1 (StateP, WkMP, WkPP, SemC) *In a composite physical system of $N \geqslant 2$ similar particles of arbitrary spin, all particles are categorically weakly discernible in every admissible physical state, pure and mixed, for every infinite-dimensional Hilbert-space.*

Proof. To deal with spin, we need SymP. The actual proof of the categorical weak discernibility for all particles having non-zero spin-magnitude is at bottom a notational variant of the proof of Theorem 1. Let us sketch how this works for $N = 2$. We begin with the following Hilbert-space for a single particle:

$$\mathcal{H}_s \equiv \big(L^2(\mathbb{R}^3)\big)^{2s+1} , \tag{18}$$

which is the space of *spinorial wave-functions* Ψ, i.e. column-vectors of $2s + 1$-entries, each entry being a complex wave-function of $L^2(\mathbb{R}^3)$. Notice that $\mathcal{H}_s$ is an $(2s + 1)$-fold Cartesian-product set, which becomes a Hilbert-space by carrying the Hilbert-space properties of $L^2(\mathbb{R}^3)$ over to $\mathcal{H}_s$. For instance, the inner-product on $\mathcal{H}_s$ is just the sum of the inner-products of the components of the spinors:

$$\langle\Psi|\Phi\rangle = \sum_{k=1}^{2s+1} \langle\Psi_k|\Phi_k\rangle , \tag{19}$$

where Ψ_k is the k-th entry of Ψ (and similarly for Φ_k), which provides the norm of $\mathcal{H}_s$, which in turn generates the norm-topology of $\mathcal{H}_s$, etc. The degenerate case of the spinor having only one entry is the case of $s = 0$, which we treated in Theorem 1. So we proceed here with $s > 0$. In particular, the number of entries $2s + 1$ is even iff the particles have half-integer spin, and odd iff the particles have integer spin, in units of $\hbar$.

Let $\mathbf{e}_k \in \mathbb{C}^{2s+1}$ be such that its k-th entry is 1 and all others 0. They form the standard basis for $\mathbb{C}^{2s+1}$ and are the eigenvectors of the z-component of the spin-operator $\widehat{S}_z$, whose eigenvalues are traditionally denoted by m (in the terminology of atomic physics: 'magnetic quantum number'):

$$\widehat{S}_z \mathbf{e}_k = m_k \mathbf{e}_k , \tag{20}$$

where $m_1 = -s$, $m_2 = -s + 1$, …, $m_{s-1} = s - 1$, $m_s = +s$. Let $\phi_1, \phi_2, \ldots$ be a basis for $L^2(\mathbb{R}^3)$; then this is a basis for single-particle spinor space $\mathcal{H}_s$ (18):

$$\big\{\mathbf{e}_k\phi_m \in \mathcal{H}_s \mid k \in \{1, 2, \ldots, 2s + 1\},\ m \in \mathbb{N}^+\big\} . \tag{21}$$

Recall that the linear momentum-operator $\widehat{P}$ and the Cartesian position operator $\widehat{Q}$ on an arbitrary complex wave-function $\phi \in L^2(\mathbb{R}^3)$ are the differential-operator times $-\mathrm{i}\hbar$ and the multiplication-operator, respectively:

$$\widehat{P} : \mathcal{D}_P \to L^2(\mathbb{R}^3),\ \phi \mapsto \widehat{P}\phi, \quad \text{where } \big(\widehat{P}\phi\big)(\boldsymbol{q}) \equiv -\mathrm{i}\hbar \frac{\partial\phi(\boldsymbol{q})}{\partial\boldsymbol{q}} \tag{22}$$

and

$$\widehat{Q} : \mathcal{D}_Q \to L^2(\mathbb{R}^3),\ \phi \mapsto \widehat{Q}\phi, \quad \text{where } \big(\widehat{Q}\phi\big)(\boldsymbol{q}) \equiv (q_x + q_y + q_z)\phi(\boldsymbol{q})\ , \tag{23}$$

where domain $\mathcal{D}_P = C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and domain $\mathcal{D}_Q \subset L^2(\mathbb{R}^3)$ consist of all wave-functions ψ such that $|\boldsymbol{q}|^2\psi(\boldsymbol{q}) \to 0$ when $|\boldsymbol{q}| \to \infty$. The action of $\widehat{P}$ and $\widehat{Q}$ is straightforwardly extended to arbitrary spinorial wave-functions by letting the operators act component-wise on the $2s+1$ components. The canonical commutator of $\widehat{P}$ and $\widehat{Q}$ (17) then carries over to spinor space $\mathcal{H}_2$ (18). We can now appeal to the general Lemma 1 and conclude that the two arbitrary spin-particles are categorically and weakly discernible. **Q.e.d.**

Another possibility to finish the proof is more-or-less to repeat the proof of Lemma 1 but now with spinorial wave-functions. For step [S1], the case $N = 2$, the state space of the composite system becomes:

$$\mathcal{H}_s^2 \equiv \big(L^2(\mathbb{R}^3)\big)^{2s+1} \otimes \big(L^2(\mathbb{R}^3)\big)^{2s+1}\ , \tag{24}$$

where the spinorial wave-functions now have $(2s+1)^2$ entries — $(2s+1)^N$ for N spin-s particles. A basis for $\mathcal{H}_s^2$ (24) is

$$\big\{\mathbf{e}_k\phi_m \otimes \mathbf{e}_j\phi_l \in \mathcal{H}_s^2 \mid k, j \in \{1, 2, \ldots, 2s+1\},\ m, l \in \mathbb{N}^+\big\}\ . \tag{25}$$

An arbitrary spinorial wave-function $\Psi \in \mathcal{H}_s^2$ of the composite system can then be expanded as follows:

$$\Psi(\boldsymbol{q}_1, \boldsymbol{q}_2) = \sum_{k,j=1}^{2s+1} \sum_{m,l=1}^{\infty} \frac{\gamma_{ml}}{(2s+1)}\, \mathbf{e}_k\phi_m(\boldsymbol{q}_1) \otimes \mathbf{e}_j\phi_l(\boldsymbol{q}_2)\ , \tag{26}$$

where the γ_{ml} form a squarely-summable sequence, i.e. a Hilbert-vector in $\ell^2(\mathbb{N})$, of norm 1.

With the usual definitions,

$$\widehat{P}_{\mathit{1}} \equiv \widehat{P} \otimes \mathit{1}, \quad \widehat{P}_{\mathit{2}} \equiv \mathit{1} \otimes \widehat{P}, \quad \widehat{Q}_{\mathit{1}} \equiv \widehat{Q} \otimes \mathit{1}, \quad \widehat{Q}_{\mathit{2}} \equiv \mathit{1} \otimes \widehat{Q}\ , \tag{27}$$

one obtains all the relevant commutators on $\mathcal{H}_s^2$ by using expansion (26). The discerning relation (8) on the direct-product spinor-space $\mathcal{H}_s^2$ then becomes

$$\mathsf{C}(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \forall\, \Psi \in \mathcal{D} :\ \big[\widehat{P}_{\boldsymbol{a}},\, \widehat{Q}_{\boldsymbol{b}}\big]\Psi = -\mathrm{i}\hbar\Psi\ , \tag{28}$$

where $\mathcal{D} \subset \mathcal{H}_s^2$ is the domain of the commutator. Etc.

We close this Section with a number of systematic remarks.

Remark 1. Notice that in contrast to the proof of Theorem 1, the proof Corollary 1 relies, besides on StateP, WkMP, WkPP and SemC, on the Symmetrisation Postulate (SymP) *only in so far as* that without this postulate the distinction between integer and half-integer spin particles makes little sense and, more importantly, the tacit claim that this distinction exhausts all possible composite systems of similar particles is unfounded. Besides this, SymP does not perform any deductive labour in the proof. Specifically, the distinction between Bose-Einstein and Fermi-Dirac states never enters the proof, which means that any restriction on Hilbert-rays and on statistical operators, as SymP demands, leaves the proof valid: the theorem holds for all particles in all sorts of states, fermions, bosons, quons, parons, quarticles, anyons and what have you.

Remark 2. The proofs of Theorem 1 and Corollary 1 exploit the non-commutativity of the physical magnitudes, which is one of the algebraic hall-marks of quantum physics. Good thing. The physical meaning of relation C (28) can be understood as follows: momentum and position pertain to two particles differently from how they pertain to a single particle. Admittedly this is something we already knew for a long time, since the advent of QM. What we didn't know, but do know now, is that this old knowledge provides the ground for discerning similar particles weakly and categorically.

Remark 3. The spinorial wave-function Ψ must lie in the domain $\mathcal{D}$ of the commutator of the unbounded operators $\widehat{P}$ and $\widehat{Q}$, which domain is a proper subspace of $\mathcal{H}_s^2$ so that the members of $\mathcal{H}_s^2 \backslash \mathcal{D}$ fall outside the scope of relation C (8). The domain $\mathcal{D}$ does however lies *dense* in $\mathcal{H}_s^2$, even the domain of all polynomials of $\widehat{P}$ and $\widehat{Q}$ does so (the non-Abelian ring on $\mathcal{D}$ they generate) — $\mathcal{D}$ is the Schwarz-space of all complex wave-functions that are continuously differentiable and fall off exponentially. This means that every wave-function that does not lie in Schwarz-space can be approximated with *arbitrary* accuracy by means of wave-functions that do lie in Schwarz-space. This is apparently good enough for physics. Then it is good enough for us too.

Remark 4. A special case of Theorem 1 is that two bosons in symmetric direct-product states, say

$$\Psi(\boldsymbol{q}_1, \boldsymbol{q}_2) = \phi(\boldsymbol{q}_1)\phi(\boldsymbol{q}_2) \ , \tag{29}$$

are also weakly discernible. This seems a hard nut to swallow. If two bosons in state (29) are *discernible*, then something must have gone wrong. Perhaps we attach too much metaphysical significance to a mathematical result?

Our position is the following. The weak discernibility of the two bosons in state (29) is a deductive consequence of a few postulates of QM. Rationality dictates that if one accepts those postulates, one should accept every consequence of those postulates. This is part of *what it means to accept deductive logic*, which we do accept. We admit that the discernibility of two bosons in state (29) is an unexpected if not bizarre consequence. But in comparison to other bizarre consequences of QM, such as inexplicable correlations at a distance (EPR), animate beings that are neither dead nor alive (Schrödinger's immortal cat), kettles of water on a seething fire that will never boil (quantum Zeno), an anthropocentric and intentional concept taken as primitive (measurement), states of matter defying familiar states of aggregation (BE-condensate), in comparison to all that, the weak discernibility of bosons in direct-product states is not such a hard nut to swallow. Get real, it's peanuts.

Remark 5. Every 'realistic' quantum-mechanical model of a physical system, whether in atomic physics, nuclear physics or solid-state physics, employs wave-functions. This means that *now, and only now*, we can conclude that the similar elementary particles of the real world are categorically and weakly discernible. Conjecture 1 of Muller & Saunders (2008: 537) has been proved.

Parenthetically, do finite-dimensional Hilbert-spaces actually have applications at all? Yes they have, in quantum optics and even more prominently in quantum information theory. There one chooses to pay attention to spin-degrees of freedom only and ignores all others — position, linear momentum, energy. This is not to deny there are physical magnitudes such as position, momentum or energy, or that these physical magnitudes do not apply in the quantum-information-theoretic models. Of course not. Ignoring these physical magnitudes is a matter of expediency if one is not interested in them. Idealisation and approximation are part and parcel of science. No one would deny that quantum-mechanical models using spinorial wave-functions in infinite-dimensional Hilbert-space match physical reality *better* — if at all — than finite-dimensional models do that *only* consider spin, and it is for those better models that we have proved our case.

Nevertheless, we next proceed to prove the discernibility of elementary particles for finite-dimensional Hilbert-spaces.

4 To Discern in Finite-Dimensional Hilbert-Spaces

In the case of finite-dimensional Hilbert-spaces, considering $\mathbb{C}^d$ suffices, because every d-dimensional Hilbert-space is isomorphic to $\mathbb{C}^d$ ($d \in \mathbb{N}^+$). The proof is a vast generalisation of the total-spin relation T of Muller & Saunders (2008: 535).

Theorem 2 (StateP, WkMP, StrPP, SymP) *In a composite physical system of $N \geqslant 2$ similar particles, all particles are categorically weakly discernible in every physical state, pure and mixed, for every finite-dimensional Hilbert-space by only using their spin degrees of freedom.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{j}$ be particle-variables, ranging over the set $\{\boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{N}\}$ of N particles. We proceed again Step-wise, as follows.

[S1] Case for $N = 2$, pure states.
[S2] Case for $N = 2$, mixed states.
[S3] Case for $N > 2$, all states.

[S1]. *Case for $N = 2$, pure states.* We begin by considering two similar particles, labeled $\boldsymbol{1}, \boldsymbol{2}$, of spin-magnitude $s\hbar$, which is a positive integer or a half-integer; $\boldsymbol{a}$ and $\boldsymbol{b}$ are again variables over this set. The single particle Hilbert-space is $\mathbb{C}^{2s+1}$, which is isomorphic to every $(s2+1)$-dimensional Hilbert-space; for N-particles the associated Hilbert-space is the N-fold $\otimes$-product of $\mathbb{C}^{2s+1}$. According to SymP, when we have considered integer and half-integer spin particles, we have considered all particles.

We begin by considering the spin-operator of a single particle acting in $\mathbb{C}^{2s+1}$:

$$\widehat{\mathbf{S}} = \widehat{S}_x + \widehat{S}_y + \widehat{S}_z \,, \tag{30}$$

where $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ are the three spin-operators along the three perpendicular spatial directions (x, y, z). The operators $\widehat{\mathbf{S}}^2$ and S_z are self-adjoint and commute and therefore have a common set of orthonormal eigenvectors $|s, m\rangle$; their eigenvector-equations are:

$$\widehat{\mathbf{S}}^2 |s, m\rangle = s(s+1)\hbar^2 |s, m\rangle \quad \text{and} \quad \widehat{S}_z |s, m\rangle = m\hbar\, |s, m\rangle \,, \tag{31}$$

where eigenvalue $m \in \{-s, -s+1, \ldots, s-1, +s\}$ (see e.g. Cohen-Tannoudji *et al.* (1977: Ch. X) or Sakurai (1995: Ch. 3). Next we consider two particles.

The total spin operator of the composite system is

$$\widehat{\mathbf{S}} \equiv \widehat{\mathbf{S}}_{\mathit{1}} + \widehat{\mathbf{S}}_{\mathit{2}} \,, \quad \text{where} \quad \widehat{\mathbf{S}}_{\mathit{1}} \equiv \widehat{\mathbf{S}} \otimes \mathit{1} \,, \quad \widehat{\mathbf{S}}_{\mathit{2}} \equiv \mathit{1} \otimes \widehat{\mathbf{S}} \,, \tag{32}$$

and its z-component is

$$\widehat{S}_z = \widehat{S}_z \otimes \mathit{1} + \mathit{1} \otimes \widehat{S}_z \,, \tag{33}$$

which all act in $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$. The set

$$\{\widehat{\mathbf{S}}_{\mathit{1}}, \widehat{\mathbf{S}}_{\mathit{2}}, \widehat{\mathbf{S}}\,, \widehat{S}_z\} \tag{34}$$

is a set of commuting self-adjoint operators. These operators therefore have a common set of orthonormal eigenvectors $|s; S, M\rangle$. Their eigenvector equations are:

$$\begin{aligned}
\widehat{\mathbf{S}}_{\mathit{1}}^2 |s;\, S, M\rangle &= s(s+1)\hbar^2\, |s;\, S, M\rangle \,, \\
\widehat{\mathbf{S}}_{\mathit{2}}^2 |s;\, S, M\rangle &= s(s+1)\hbar^2\, |s;\, S, M\rangle \,, \\
\widehat{\mathbf{S}}^2 |s;\, S, M\rangle &= S(S+1)\hbar^2\, |s;\, S, M\rangle \,, \\
\widehat{S}_z\, |s;\, S, M\rangle &= M\hbar\, |s;\, S, M\rangle \,.
\end{aligned} \tag{35}$$

One easily shows that $S \in \{0, 1, \ldots, 2s\}$ and $M \in \{-S, -S+1, \ldots, S-1, S\}$.

We note that every vector $|\phi\rangle \in \mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$ has a unique expansion in terms of these orthonormal eigenvectors $|s; S, M\rangle$, because they span this space:

$$|\phi\rangle \;=\; \sum_{S=0}^{2s} \sum_{M=-S}^{+S} \gamma(M, S)|s;\, S, M\rangle\;, \tag{36}$$

where $\gamma(M, S) \in \mathbb{C}[0, 1]$ and their moduli sum to 1. Since the vectors $|s; m, m'\rangle \equiv |s, m\rangle \otimes |s, m'\rangle$ also form a basis of $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$, so that

$$|\phi\rangle \;=\; \sum_{m=-s}^{+s} \sum_{m'=-s}^{+s} \alpha(m, m';\, s)|s;\, m, m'\rangle\;, \tag{37}$$

where $\alpha(m, m'; s) \in \mathbb{C}[0, 1]$ and their moduli sum to 1, these two bases can be expanded in each other. The expansion-coefficients $\alpha(m, m'; s)$ of the basis-vector $|s; S, M\rangle$ are the well-known 'Clebsch-Gordon coefficients'. See for instance Cohen-Tannoudji (1977: 1023).

Let us now proceed to prove Theorem 2. Consider the following categorical 'Total-spin relation':

$$\mathsf{T}(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \forall\, |\phi\rangle \in \mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1} :\; \big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\big)^2\, |\phi\rangle \;=\; 4s(s+1)\hbar^2\, |\phi\rangle\;. \tag{38}$$

One easily verifies that relation T (38) meets Req1 and Req2.

We now prove that relation T (38) discerns the two fermions weakly.

Case 1: $\boldsymbol{a} = \boldsymbol{b}$. We then obtain the spin-magnitude operator of a single particle, say $\boldsymbol{a}$:

$$\big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{a}}\big)^2\, |s;\, S, M\rangle \;=\; 4\widehat{\mathbf{S}}_{\boldsymbol{a}}^2\, |s;\, S, M\rangle \;=\; 4s(s+1)\hbar^2\, |s;\, S, M\rangle\;, \tag{39}$$

which extends to arbitrary $|\phi\rangle$ by expansion (36):

$$\big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{a}}\big)^2\, |\phi\rangle \;=\; 4s(s+1)\hbar^2\, |\phi\rangle\;. \tag{40}$$

By WkPP, the composite system then possesses the following quantitative physical property (when substituting $\boldsymbol{1}$ or $\boldsymbol{2}$ for $\boldsymbol{a}$):

$$\big\langle 4\widehat{\mathbf{S}}_{\boldsymbol{a}}^2,\, 4s(s+1)\hbar^2 \big\rangle\;. \tag{41}$$

This property (41) *is* a relation between the constituent parts of the system, namely T (38), and this relation is reflexive: $\mathsf{T}(\boldsymbol{a}, \boldsymbol{a})$ for every $\boldsymbol{a}$ due to (40).

Case 2: $\boldsymbol{a} \neq \boldsymbol{b}$. The basis states $|s; S, M\rangle$ are eigenstates (35) of the total spin-operator $\widehat{\mathbf{S}}$ (32):

$$\big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\big)^2\, |s;\, S, M\rangle \;=\; S(S+1)\hbar^2\, |s;\, S, M\rangle\;, \tag{42}$$

which does *not* extend to arbitrary vectors $|\phi\rangle$ but only to superpositions of basis-vectors having the same value for S, that is, to vectors of the form:

$$|s;\,S\rangle \;=\; \sum_{M=-S}^{+S} \gamma(M,S)|s;\,S,M\rangle\ . \tag{43}$$

Since S is maximally equal to $2s$, the eigenvalue $S(S+1)$ belonging to vector $|s;S\rangle$ (43) is always smaller than $4s(s+1) = 2s(s+1) + 2s$, because $s > 0$. Therefore relation T (38) fails for $\boldsymbol{a} \neq \boldsymbol{b}$ for all S:

$$\left(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\right)^2 |s;\,S,M\rangle \;\neq\; s(s+1)\hbar^2\,|s;\,S,M\rangle\ . \tag{44}$$

The composite system does indeed *not* possess, by SemC (4), the following *two* quantitative physical properties of the composite system (substitute ***1*** for $\boldsymbol{a}$ and ***2*** for $\boldsymbol{b}$ or conversely):

$$\left\langle \left(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\right)^2,\; s(s+1)\hbar^2 \right\rangle\ , \tag{45}$$

which is expressed by predicate T as a relation between its constituent parts, ***1*** and ***2***, because the system does possess this property according to WkPP:

$$\left\langle \left(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\right)^2,\; S(S+1)\hbar^2 \right\rangle\ . \tag{46}$$

However, superpositions of basis-vectors having a different value for S, such as

$$\tfrac{1}{\sqrt{2}}\big(|s;\,0,0\rangle + |s;\,1,M\rangle\big)\ , \tag{47}$$

where M is -1, 0 or $+1$, are *not* eigenstates of the total spin-operator (32). Precisely for these states we need to appeal to StrPP, because according to the converse of WkPP this is sufficient to conclude that the composite system does not possess physical property (45), so that also for these states relation $\mathsf{T}(\boldsymbol{a},\boldsymbol{b})$ fails for $\boldsymbol{a} \neq \boldsymbol{b}$. From this fact and the theorem of predicate logic (14), we then conclude that T is symmetric (Req2). Since the operators involved correspond to physical magnitudes, e.g. spin, relation T (38) is physically meaningful (Req1) and hence is admissible, because it also meets Req2 (T is reflexive and symmetric).

Therefore total-spin-relation T (38) discerns the two spin-s particles weakly. Since no probability measures occur in the *definiens* of T; it discerns them also categorically.

[S2]. *Case for $N = 2$, mixed.* The extension from pure to mixed states runs as before, as in step [S2] of the proof of Lemma 1. There is however one subtle point we need to take care of.

Case 1: $\boldsymbol{a} = \boldsymbol{b}$. Rewriting relation T (38) for 1-dimensional projectors is easy. Since the spin s of the constituent particles is fixed, the 1-dimensional projector that projects on the ray that contains $|s;\, S, M\rangle$ is an eigenoperator (eigenstate) of $(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{a}})^2$ having the same eigenvalue $4s(s+1)$ (39). Consequently, every (convex) sum of 1-dimensional projectors that project on vectors with the same value of S has this same eigenvalue and we proceed as before in [S2] of Lemma 1, by an appeal to WkPP and a generalisation of T (38) to mixed states:

$$\mathsf{T}(\boldsymbol{a}, \boldsymbol{b}) \quad \text{iff} \quad \forall\, W \in \mathcal{S}(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}) : \; \big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\big)^2\, W \;=\; 4s(s+1)\hbar^2\, W \,. \tag{48}$$

For (convex) sums of projectors that project on vectors of different value of S, we need StrPP again, as in step [S1] above. Relation $\mathsf{T}(\boldsymbol{a}, \boldsymbol{a})$ (48) holds also for mixed states.

Case 2: $\boldsymbol{a} \neq \boldsymbol{b}$. The 1-dimensional projector on $|s; S, M\rangle$ now is an eigenoperator (eigenstate) of $\big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\big)^2$ having eigenvalue $S(S+1)\hbar^2$ (35). Since $S \leqslant 2s$ for every S, this eigenvalue is necessarily smaller than $4s(s+1)$ for all S. Then either every convex mixture of the 1-dimensional projectors has an eigenvalue smaller than $4s(s+1)$ too, or it is not an eigenstate of $\big(\widehat{\mathbf{S}}_{\boldsymbol{a}} + \widehat{\mathbf{S}}_{\boldsymbol{b}}\big)^2$ at all (when the mixture consists of projectors on different states $|s; S, M\rangle$ and $|s; S', M'\rangle$, $S \neq S'$). In virtue of StrPP, relation T (38) then does *not* hold for its parts (for $\boldsymbol{a} \neq \boldsymbol{b}$), for *all* states, mixed and pure, because the system does not possess the required physical property.

So T (48) is reflexive and symmetric (Req2) and certainly physically meaningful (Req1). In conclusion two similar particles in in every finite-dimensional are categorically weakly discernible in all admissible states, both pure and mixed.

[S3]. *Case for* $N > 2$*, all states.* Consider a subsystem of two particles, say $\boldsymbol{a}$ and $\boldsymbol{b}$, of the N-particle system. We can consider these two to form a composite system and then repeat the proof we have just given, in [S1] and [S2], to show they are weakly and categorically discernible. When we can discern an arbitrary particle, say $\boldsymbol{a}$, from every other particle, we have discerned all particles. **Q.e.d.**

We end this Section again with a few more systematic remarks.

Remark 1. In our proofs we started with N particles. Is it not circular, then, to prove they are discernible because to *assume* they are *not identical* (for if they were, we would have single particle, and not $N > 1$ particles), implies we are somehow tacitly *assuming* they are discernible? Have we committed the fallacy of propounding a *petitio principii*?

No we have not. We assume the particles are *formally* discernible, e.g. by their labels, but then demonstrate on the basis of a few postulates of QM that they are *physically* discernible. Or in other words, we assume the particles are quantitatively not-identical and we prove they are qualitatively not-identical. Or still in other words, we assume numerical diversity and prove weak qualitative diversity. See further Muller & Saunders (2008: 541–543) for an elaborate discussion of precisely this issue.

Remark 2. Of course Theorem 1 implies probabilistic versions. The *Probability Postulate* (ProbP) of QM gives the Born probability measure over measurement outcomes for pure states and gives Von Neumann's extension to mixed states, which is the trace-formula. By following the strategy of Muller & Saunders (2008: 536–537) to carry over categorical proofs to probabilistic proofs, one easily proves the probabilistic weak discernibility of similar particles, notably then without using WkPP and SemC (4).

Remark 3. In contrast to Theorem 1, Theorem 2 relies on StrPP, which arguably is an empirically superfluous postulate. StrPP also leads almost unavoidably to nothing less than the Projection Postulate (see Muller & Saunders 2008: 514). Foes of the Projection Postulate are not committed to Theorem 2. They will find themselves metaphysically in the following situation (provided they accept the whiff of interpretation WkPP): similar elementary particles in infinite-dimensional Hilbert-spaces are weakly discernible, in certain classes of states in finite-dimensional Hilbert-spaces they are also weakly discernible, fermions in finite-dimensional Hilbert-spaces are weakly discernible in all admissible states when there always is a maximal operator of physical significance (see Introduction), but for other classes of states in finite-dimensional Hilbert-spaces the jury is still out.

For those who have no objections against StrPP, all similar particles in all kinds of Hilbert-spaces in all kinds of states are weakly discernible. This may be seen as an argument in favour of StrPP: it leads to a uniform nature of elementary particles when described quantum-mechanically and the proofs make no distinction between fermions and bosons.

Remark 4. The so-called *Second Underdetermination Thesis* says roughly that the physics underdetermines the metaphysics — the *First Underdetermination Thesis* then is the familiar Duhem-Quine thesis of the underdetermination of theory by all actual or by all possible data; see Muller (2009). *Naturalistic metaphysics*, as recently has been vigorously defended by Ladyman & Ross (2007: 1–65), surely follows scientific theory wherever scientific theory leads us, without prejudice, without clinging to so-called common sense, and without tacit adherence to what they call *domesticated metaphysics*. Well, QM leads us by means of mathematical proof to the metaphysical statements (if they are metaphysical) that similar elementary particles are categorical (and by implication probabilistic) relationals, more specifically weak discernibles. Those who have held that QM underdetermines the metaphysics in this regard (see references in the Introduction), in this case the nature of the elementary particle, are guilty of engaging in unnatural metaphysics (for elaboration, see Muller (2009: Section 4)).

5 Conclusion: Leibniz Reigns

We have demonstrated that for every set $\mathcal{S}_N$ of N similar particles, in infinite-dimensional and finite-dimension Hilbert-spaces, in all their physical states, pure and mixed, similar particles can be discerned by physically meaningful and permutation-invariant means, and therefore are *not* physically indiscernible:

$$\text{QM}^- \ \vdash\ \forall\, N \in \{2, 3, \ldots\},\ \forall\, \boldsymbol{a}, \boldsymbol{b} \in \mathcal{S}_N :\ \boldsymbol{a} \neq \boldsymbol{b} \ \longrightarrow\ \neg\mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b})\ , \tag{49}$$

where QM^- now stands for StP, WkMP, StrPP and SymP, which is logically the same as having proved PII (5):

$$\text{QM}^- \ \vdash\ \forall\, N \in \{2, 3, \ldots\},\ \forall\, \boldsymbol{a}, \boldsymbol{b} \in \mathcal{S}_N :\ \mathsf{PhysInd}(\boldsymbol{a}, \boldsymbol{b}) \ \longrightarrow\ \boldsymbol{a} = \boldsymbol{b}\ , \tag{50}$$

and by theorem of logic (1) as having disproved IT. Hence

$$\text{QM}^- \ \vdash\ \text{PII} \ \wedge\ \neg\text{IT}\ . \tag{51}$$

Therefore all claims to the contrary, that QM refutes PII, or is inconsistent with PII, or that PII cannot be established (see the references in Section 1 for propounders of these claims) find themselves in heavy weather. Quantum-mechanical particles are *categorical weak discernibles*, and therefore not *indiscernibles* as propounders of IT have claimed. Similar elementary particles are like points on a line, in a plane or in Euclidean space: absolutely indiscernible yet not identical (there is more than one of them!). Points on a line are categorical relationals, categorical weak discernibles to be precise. Elementary particles are exactly like points in this regard.

Leibniz is back from exile and reigns over all quantum-mechanically possible worlds, *salva veritate*.

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