View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by PhilSci Archive

Disproof of Bell's Theorem: Further Consolidations

Joy Christian*

Perimeter Institute, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada, and
Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, England

The failure of Bell's theorem for Clifford algebra valued local variables is further consolidated by proving that the conditions of remote parameter independence and remote outcome independence are duly respected within the recently constructed exact, local realistic model for the EPR-Bohm correlations. Since the conjunction of these two conditions is equivalent to the locality condition of Bell, this provides an independent geometric proof of the local causality of the model, at the level of microstates. In addition to local causality, the model respects at least seven other conceptual and operational requirements, arising either from the predictions of quantum mechanics or the premises of Bell's theorem, including the Malus's law for sequential spin measurements. Since the agreement between the predictions of the model and those of quantum mechanics is quantitatively precise in all respects, the ensemble interpretation of the entangled singlet state becomes amenable.

PACS numbers: 03.65.Ud, 03.67.-a, 02.10.Ud

arXiv:0707.1333v1 [quant-ph] 10 Jul 2007

I. INTRODUCTION

One of the sources of the geometrical beauty, internal coherence, and empirical success of general relativity is its strict adherence to local causality [1]. Quantum theory, on the other hand, is peculiarly defiant of this notion, as was stressed many years ago by Einstein, Podolsky, and Rosen (EPR) [2]. Worse still, any hope of "completing" quantum theory into a realistic, locally causal theory in a manner espoused by EPR is generally believed to have long been dashed by Bell's theorem and its variants [3][4]. Indeed, these theorems are breathtakingly ambitious in their scope to frustrate any attempt to the contrary: *no physical theory which is realistic as well as local in a specified sense can reproduce all of the statistical predictions of quantum mechanics* [5]. In a recent paper [6], however, the legitimacy of this claim was put in doubt by means of an exact, deterministic, local realistic model for the EPR-Bohm correlations on which Bell's theorem rests, without appealing to either remote contextuality or backward causation. In particular, it was shown that the much studied CHSH inequality [7] in this context is violated within this model and extended to the extrema of $\pm\,2\sqrt{2}\,$, in *exactly* the same manner as it is within quantum mechanics. Despite these compelling features, however, since it stands against the received wisdom of some four decades, the model has met with a certain predisposed skepticism. In what follows this skepticism is systematically addressed, and proven to be entirely unwarranted. This is accomplished by demonstrating that, not only the model adheres *strictly* to the notion of local causality as prescribed by Bell, but also the agreement between the predictions of the model and those of quantum mechanics is *quantitatively* precise in *all* relevant aspects of physics, within the context of his theorem.

A. Exact, Deterministic, Locally Causal Model for the EPR-Bohm Correlations

The central strategy of the proposed model of Ref.[6] is to make use of the Clifford algebra valued observables in a manner that renders the non-commuting parts of their products to contribute only counterfactually. The complete state describing the singlet spin system is taken to be a unit trivector $\boldsymbol{\mu}$, which is an element of the Clifford algebra $Cl_{3,0}$ of the subspaces of the Euclidean space $\mathrm{I\!R}^3$. It can be viewed as a unit volume element in $\mathrm{I\!R}^3$, assembled by three ordinary vectors of finite lengths and arbitrary directions. Since every trivector in the algebra $Cl_{3,0}$ differs only by its volume and orientation from the standard trivector I composed of a right-handed frame of orthonormal vectors, $\boldsymbol{\mu}$ differs from I only by the sign of its handedness: $\boldsymbol{\mu}=\pm\,I$. Thus, the local hidden variable of the model is essentially the intrinsic freedom of choice in the initial orientation of the volume element $\boldsymbol{\mu}$, which is a pseudoscalar, dual to a scalar in $\mathrm{I\!R}^3$, and hence commutes with every other element of $Cl_{3,0}$. Next, the actual spin observables

*Electronic address: joy.christian@wolfson.oxford.ac.uk

in the model are taken to be the projections $\boldsymbol{\mu}\cdot\mathbf{n} =: A_{\mathbf{n}}(\boldsymbol{\mu})$ of the trivector $\boldsymbol{\mu}$ along the unit directions $\mathbf{n}$, which are simply the directions of the orientations of spin analyzers. In other words, the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$ are the unit, non-commuting bivectors $\boldsymbol{\mu}\cdot\mathbf{n}$, isomorphic to the familiar quaternionic numbers [8]. Finally, the Clifford product of two such observables, say $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $A_{\mathbf{b}}(\boldsymbol{\mu})$, satisfies the following identity, which plays a central role in the model:

$$(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}). \tag{1}$$

This identity is simply a more general expression of the bivector subalgebra within the algebra $Cl_{3,0}$ (cf. Eq. (2.65) of Ref.[8]). For notations, conventions, and further background, the reader is urged to consult Ref.[6].

B. Eight Essential Requirements Satisfied by the Local Model

As elementary as this model seems to be, it respects at least *eight* different conceptual and operational requirements, arising from either the predictions of quantum mechanics or the premises of Bell's theorem. For convenience, let us begin our analysis by anthologizing these requirements, without any attempt to iron out redundancies:

(1) Mathematical representation of the physical quantities in a proposed local model for the EPR-Bohm correlations must not be *ad hoc* [9], but a part of at least operationally well-motivated theoretical framework. It will be clear from the discussion in the next section that our choice of representation for the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$ as elements of the Clifford algebra $Cl_{3,0}$ amply satisfies this requirement. Since the pioneering works of Grassmann and Clifford, there has been a sustained impetus to reformulate *all* physical quantities within the Clifford algebraic framework [10][11][12]. What is more, it is known that many of the quantum mechanical observables find natural expressions within this framework [8][10]. The physical motivations behind our own choice of the local realistic observables within this framework has been made clear in Ref.[6]. Below we will discuss why this choice is also operationally well motivated.

(2) The actual values of the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$, for all $\mathbf{n}$ and $\boldsymbol{\mu}$, must lie in the interval $[-1, +1]$. From the discussion below, it will be clear that this requirement is necessarily satisfied by the observables of our model, despite the non-commutativity of their products. The only non-zero values the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$ can possibly yield in any experiment correspond to a *bit* of information. Thus $A_{\mathbf{n}}(\boldsymbol{\mu})$ are truly dichotomic observables, with binary values $\pm\,1$.

(3) The classical ensemble average of a single observable $A_{\mathbf{n}}(\boldsymbol{\mu})$, taken as a dispersion-free counterpart [13] of the corresponding quantum mechanical spin operator $\boldsymbol{\sigma}\cdot\mathbf{n}$, must be equal to the quantum mechanical expectation value of the operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ in the singlet spin state $|\Psi_{\mathbf{n}}\rangle$. In the notations of Ref.[6] this statement can be expressed as

$$\mathcal{E}_{h.v.}(\mathbf{n}) \,=\, \int_{\mathcal{V}_3} A_{\mathbf{n}}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}\cdot\mathbf{n}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0\,. \tag{2}$$

It is demonstrated in Ref.[6] that this result—which turns out to be a straightforward consequence of the dichotomic nature of the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$—holds exactly within our local model.

(4) The classical ensemble average of a joint observable $(A_{\mathbf{a}}\,B_{\mathbf{b}})(\boldsymbol{\mu})$, satisfying the factorizability condition

$$(A_{\mathbf{a}}\,B_{\mathbf{b}})(\boldsymbol{\mu}) \,=\, A_{\mathbf{a}}(\boldsymbol{\mu})\,B_{\mathbf{b}}(\boldsymbol{\mu})\,, \tag{3}$$

and taken as a dispersion-free counterpart of the corresponding quantum mechanical spin operator $\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}$, must be equal to the quantum mechanical expectation value of this operator in the singlet state $|\Psi_{\mathbf{n}}\rangle$, giving

$$\mathcal{E}_{h.v.}(\mathbf{a},\,\mathbf{b}) \,=\, \int_{\mathcal{V}_3} A_{\mathbf{a}}(\boldsymbol{\mu})\;B_{\mathbf{b}}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,. \tag{4}$$

As demonstrated in Ref.[6], this result—which follows almost trivially from the identity (1)—holds *exactly* within our local model. It is also worth stressing here that the condition $\mathcal{E}_{c.v.}(\mathbf{n},\,\mathbf{n}) = -\,1$, which was adapted by EPR for the special and ideal case of perfect correlations, and which also played a crucial role in the original theorem of Bell concerning only deterministic local hidden variable theories [3], also holds exactly within our local model.

(5) In conjunction with the factorizability condition (3) above, the actual values of the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$, $B_{\mathbf{b}}(\boldsymbol{\mu})$, and $(A_{\mathbf{a}}\,B_{\mathbf{b}})(\boldsymbol{\mu})$—for all $\mathbf{a}$, $\mathbf{b}$, and $\boldsymbol{\mu}$—must respect the following relationship among themselves:

$$v(A,\,B) \,=\, A\,B\,, \tag{5}$$

where A and B, respectively, are the actual values that could be found if the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ are measured in some experiment, and $v(A,\,B)$ is the corresponding value of the joint observable $(A_{\mathbf{a}}\,B_{\mathbf{b}})(\boldsymbol{\mu})$. In the subsection III below, this condition—which is simply the celebrated locality condition of Bell [3][14]—will be shown to hold rigorously within our local model. It can be restated in two conceptually distinct parts as follows:

(5 a) A proposed local model must satisfy the condition of remote parameter independence [5]. The deterministic version of this part of the locality condition states that, for a given microstate $\boldsymbol{\mu}$, the outcome of an experiment at a local station **1** must be independent of the chosen experimental settings at the remote station **2**, and vice versa. In the subsection III A below, this condition will be shown to hold rigorously within our local model.

(5 b) A proposed local model must satisfy the condition of remote outcome independence [5]. The deterministic version of this part of the locality condition states that, for a given microstate $\boldsymbol{\mu}$, the outcome of an experiment at a local station **1** must be independent of the outcome of an experiment at the remote station **2**, and vice versa. In the subsection III B below, this condition will be shown to hold rigorously within our local model.

(6) The choice of the measurement settings within a proposed local model for the EPR-Bohm correlations, such as our vectors $\mathbf{a}$ and $\mathbf{b}$, should not in any way be constrained by the microstates $\boldsymbol{\mu}$. More pertinently, the distribution $\boldsymbol{\rho}(\boldsymbol{\mu})$ of the microstates must be independent of the measurement settings $\mathbf{a}$ and $\mathbf{b}$ [15]. As discussed in Ref.[6], this requirement, accommodating the "free will" of the experimenter, is well respected within our local model.

(7) A proposed model for the EPR-Bohm correlations must violate the CHSH inequality in precisely the same quantitative manner as do the corresponding nonlocal predictions of quantum mechanics. Namely, in the standard notations [14], the correlation functions (4) must satisfy the Tsirel'son inequality [16]:

$$|\,\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\,|\,\leqslant\,2\sqrt{2}\,. \tag{6}$$

That this inequality is exactly predicted by our local model has been already exhibited twice. In Ref.[6] it was shown to hold in general terms within the model. This demonstration, however, proved to be too abstract for some readers, and hence, in response [17], the inequality was again derived in terms of the cosines of relative angles between the four spin analyzers. In section IV below, we shall once again derive this inequality in a different manner, to exhibit the model's precise quantitative agreement with the corresponding predictions based on the entangled singlet state.

(8) A complete local model for the EPR-Bohm correlations must also reproduce the Malus's law for sequential spin measurements, since quantum mechanics respects this law. In our notations, this requirement can be stated as

$$\int_{\mathcal{V}_3} A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})\;\;d\boldsymbol{\rho}(\boldsymbol{\mu})\,=\,\langle\, s_{\mathbf{p}}=\pm 1\,|\,\boldsymbol{\sigma}\cdot\mathbf{a}\,|\,s_{\mathbf{p}}=\pm 1\,\rangle\,=\,\mathbf{a}\cdot\mathbf{p}\,, \tag{7}$$

where the observable $A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})$ now depends on both the polarizer $\mathbf{p}$ and the analyzer $\mathbf{a}$. In section V below, this relation, as well as its extension to sequential spin measurements, will be shown to hold exactly within our model.

The above results complement and strengthen the disproof of Bell's theorem presented in Ref.[6]. More precisely, if we take Bell's theorem to be the statement that: *it is impossible to construct a deterministic local hidden variable theory based on the observables* $A_{\mathbf{n}}(\boldsymbol{\mu})$ *and probability distribution* $\boldsymbol{\rho}(\boldsymbol{\mu})$ *such that all eight of the requirements listed above are satisfied in conjunction*, then the local realistic model constructed in Ref.[6] decisively disproves this theorem.

To better appreciate this claim, let us first have a closer look at the basic observables of our model.

II. OPERATIONAL ADEQUACY OF THE OBSERVABLES $A_{\mathbf{n}}(\boldsymbol{\mu})$, AND A THEORY OF THEIR MEASUREMENT:

As we noted above, the local observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ of our model are unit bivectors, with binary values ± 1. Some readers and commentators have found the use of non-commuting bivectors as observables less than compelling. In the proof of Bell's theorem, they allege, locality requires that the observables such as $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ must simply be ordinary functions—maps, if you like, from whatever valued hidden variables $\boldsymbol{\mu}$ one may wish to consider, to commuting real numbers. Indeed, they argue, in the derivation of CHSH inequality one only needs to assume that the outcomes $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$—for all $\mathbf{a}$, $\mathbf{b}$, and $\boldsymbol{\mu}$—are commuting real numbers, lying in the closed interval $[-1,\,+1]$. In this section we point out that the above charge against our local model is based on a failure to appreciate the Clifford algebraic concepts employed in the model, and that it overplays the necessity of commuting numbers within the operational contexts of Bell's theorem. In the following section we then demonstrate that the relativistic local causality—as famously crystallized by Bell in his locality (or factorizability) condition [15]—is rigorously respected within our model, despite the apparent non-commutativity of our observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$.

To this end, let us first recall that, since the pioneering vision of Grassmann circa 1844, the very purpose of the Clifford algebra $Cl_{3,0}$ has been to treat *all* of the subspace elements of $\mathbb{R}^3$—namely, scalars, vectors, bivectors, and trivectors—on equal footing, and use them as *direct* elements of geometric computations [8][10][11][12]. This provides a powerful computational framework that utilizes subspace elements across dimensions, from 0-dimensional scalars to 3-dimensional trivectors, all equally respected as "directed numbers", and unified by the *invertible* Clifford product of immense power and versatility. In particular, within this framework scalars are just as much a part of the algebra as vectors, bivectors, and trivectors are [12]. Indeed, a generic multivector in this 8-dimensional linear space $Cl_{3,0}$ includes a scalar, and is written as a linear combination of that scalar, plus a vector, a bivector, and a trivector:

$$\boldsymbol{\xi} = \text{scalar} + \text{vector} + \text{bivector} + \text{trivector}. \tag{8}$$

What is more, the familiar tools such as complex numbers and quaternions are now absorbed within a *real* vector space, allowing one to treat them as *reals*, in every sense of the word. Even the differential forms, symplectic forms, tensors, spinors, and matrices are not spared from assimilation and simplification by this vastly versatile framework. Indeed, the modern program of Clifford algebra is anything but modest. It aims to provide nothing less than a *universal* language for all of mathematics, physics, and engineering [10][12][18]. Unfortunately, as mentioned above, unfamiliarity with this empowering and unifying framework has given rise to some misplaced questions and unsubstantiated presumptions about our counterexample to Bell's theorem. In what follows, we answer these questions by first spelling out exactly how do the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$ in our local model represent what is actually observed in a Stern-Gerlach type experiment, and how do they do so more faithfully than the observables of Bell's own local model.

As we noted above, the local realistic physical picture underlying our model is as follows: By construction, the complete microstates are the *unit* volume elements $\boldsymbol{\mu}$, with unspecified orientations—i.e., the initial handedness of the trivectors $\boldsymbol{\mu}$ is taken to be unspecified. Given two arbitrary unit vectors $\mathbf{a}$ and $\mathbf{b}$, specifying, say, the "z directions" of two Stern-Gerlach apparatuses, the two remote spin observables for a given microstate are represented by the bivectors $A_{\mathbf{a}}(\boldsymbol{\mu}) = \boldsymbol{\mu}\cdot\mathbf{a}$ and $B_{\mathbf{b}}(\boldsymbol{\mu}) = \boldsymbol{\mu}\cdot\mathbf{b}$, respectively. Now, since any bivector has an intrinsic sense of rotation, spin is routinely represented in Clifford algebra by a bivector (see, e.g., Eq. (88) of Ref.[10]). Moreover, the bivectors $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ of our model are necessarily of *unit* magnitude, which can be checked by evaluating their norm: $||A_{\mathbf{n}}(\boldsymbol{\mu})||^2 = ||\boldsymbol{\mu}\cdot\mathbf{n}||^2 = (\mp I\,\mathbf{n})(\pm I\,\mathbf{n}) = \mathbf{n}\,\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = 1$. Thus, the magnitudes of $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ in our model are universally fixed to be unity, and their directions—which are simply the normal directions to their planes—are defined by the "z" directions of the Stern-Gerlach magnets themselves. This is because, just as the operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ represents a projection of the spin operator $\boldsymbol{\sigma}$ along the direction $\mathbf{n}$ in quantum mechanics, the bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ represents a projection of the volume element $\boldsymbol{\mu}$ along the direction $\mathbf{n}$ in our local model. And since geometrically this projection gives rise to a bivector $\boldsymbol{\mu}\cdot\mathbf{n}$—which, in turn, is simply a plane segment with an intrinsic sense of rotation perpendicular to the direction $\mathbf{n}$—the geometrical picture—reminiscent of a spinning wheel of a miniscule gyroscope with its axis of rotation $\mathbf{n}$ aligned to the direction z of the Stern-Gerlach apparatus—can hardly be more compelling.

In fact, the observables $\boldsymbol{\mu}\cdot\mathbf{n}$ are even operationally no less compelling. To appreciate this, let us recall that, when considered by itself, each element of Clifford algebra $Cl_{3,0}$ is *completely* specified by three properties, and three properties *only* [10][11][12]. These are (1) its magnitude, (2) its direction, and (3) its orientation (or sense). In the case of our bivector $\boldsymbol{\mu}\cdot\mathbf{n}$, the first of these three properties is universally fixed, once and for all. As we calculated above, its magnitude—which is simply the area of the plane segment it represents—is universally fixed to be unity. And even the direction of this bivector—which is simply the direction of its dual vector $\mathbf{n}$—is fixed. As discussed above, it is defined by the direction of the spin analyzer itself, once selected by the experimenter. Thus, once a choice of the direction $\mathbf{n}$ is made, the direction of the bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ is no longer unspecified. Consequently, the only unspecified property of the bivector—i.e., the only unknown property that a spin analyzer can possibly reveal—is its orientation, inherited from $\boldsymbol{\mu}$, which, for a bivector, is simply its sense of rotation—i.e., whether it is spinning counterclockwise about $\mathbf{n}$ or clockwise—or, in the dual picture, whether it is spinning "up" or "down" along $\mathbf{n}$. It is crucial to note here that a given bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ cannot be spinning either "up" or "down", or in any other way, about any other direction but $\mathbf{n}$. This is simply because the very meaning of a particular bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ is defined by the direction $\mathbf{n}$.

It should be clear from the above picture that the outcome of a measurement of the observable $A_{\mathbf{n}}(\boldsymbol{\mu})$ cannot yield anything but the binary values ± 1. It will yield $+1$ if the bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ is spinning counterclockwise about the direction $\mathbf{n}$, and -1 if the bivector is spinning clockwise about $\mathbf{n}$. Moreover, it will yield null result (i.e., zero) if the spin analyzer is not aligned with the direction $\mathbf{n}$ of the bivector. This is because a projection onto (or inner product with) the direction $\mathbf{n}$ of any arbitrary bivector different from $\boldsymbol{\mu}\cdot\mathbf{n}$ would not yield another bivector but an ordinary vector, in the direction *orthogonal* to the original direction $\mathbf{n}$ (cf. pp $30-33$ of Ref.[8]). It is also worth stressing here that the above picture is not meant to suggest that spin is somehow "created" by the spin analyzer, but simply that only the component $\boldsymbol{\mu}\cdot\mathbf{n}$ of the trivector $\boldsymbol{\mu}$—obtained by a projection along the direction $\mathbf{n}$ of the analyzer—will pass through that analyzer. And when this particular component of $\boldsymbol{\mu}$ does pass through, it can only trigger either the "spin up" detector or the "spin down" detector, for the resulting bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ could only be spinning "up" or "down" along the direction $\mathbf{n}$. Thus, when probed individually, no scrutiny by means of a spin analyzer can extract anything but a *bit* of information from the observables $A_{\mathbf{n}}(\boldsymbol{\mu})$, since, apart from the sense of their rotation, no other properties of these observables are unspecified. It should now be fairly clear how the binary value assignment of these observables, along with the value *zero* for a possible null result, comes about. It is, in fact, no different from that illustrated by Clauser and Shimony in Figure 1 of their celebrated report [14]. Thus, our observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ represent quite faithfully what is actually observed in a Stern-Gerlach type experiment.

To be sure, a diehard nonlocalist may continue to see here an opportunity to dissent. For, although our dichotomic observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ can only yield the binary values ± 1 as demanded by quantum mechanical predictions, there still remains a psychological problem of seeing bivectors where one is used to seeing commuting real numbers. It would be a naive mistake, however, to read too much into this bivectorial representation of the observable physical quantities. To begin with, as we have already noted, the perspective on the relation between scalars and bivectors is radically different in Clifford algebra from the conventional perspective fostered by the vector algebra. In particular, scalars, vectors, bivectors, and trivectors are all intimately linked in Clifford algebra by the invertible geometric product of almost magical significance. What is more, the bivectors in the Clifford algebra $Cl_{3,0}$ are in fact isomorphic to the familiar quaternionic numbers. More precisely, left-handed set of orthonormal bivectors is isomorphic to the right-handed set of pure quaternionic numbers [8]. Consequently, the observable physical quantities within our model can be equivalently thought of as either "real bivectors" or "complex quaternions". In other words, there is northing intrinsically either bivectorial or complex about the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$. These are simply their representation-dependent features, and hence cannot be expected to have deeper physical significance. The feature that remains truly independent of a particular representation is the *non-commutativity* of their products, and hence it is this non-commutativity that has a genuine physical significance. It would play a decisive role in what follows.

Is it, however, reasonable to use non-commuting numbers such as quaternions in a local realistic theory? Of course it is. Aerospace engineers routinely use non-commuting quaternions in applications to rotations in the ordinary Euclidean space, precisely because they do not commute. Moreover, this lack of commutativity of quaternions merely reflects the fact that the basis vectors of Euclidean space can be chosen to be orthogonal—a fact of geometry, not of dynamics [8]. Thus, *a priori*, the local realists are by no means obliged to remain unimaginative and consider only the commuting real numbers for their theories. Furthermore, let us not forget—as so often unguarded preoccupation with operationalism makes us do—that physically the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ are supposed to have more significance than just representing the results of spin measurements. They must, in fact, also represent the *dynamical variables* of a yet to be discovered physical theory, which, when measured, should reproduce the binary outcomes ± 1. That is to say, the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ are supposed to be the dispersion-free counterparts of the quantum mechanical spin operators $\boldsymbol{\sigma}\cdot\mathbf{a}$ and $\boldsymbol{\sigma}\cdot\mathbf{b}$, in addition to having operational significance of representing the measurement outcomes ± 1 [13]. Indeed, as Bell himself emphasized: "In a complete physical theory of the type envisaged by Einstein, the hidden variables would have dynamical significance and laws of motion..." [3]. If, then, one of these hidden variables is allowed to be a Clifford algebra valued variable $\boldsymbol{\mu}$ (and this much the nonlocalists must allow if they are to remain in the game), then, inevitably, the resulting observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$, in addition to having operational attributes, would also inherit the Clifford-algebraic attributes from $\boldsymbol{\mu}$, and hence be non-commuting. *A priori*, then, there seems to be no reason for excluding non-commuting observables from all conceivable local realistic theories, so long as they do not contradict any other premises of Bell's theorem, or the predictions of quantum mechanics.

Could it be, however, that one of the premises of Bell's theorem, such as locality, actually precludes non-commuting observables from consideration, if only *a posteriori*? As we shall soon see, the answer to this question is: *No!* At least not *counterfactually* (in a sense that will become clear soon). In the following section—by proving the conditions of remote parameter independence and remote outcome independence [5]—we shall explicitly show that the model constructed in Ref.[6] is strictly locally causal, even at the level of the individual, uncontrollable, microstates.

III. LOCAL CAUSALITY OF THE MODEL AT THE LEVEL OF MICROSTATES

In the original formulation of his theorem [3], Bell considered a joint spin observable, such as the product $(A_{\mathbf{a}}\,B_{\mathbf{b}})$ in our notation, which is a single observable of an EPR-Bohm type pair of particles, but requiring two distinct operations for its measurement. Since in a deterministic hidden variable theory this observable should have a definite value, say $(A_{\mathbf{a}}\,B_{\mathbf{b}})(\lambda)$, for each microstate λ, Bell required the joint observable $(A_{\mathbf{a}}\,B_{\mathbf{b}})$ to satisfy the locality condition [14]

$$(A_{\mathbf{a}}\,B_{\mathbf{b}})(\lambda)\;=\;A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\,, \tag{9}$$

which states that the value of the product observable $(A_{\mathbf{a}}\,B_{\mathbf{b}})$ is necessarily equal to the product of the values of the two individual observables $A_{\mathbf{a}}$ and $B_{\mathbf{b}}$, if the hidden variable theory in question is to be locally causal in addition to being realistic and deterministic. Moreover, it is evident from the notation used in the above equation that, once the microstate λ is specified and the particles have separated, measurement outcomes of the local observable $A_{\mathbf{a}}$ do not depend on the remote parameter $\mathbf{b}$, but only upon the microstates λ and the local parameter $\mathbf{a}$, and likewise for the observable $B_{\mathbf{b}}$. In other words, the above factorizability condition requires that the actual values of the three observables $A_{\mathbf{a}}$, $B_{\mathbf{b}}$, and $(A_{\mathbf{a}}\,B_{\mathbf{b}})$—for all $\mathbf{a}$, $\mathbf{b}$, and λ—must satisfy the following relationship among themselves:

$$v(A,\,B)\;=\;A\,B\,, \tag{10}$$

where, respectively, $A = -1$, 0, or $+1$, and $B = -1$, 0, or $+1$ are the values that could be found if the observables $A_{\mathbf{a}}$ and $B_{\mathbf{b}}$ are measured in some experiment, and $v(A,\,B)$ is the corresponding value of the joint observable $(A_{\mathbf{a}}\,B_{\mathbf{b}})$.

For any deterministic local hidden variable theory, the factorizability condition (9) stated above is both necessary and sufficient to guarantee the local causality of the theory. Therefore our adaptation in Ref.[6] of Bell's original local realistic framework based on this condition is perfectly adequate. Unfortunately, the apparent non-commutativity of our observables has raised suspicion that perhaps this condition is not satisfied within our model after all, and perhaps there is some subtle form of nonlocality lurking beneath the surface, especially because the model reproduces the quantum mechanical correlations so exactly. Such a suspicion, however, as we shall soon see, is without merit.

To be sure, the non-commutativity of observables plays a central role in our model, and it is indeed hiding in the equation (1) above (or in the equation (17) of Ref.[6]). It can be made explicit by writing

$$[\,\boldsymbol{\mu}\cdot\mathbf{a},\,\boldsymbol{\mu}\cdot\mathbf{b}\,]\;=\;-\,2\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\;=\;-\,2\,(\boldsymbol{\mu}\cdot\mathbf{z})\,\sin\theta_{\mathbf{ab}}\,, \tag{11}$$

where $\theta_{\mathbf{ab}}$ is the angle from $\mathbf{a}$ to $\mathbf{b}$ about the unit direction $\mathbf{z}\equiv(\mathbf{a}\times\mathbf{b})/\sin\theta_{\mathbf{ab}}$. Admittedly, at first sight such a blatant non-commutativity does seem to be in conflict with the locality condition (10), since this condition implies

$$v(A,\,B)\,-\,v(B,\,A)\;=\;A\,B\,-\,B\,A\;=\;[\,A,\,B\,]\;=\;0. \tag{12}$$

This apparent conflict evaporates, however, as soon as it is appreciated that the numbers A and B here are the *actual* values of the observables $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$, whereas the observable $\boldsymbol{\mu}\cdot\mathbf{z}$ appearing on the RHS of the equation (11) can contribute to physics only *counterfactually.* This is because the direction $\mathbf{z}$ defining the observable $\boldsymbol{\mu}\cdot\mathbf{z}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and hence it is necessarily *exclusive* to at least one of them. Consequently, in any EPR-Bohm type experiment, simultaneous measurements of either the pair $\{\boldsymbol{\mu}\cdot\mathbf{a},\,\boldsymbol{\mu}\cdot\mathbf{z}\}$ or the pair $\{\boldsymbol{\mu}\cdot\mathbf{b},\,\boldsymbol{\mu}\cdot\mathbf{z}\}$ of observables would be *impossible.* That is to say, a measurement of any one member of one of the pairs at a chosen station would preclude the measurement of the second member of the same pair at the same station. This, in turn, means that simultaneous measurements of the observables $\boldsymbol{\mu}\cdot\mathbf{a}$, $\boldsymbol{\mu}\cdot\mathbf{b}$, and $\boldsymbol{\mu}\cdot\mathbf{z}$, even using both stations, would require experimental arrangements along at least two mutually exclusive directions, rendering the joint measurement of all three of them impossible. Simply put, as far as we know, a spin analyzer at a given station cannot be aligned to two mutually exclusive directions at the same time. And even if the inhabitants of some advanced civilization may manage to align their analyzers along two mutually exclusive Euclidean directions, the detector along one of these two directions would not respond, because the second particle would be near the remote end of the EPR experiment. In other words, for a given pair of particles, whenever the two observables, say $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$, are found to have non-zero values, such as the values A and B above, the only value the third observable $\boldsymbol{\mu}\cdot\mathbf{z}$ could possibly be found to have is *zero*, corresponding to non-detection, regardless of the station that could have been used to measure it. Thus, for any physically realizable experiment the RHS of the equation (11) would necessarily vanish. Consequently, in any actual experiment, the commutativity condition (12) is necessarily satisfied within our model, despite the manifest non-commutativity of our observables. Counterfactually, on the other hand, the RHS of the equation (11) plays a crucial role in generating the EPR-Bohm correlations within our model, as we shall see in section IV below.

Together with the analysis in Ref.[6], the above argument is adequate enough to reaffirm the local causality of our model. A deterministic, local realistic model such as ours, however, should be able to do better than relying on an operationally flavored argument for its consistency. Let us, therefore, investigate exactly how the local causality is maintained within our model, independently of what any experimental constraint forbids us to do. For this purpose, it is convenient to analyze the equation (11) above in terms of the conditions of remote parameter independence and remote outcome independence [5]. It is well known that Bell's locality condition (9) is equivalent to the conjunction of these two conditions. More significantly, these conditions allow us to demonstrate that the non-commutativity of the observables within our local model is quite harmless, and reflects merely the intrinsic geometrical features of rotations in the physical space. To this end, let us first focus on the condition of remote parameter independence.

A. Remote Parameter Independence within the Local Model

The deterministic version of this condition states that, for a given microstate $\boldsymbol{\mu}$, the outcome of an experiment at a local station **1** must not dependent on the chosen experimental settings at the remote station **2**, and vice versa. Now, our model is a *deterministic* hidden variable model. Nevertheless, it is convenient to state the *probabilistic* version of this condition in a symbolic form, because it makes the concepts involved more transparent:

$$P_{\boldsymbol{\mu}}(A\,|\,\mathbf{a},\,\mathbf{b}') \,=\, P_{\boldsymbol{\mu}}(A\,|\,\mathbf{a},\,\mathbf{b}) \tag{13}$$

$$\text{and}\quad P_{\boldsymbol{\mu}}(B\,|\,\mathbf{a}',\,\mathbf{b}) \,=\, P_{\boldsymbol{\mu}}(B\,|\,\mathbf{a},\,\mathbf{b}), \tag{14}$$

where $\mathbf{a}$ and $\mathbf{a}'$ and $\mathbf{b}$ and $\mathbf{b}'$ are mutually exclusive directions at stations $\mathbf{1}$ and $\mathbf{2}$, respectively. The question we now wish to address is whether the deterministic version of these conditions hold within our local model, despite the non-commutativity of observables expressed in equation (11). To answer this question, let us rewrite equation (11) as

$$[A_{\mathbf{a}}(\boldsymbol{\mu}),\,B_{\mathbf{b}}(\boldsymbol{\mu})] \,=\, -\,2\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\,, \tag{15}$$

and use the fact that the Clifford product is invertible in general. This allows us to express the equation as

$$A_{\mathbf{a}}(\boldsymbol{\mu}) \,=\, B_{\mathbf{b}}(\boldsymbol{\mu})\,A_{\mathbf{a}}(\boldsymbol{\mu})\,B^{-1}_{\mathbf{b}}(\boldsymbol{\mu}) \,-\, 2\,\{\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\}\,B^{-1}_{\mathbf{b}}(\boldsymbol{\mu})\,. \tag{16}$$

At first sight, it appears from this relation that—within our model—a remote observable such as $B_{\mathbf{b}}(\boldsymbol{\mu})$ can indeed influence a local observable $A_{\mathbf{a}}(\boldsymbol{\mu})$, and vice versa. In fact, despite the manifest functional independence from $\mathbf{b}$ of the observable $A_{\mathbf{a}}(\boldsymbol{\mu})$, it appears that even the choice of a remote parameter $\mathbf{b}$ can affect $A_{\mathbf{a}}(\boldsymbol{\mu})$ in a direct manner, thereby violating the basic principles of relativity. This, however, is not the case. Despite appearances, the observable $A_{\mathbf{a}}(\boldsymbol{\mu})$ cannot be influenced by either the remote parameter $\mathbf{b}$ or the remote outcome B. In fact, the above relation is simply a geometrical statement expressing exactly the opposite. In particular, it asserts that any changes in the parameter $\mathbf{b}$ on the RHS of equation (16)—such as from $\mathbf{b}$ to $\mathbf{b}'$—cannot affect the observable $A_{\mathbf{a}}(\boldsymbol{\mu})$ on its LHS. The situation is analogous to that in gauge invariance: A simultaneous change of "gauge" in the two "gauge-dependent" terms on the RHS of equation (16) does not affect their "gauge invariant" sum on the LHS. To appreciate this fact, let us note that the condition of remote parameter independence can be said to hold in our model if the LHS of equation (16) can be shown to be unaffected by the changes in the parameter $\mathbf{b}$ on its RHS, in line with the condition (13) above. That is to say, remote parameter independence is said to hold in our model if the following equality holds:

$$B_{\mathbf{b}'}(\boldsymbol{\mu})\,A_{\mathbf{a}}(\boldsymbol{\mu})\,B^{-1}_{\mathbf{b}'}(\boldsymbol{\mu}) \,-\, 2\,\{\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}')\}\,B^{-1}_{\mathbf{b}'}(\boldsymbol{\mu}) \,=\, B_{\mathbf{b}}(\boldsymbol{\mu})\,A_{\mathbf{a}}(\boldsymbol{\mu})\,B^{-1}_{\mathbf{b}}(\boldsymbol{\mu}) \,-\, 2\,\{\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\}\,B^{-1}_{\mathbf{b}}(\boldsymbol{\mu}). \tag{17}$$

This apparently complicated equality can be greatly simplified by using the relation $B^{-1}_{\mathbf{b}}(\boldsymbol{\mu}) \,=\, -\,B_{\mathbf{b}}(\boldsymbol{\mu})$ (which is true for any bivector), and then substituting the explicit expressions for the observables appearing in it. This gives

$$-\,\boldsymbol{\mu}\,\mathbf{b}'\,\boldsymbol{\mu}\,\mathbf{a}\,\boldsymbol{\mu}\,\mathbf{b}' \,+\, 2\,\boldsymbol{\mu}(\mathbf{a}\times\mathbf{b}')\,\boldsymbol{\mu}\mathbf{b}' \,=\, -\,\boldsymbol{\mu}\,\mathbf{b}\,\boldsymbol{\mu}\,\mathbf{a}\,\boldsymbol{\mu}\,\mathbf{b} \,+\, 2\,\boldsymbol{\mu}(\mathbf{a}\times\mathbf{b})\,\boldsymbol{\mu}\mathbf{b}\,. \tag{18}$$

By elementary manipulations and a use of the triple cross product identity, this equality can be further reduced to

$$\boldsymbol{\mu}\{\mathbf{b}'\,\mathbf{a}\,\mathbf{b}' \,-\, 2\,\mathbf{b}'(\mathbf{a}\cdot\mathbf{b}') \,+\, 2\,\mathbf{a}\} \,=\, \boldsymbol{\mu}\{\mathbf{b}\,\mathbf{a}\,\mathbf{b} \,-\, 2\,\mathbf{b}(\mathbf{a}\cdot\mathbf{b}) \,+\, 2\,\mathbf{a}\}\,, \tag{19}$$

which—*nota bene*—involves Clifford products of ordinary vectors. Now this equality would hold if the relation

$$\mathbf{b}'\,\mathbf{a}\,\mathbf{b}' \,-\, 2\,\mathbf{b}'(\mathbf{a}\cdot\mathbf{b}') \,=\, \mathbf{b}\,\mathbf{a}\,\mathbf{b} \,-\, 2\,\mathbf{b}(\mathbf{a}\cdot\mathbf{b}) \tag{20}$$

is satisfied. But this relation is indeed satisfied, because both sides of it are simply elementary geometric expressions of one and the same vector in the Euclidean space. To appreciate this, first note that $\mathbf{b}\,\mathbf{a}\,\mathbf{b}$ is simply a reflection of $\mathbf{a}$ across $\mathbf{b}$, in the plane defined by $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{b}\,(\mathbf{a}\cdot\mathbf{b})$ is the projection of $\mathbf{a}$ along $\mathbf{b}$ (cf. Fig. 9 of Ref.[10]). It is then easy to realize that the above equality does hold, with both sides being simply equal to $-\,\mathbf{a}$. Consequently, the equality (17) also holds, and hence the remote parameter independence within our model is indeed satisfied.

B. Remote Outcome Independence within the Local Model

But perhaps the remote parameter independence of the model was never in doubt. It is fairly obvious from the bivectorial definition of the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ that they have nothing to do with the remote contexts $\mathbf{b}$ and $\mathbf{a}$, respectively. Perhaps, then, the model is nonlocal in a more subtle manner, and violates somehow the condition of remote *outcome* independence, in a manner reminiscent of quantum mechanics. After all, it does reproduce the relevant predictions of quantum mechanics *exactly*. We shall soon see, however, that, on the contrary, our model strictly respects remote outcome independence, and does so unequivocally. To appreciate this fact, let us first recall the precise meaning of the condition of remote outcome independence. The deterministic version of this condition states that, for a given microstate $\boldsymbol{\mu}$, the outcome of an experiment at a local station **1** must not dependent on the outcome at the remote station **2**, and vice versa. Again, for the reasons of conceptual transparency, despite the fact that ours is a *deterministic* hidden variable model, we restate this condition symbolically in its *probabilistic* form:

$$P_{\boldsymbol{\mu}}(A=+1\,|\,\mathbf{a},\,\mathbf{b},\,B=+1) \;=\; P_{\boldsymbol{\mu}}(A=+1\,|\,\mathbf{a},\,\mathbf{b},\,B=-1)\,, \tag{21}$$

$$P_{\boldsymbol{\mu}}(A=-1\,|\,\mathbf{a},\,\mathbf{b},\,B=+1) \;=\; P_{\boldsymbol{\mu}}(A=-1\,|\,\mathbf{a},\,\mathbf{b},\,B=-1)\,, \tag{22}$$

and similar equalities with A and B interchanged. It is worth emphasizing here that this condition does not preclude correlations between the outcomes A and B at the two ends of an EPR experiment. Rather, it asserts that, given a complete state $\boldsymbol{\mu}$, the outcome at one end of the experiment provides no *additional* information concerning the outcome at the other end, and vice versa [5]. Again, the question we wish to address here is whether the deterministic versions of the above equalities hold within our local model. To answer this question, let us rewrite equation (11) again as

$$A_{\mathbf{a}}(\boldsymbol{\mu}) \;=\; -\,B_{\mathbf{b}}(\boldsymbol{\mu})\,A_{\mathbf{a}}(\boldsymbol{\mu})\,B_{\mathbf{b}}(\boldsymbol{\mu}) \;+\; 2\,C_{\mathbf{z}}(\boldsymbol{\mu})\,B_{\mathbf{b}}(\boldsymbol{\mu})\,\sin\theta_{\mathbf{ab}}\,, \tag{23}$$

where $C_{\mathbf{z}}(\boldsymbol{\mu}) := \boldsymbol{\mu}\cdot\mathbf{z}$, and we have again used the fact that $B^{-1}_{\mathbf{b}}(\boldsymbol{\mu}) \;=\; -\,B_{\mathbf{b}}(\boldsymbol{\mu})$. As we have already noted, at first sight this relation seems to imply that the measurement outcomes of the remote observable $B_{\mathbf{b}}(\boldsymbol{\mu})$ can influence those of the local observable $A_{\mathbf{a}}(\boldsymbol{\mu})$, and vice versa. Again, as a first step towards proving the contrary, we observe that the condition of remote outcome independence can be said to hold in our model if the LHS of equation (23) can be shown to be unaffected by the changes in the measurement outcomes of the observable $B_{\mathbf{b}}(\boldsymbol{\mu})$ on its RHS, in line with, say, the condition (21) above. That is to say, remote outcome independence is said to hold in our model if the equality

$$-\,B^{(+)}_{\mathbf{b}}(\boldsymbol{\mu})\,A^{(+)}_{\mathbf{a}}(\boldsymbol{\mu})\,B^{(+)}_{\mathbf{b}}(\boldsymbol{\mu}) + 2\,C^{(+)}_{\mathbf{z}}(\boldsymbol{\mu})\,B^{(+)}_{\mathbf{b}}(\boldsymbol{\mu})\,\sin\theta_{\mathbf{ab}} \;=\; -\,B^{(-)}_{\mathbf{b}}(\boldsymbol{\mu})\,A^{(+)}_{\mathbf{a}}(\boldsymbol{\mu})\,B^{(-)}_{\mathbf{b}}(\boldsymbol{\mu}) + 2\,C^{(-)}_{\mathbf{z}}(\boldsymbol{\mu})\,B^{(-)}_{\mathbf{b}}(\boldsymbol{\mu})\,\sin\theta_{\mathbf{ab}} \tag{24}$$

holds, together with the three other analogous equalities. Here the signs in parentheses over the observables indicate whether the corresponding bivector is rotating counterclockwise (+) or clockwise (−). Clearly, the signs over the observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ are determined by the choice of the equality, but it may not be obvious how the signs over the third observable $C_{\mathbf{z}}(\boldsymbol{\mu})$ have come about. They are, in fact, a result of the geometrical fact that the bivector corresponding to $C_{\mathbf{z}}(\boldsymbol{\mu})$ would be rotating counterclockwise (+) when the other two bivectors are rotating in the same sense (either both counterclockwise or both clockwise), and it would be rotating clockwise (−) when the other two are rotating in the opposite senses to each other. This is because $\boldsymbol{\mu}\cdot\mathbf{z}$ is not an independent bivector, but defined by the direction resulting from the cross product of the directions defining the other two bivectors. Thus, for example, when $A_{\mathbf{a}}(\boldsymbol{\mu})$ happens to be equal to $(+I)(+\mathbf{a})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ happens to be equal to $(-I)(+\mathbf{b}) = (+I)(-\mathbf{b})$, then $C_{\mathbf{z}}(\boldsymbol{\mu})$ would clearly be equal to $(+I)(-\mathbf{z}) = (-I)(+\mathbf{z})$, and hence it would be rotating in the clockwise sense about the direction $+\mathbf{z}$. In short, when $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$ are both spinning "up", then $\boldsymbol{\mu}\cdot\mathbf{z}$ would be spinning "up" as well, but when, say, $\boldsymbol{\mu}\cdot\mathbf{a}$ is spinning "up" and $\boldsymbol{\mu}\cdot\mathbf{b}$ is spinning "down", then $\boldsymbol{\mu}\cdot\mathbf{z}$ would be spinning down, along $+\mathbf{z}$, and so on.

Once these geometrical facts are appreciated, it is simply a matter of counting the signs to realize that the above equality does hold within our model, and so do the other three equalities. This proves that the observable $A_{\mathbf{a}}(\boldsymbol{\mu})$ and its measurement outcomes on the LHS of equation (23) are unaffected, not only by the measurement outcomes of the remote observable $B_{\mathbf{b}}(\boldsymbol{\mu})$, but also by whether or not the latter is actually measured [19]. This, in turn, means that the condition of remote outcome independence also holds within our model. What is more, by simultaneously changing the parameter $\mathbf{b}$ and the outcome B in the equation (23) above, it is easy to prove that the two independence conditions hold within our model not only separately, but also in conjunction. This, then, completes the purely geometric proof of the local causality of our model, at the level of individual microstates. As an immediate implication, the model allows a common cause explanation of the EPR-Bohm correlations, as we now proceed to demonstrate.

IV. STATISTICAL INTERPRETATION OF THE ENTANGLED SINGLET STATE

As is well known, Einstein wished to interpret the entangled quantum state as describing an ensemble of micro systems rather than an individual quantum system [20]. Such a statistical interpretation of the entangled quantum state would be justified if a quantitatively precise account of the predictions based on such a state can be provided, in terms of an ensemble of sub-quantum microstates. Here we shall provide such an account for the entangled singlet state. To be sure, the derivations of the equation (19) of Ref.[6] (or of the equation (5) of Ref.[17]) may well thought to be sufficient for such an account, but it is instructive to see how well the account holds up when extended to as intricate a scenario as an actual CHSH type experiment [7]. We shall see that the non-commutativity of observables displayed in the equation (11) above plays a crucial role in providing the quantitative aspects of this account.

To this end, let us reconsider the familiar CHSH string of expectation values [14]:

$$\mathcal{E}(\mathbf{a},\, \mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\, \mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}')\,. \tag{25}$$

For the microstates $\boldsymbol{\mu}$ of our local model this string can be rewritten using the notations of Ref.[6] as

$$\mathcal{E}(\mathbf{a},\, \mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\, \mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}') \,=\, \int_{\mathcal{V}_3} \mathcal{F}_{c.v.}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu})\,, \tag{26}$$

with the corresponding local realistic function $\mathcal{F}_{c.v.}(\boldsymbol{\mu})$ defined as

$$\mathcal{F}_{c.v.}(\boldsymbol{\mu}) := A_{\mathbf{a}}(\boldsymbol{\mu})\,\{\,B_{\mathbf{b}}(\boldsymbol{\mu}) + B_{\mathbf{b}'}(\boldsymbol{\mu})\,\} + A_{\mathbf{a}'}(\boldsymbol{\mu})\,\{\,B_{\mathbf{b}}(\boldsymbol{\mu}) - B_{\mathbf{b}'}(\boldsymbol{\mu})\,\}. \tag{27}$$

If we now use the fact that the observables $A^2_{\mathbf{a}}(\boldsymbol{\mu})$, $A^2_{\mathbf{a}'}(\boldsymbol{\mu})$, $B^2_{\mathbf{b}}(\boldsymbol{\mu})$, and $B^2_{\mathbf{b}'}(\boldsymbol{\mu})$ are all equal to -1 (because they are unit bivectors), then the square of the function $\mathcal{F}_{c.v.}(\boldsymbol{\mu})$ simplifies to (cf. Ref.[16])

$$\mathcal{F}^2_{c.v.}(\boldsymbol{\mu}) \,=\, 4 \,+\, [\,A_{\mathbf{a}}(\boldsymbol{\mu}),\, A_{\mathbf{a}'}(\boldsymbol{\mu})\,]\,[\,B_{\mathbf{b}'}(\boldsymbol{\mu}),\, B_{\mathbf{b}}(\boldsymbol{\mu})\,], \tag{28}$$

provided we assume that both A's commute with both B's, and vice versa:

$$[\,A_{\mathbf{n}}(\boldsymbol{\mu}),\, B_{\mathbf{n}'}(\boldsymbol{\mu})\,] \,=\, 0\,, \quad \forall\ \mathbf{n} \text{ and } \mathbf{n}'. \tag{29}$$

Using equation (18) of Ref.[6] it is easy to see that the vanishing of the above commutator does hold for our observables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$, at least upon averaging over $\boldsymbol{\mu}$, which is sufficient for the purposes of this section. Next, using the identity (1), the Clifford product of the local commutators appearing in equation (28) can be worked out as[1]

$$[\,A_{\mathbf{a}}(\boldsymbol{\mu}),\, A_{\mathbf{a}'}(\boldsymbol{\mu})\,]\,[\,B_{\mathbf{b}'}(\boldsymbol{\mu}),\, B_{\mathbf{b}}(\boldsymbol{\mu})\,] \,=\, 4\,[\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}')]\,[\,\boldsymbol{\mu}\cdot(\mathbf{b}'\times\mathbf{b})]. \tag{30}$$

Averaging this product over the uncontrollable microstates $\boldsymbol{\mu}$ (just as in equation (19) of Ref.[6]) then gives

$$\int_{\mathcal{V}_3} [\,A_{\mathbf{a}}(\boldsymbol{\mu}),\, A_{\mathbf{a}'}(\boldsymbol{\mu})\,]\,[\,B_{\mathbf{b}'}(\boldsymbol{\mu}),\, B_{\mathbf{b}}(\boldsymbol{\mu})\,]\; d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, -\,4\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})\,. \tag{31}$$

This relation is of course equivalent to averaging over both sides of the equation (28):

$$\int_{\mathcal{V}_3} \mathcal{F}^2_{c.v.}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, 4 + \int_{\mathcal{V}_3} [\,A_{\mathbf{a}}(\boldsymbol{\mu}),\, A_{\mathbf{a}'}(\boldsymbol{\mu})\,]\,[\,B_{\mathbf{b}'}(\boldsymbol{\mu}),\, B_{\mathbf{b}}(\boldsymbol{\mu})\,]\; d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, 4 \,-\, 4\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b}). \tag{32}$$

From this average, using the variance inequality

$$\left|\int_{\mathcal{V}_3} \mathcal{F}_{c.v.}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu})\right|^2 \,\leq\, \int_{\mathcal{V}_3} \mathcal{F}^2_{c.v.}(\boldsymbol{\mu})\;\, d\boldsymbol{\rho}(\boldsymbol{\mu})\,, \tag{33}$$

we finally arrive at the violations of the CHSH inequality within our local model:

$$|\,\mathcal{E}(\mathbf{a},\, \mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\, \mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}')\,| \,\leq\, \sqrt{4 \,+\, 4\,|\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})\,|\,} \,\leq\, 2\sqrt{2}\,. \tag{34}$$

This result is in quantitatively precise agreement with the corresponding prediction of quantum mechanics [16][21]. That there is such a precise agreement between the prediction of our local model and that of quantum mechanics was already demonstrated in Ref.[17]. To appreciate it further, let us calculate the expectation value in the singlet state of the quantum mechanical analogue of the equation (28), by means of the so-called Bell operator:

$$\mathcal{B}_{op} \,=\, \boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b} \,+\, \boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}' \,+\, \boldsymbol{\sigma}_1\cdot\mathbf{a}'\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b} \,-\, \boldsymbol{\sigma}_1\cdot\mathbf{a}'\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}'. \tag{35}$$

The square of this operator is of course well known [22], and works out to be

$$\mathcal{B}^2_{op} \,=\, 4\,\mathbb{1} \,+\, 4\,\{\boldsymbol{\sigma}_1\cdot(\mathbf{a}{\times}\mathbf{a}')\}\otimes\{\boldsymbol{\sigma}_2\cdot(\mathbf{b}{\times}\mathbf{b}')\} \tag{36}$$

(where $\mathbb{1}$ is the identity operator), with its expectation value in the entangled singlet state being

$$\langle\,\mathcal{B}^2_{op}\,\rangle \,=\, 4 \,+\, 4\,\langle\,\{\boldsymbol{\sigma}_1\cdot(\mathbf{a}{\times}\mathbf{a}')\}\otimes\{\boldsymbol{\sigma}_2\cdot(\mathbf{b}{\times}\mathbf{b}')\}\,\rangle \,=\, 4 \,-\, 4\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}\times\mathbf{b}'). \tag{37}$$

Once again using the variance inequality, we arrive at the quantum mechanical violations of the CHSH inequality [16]:

$$|\,\langle\,\mathcal{B}_{op}\,\rangle\,| \,\leq\, \sqrt{4 \,+\, 4\,|\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}\times\mathbf{b}')\,|\,} \,\leq\, 2\sqrt{2}\,. \tag{38}$$

Comparing this inequality with the local realistic inequality (34) we see that the agreement between the statistical predictions of quantum mechanics and those of our local model is both exact and complete. Thus, our local model unequivocally endorses the statistical interpretation of the entangled singlet state, as anticipated by Einstein [20].

[1] Equations (30) and (31) were first worked out by Michael Seevinck in a private correspondence regarding Ref.[6].

V. DERIVATION OF THE MALUS'S LAW FOR SEQUENTIAL SPIN MEASUREMENTS:

For the above statistical interpretation of the entangled singlet state to be valid beyond doubt, we must also show that Malus's law is respected within our local model, since it is respected within quantum mechanics. In this section we show that this law holds within our local model just as exactly as it does within quantum mechanics.

Suppose we have a subensemble of spin one-half particles, prepared with a definite spin by a polarizer, which we denote by a unit vector $\mathbf{p}$. Suppose next we want to calculate the expected value of the result of a measurement of the spin component $\boldsymbol{\sigma}\cdot\mathbf{a}$, where $\mathbf{a}$ is a unit direction along an analyzer. Without loss of generality, we shall choose the spins to be selected "up" along the direction $\mathbf{p}$, with the value $s_{\mathbf{p}} = +1$. In the language of our model this selection can be stated as $\boldsymbol{\mu}\cdot\mathbf{p} = +I\,\mathbf{p}$, which means that the subensemble of spins prepared by means of projections of the microstates $\boldsymbol{\mu}$ along the direction $\mathbf{p}$ will all be spinning "up" in the counterclockwise sense. Then, what we want to calculate is the expected value of finding the spin along the direction $\mathbf{a}$. Since we already know the result of this calculation from quantum mechanics, we can rephrase our question in the local realistic terms as follows [23]:

$$\int_{\mathcal{V}_3} A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})\;\,d\boldsymbol{\rho}(\boldsymbol{\mu}) \;\overset{?}{=}\; \langle\, s_{\mathbf{p}}=+1\,|\,\boldsymbol{\sigma}\cdot\mathbf{a}\,|\,s_{\mathbf{p}}=+1\,\rangle \;=\; \mathbf{a}\cdot\mathbf{p}\,, \tag{39}$$

where the RHS is the quantum mechanical expectation value, and the LHS is the average over the microstates of a dichotomic observable $A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu}) := (\boldsymbol{\mu}\cdot\mathbf{a})\;(\boldsymbol{\mu}\cdot\mathbf{p})$. The latter is of course a product of two unit bivectors, producing a quaternion of unit norm, and, as we discussed in sections II and III, can only yield binary values ± 1 in any actual observation. Thus, it can serve as a dichotomic observable for the purpose at hand. Now, since, as a physical variable, $A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})$ has been designed to be sensitive to what happens to the particles at the polarizer, the initial orientations of the bivectors $\boldsymbol{\mu}\cdot\mathbf{a}$ along the direction $\mathbf{a}$ will not be evenly balanced between $+I\,\mathbf{a}$ and $-I\,\mathbf{a}$. Indeed, by preparation, the spins within the initial ensemble have been filtered out to be spinning only "up" along the direction $\mathbf{p}$, which may not be equal to the direction $\mathbf{a}$ in general, and hence initially the spins will not be spinning "up" along the direction $\mathbf{a}$, unless, of course, $\mathbf{a}=\mathbf{p}$. In other words, the initial probability of finding the spin "up" along the direction $\mathbf{a}$ is strictly zero. Consequently, in analogy with Bell's own derivation of the Malus's law (cf. Ref.[3], Eqs. (4) to (7)), where he preselects an appropriate subensemble of spins with definite polarization by imposing the condition $\boldsymbol{\lambda}\cdot\mathbf{p}>0$ on his "hidden" observable), we must preselect an appropriate subensemble for our scenario by assigning the weight *zero* to the initial probability of orientations $(\pm\,I\,\mathbf{a})\;(-\,I\,\mathbf{p})$ and $(+\,I\,\mathbf{a})\;(+\,I\,\mathbf{p})$ of the variable $A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})$, before taking the average over the microstates $\boldsymbol{\mu}$. This, in turn, means—equivalently—that we must assign the weight *one* to the initial probability of orientation $(-\,I\,\mathbf{a})\;(+\,I\,\mathbf{p})$ of the variable $A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})$. With these physical constraints in place, the net expected value—as prescribed in equation (39) above—can be easily calculated as follows:

$$\begin{aligned}\int_{\mathcal{V}_3} A(\mathbf{a},\,\mathbf{p},\,\boldsymbol{\mu})\;\,d\boldsymbol{\rho}(\boldsymbol{\mu}) \;&=\; \int_{\mathcal{V}_3} (-\,I\,\mathbf{a})\;(+\,I\,\mathbf{p})\;\,d\boldsymbol{\rho}(\boldsymbol{\mu}) \;=\; -\,I^2\int_{\mathcal{V}_3} \mathbf{a}\,\mathbf{p}\;\,d\boldsymbol{\rho}(\boldsymbol{\mu})\\ &=\int_{\mathcal{V}_3} \mathbf{a}\,\mathbf{p}\;\,d\boldsymbol{\rho}(\boldsymbol{\mu}) \;=\; \mathbf{a}\cdot\mathbf{p} \,+\, \int_{\mathcal{V}_3} \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{p})\;\,d\boldsymbol{\rho}(\boldsymbol{\mu})\\ &=\mathbf{a}\cdot\mathbf{p}\,+\,0\,,\end{aligned} \tag{40}$$

where the second and third lines follow from a use of the equations (17) to (19) of Ref.[6]. Needless to say, we would arrive at the same result for the case where the initial subensemble is preselected to be "spin down" along $\mathbf{p}$, with the value $s_{\mathbf{p}} = -1$ and orientation $(+\,I\,\mathbf{a})\;(-\,I\,\mathbf{p})$. This makes it evident that the quantum version of the Malus's law,

$$\langle\, s_{\mathbf{p}}=\pm 1\,|\,\boldsymbol{\sigma}\cdot\mathbf{a}\,|\,s_{\mathbf{p}}=\pm 1\,\rangle \;=\; \mathbf{a}\cdot\mathbf{p}\,, \tag{41}$$

holds *exactly* within our model. Moreover, the foregoing steps from polarizer to analyzer can be repeated at will for a sequence of measurements. For this purpose a new state of the system for each subsequent measurement must be prepared (cf. Ref.[23]). We shall again assume that the apparatus only lets through the "spin up" particles. Then, unlike the complicated procedure that must be followed to prepare the new state in the case of Bell's model [23], in our case all one has to do is to rotate the polarizer in the direction of the analyzer until the new direction of the polarizer becomes $\mathbf{p}'=\mathbf{a}$. In other words, after the first measurement, all one has to do is to prescribe that the apparatus defines $\mathbf{p}'=\mathbf{a}$ as the new polarization direction. Then the new distribution of hidden variables will be identical to the one before the first measurement, apart from being rotated to the new direction $\mathbf{p}'$. A second measurement by an analyzer $\mathbf{a}'$ on a similarly preselected subensemble $(-\,I\,\mathbf{a}')\;(+\,I\,\mathbf{p}')$ will then yield the expected value

$$\int_{\mathcal{V}_3} A(\mathbf{a}',\,\mathbf{p}',\,\boldsymbol{\mu})\;\,d\boldsymbol{\rho}(\boldsymbol{\mu}) \;=\; \langle\, s_{\mathbf{p}'}=\pm 1\,|\,\boldsymbol{\sigma}\cdot\mathbf{a}'\,|\,s_{\mathbf{p}'}=\pm 1\,\rangle \;=\; \mathbf{a}'\cdot\mathbf{p}', \tag{42}$$

again in agreement with the prediction of quantum mechanics. Thus we see that the quantum mechanical predictions for the sequential measurements of spin-components along arbitrary directions are duly reproduced by our model.

VI. CONCLUDING REMARKS

Contrary to the received wisdom, Bell's theorem is not a threat to local realism. Neither is it a curb on determinism. The counterexample constructed in Ref.[6] provides a fully deterministic, common cause explanation of the EPR-Bohm correlations. In fact, it is hard to imagine a more simple common cause than the one on which the counterexample is based—namely, the intrinsic freedom of choice in the initial orientation of the orthogonal directions in the Euclidean space. In the present paper we have further consolidated the conclusions of Ref.[6] by demonstrating that the exact, locally causal model for the EPR-Bohm correlations constructed therein satisfies at least eight essential requirements, arising from either the predictions of quantum mechanics or the premises of Bell's theorem. These requirements, as listed in the Introduction, include the locality condition of Bell, and hence by respecting them our model fully endorses the view that the quantum mechanical description of reality is incomplete [2]. Moreover, since this view is reinforced by three different local realistic derivations of the violations of the CHSH inequality [6][17], and since all three of them agree with the corresponding predictions of quantum mechanics in *quantitatively* precise manner, the statistical interpretation of the entangled singlet state becomes the most natural interpretation of this state, as anticipated by Einstein. It is therefore hoped that—strengthened by the results of the present paper—the counterexample of Ref.[6] would rejuvenate the search for a unified, locally causal basis for the whole of physics, as envisaged by Einstein [20].

Acknowledgments

This paper has been inspired and shaped by the largely skeptical questions and comments concerning Ref.[6] from many friends and colleagues, including Paul Busch, Marek Czachor, Laurent Freidel, Richard Gill, Lucien Hardy, Marc Holman, Matthew Leifer, Owen Maroney, Peter Morgan, Marcin Pawlowski, Markus Penz, Kevin Resch, Michael Seevinck, Abner Shimony, Lee Smolin, Rafael Sorkin, Ward Struyve, Chris Timpson, Gregor Weihs, and Hans Westman. The research for this paper was carried out during a visiting appointment at the Perimeter Institute for Theoretical Physics, Canada, whose generous hospitality and support are gratefully acknowledged.

References

[1] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] J. S. Bell, Physics **1**, 195 (1964).
[4] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990); L. Hardy, Phys. Rev. Lett. **71**, 1665 (1993).
[5] A. Shimony, in Stanford Encyclopedia of Philosophy, URL http://plato.stanford.edu/entries/bell-theorem/
[6] J. Christian, *Disproof of Bell's Theorem by Clifford Algebra Valued Local Variables*, arXiv:quant-ph/0703179
[7] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[8] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).
[9] J. S. Bell, in *Foundations of Quantum Mechanics: Proceedings of the International School of Physics 'Enrico Fermi'*, Course IL, edited by B. d'Espagnat (Academic Press, New York, 1971) pp 171-181.
[10] D. Hestenes, Am. J. Phys, **71**, 104 (2003).
[11] D. Hestenes, *New Foundations for Classical Mechanics*, Second Edition (Kluwer, Dordrecht, 1999); T. G. Vold, Am. J. Phys, **61**, 491 (1993); D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordrecht, 1984).
[12] L. Dorst, D. Fontijne, and S. Mann, *Geometric Algebra for Computer Science* (Morgan Kaufmann, San Francisco, 2007).
[13] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[14] J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).
[15] J. S. Bell, in *Between Science and Technology*, edited by A. Sarlemijn and P. Kroes (Elsevier, Amsterdam, 1990).
[16] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980); L. J. Landau, Phys. Lett. A **120**, 54 (1987).
[17] J. Christian, *Disproof of Bell's Theorem: Reply to Critics*, arXiv:quant-ph/0703244
[18] J. Lasenby, A. N. Lasenby, and C. J. L. Doran, Phil. Trans. R. Soc. Lond. A **358**, 21 (2000).
[19] A. Einstein, Dialectica **2**, 320 (1948).
[20] A. Einstein, *Physics and Reality*, in The Journal of the Franklin Institute, **221**, No. 3 (March, 1936).
[21] M. Seevinck and J. Uffink, arXiv:quant-ph/0703134
[22] S. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).
[23] J. F. Clauser, Am. J. Phys. **39**, 1095 (1971).