On Minkowskian Branching Structures

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Abstract

We introduce the notion of a Minkowskian Branching Structure ("MBS" for short). Then we prove some results concerning the phenomenon of funny business in its finitary and infinitary variants.

1 Branching Space-Times

The theory of Branching Space-Times (BST), as presented by Nuel Belnap in 1992 ([2]), combines objective indeterminism and relativity in a rigorous way. Its primitives are a nonempty set W (called "Our World", interpreted as the set of all possible point events) and a partial ordering $\leq$ on W, interpreted as a "causal order" between point events.

There are no "Possible Worlds" in this theory; there is only one world, Our World, containing all that is (timelessly) possible. Instead, a notion of "history" is used, as defined below:

Definition 1 *A set $h \subseteq W$ is upward-directed iff $\forall e_1, e_2 \in h\ \exists e \in h$ such that $e_1 \leq e$ and $e_2 \leq e$.*

A set h is maximal with respect to the above property iff $\forall g \in W$ such that $g \supsetneq h$ g is not upward-directed.

A subset h of W is a history iff it is a maximal upward-directed set.

A very important feature of BST is that histories are closed downward: if $e_1 \leq e_2$ and $e_1 \notin h$, then $e_2 \notin h$. In other words, there is no backward branching among histories in BST. No two incompatible events are in the past of any event; equivalently: the past of any event is "fixed", containing only compatible events.

We will now give the definition of a BST model; for more information about BST in general see [1].

Definition 2 *$\langle W, \leq \rangle$ where W is a nonempty set and $\leq$ is a partial ordering on W is a model of BST if and only if it meets the following requirements:*

1. *The ordering $\leq$ is dense.*
2. *$\leq$ has no maximal elements.*
3. *Every lower bounded chain in W has an infimum in W.*
4. *Every upper bounded chain in W has a supremum in every history that contains it.*
5. *(Prior choice principle) For any lower bounded chain $O \in h_1 - h_2$ there exists a point $e \in W$ such that e is maximal in $h_1 \cap h_2$ and $\forall e' \in O\ e < e'$.*

2 Introducing Minkowskian Branching Structures

In different models of BST histories can be space-times with various metrics (or even with no metrics). What we would like to call a Minkowskian Branching Structure ("MBS"[1] for short) is a model of BST in which histories are as close as possible to the Minkowski space-time. Apart from the standard metric, this approach will provide us with a straightforward notion of an instant. This part of our work is based on Müller's theory from [4].

The points of the Minkowskian space-time are elements of $\mathbb{R}^4$, e.g. $x = \langle x^0, x^1, x^2, x^3 \rangle$, where the first element of the quadruple is the time coordinate. The Minkowskian space-time distance is a function $D^2_M : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ defined as follows (for $x, y \in \mathbb{R}^4$):

$$D^2_M(x, y) := -(x^0 - y^0)^2 + \sum_{i=1}^{3} (x^i - y^i)^2 \tag{1}$$

The natural ordering on the Minkowski space-time, call it "Minkowskian ordering $\leq_M$", is defined as follows ($x, y \in \mathbb{R}^4$):

$$x \leq_M y \text{ iff } D^2_M(x, y) \leq 0 \text{ and } x^0 \leq y^0 \tag{2}$$

We will say that two points $x, y \in \mathbb{R}^4$ are space-like related ("SLR" for short) iff neither $x \leq_M y$ nor $y \leq_M x$. Naturally, $x <_M y$ iff $x \neq y$ and $x \leq_M y$.

[1] Although the structure we present here bases on work of Müller, he has never used the term "Minkowskian Branching Structure" in print.

Now we need to provide a framework for "different ways in which things can happen" and for filling the space-times with content. For the first task we will need a set Σ of labels $\sigma, \eta, ...$. (In contrast to Müller ([4]), we allow for any cardinality of Σ). For the second task, we will use a so called "state" function $S : \Sigma \times \mathbb{R}^4 \to P$, where P is a set of point properties (on this we just quote Müller saying "finding out what the right P is is a question of physics, not one of conceptual analysis").

One could ask about the reasons for an extra notion of a "scenario". Why don't we start immediately with "histories"? This is equivalent to the question: Why don't we build histories out of points from $\mathbb{R}^4 \times P$? The reason is that a member of BST's Our World has a fixed past. If two different trains of events lead to exactly the same event $E \in \mathbb{R}^4 \times P$, the situation gives rise to two different point events, two different members of W. In contrast, states can reconverge: for a point $\langle x, p_0 \rangle$ from $\mathbb{R}^4 \times P$ there can exist two different points $\langle y, p_1 \rangle$ and $\langle y, p_2 \rangle$ from $\mathbb{R}^4 \times P$ such that $y <_M x$. If scenarios were histories, this would, as a case of backward branching, contradict the fact that histories are closed downward - so the set $\mathbb{R}^4 \times P$ is not a good candidate for the set of the "building blocks" of the MBS version of W.

The idea behind the concept of scenario is that every scenario corresponds to a $\mathbb{R}^4$ space filled with content, where the content derives from the elements of P. Assuming a certain state function S is given, for any $\sigma, \eta \in \Sigma$ the set $C_{\sigma\eta} \subset \mathbb{R}^4$ is the set of "splitting points" between scenarios σ and η, intuitively: the set of points in which a choice between the two scenarios is made. All members of $C_{\sigma\eta}$ have to be space-like related. Of course a choice between σ and η is a choice between η and σ, so $C_{\sigma\eta} = C_{\eta\sigma}$. In the former section we have mentioned a BST postulate of historical connection: any two different histories have a nonempty intersection. We take over this idea by saying that any two different scenarios must split at some point, which is equivalent to saying that they share a common root. Formally: $\forall \sigma, \eta \in \Sigma \; (\sigma \neq \eta \Rightarrow C_{\sigma\eta} \neq \emptyset)$.

The next requirement considers triples of scenarios. Any set $C_{\sigma\eta}$ determines a region in which both scenarios coincide: namely, that part of $\mathbb{R}^4$ that is not in the Minkowskian sense strictly above any point from $C_{\sigma\eta}$. Following Müller we call it the region of overlap $R_{\sigma\eta}$ between scenarios σ, η defined as below:

$$R_{\sigma\eta} := \{x \in \mathbb{R}^4 | \neg \exists y \in C_{\sigma\eta} \; y <_M x\} \tag{3}$$

(Of course it follows that for any $\sigma, \eta \in \Sigma C_{\sigma\eta} \subseteq R_{\sigma\eta}$.) Assuming the sets $C_{\sigma\eta}$ and $C_{\eta\gamma}$ are given, we get two regions of overlaps $R_{\sigma\eta}$ and $R_{\eta\gamma}$. At the points in the intersection of those two regions σ coincides with η and η coincides with γ, therefore by transitivity of coincidence σ coincides with γ. In general we can say that for any $\sigma, \eta, \gamma \in \mathbb{R}^4$

$$R_{\sigma\gamma} \supseteq R_{\sigma\eta} \cap R_{\eta\gamma} \tag{4}$$

which translated to a requirement on sets of splitting points is

$$\forall x \in C_{\sigma\gamma} \exists y \in C_{\sigma\eta} \cup C_{\eta\gamma} y \leq_M x. \tag{5}$$

In his paper Müller put another requirement on $C_{\sigma\eta}$: finitude. The motivation was to exclude splitting along a "simultaneity slice". The strong requirement of finitude excludes however many more types of situations, in which splitting is not continuous or happens in a region of space-time of a finite diameter. In the present paper we drop this requirement, not putting any restrictions on the cardinality of $C_{\sigma\eta}$ for any $\sigma, \eta \in \Sigma$.

Each state function assigns to each pair $\langle$a label from Σ, a point from $\mathbb{R}^4\rangle$ an element of P. Colloquially, the state functions tells us what happens at a certain point of the space-time in a given scenario. We can look at the situation from a slightly different perspective: every label σ is assigned a mapping S_σ from $\mathbb{R}^4$ to P.

We now proceed to construct the elements of MBS version of Our World; they will be equivalence classes of a certain relation $\leq_S$ on $\Sigma \times \mathbb{R}^4$. For convenience, we write the elements of $\Sigma \times \mathbb{R}^4$ as x_σ where $x \in \mathbb{R}^4, \sigma \in \Sigma$. The idea is to "glue together" points in regions of overlap; hence the relation is defined as below:

$$x_\sigma \equiv_S y_\eta \text{ iff } x = y \text{ and } x \in R_{\sigma\eta} \tag{6}$$

Müller provides a simple proof of the fact that $\equiv_S$ is an equivalence relation on $\Sigma \times \mathbb{R}^4$; therefore we can produce a quotient structure. The result is the set B being the MBS version of Our World:

$$B := (\Sigma \times \mathbb{R}^4)/\equiv_S \; = \{[x_\sigma] | \sigma \in \Sigma, x \in \mathbb{R}^4\}. \tag{7}$$

where $[x_\sigma]$ is the equivalence class of x with respect to the relation $\equiv_S$:

$$[x_\sigma] = \{x_\eta | x_\sigma \equiv_S x_\eta\}. \tag{8}$$

Next, we define a relation $\leq_S$ on B:

$$[x_\sigma] \leq_S [y_\eta] \text{ iff } x \leq_M y \text{ and } x_\sigma \equiv_S x_\eta \tag{9}$$

which (as Müller shows) is a partial ordering on B.

The goal would now be to prove that $\langle B, \leq_S\rangle$ is a model of BST. To do so, and in particular to prove the prior choice principle and requirement no. 4 from definition 2, we need to know more about the shape of the histories in MBS - that they are the intended ones.

2.1 The shape of MBS histories

We would like histories, that is: maximal upward-directed sets, to be sets of equivalence classes $[x_\sigma]$ (with respect to $\equiv_S$) for $x \in \mathbb{R}^4$ for some $\sigma \in \Sigma$. In other words, we wish to be able to identify a history just by specifying a scenario to which it is assigned. This is Müller's Lemma 3 and our

Theorem 3 *Every history in a given MBS is of the form* $h = \{[x_\sigma] | x \in \mathbb{R}^4\}$ *for some* $\sigma \in \Sigma$.

The problem is that, aside from minor brushing up required by the proof of the "right" direction, the proof of the "left" direction supplied in [4] needs to be fixed as it does not provide adequate reasons for nonemptiness of an essential intersection $\bigcap \Sigma_h(z_i)$. More on that below. Let us divide the above theorem into two lemmas (4 and 8) corresponding to the directions and prove the "right" direction first. Until we prove the theorem we refrain from using the term "history" and substitute it with a "maximal upward-directed set" for clarity.

Lemma 4 *If* h *is of the form* $h = \{[x_\sigma] | x \in \mathbb{R}^4\}$ *for some* $\sigma \in \Sigma$ *than* h *is a maximal upward-directed subset of* B.

Proof: Let us consider $e_1, e_2 \in h, e_1 = [x_\sigma], e_2 = [y_\sigma]$. Since $x, y \in \mathbb{R}^4$ there exists a $z \in \mathbb{R}^4$ such that $x \leq_M z$ and $y \leq_M z$. Therefore $[x_\sigma] \leq_S [z_\sigma]$ and $[y_\sigma] \leq_S [z_\sigma]$, and so h is upward-directed.

For maximality, consider a $g \subseteq B, g \supsetneq h$ and assume g is upward-directed. It follows that there exists a point $[x_\eta] \in g - h$ such that $[x_\eta] \neq [x_\sigma] \in h$. Since both points belong to g which is upward-directed, there exists $[z_\alpha] \in g$ (note that we are not allowed to choose σ as the index at that point) such that $[x_\eta] \leq_S [z_\alpha]$ and $[x_\sigma] \leq_S [z_\alpha]$. Therefore $x_\eta \equiv_S x_\alpha \equiv_S x_\sigma$, and so we arrive at a contradiction by concluding that $[x_\eta] = [x_\sigma]$. **Q.E.D.**

The proof of the other direction is more complex and, what might be surprising, involves a topological postulate. First, we will need a simple definition:

Definition 5 *For a given maximal upward-directed set* h *and a point* $x \in \mathbb{R}^4$, $\Sigma_h(x) := \{\sigma \in \Sigma | [x_\sigma] \in h\}$.

Consider now a given maximal upward-directed set $h \subseteq B$. With every lower bounded chain $L \subset \mathbb{R}^4$ we would like to associate a topology (called "chain topology") on the set of $\Sigma_h(\inf(L))$. We define the topology by describing the whole family of closed sets, which is equal to $\{\emptyset, \Sigma_h(\inf(L))\} \cup$

$\{\Sigma_h(l)|l \in L\} \cup \{\cap\{\Sigma_h(l)|l \in L\}\}$. (Because L is a chain it is evident that the family is closed with respect to intersection and finite union). The postulate runs as follows:

Postulate 6 *For every maximal upward-directed set $h \subseteq B$ and for every lower bounded chain $L \subset \mathbb{R}^4$ the "chain topology" described above is compact.*

It is easily verifiable that in such a topology $\{\Sigma_h(l)|l \in L\}$ is a centred family of closed sets (every finite subset of it has a nonempty intersection). Together with the above postulate we get a

Corollary 7 *For every maximal upward-directed set $h \subseteq B$ and for every chain $L \subset \mathbb{R}^4$, $\bigcap\{\Sigma_h(l)|l \in L\} \neq \emptyset$.*

Lemma 8 *If h is a maximal upward-directed subset of B then h is of the form $h = \{[x_\sigma]|x \in \mathbb{R}^4\}$ for some $\sigma \in \Sigma$.*

To turn next to the proof, its structure mimics proof of Müller's (see [4]). It is divided into three parts, the first and the last being reproduced here. On the other hand, the second part contains an error (as stated above, the statement that $\bigcap \Sigma_h(z_i) \neq \emptyset$ is not properly justified) and bears on an assumption that for every history h and point $x \in \mathbb{R}^4$ the set $\Sigma_h(x)$ is at most countably infinite. We wish both to drop this assumption and correct the proof using the above topological postulate.

Proof: Suppose that h is a maximal upward-directed subset of B. In order to prove the lemma, we will prove the following three steps:

1. If for some $\sigma, \eta \in \Sigma$ both $[x_\sigma] \in h$ and $[x_\eta] \in h$, then $x_\sigma \equiv_S x_\eta$.
2. There is a $\sigma \in \Sigma$ such that for every η, if $[x_\eta] \in h$, then $x_\eta \equiv_S x_\sigma$.
3. With the σ from step 2, $h = \{[x_\sigma]|x \in \mathbb{R}^4\}$.

Ad. 1. Since h is maximal by assumption, there exists a $[y_\gamma] \in h$ such that $[x_\sigma] \leq_S [y_\gamma]$ and $[x_\eta] \leq_S [y_\gamma]$. These last two facts imply that $x_\sigma \equiv_S x_\gamma \equiv_S x_\eta$, so by transitivity of $\leq_S$ we get $x_\sigma \equiv_S x_\eta$.

Ad. 2. Assume the contrary: $\forall\sigma \in \Sigma\ \exists[x_\eta] \in h, x_\eta \not\equiv_S x_\sigma$.

Take a point $[y_\alpha] \in h$. Accordingly, $\Sigma_h(y) \neq \emptyset$. If $\sigma \notin \Sigma_h(y)$ then $[y_\sigma] \notin h$, so in particular $[y_\sigma] \neq [y_\alpha]$. Therefore in our search for the "proper" scenario needed by the lemma we can confine ourselves to the set $\Sigma_h(y)$ only.

For each scenario $\sigma_\alpha \in \Sigma_h(y)$ we define a set $\Theta_\alpha = \{[x_\eta] \in h|x \in \mathbb{R}^4, \eta \in \Sigma_h(y), x_{\sigma_\alpha} \not\equiv_S x_\eta\}$, which by our assumption is never empty. Colloquially, it is a set of the points that make the scenario a wrong candidate for the proper scenario from our lemma - the scenario "doesn't fit" the history at those points. For each scenario σ_α we would like to choose a single element of Θ_α, and to that end we employ a choice function S defined on the set of subsets of $\{[x_\eta] | x \in \mathbb{R}^4, \eta \in \Sigma_h(y)\}$ (any Θ_α is an example of such a subset) such that $S(\Theta_\alpha) \in \Theta_\alpha$, naming the element chosen by it as follows: $S(\Theta_\alpha) := [x_\alpha \eta_\alpha]$. From the above construction we get that $[x_\alpha \eta_\alpha] \in h$ and $x_\alpha \eta_\alpha \not\equiv_S x_\alpha \sigma_\alpha$.

Observe that we will arrive at a contradiction if we prove that

$$\bigcap_{\sigma_\alpha \in \Sigma_h(y)} \Sigma_h(x_\alpha) \neq \emptyset \tag{10}$$

(since for any $\sigma_\beta \in \Sigma_h(y)$ $\sigma_\beta \notin \Sigma_h(x_\beta)$). We will construct a vertical chain $L = \{[z_0\gamma_0], [z_1\gamma_1], \ldots ,[z_\omega\gamma_\omega], \ldots\}$ of points in h. We want it to be vertical in order for it (in case it does not have an upper bound itself) to contain an upper bound of any point in B. First, we define a function "sup" which given two points $[a_\sigma], [b_\eta] \in B$ will produce a point $c \in \mathbb{R}^4$ such that c has the same spatial coordinates as a but is above b. In other words, if $x = \langle x^0, x^1, x^2, x^3 \rangle \in \mathbb{R}^4$, $y = \langle y^0, y^1, y^2, y^3 \rangle \in \mathbb{R}^4$, $[x_\alpha], [y_\beta] \in B$, $\sup([x_\alpha], [y_\beta]) := \langle x^0 + (\sum_1^3 (x^i - y^i)^2)^{1/2},\ x^1, x^2, x^3 \rangle \in \mathbb{R}^4$. Notice that sup is not commutative.

We proceed to define the above mentioned chain L in the following way:

1.$[z_0\gamma_0] = [\sup([y_\alpha], [x_0\eta_0])\gamma_0]$.

$[z_1\gamma_1] = [\sup([z_0\gamma_0], [x_1\eta_1])\gamma_1]$.

Generally, $[z_{\sigma+1}\gamma_{\sigma+1}] = [\sup([z_\sigma\gamma_\sigma], [x_{\sigma+1}\eta_{\sigma+1}])\gamma_{\sigma+1}]$

2. Suppose ρ is a limit number. Define $A_\rho := \{[z_\beta\gamma_\beta] \in h | \gamma_\beta \in \Sigma_h(y), \beta < \rho\}$. We need to distinguish two cases:

a) A_ρ is upper bounded with respect to $\leq_S$. Then it has to have "vertical" upper bounds $[t_\delta]$ with spatial coordinates $t^i = z_0^i$ $(i = 1, 2, 3)$. In this case, we employ the above defined function S to choose one of those upper bounds:

$$S(\{[t_\delta] \in h \mid \forall \beta < \rho\ [z_\beta\gamma_\beta] \leq_S [t_\delta] \wedge t^i = z_0^i (i = 1, 2, 3)\}) := [t_\rho\gamma_\rho]. \tag{11}$$

Then we put $z_\rho := \sup([t_\rho\gamma_\rho], [x_\rho\eta_\rho])$, arriving at $[z_\rho\gamma_\rho]$ as the next element of our chain L.

b) if A_ρ is not upper bounded with respect to $\leq_S$, the set

$$B_\rho = \{[t_\delta] \in A_\rho | [x_\rho\gamma_\rho] \leq_S [t_\delta]\} \tag{12}$$

is not empty (because A_ρ is vertical). Therefore we put $[z_\rho\gamma_\rho] := S(B_\rho)$, arriving at the next element of our chain L.

Notice that in our chain it might happen that while $\alpha < \beta$, $[z_\beta\gamma_\beta] \leq_S [z_\alpha\gamma_\alpha]$, but $[z_0\gamma_0]$ is a lower bound of L.

Since in general $[x_\alpha] \leq_S [y_\beta]$ implies $x \leq_M y$, we can transform our chain L of points in B into a chain $L^M = \{z_0, z_1, ..., z_\omega, ...\}$ of points in $\mathbb{R}^4$. L

is lower bounded (by z_0), so our postulate 6 applies. By employing it and corollary 7 we infer that

$$\bigcap_{\alpha \in \Sigma_h(y)} \{\Sigma_h(z_\alpha) | z_\alpha \in L^M\} \neq \emptyset \tag{13}$$

By our construction of the chain L, for all α it is true that $[x_\alpha \eta_\alpha] \leq_S [z_\alpha \gamma_\alpha]$. Therefore $x_\alpha \leq_M z_\alpha$, from which we conclude that $\Sigma_h(z_\alpha) \subseteq \Sigma_h(x_\alpha)$. Thus, if

$$\bigcap_{\alpha \in \Sigma_h(y)} \{\Sigma_h(z_\alpha) | z_\alpha \in L^M\} \neq \emptyset, \tag{14}$$

then

$$\bigcap_{\alpha \in \Sigma_h(y)} \Sigma_h(x_\alpha) \neq \emptyset, \tag{15}$$

which is the equation 10 that we tried to show. Therefore we arrive at a contradiction and part 2 of the proof is complete.

Ad. 3. We have shown that there is a scenario $\sigma \in \Sigma$ such that all members of h can be identified as $[x_\sigma]$ for some $x \in \mathbb{R}^4$. What remains is to show that the history cannot "exclude" some regions of $\{\sigma\} \times \mathbb{R}^4$, that is: to prove that for all $x \in \mathbb{R}^4, [x_\sigma] \in h$. But in lemma 4 we have shown that $\{[x_\sigma] | x \in \mathbb{R}^4\}$ is a maximal upward-directed subset of B, so any proper subset of it cannot be maximal upward-directed. **Q.E.D.**

By showing lemmas 4 and 8 we have proven theorem 3.

2.2 The importance of the topological postulate

So far it might seem that our topological postulate 6 is just a handy trick for proving the lemma 8. To show its importance we will now prove that its falsity leads to the falsity of the lemma, and then present an example of a structure in which the lemma does not hold.

Theorem 9 *If the postulate 6 is false, then lemma 8 is also false.*

Proof: Assume that our topological postulate does not hold. Therefore there exists a maximal upward-directed set $h \subseteq B$ and a lower bounded chain $L \subset \mathbb{R}^4$ such that the chain topology is not compact. This is by rules of topology equivalent to the fact that there is a centred family of closed sets with an empty intersection. But all closed sets in the chain topology form a chain with respect to inclusion. Of course, if a part of a chain has an empty intersection, a superset of the part also has an empty intersection. We infer that

$$\bigcap_{x \in L} \Sigma_h(x) = \emptyset \tag{16}$$

from which, by definition 5, we get that

$$\neg \exists \sigma \in \Sigma : \forall x \in L \; [x_\sigma] \in h \tag{17}$$

so there is no scenario σ such that $h = \{[x_\sigma] | x \in \mathbb{R}^4\}$. Thus, lemma 8 is false. **Q.E.D.**

We will now show a situation in which lemma 8 does not hold. The construction resembles the M_1 structure from [6]. By fixing two spatial dimensions we will restrict ourselves to $\mathbb{R}^2$, the first coordinate representing time.

As usual, Σ is the set of all scenarios of a world B. Let C be the set of all splitting points:

$$C := \bigcup_{\sigma, \eta \in \Sigma} C_{\sigma\eta}$$

We put

$$C := \{\langle 0, n \rangle | n \in \mathbb{N} \cup \{0\}\} \tag{18}$$

The idea is that all splitting points are binary: any scenario passing through a given splitting point can go either "left" or "right". Since there are as many splitting points as natural numbers, we can identify Σ with a set of 01-sequences. Another requirement on Σ is that it contains only the sequences with finitely many 0s. Let G be a subset of Σ containing only the sequence without any 0s and all sequences that have all their 0s in the beginning. The elements of G will be labeled as below:

$$\begin{aligned} \sigma_0 &= 1111..... \\ \sigma_1 &= 01111.... \\ \sigma_2 &= 00111.... \\ \sigma_3 &= 00011.... \end{aligned}$$

Let us next consider a sequence Z_i^M of points in $\mathbb{R}^2$ such that for all $i \in \mathbb{N}$ $z_i = \langle i - 1/2, 0 \rangle$. This way, a given $z_i \in Z_i^M$ is in the Minkowskian sense above all splitting points $\langle 0, n \rangle | n < i$ and above no other splitting points.

Consider now a sequence Z_i in B, $Z_i = \{[z_i \sigma_i] | i \in \mathbb{N}\}$. We will now show that Z_i is a chain. Take any $[z_m \sigma_m], [z_n \sigma_n] \in Z_i$ such that $m \neq n$. Either $m < n$ or $n < m$; suppose $m < n$ (the other case is analogous). Since $m < n$, $z_m \leq_M z_n$. $z_m \in R_{\sigma_m \sigma_n}$ since it is not above any splitting points between

σ_m and σ_n. Therefore $z_m\sigma_m \equiv_S z_m\sigma_n$, so $[z_m\sigma_m] \leq_S [z_n\sigma_n]$. We have shown that any two elements of Z_i are comparable by $\leq_S$. Therefore, Z_i is a chain in B, thus being an upward-directed subset of B.

The set of all upward-directed subsets of B meets the requirements of Kuratowski-Zorn Lemma, since a set-theoretical sum of any chain subset of it is also an upward-directed subset of B and is an upper bound of the chain with respect to inclusion. Therefore, there exists a maximal upward-directed subset of B (a history h^*) such that $Z_i \subseteq h^*$. But lemma 8 is false with respect to this history, since for all $\sigma \in \Sigma$, $h^* \neq \{[x_\sigma] | x \in \mathbb{R}^2\}$! Suppose to the contrary, that for a certain $\sigma \in \Sigma$ $h = \{[x_\sigma] | x \in \mathbb{R}^2\}$. As a member of Σ, σ has to contain a "1" at some point k (starting with 0). Then both $[z_{k+1}\sigma_{k+1}] \in h^*$ and $[z_{k+1}\sigma] \in h^*$, so $z_{k+1} \in R_{\sigma_k \sigma_{k+1}}$. But $C_{\sigma_k \sigma_{k+1}} \ni \langle 0, k\rangle \leq_M z_{k+1}$, so $z_{k+1} \notin R_{\sigma_k \sigma_{k+1}}$ and thus we arrive at a contradiction.

We will now show that our topological postulate 6 is not met in this situation. Consider a chain $Z := Z_i^M \cup \{\langle -1, 0\rangle\}$. Note that $\langle -1, 0\rangle = inf(Z)$. Consider next the chain topology on $\Sigma_{h^*}(\langle -1, 0\rangle)$ (as defined in the last section) with Z as the original chain. $\{\Sigma_{h^*}(z_i)\}$ is a centred family of closed sets, but its intersection is empty as Σ does not contain a scenario corresponding to the sequence comprised of 0s only. Therefore we arrived at a contradiction with our corollary 7, so the postulate 6 is not met: the chain topology is *not* compact.

2.3 BST models and MBS

Having proven theorem 3 we can adopt Müller's proof (from [4]) of the fact that $\langle B, \leq_S\rangle$ meets all the requirements in definition 2 and conclude that it is a model of BST. We keep in mind, though, that we have introduced a new postulate 6 into the proof and shown that it is not trivial (not always true). We will demand from the structures we would like to call "Minkowskian Branching Structures" to meet our topological postulate. This way, a MBS is a special kind of a BST model: its Our World and ordering $\leq$ are constructed as respectively B and $\leq_S$ as proposed by Müller, and furthermore our postulate 6 is true in the model.

2.4 Splitting points and choice points

Since it purports to establish that "For histories h_σ, $h_\eta \subset B$ the set $C_{\sigma,\eta}$ is the set of choice points", Lemma 4 in Müller seems to require reformulation. A splitting point, as a member of $\mathbb{R}^4$, is not a member of B, and thus is not a choice point.

An obvious move would be to observe that every splitting point x for scenarios σ and η in a sense "generates" a choice point for histories h_σ and h_η. That is, if $x \in C_{\sigma\eta}$ then $[x_\sigma]$ is maximal in $h_\sigma \cap h_\eta$.

What might not be as evident is that, since we have dropped the requirement of finitude of every $C_{\sigma\eta}$, the converse is not true: in some cases there are choice points which are not "generated" in the above way by any splitting points. We will now try to persuade the reader that this is indeed the case. The idea is to use sequences of generated splitting points convergent to the same point. The argument is simple in $\mathbb{R}^2$ as we need only two sequences, but gets more complicated as the number of dimensions increases. (For convenience, in the below argument we use symbols "$>_S$" and "$>_M$" defined in the natural way basing on respectively "$\leq_S$" and "$\leq_M$".)

Definition 10 *1.* $SC_{\sigma\eta} := \{[c_\sigma] | c \in C_{\sigma\eta}\}$

$$2. \boldsymbol{C}_{\sigma\eta} := \{[x_\gamma] : \begin{array}{ll} (1) & [x_\gamma] \in h_\sigma \cap h_\eta \textit{ and} \\ (2) & \forall z \in \mathbb{R}^4 \forall \alpha \in \Sigma([z_\alpha] >_S [x_\gamma] \Rightarrow [z_\alpha] \notin h_\sigma \cap h_\eta) \end{array}$$

"$SC_{\sigma\eta}$" is to be read as "The set of generated choice points for histories h_σ and h_η".

"$\mathbf{C}_{\sigma\eta}$" is to be read as "The set of choice points for histories h_σ and h_η".

It is of course irrelevant whether we choose σ or η in square brackets in the definition of the set of generated choice points, since if $c \in C_{\sigma\eta}$ then $c_\sigma \equiv_s c_\eta$ and thus $[c_\sigma] = [c_\eta]$.

Theorem 11 *For some* $C_{\sigma\eta}$, $SC_{\sigma\eta} \subsetneq \boldsymbol{C}_{\sigma\eta}$.

Proof sketch. Again, by fixing two spatial dimensions we will restrict ourselves to $\mathbb{R}^2$. Let $x = (0, 0)$. Let $C_1 = \{(0, 1/n) | n \in \mathbb{N}\backslash\{0\}\}$ and $C_2 = \{(0, -1/n) | n \in \mathbb{N}\backslash\{0\}\}$. Let $C_{\sigma\eta} = C_1 \cup C_2$. As $x \notin C_{\sigma\eta}$, it is evident that $[x_\sigma] \notin SC_{\sigma\eta}$. We will show that $[x_\sigma] \in \mathbf{C}_{\sigma\eta}$, thus proving the theorem.

We have to show that $[x_\sigma]$ meets conditions (1) and (2) from the above definition. As for (1), $\forall c \in C_{\sigma\eta}$ x SLR c , so $x \in R_{\sigma\eta}$. It follows that $x_\sigma \equiv_S x_\eta$ and finally (as it is obvious that $[x_\sigma] \in h_\sigma$) that $[x_\sigma] \in h_\sigma \cap h_\eta$.

Now for (2). Consider $[z_\alpha]$ such that (a) $[z_\alpha] >_S [x_\sigma]$. By definition of $>_S$, $z >_M x$ and $x_\alpha = x_\sigma$. Let $z = (z_0, z_1)$ (the first coordinate is temporal). We distinguish two cases: either the spatial coordinate z_1 is equal to 0 or it's something else.

If $z = (z_0, 0)$, take $k \in \mathbb{R}$, $k < z_0$ such that $(0, k) \in C_{\sigma\eta}$ (such k exists since C_1 converges to $(0, 0)$). (*) Since $D^2_M(z, (0, k)) = k - z_1 < 0$, it follows that $x >_M (0, k) \in C_{\sigma\eta}$.

On the other hand, if $z_1 \neq 0$, consider v defined as follows:

$$v := \begin{cases} 1 & \text{if } z_1 \geq 1 \\ z_1 & \text{if } z_1 \in (0,1) \cup (-1,0) \\ -1 & \text{if } z_1 \leq -1 \end{cases}$$

We choose $(0,k) \in C_{\sigma\eta}$ such that $0 < k \leq v$ (if v is positive) or $v \leq k < 0$ (if v is negative). It is always possible to find such a point since both C_1 and C_2 converge to $(0,0)$. We have to prove that (b) $z >_M (0,k)$.

From (a) we know that (c) $z >_M (0,0)$. To arrive at (b) it suffices to show that (d) $z >_M (0,v)$. From (c) it follows that (e) $z_0 \geq z_1$. We have two cases to consider. First, if (f) $z_1 \geq 1$ or $z_1 \leq -1$, $D^2_M(z,(0,v)) = -z_0^2 + (z_1 - 1)^2 = -z_0^2 + z_1^2 + 1 - 2z_1$, which (by (f) and (e)) is below 0, which fact is equivalent to (d). Second, if $z_1 \in (0,1) \cup (-1,0)$, $D^2_M(z,(0,v)) = -z_0^2 + (z_1 - z_1)^2 = -z_0^2$ which is of course negative, so again we arrive at (d).

From (c) and (d) and from the requirement on choosing $(0,k)$ we get the needed result (b).

Since $z >_M (0,k) \in C_{\sigma\eta}$, it is true that $z \notin R_{\sigma\eta}$ and thus $[z_\alpha] \notin h_\sigma \cap h_\eta$. We have thus proven that $[x_\sigma]$ fulfills condition (2).

Unfortunately already in $\mathbb{R}^3$ the construction fails at point (*). To overcome the problem we would have to use four sequences of splitting points convergent to $(0,0,0)$ (intuitively situated at the arms of the coordinate system). To deal with the situation in $\mathbb{R}^4$ we would have to similarly introduce six sequences convergent to $(0,0,0,0)$. We don't dwell into the details here as the point being made doesn't seem to be significant enough in proportion to the arduous complexity of the argument.

Conjecture 12 *For any scenarios $\sigma, \eta \in \Sigma$, the set $\boldsymbol{C}_{\sigma\eta}$ contains exclusively points which belong to $SC_{\sigma\eta}$ or points $[x_\alpha]$ such that x is a limit of a sequence of points belonging to $C_{\sigma\eta}$.*

3 Funny business

The rest of the paper will concern the funny business phenomenon in its finitary and infinitary variants. Funny business in BST is to resemble cases of EPR. Roughly speaking, the idea is that there exist space-like related point events (ie events that cannot influence each other) whose outcomes are correlated - certain combination of outcomes cannot occur. This amounts to saying that a certain branch of histories is missing in the model. The most common example: consider two binary (+/-) SLR choice points e_1 and e_2. Combinatorics dictate that there should be four branches of histories that pass through both of those points: those that give + on e_1 and + on e_2, those that give + on e_1 and - on e_2, and so on. Funny business occurs if one (or more) of those branches is empty.

Note how similar the above example is to what Aristotle writes in Physics (II, 4, 195b): "Some people (...) say that nothing happens by chance, but that everything which we ascribe to chance or spontaneity has some definite cause, e.g. coming 'by chance' into the market and finding there a man whom one wanted but did not expect to meet is due to one's wish to go and buy in the market"[2]. Determinist connotations of the first part of the quote aside, the second part seems to describe a belief in EPR - like phenomena: e_1 and e_2 from the above example could be points in which two people in distant (SLR) parts of the city make their decisions whether or not to go to the market, and the missing history would be the one in which in e_2 the decision is positive, while in e_1 it is negative. So, if the person in e_2 decides to go the market, there was only an illusion of choice in e_1, because the person from e_1 was bound to go to the market.

To properly define funny business, we will need a few more formal notions.

Definition 13 *Hist is the set of all histories in the model.*

H_e is the set of all histories to which point event e belongs.

Next we will need a notion of an "elementary possibility at e", which will be an element of a partition of H_e. The partition is a a certain equivalence relation $\equiv_e$ on H_e which is to convey the sense of "being undivided in e" - sharing a point above e.

Definition 14 *Consider $h_1, h_2 \in H_e$. $h_1 \equiv_e h_2$ iff $\exists e^* > e$ such that $e^* \in h_1 \cap h_2$. $h_1 \perp_e h_2$ iff it is not the case that $h_1 \equiv_e h_2$.*

Suppose $h \in H_e$. $\Pi_e\langle h\rangle) \subseteq H_e$ is an elementary possibility in e iff it is the equivalence class of the history h w.r.t. the relation $\equiv_e$. If $x \in W$ and $e < x$, by $\Pi_e\langle x\rangle$ we mean the elementary possibility in e to which a history $h \in H_x$ belongs.

(As noted in [1], $\equiv_e$ is an equivalence relation due to the BST postulates.)

Following the existing literature of the subject we will define Π_e as the set of all elementary possibilities at e. We will now give our definitions of funny business in its two variants. They are to resemble the definitions of modal funny business in the literature of the subject (see [3], [5]) - namely, a history that is combinatorially possible is missing. For a given infinite set S of pairwise SLR points that is a subset of a history, we will consider functions f which, given a point $e \in S$ as an argument, produce an elementary possibility

[2]Translated by R. P. Hardie and R. K. Gaye

from Π_e. Colloquially speaking, if all points in S are binary choice points, a function f will give us all information as to whether "turn left" or "right" in any of those points. Formally,

$$\prod_{e \in S} \Pi_e = \{f : S \to \bigcup_{e \in S} \Pi_e : \forall e_k \in S\ f(e_k) \in \Pi_{e_k}\} \tag{19}$$

The definitions (for future convenience they define "NO funny business" rather then "funny business") run as follows:

Definition 15 *Assume S is an infinite set of pairwise SLR points such that there exists a history h for which $S \subset h$. Consider a function $f \in \prod_{e \in S} \Pi_e$.*

*$\langle S, f\rangle$ **do not constitute a case of finitary funny business** iff for any finite family of sets $\{A_1, A_2, ..., A_k\}$ such that $\forall i \leq k A_i \subseteq S$ if $\forall i \bigcap\{f(e) : e \in A_i\} \neq \emptyset$ then $\bigcap\{f(e) : e \in \bigcup_{1 \leq i \leq k} A_i\} \neq \emptyset$.*

*$\langle S, f\rangle$ **do not constitute a case of infinitary funny business** iff $\bigcap\{f(e) : e \in S\} \neq \emptyset$.*

*S **does not give rise to (in)finitary funny business** iff $\forall f \in \prod_{e \in S} \Pi_e$ $\langle S, f\rangle$ do not constitute a case of (in)finitary funny business.*

For brevity, from now on instead of "finitary funny business" we will usually write "FINFB" and instead of "infinitary funny business" we will usually write "INFFB".

3.1 M_2

In [6] a certain BST structure named M_2 was introduced in which FINFB was absent, whereas INFFB was present. We will now briefly reproduce its definition, because it is an interesting example of funny business and we will use it in our last theorem. For a detailed discussion and a proof that M_2 is a BST model with the above properties, see [6].

M_2 is a pair $\langle W, <\rangle$. W is a union of four sets: $W_0 = (-\infty, 0]$, $W_1 = (0, 1] \times \mathbb{N}$, $W_2 = (1, 2) \times \mathbb{N} \times \{0, 1\}$ and $W_3 = [2, \infty) \times \mathbb{F}$ where $\mathbb{F}$ is the set of all functions $f : \mathbb{N} \to \{0, 1\}$ such that for only finitely many $n \in \mathbb{N}$, $f(n) = 0$.

The strict partial ordering $<$ is the transitive closure of the following for relations:

- For e, e_1 from the same W_i: $e < e_1$ iff the first coordinate of e is smaller than that of e_1 and the other coordinates are the same.

- $x < (y, n)$ for every $x \in W_0$ and $(y, n) \in W_1$.

- For $(x, n) \in W_1$ and $(y, m, i) \in W_2 : (x, n) < (y, m, i)$ iff $n = m$.
- For $(x, n, i) \in W_2$ and $(y, f) \in W_3 : (x, n, i) < (y, f)$ iff $(f(n) = i$.

The structure has a countable set histories and also a countable set of binary choice points $\{\langle 1.n \rangle : n \in \mathbb{N}\}$.

Note that we encounter difficulties when trying to directly "convert" M_2 into a MBS. This is because in M_2 a point above some two choice points is always above an infinite number of choice points. It seems that in MBS' one could achieve this by employing sets of choice points that would be dense or contain a convergent sequence. We hope this matter will be a subject of further studies.

3.2 Results

It's obvious that if S gives rise to FINFB it also gives rise to INFFB. In the remaining part of the paper we will try to establish other connections between the two notions. The guiding principle is to find a set of conditions in which the two variants of funny business are equivalent.

A simple corollary of the definition 15 is given below:

Corollary 16 *Suppose that A is a finite subset of S. Then, if S does not give rise to FINFB,* $\bigcap\limits_{e \in A} \{f(e)\} \neq \emptyset$.

The corollary stems from the fact that any finite set is a union of a finite family of singletons. Note also that it is unfortunate to say that the above A does not give rise to INFFB, since it is finite. Still, the corollary shows that for finite sets the two variants of funny business are equivalent.

We will now prove the following:

Theorem 17 *Assume that S is an infinite set of pairwise SLR points such that for some history h, $S \subset h$.*

If there exist sets A_1, A_2 such that $A_1 \cup A_2 = S$ and none of them gives rise to INFFB, then (if S gives rise to INFFB, then S gives rise to FINFB).

Proof: From the first antecedent we get that $\forall f \in \prod\limits_{e \in A_1} : \bigcap\{f(e) : e \in A_1\} \neq \emptyset$ and a similar result for A_2. From the second antecedent we get that $\exists g \in \prod\limits_{e \in S} : \bigcap\{f(e) : e \in S\} = \emptyset$. We can of course think of the function g defined on S as a union of two functions defined respectively on A_1 and A_2. Thus, we see that $\langle S, g \rangle$ constitute a case of FINFB because $\bigcap\{g(e) : e \in A_i\} \neq \emptyset$ and $\bigcap\{g(e) : e \in A_2\} \neq \emptyset$ while $\bigcap\{g(e) : e \in A_1 \cup A_2\} = \emptyset$. Therefore S gives rise to FINFB. **Q.E.D.**

The above theorem yields us the following simple corollary:

Corollary 18 *Assume S is an infinite set of pairwise SLR points such that for some history h, $S \subset h$. Then, if S does not give rise to FINFB and there exists a cofinite subset of S which does not give rise to INFFB, then the whole set S does not give rise to INFFB.*

We will now introduce two postulates and prove a few theorems about how they relate to FINFB and INFFB.

A certain structure called M_1 (see [6]) provides a situation in which we rule out a history from appearing in our model only to see it re-inserted "by force" by Kuratowski-Zorn Lemma. Our first postulate stems from our thoughts on how to prevent such a situation.

Postulate 19 ***(Postulate A)*** *There exist 1) a set $S \subset W$ which is an infinite set of pairwise SLR points such that for some history h $S \subset h$ and 2) a function $f \in \prod_{e \in S} \Pi_e$ such that*

$$\exists e \in S \, \forall h \in Hist \, \forall x \in W : \\ (x \notin h \ \vee \ \neg(x > e) \ \vee \ h \notin f(e) \ \vee \ \exists e_1 \in S (h \notin f(e_1) \wedge \neg(x \ SLR \ e_1)))$$

We got the idea for the second postulate by investigating M_2 and trying to understand why it contains a case of INFFB. We elaborate a bit on this in theorem 25 below.

Postulate 20 ***(Postulate B)*** *There exists a set $X \subset W$ such that: X is infinite, for any two different points from X there exists a history to which only one of them belongs, for every finite subset $A \subset X$ there exists a history h such that $A \subset h$ and there is no history h such that $X \subset h$.*

The theorems we will show are summarized in the list below:

1. (Theorem 21)$Postulate A \Rightarrow$INFFB
2. (Theorem 22)$Postulate B \wedge Supplement \Rightarrow$ INFFB
3. (Theorem 23)Given that the BST model has space-time points, NOFINFB* $\wedge \neg(Post.A) \wedge \neg(Post.B) \Rightarrow$NOINFFB
4. (Theorem 24)$Postulate A \nRightarrow$ FINFB
5. (Theorem 25)$\neg(Postulate A) \wedge Postulate B \wedge Supplement \nRightarrow$ FINFB

The nature of the *Supplement* mentioned above will become clear in the course of the proof of theorem 22. NOFINB* is NOFINFB plus a condition stating that no doubleton such that at least one element of it belongs to an infinite set S of pairwise SLR points (such that for some history h $S \subset h$) gives rise to FINFB.

As for "space-time points" mentioned in theorem 23, in its proof we want to be able to say that something happens "in the same space-time point" in different histories. A triple $\langle W, \leq, i\rangle$ is a "BST model with space-time points" (BST+S) iff $\langle W, \leq\rangle$ is a BST model and s (from the expression "space-time point") is an equivalence relation on W such that 1) for each history h in W and for each equivalence class $s(x)$, $x \in W$, the intersection $h \cap s(x)$ contains exactly one element and 2) s respects the ordering: for equivalence classes $s(x), s(y)$ and histories h_1, h_2, $s(x) \cap h_1 = s(y) \cap h_1$ iff $s(x) \cap h_2 = s(y) \cap h_2$, and the same for "$<$" and "$>$". As Müller shows in [5], not every BST model can be extended to a BST+S model, so our theorem is not as general as we would ideally prefer.

Observe now that if Postulate A is false, then for any infinite pairwise SLR set S such that for some history h $S \subset h$ and for any function $f \in \prod_{e\in S} \Pi_e$ we can define a function $F : S \to Hist \times W$ in the following way ($e \in S$):

$$F(e) := \langle h, x\rangle : (x > e \wedge x \in h \wedge h \in f(e) \wedge \forall_{e' \in S}(h \notin f(e') \Rightarrow e' \; SLR \; x)) \tag{20}$$

(Of course many different functions meeting this requirement might exist as there might be many equally good candidates for h and x such that for a given e $F(e) = \langle h, x\rangle$. What is important for us is that, when Postulate A is false, such functions *do* exist; we will just choose one.)

Theorem 21 *Suppose Postulate A is true due to some $S \subset W$ and $f \in \prod_{e\in S} \Pi_e$. Then $\langle S, f\rangle$ constitute a case of INFFB.*

Proof: Suppose the contrary: $\bigcap\{f(e)|e \in S\} \neq \emptyset$. Hence, there must be a history (a) $h^* \in \bigcap\{f(e)|e \in S\}$. Suppose $e^* \in S$ is one of the points of which the existential formula in Postulate A is true. Since it follows that $h^* \in f(e^*)$, it is true for e^* that

$$\forall x \in W(x \notin h^* \;\vee\; \neg(x > e^*) \;\vee\; \exists e_1 \in S(h^* \notin f(e_1) \wedge \neg(x \; SLR \; e_1))). \tag{21}$$

Again, since $h^* \in f(e^*)$ and there are no maximal elements in the model (see point 2 of definition 2), we can find a point x^* such that $x^* > e^*$ and $x^* \in h^*$. In other words, for this x^* two elements of the above alternative are false - so the third one must be true. But it also is false, since one of the conjuncts is always false: namely, because of (a) it can't be true for any $e_1 \in S$ that $h^* \notin f(e_1)$. So the whole alternative is false for x^*, and thus we arrive at a contradiction. Therefore $\bigcap\{f(e)|e \in S\} = \emptyset$ so $\langle S, f\rangle$ constitute a case of INFFB. **Q.E.D.**

Let us prepare for the next theorem (22). Suppose that Postulate B is true due to a certain set X. Our goal is to find a set S and a function f such that $\langle S, f\rangle$ constitute a case of INFFB. Let $H = \bigcup_{e \in X} H_x$. For each $x \in X$ consider $C(x) = \{c \in W : c < x \wedge \exists h \in H_x\ \exists h' \in Hist - H_x\ h \perp_c h'\}$, a set of choice points below x. Let S^* be the sum of all $C(x)$ for $x \in X$. We need to make sure that all chains in S^* are upper bounded and that there is a history h^* such that $S^* \subseteq h^*$. This will be our *Supplement*.

Supplement: Every chain in S^* is upper bounded. Also, $\exists h^* \in Hist : S^* \subseteq h^*$.

We can now properly formulate our next theorem.

Theorem 22 *Suppose Postulate B and Supplement (as defined above) are true. Then there exists a case of INFFB in the model, ie there exists an infinite set S of pairwise SLR points such that there exists a history $h : S \subset h$ and a function $f \in \prod_{e \in S} \Pi_e$ such that $\langle S, f\rangle$ constitute a case of INFFB.*

Proof: We proceed as above until we reach the point in which we have to invoke the *Supplement*. So, assume the *Supplement* is true. Let us put

$$S := \{sup_{h^*}(l)|\text{l is a maximal chain in } S^*\} \tag{22}$$

Thanks to point 4 of definition 2 we get that $S \subset h^*$. From its definition, S is also pairwise SLR. Note that it is possible that for some e, $e \in S$ but for any $x \in X$ $e \notin C(x)$.

We proceed to define a function $f \in \prod_{e \in S} \Pi_e$. If $e \in C(x)$, then $e < x$ and we put $f(e) = \Pi_e\langle x\rangle$. If, on the other hand, $e \notin S^*$, we put $f(e) = \Pi_e\langle h^*\rangle$. We will show that $\bigcap\{f(e) : e \in S\} = \emptyset$.

Suppose the contrary. Then, there exists a history h such that $h \in \bigcap_{e \in S} f(e)$, so $\forall_{e \in S} h \in f(e)$. By definition of the function f we get that $\forall_{e \in S \cap S^*} h \in \Pi_e\langle x\rangle \wedge \forall_{e \in S - S^*} h \in \Pi_e\langle h^*\rangle$. It follows that $\forall_{e \in S \cap S^*}(e \in C(x) \Rightarrow h \equiv_e h_x)$.

We will show that $X \subseteq h$. Suppose that $\exists_{x \in X}\ x \notin h$ and $x \in h_x \in H_x$. Then by PCP $\exists e : h \perp_e h_x$, so $h \notin \Pi_e\langle x\rangle$. But $e \in C(x)$, so either $e \in S$

or e is below some point from S. Therefore $h \in \Pi_e\langle x\rangle$, so we arrive at a contradiction, proving that $X \in h$, which in turn contradicts Postulate B. Therefore $\bigcap\{f(e) : e \in S\} = \emptyset$ and so $\langle S, f\rangle$ constitute a case of INFFB. **Q.E.D.**

We will now prove our main theorem. It turns out that if we suppose NOFINFB*, negations of Postulates A and B are sufficient to guarantee that there is no INFFB in the model.

Theorem 23 *Suppose our model is a BST+S model and NOFINFB* is true. Suppose that both Postulates A and B are false. Then no infinite set S of pairwise SLR points such that for some history h $S \subset h$ gives rise to INFFB.*

Proof: Consider a set S meeting the requirements from the theorem. We will show that for no function $f \in \prod_{e\in S} \Pi_e$ $\langle S, f\rangle$ constitute a case of INFFB. So, consider a function $f \in \prod_{e\in S} \Pi_e$. We will prove that there is a history $h \in \bigcap\{f(e) : e \in S\}$. Consider S as naturally indexed by its cardinality.

Since Postulate A is false, consider a function $F : S \to Hist \times W$ defined as in 20. Take $e_0 \in S$. For some $x_0 \in W$ and $h_0 \in Hist$ we have that $F(e_0) = \langle h_0, x_0\rangle$. Consider $S_0 := \{e \in S : h_0 \in f(e) \wedge x_0 > e\}$. If $S_0 = S$, we have completed the proof and h_0 is the desired history.

Otherwise, the construction guarantees that x_0 SLR $(S - S_0)$. Take a point from $S - S_0$ (say, a point e_i such that i is the minimal index in the set of indexes of points from $S - S_0$) and call it e_1. So, for some $x'_1 \in W$ and $h'_1 \in Hist$ we have that $F(e_1) = \langle h'_1, x'_1\rangle$. From NOFINFB* (applied to SLR points x_0 and e_1) we get that $H_{x_0} \cap \Pi_{e_1}(h'_1) \neq \emptyset$ so there is a history h_1 belonging to the intersection. Thanks to the fact that our model is by assumption a BST+S model, we can take a point $x_1 := s(x'_1) \cap h_1$. Accordingly, $x_0, x_1 \in h_1$. We define $\Sigma_1 := \{x_0, x_1\}$. Take $S_1 := \{e \in S - S_0 : h_1 \in f(e) \wedge x_1 > e\}$. On the occasion that $S = S_0 \cup S_1$ we have completed the proof and h_1 is the desired history. If not, we continue similarly with a point $e_2 \in S - (S_0 \cup S_1)$.

The above two steps should give us an idea of what to do while moving from e_k to $e_k + 1$. Suppose we finished the k-th step and accordingly we have the sets S_k and Σ_k and the history h_k. If $S - \bigcup_{0\le i\le k} S_i \neq \emptyset$, the theorem is not proven yet, so we take a point from $S - \bigcup_{0\le i\le k} S_i$ and label it e_{k+1}. So, for some $x'_{k+1} \in W$ and $h'_{k+1} \in Hist$ we have that $F(e_{k+1}) = \langle h'_{k+1}, x'_{k+1}\rangle$. Since Σ_k is finite, there is (thanks to NOFINFB*) a history h_k such that $\Sigma_k \subset h_k$. We will label the set of all such histories as H_{Σ_k}. From NOFINFB* we get (since Σ_k has an upper bound) that $H_{\Sigma_k} \cap \Pi_{e_{k+1}}(h'_{k+1}) \neq \emptyset$ so there is a history h_{k+1} belonging to the intersection. Take $x_{k+1} := s(x'_{k+1}) \cap h_{k+1}$ and put $\Sigma_{k+1} = \Sigma_k \cup \{x_{k+1}\}$. Of course $\Sigma_{k+1} \subset h_{k+1}$. Define $S_{k+1} := \{e \in S - \bigcup_{0 \leq i \leq k} S_i : h_{k+1} \in f(e) \wedge x_{k+1} > e\}$. On the occasion that $S = \bigcup_{0 \leq i \leq k+1} S_i$ we have completed the proof and h_{k+1} is the desired history. If not, we continue similarly with a point $e_{k+2} \in S - \bigcup_{0 \leq i \leq k+1} S_i$.

Let us now move to the limit case. Consider the set $\bigcup_{k<\omega} \Sigma_k$. It possesses the following properties:

- For every finite subset A of it there exists a history $h : A \subset h$ (since it is finite, A has to be a subset of Σ_k for some $k < \omega$, and so $A \subset h_k$)
- For any $x, y \in \bigcup_{k<\omega} \Sigma_k$ s.t. $x \neq y$ $\exists h : (x \notin h \vee y \notin h)$ (this follows from the fact that there has to be a $k < \omega$ such that one member of the doubleton $\{x, y\}$ (say y) belongs to Σ_k and the other to Σ_{k+1}, the fact that y and Σ_k are above respectively S_{k+1} and $\bigcup_{i=0}^{k} S_i$, both of which are subsets of S, and finally from NOFINFB* applied to S_{k+1} and $\bigcup_{i=0}^{k} S_i$)
- It is infinite (since $\forall_{i,j}(i \neq j \Rightarrow \Sigma_i \neq \Sigma_j)$).

Therefore, the set is of the kind that Postulate B speaks about. Since we assumed its negation, we infer that there is a history $h^* \in Hist$ such that $\bigcup_{k<\omega} \Sigma_k \subset h^*$. If $S = \bigcup_{k<\omega} S_k$, the theorem is proven and h^* is the desired history.

Suppose that $S - \bigcup_{k<\omega} S_k \neq \emptyset$. Take a point $e_\omega \in S - \bigcup_{k<\omega} S_k$. So, for some $x'_\omega \in W, h'_\omega \in Hist$ it is so that $F(e_\omega) = \langle h'_\omega, x'_\omega \rangle$. Now we will distinguish two cases. First, if $\bigcup_{k<\omega} \Sigma_k$ has an upper bound, we proceed as before, producing accordingly a history h_ω and sets Σ_ω and S_ω. Second, if $\bigcup_{k<\omega} \Sigma_k$ does not have an upper bound, consider sets $A_1 := \{e_i : 0 \leq i < \omega\}$ and $A_2 := \{e_\omega\}$ From the construction it follows that $h^* \in \bigcap_{e \in A_1} f(e)$ and $h'_\omega \in \bigcap_{e \in A_2} f(e)$. So, by NOFINFB*, $\bigcap_{e \leq \omega} f(e) \neq \emptyset$, so there is a history h_ω belonging to the intersection. Let $x_\omega := s(x'_\omega) \cap h_\omega$. Let $\Sigma_\omega := \{x_\omega\} \cup \{s(x) \cap h_\omega : x \in \bigcup_{k<\omega} \Sigma_k\}$. Let $S_\omega := \{e \in S - \bigcup_{k<\omega} \Sigma_k : h_\omega \in f(e) \wedge x_\omega > e\}$.

If $S = \bigcup_{k \leq \omega} S_k$ we have completed the proof and h_ω is the desired history.

If not, we continue similarly with points from $S - \bigcup_{k \leq \omega} S_k$. Since we have given instructions on what to do with e point e_i whether i is a limit number or not (the above case with ω is easily generalized), we are bound to arrive at a desirable history $h \in \bigcap\{f(e) : e \in S\}$. **Q.E.D.**

The last two theorems are to show that the first two theorems from the list above are not useless: since FINFB leads to INFFB, we need to make sure that neither Postulate A alone nor the combination of conditions from 22 yields FINFB.

Theorem 24 *If a set $S \subset W$ is an infinite set of pairwise SLR points such that for some history h $S \subset h$ and a function $f \in \prod_{e \in S} \Pi_e$ satisfies this condition:*

$$\begin{aligned}&\exists e \in S \, \forall h \in Hist \, \forall x \in W : \\ &\quad x \notin h \ \vee \ \neg(x > e) \ \vee \ h \notin F(e) \ \vee \ \exists e_1 \in S(h \notin f(e_1) \wedge \neg(x \; SLR \; e_1))\end{aligned}$$

then it does not follow that $\langle S, f \rangle$ constitute a case of FINFB.

Proof sketch. Our example will take place in a MBS. Unfortunately, during the construction we have run into similar problems as with theorem 11: namely, we can present a proper set if we restrict ourselves to $\mathbb{R}^2$, while the $\mathbb{R}^4$ case involves an intuitive extension of our idea which unfortunately would be formally painful. Thus we will show the $\mathbb{R}^2$ case. The second coordinate is spatial. (By "(a, b)" we will sometimes mean "a point in $\mathbb{R}^2$"or "a segment of $\mathbb{R}$", but it will always be clear from the context.)

Let $S_1 = \{(0, x) \in \mathbb{R}^2 : x \in (0, 1)\}$ be a dense segment of splitting points. Suppose all choice points generated by S_1 are binary and label one possibility "0" and the other "1". Assume that each scenario from Σ corresponds to a history belonging to only a finite number of "0"-possibilities (in harmony with lemma 8, there are no other histories). Put $B := (\Sigma \times \mathbb{R}^2)/\equiv_S$.

The set of choice points generated by S_1 will be called S. $S = \{[(0, x)_\sigma] : x \in (0, 1), \sigma \in \Sigma\}$. Consider a function $f \in \prod_{e \in S} \Pi_e$ such that

$$f([(0, x)_\sigma]) = \begin{cases} 1 & \text{if } x \geq 1/2 \\ 0 & \text{if } x < 1/2 \end{cases}$$

The point that will make Postulate A true is $[(0, 1/2)_\sigma]$. It is because it is true that

$$\begin{aligned}&\forall x \in W \, \forall h \in Hist : \\ &\quad (x \in h \wedge h \in f([(0, 1/2)_\sigma]) \wedge x > [(0, 1/2)_\sigma]) \Rightarrow \exists e_1 \in S(h \notin f(e_1) x > e_1)\end{aligned}$$

which we arrive at by transforming Postulate A. And the above is true because any point above $[(0, 1/2)_\sigma]$ is also above an infinite number of points $[(0, x)_\sigma]$ such that $x \in (0, 1/2)$. Any history has to belong to the "1"-possibility in some of those points, contrary to what function f dictates.

Now we have to show that $\langle S, f\rangle$ do not constitute a case of FINFB. Consider $A, B \subset S$. If $\{[(0, x)_\sigma] : x \in (0, 1/2)\} \cap A$ is infinite or $\{[(0, x)_\sigma] : x \in (0, 1/2)\} \cap B$ is infinite, then from our assumption about the histories in our model we infer that $\bigcap\{f(e) : e \in A\} = \emptyset$ (resp. $\bigcap\{f(e) : e \in A\} = \emptyset$), so the antecedent from the definition of NOFINFB is false. In the other case, if both $\{[(0, x)_\sigma] : x \in (0, 1/2)\} \cap A$ and $\{[(0, x)_\sigma] : x \in (0, 1/2)\} \cap B$ are finite, then $\{[(0, x)_\sigma] : x \in (0, 1/2)\} \cap (A \cup B)$ is finite, therefore (again, by our assumption about the histories in the model) $\bigcap\{f(e) : e \in A \cup B\} \neq \emptyset$, so the consequent from the definition of NOFINFB is false. Therefore, $\langle S, f\rangle$ do not constitute a case of FINFB.

Theorem 25 *Suppose Postulate A is false in our model and X is the set whose existence is entailed by Postulate B. Suppose also that the Supplement is true for X. It does not follow that S gives rise to FINFB.*

Proof by observation: M_2 provides us with an appropriate set. Consider $X := \{\langle 3/2, n, 0\rangle : n \in \mathbb{N}\}$. X is exactly of the kind required by Postulate B, the *Supplement* is true because S is the countable set of choice points in M_2, and, as noted in [6], there is no funny business in M_2.

4 Conclusion and open problems

In this paper we have introduced the notion of a Minkowskian Branching Structure, based on Müller's work from [4]. In the second part of the paper we have shown some results concerning finitary and infinitary funny business. We have found two situations in which INFFB is not equivalent to FINFB: one involving Postulate A, the other involving Postulate B. M_2 is a structure in which Postulate B is true. We suspect that in all MBS' this postulate is false and that is the reason for which M_2 is not "translatable" to a MBS. Also, it seems to be true that truth of Postulate A can be achieved in MBS' by using dense sets of pairwise SLR points or similar sets containing a convergent sequence. Let us end by noting a few problems which we hope to become subject to future investigation.

- Prove theorem 23 for BST models without space-points.
- Find conditions on X from Postulate B that would impose truth of the *Supplement*.

- Investigate the reasons for which M_2 cannot be directly reproduced in any MBS; try to find a MBS in which INFFB would be present but from which NOFINFB would be absent.

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