Only countable common cause systems exist

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April 22, 2009

Abstract

In this paper we give a positive answer to a problem posed by G. Hofer-Szabó and M. Rédei (2004) regarding the existence of infinite common cause systems (CCSs). An example of a countably infinite CCS is presented, as well as the proof that no CCSs of greater cardinality exist.

1 Preliminaries

The problem we tackle in this paper arose in the sub-field of philosophy of science concerning the notion of common cause. The idea is traditionally thought to have been first put forward by Hans Reichenbach in his book *The Direction of Time* (1956). Various forms of it have been found to be of interest for different sorts of researchers, from those mainly interested in physics to those dealing with Bayesian nets. We now give all definitions needed to state the problem we will be dealing with in the next sections.

By a probability space we mean a tuple $\langle S, P\rangle$, where S is a Boolean algebra and P is a probability measure on S. Due to Stone's representation theorem we can without loss of generality view S as a field of sets. Events $A, B \in S$ are (positively) correlated if $P(A \cap B) > P(A)P(B)$.

Definition 1 *Let $A, B \in S$. An event C is said to be a* screener-off *for the pair $\{A, B\}$ if $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$. In the case where A and B are correlated we also say that C* screens off *the correlation.*

Definition 2 *Let $A, B \in S$. We say that a family of events $\{C_i\}$ satisfies the* statistical relevance *condition with regard to the pair $\{A, B\}$ if whenever $i \neq j$*

$$\big(P(A \mid C_i) - P(A \mid C_j)\big)\big(P(B \mid C_i) - P(B \mid C_j)\big) > 0$$

Definition 3 *Let $A, B \in S$.*

Then $C \in S - \{A, B\}$ is said to be a common cause of these two events if (1) both C and its complement $C^{\perp}$ are screener-offs for the pair $\{A, B\}$ and (2) the pair $\{C, C^{\perp}\}$ satisfies the statistical relevance condition with regard to $\{A, B\}$ with $P(A \mid C) > P(A \mid C^{\perp})$.

A common cause C for events A, B may be viewed as a doubleton $\{C, C^{\perp}\}$ with both elements screening off the pair and one being statistically more relevant for A and B than the other. This idea has been generalized with regard to the number of screener-offs in [3]. Recall that a partition of unity of S is a family $\{Y_i\}$ of pairwise disjoint non-empty subsets of $\mathbf{1}_S$ such that $\bigcup\{Y_i\} = \mathbf{1}_S$.

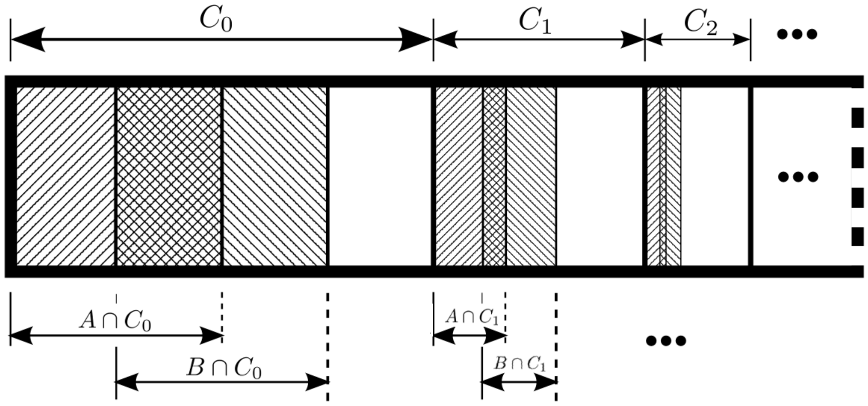
Definition 4 *A partition of unity of S is said to be a* common cause system (CCS) *for A and B if it satisfies the statistical relevance condition w.r.t. A and B and all its members are screener-offs for the pair.*

The cardinality of the partition is called the size *of the common cause system.*

It was shown in [3] that existence of a common cause system (which was then labelled 'Reichenbachian common cause system') for events $A, B \in S$ entails a correlation between those events, so it can be considered an explanation of the correlation.[1] As mentioned above, a common cause together with its complement form a CCS of size 2.

Some results regarding CCSs were published in [3] and [4]. These include the fact that for any natural n ($n \geqslant 2$) it is possible to find a probability space containing a correlation for which a CCS of size n exists. [3] asks whether infinite CCSs exist, conjecturing the positive answer. We confirm the conjecture in the following section, providing an example of a countably infinite CCS, and conclude with a proof of the non-existence of CCSs of greater size.

2 A countably infinite common cause system



Let $\langle [0,1), W, \lambda \rangle$ be the classical probability space comprising the real interval $[0,1)$, W – the set of all its Lebesgue-measurable subsets, and the Lebesgue measure λ. Put

$$C_n := \left[\frac{2^n - 1}{2^n}, \frac{2^{n+1} - 1}{2^{n+1}} \right);$$

$$C := \{C_n\}_{n \in \mathbb{N}}$$

It is evident that if $n \neq m$ ($n, m \in \mathbb{N}$), $C_n \cap C_m = \emptyset$ and that $\bigcup C = [0,1)$, so C is a countably infinite partition of $[0,1)$. Notice that for any natural n, $\lambda(C_n) = \frac{1}{2^{n+1}}$.

[1] In fact, the notion of a CCS appeared first in [2], but it was a bit different: for a partition to be a CCS for the pair $\{A, B\}$ it was enough that all its elements screened off the pair. We are not concerned with this more limited notion, since it lacks the explanatory value of the later one – one cannot deduce the correlation from screening off alone.

For any $n \in \mathbb{N}$, we want both $\lambda(A \cap C_n)$ and $\lambda(B \cap C_n)$ to be equal to $\frac{1}{(n+2)\cdot 2^{n+1}}$. To improve the clarity of the notation below, put $l_n = \frac{1}{(n+2)\cdot 2^{n+1}}$. Define

$$A := \bigcup_{n=0}^{\infty} \left[\frac{2^n - 1}{2^n}, \frac{2^n - 1}{2^n} + \, l_n\right);$$

$$B := \bigcup_{n=0}^{\infty} \left[\frac{2^n - 1}{2^n} + \frac{n+1}{n+2} \cdot l_n, \quad \frac{2^n - 1}{2^n} + \frac{n+1}{n+2} \cdot l_n + l_n\right)$$

Fix an $n \in \mathbb{N}$. From the above definitions it follows that

$$\lambda(A \mid C_n) = \frac{\lambda(A \cap C_n)}{\lambda(C_n)} = \frac{\frac{1}{(n+2)\cdot 2^{n+1}}}{\frac{1}{2^{n+1}}} = \frac{1}{n+2} = \lambda(B \mid C_n);$$

whereas

$$\lambda(A \cap B \mid C_n) = \frac{\lambda(A \cap B \cap C_n)}{\lambda(C_n)} = \frac{(1 - \frac{n+1}{n+2}) \cdot \frac{1}{(n+2)\cdot 2^{n+1}}}{\frac{1}{2^{n+1}}} =$$

$$= \frac{\frac{1}{(n+2)^2 \cdot 2^{n+1}}}{\frac{1}{2^{n+1}}} = \frac{1}{(n+2)^2}$$

and so

$$\lambda(A \cap B \mid C_n) = \lambda(A \mid C_n)\lambda(B \mid C_n),$$

which means that C satisfies the screening-off condition.

Now, fix two distinct $m, n \in \mathbb{N}$. Without loss of generality assume $m > n$. It follows that

$$\lambda(A \mid C_n) = \frac{1}{n+2} > \frac{1}{m+2} = \lambda(A \mid C_m)$$

and

$$\lambda(B \mid C_n) = \frac{1}{n+2} > \frac{1}{m+2} = \lambda(B \mid C_m).$$

Therefore, for $m, n \in \mathbb{N}$ $(m \neq n)$, the differences $\lambda(A \mid C_m) - \lambda(A \mid C_n)$ and $\lambda(B \mid C_m) - \lambda(B \mid C_n)$ have the same sign and are nonzero, so

$$\big(\lambda(A \mid C_m) - \lambda(A \mid C_n)\big)\big(\lambda(B \mid C_m) - \lambda(B \mid C_n)\big) > 0 \ (m \neq n)$$

which means that C satisfies the other condition of the definition of a CCS for $\langle A, B\rangle$. To complete the picture, from Proposition 1 of [3] it follows that events A and B are correlated.

We have shown that in the space $\langle [0,1), W, \lambda\rangle$ the countably infinite set C is a CCS for $\langle A, B\rangle$, thus giving the positive answer to the problem stated in [3].

3 Proof of nonexistence of common cause systems of greater cardinality

As we will now show, countable infinity is the limit when it comes to the cardinality of CCSs. No uncountable CCSs exist.

It is straightforward to note that the cardinality of a CCS may not exceed $2^{\aleph_0}$. For suppose $E = \{C_i\}_{i \in I}$ is a CCS for the pair of correlated events A, B in the probability space $\langle S, P \rangle$. Then the definition of a CCS requires that the function

$$f : E \ni C \mapsto P(A \mid C) \in [0, 1] \subseteq \mathbb{R}$$

be an injection, which is clearly impossible if E is of a greater cardinality.

It is however possible to prove more:

Theorem 5 *The greatest possible cardinality of a CCS is* $\aleph_0$.

This follows from the following lemma:

Lemma 6 *Let S be a Boolean algebra admitting countable joins and meets (i.e. a measurable space) and μ – a bounded measure on it. Let Π be a partition of unity in S.*

Then μ assumes a positive value for at most countably many elements of Π.

Let S, μ and Π satisfy the hypothesis of the lemma. Then it suffices to prove that if μ assumes positive values for more than countably many elements of Π, then there exists a positive real number δ with the property that for some countably infinite subset Q of Π, $\mu[Q] \subseteq [\delta, +\infty]$.

(In this case $\Sigma_{q \in Q} \mu(q)$ is divergent – the order of the summands is immaterial, because they are all positive – contradicting the assumption that μ is bounded.)

Suppose μ does indeed assume positive values for uncountably many members of Π, but no number δ possessing the property given above exists. Then for any $\eta \in (0, +\infty)$ the set $\{q \in \Pi \mid \mu(q) \geqslant \eta\}$ is finite. However,

$$\bigcup_{\eta \in \mathbb{R}^*_+} \{\pi \in \Pi \mid \mu(\pi) \geqslant \eta\} = \bigcup_{k \in \mathbb{N} - \{0\}} \{\pi \in \Pi \mid \mu(\pi) \geqslant \frac{1}{k}\}$$

would then be a countable union of finite sets, and so countable, contradicting the assumption that μ assumes positive values for uncountably many elements of Π. $\Box$

Returning to the proof the theorem, suppose that in some probability space $\langle S, P \rangle$ a CCS $\{C_i\}_{i \in I}$ of size greater than $\aleph_0$ exists. Lemma 6 entails that only countably many of the C_is may have positive probabilities. Therefore for some $k \in I$, $P(C_k) = 0$, and so C_k cannot be a screener-off because the required conditional probabilities are not defined.[2] This contradicts the assumption that $\{C_i\}_{i \in I}$ is a CCS. $\Box$

[2]The reader may prefer conditional probabilities given probability zero events to be always equal to 0, or 1 (see e.g. [1], p. 57). In these cases the proof is completed by noting that for some distinct $k, l \in I$, $P(A \mid C_k) = P(A \mid C_l)$, which violates the statistical relevance condition.

References

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[3] G. Hofer-Szabó, M. Rédei (2004). Reichenbachian Common Cause Systems, *International Journal of Theoretical Physics* 43 (2004), 1819-1826.

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[5] H. Reichenbach (1956). *The Direction of Time*, Berkeley, University of Los Angeles Press.