Quantum Probability from Subjective Likelihood: improving on Deutsch's proof of the probability rule

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Abstract

I present a proof of the quantum probability rule from decision-theoretic assumptions, in the context of the Everett interpretation. The basic ideas behind the proof are those presented in Deutsch's recent proof of the probability rule, but the proof is simpler and proceeds from weaker decision-theoretic assumptions. This makes it easier to discuss the conceptual ideas involved in the proof, and to show that they are defensible.

1 Introduction

> The mathematical formalism of the quantum theory is capable of yielding its own interpretation.
>
> Bryce DeWitt (1970)

If I were to pick one theme as central to the tangled development of the Everett interpretation of quantum mechanics, it would probably be: *the formalism is to be left alone.* What distinguished Everett's original paper both from the Dirac-von Neumann collapse-of-the-wavefunction orthodoxy and from contemporary rivals such as the de Broglie-Bohm theory was its insistence that unitary quantum mechanics need not be supplemented in any way (whether by hidden variables, by new dynamical processes, or whatever).

Many commentators on the Everett interpretation — even some, like David Deutsch, who are sympathetic to it (Deutsch 1985) — have at various points and for various reasons retreated from this claim.[1] The "preferred basis problem", for instance, has induced many to suppose that quantum mechanics must be supplemented by some explicit rule that picks out one basis as physically special. Many suggestions were made for such a rule (Barrett (1999) discusses

[1] It is perhaps worth noting that Deutsch no longer sees any need to modify the formalism, and is now happy with a decoherence-based solution to the preferred-basis problem.

several); all seem to undermine the elegance (and perhaps more crucially, the relativistic covariance) of Everett's original proposal. Now the rise of decoherence theory has produced a broad consensus among supporters of Everett (in physics, if perhaps not yet in philosophy) that the supplementation was after all not necessary. (For more details of this story, see Wallace (2002, 2003a) and references therein.)

Similarly, various commentators (notably Bell (1981), Butterfield (1996) and Albert and Loewer (1988)) have suggested that the Everett interpretation has a problem with persistence of objects (particles, cats, people, ...) over time, and some have been motivated to add explicit structure to quantum mechanics in order to account for this persistence (in particular, this is a prime motivation for Albert and Loewer's original Many-Minds theory). Such moves again undermine the rationale for an Everett-type interpretation. And again, a response to such criticism which does not require changes to the formalism was eventually forthcoming. It was perhaps implicit in Everett's original discussion of observer memory states, and in Gell-Mann and Hartle's later notion of IGUSs (Gell-Mann and Hartle 1990); it has been made explicit by Simon Saunders' work (Saunders 1998) on the analogy between Everettian branching and Parfittian fission.

In both these cases, my point is not that the critics were foolish or mistaken: Everettians were indeed obliged to come up with solutions both to the preferred-basis problem and to the problem of identity over time. However, in both cases the obvious temptation — to modify the formalism so as to solve the problem by *fiat* — has proved to be unnecessary: it has been possible to find solutions within the existing theory, and thus to preserve those features of the Everett interpretation which made it attractive in the first place.

Something similar may be going on with the other major problem of the Everett interpretation: that of understanding quantitative probability. One of the most telling criticisms levelled at Everettians by their critics has always been their inability to explain why, when all outcomes objectively occur, we should regard one as more likely than the other. The problem is not merely how such talk of probability can be meaningful; it is also how the *specific* probability rule used in quantum mechanics is to be justified in an Everettian context. Here too, the temptation is strong to modify the formalism so as to include the probability rule as an explicit extra postulate.

Here too, it may not be necessary. David Deutsch has, I believe, transformed the debate by attempting (Deutsch 1999) to derive the probability rule *within* unitary quantum mechanics, via considerations of rationality (formalised in decision-theoretic terms). His work has not so far met with wide acceptance, perhaps in part because it does not make it at all obvious that the Everett interpretation is central (and his proof manifestly fails without that assumption).

In Wallace (2003b), I have presented an exegesis of Deutsch's proof in which the Everettian assumptions are made explicit. The present paper may be seen as complementary to my previous paper: it presents an argument in the spirit of Deutsch's, but rather different in detail. My reasons for this are two-fold: firstly, I hope to show that the mathematics of Deutsch's proof can be substantially simplified, and his decision-theoretic axioms greatly weakened; secondly and perhaps more importantly, by simplifying the mathematical structure of the proof, I hope to be able to give as clear as possible a discussion of the *conceptual* assumptions and processes involved in the proof. In this way I hope that the reader may be in a better position to judge whether or not the probability problem, like other problems before it, can indeed be solved without modifications to the quantum formalism.

The structure of the paper is as follows. In section 2 I review the account of branching that Everettians must give, and distinguish two rather different viewpoints that are available to them; in section 3 I consider how probability might fit into such an account. Sections 4–5 are the mathematical core of the paper: they present an extremely minimal set of decision-theoretic assumptions and show how, in combination with an assumption which I call **equivalence**, they are sufficient to derive the quantum probability rule. The next three sections are a detailed examination of this postulate of **equivalence**. I argue that it is unacceptable as a principle of rationality for single-universe interpretations (section 6), but is fully defensible for Everettians — either via the sort of arguments used by Deutsch (section 7) or directly (sections 8–9). Section 10 is the conclusion.

2 Thinking about branching

The conceptual problem posed by branching is essentially one of transtemporal identity: given branching events in my future, how can it even make sense for me to say things like "I will experience such-and-such"? Sure, the theory predicts that people who look like me will have these experiences, but what experiences will *I* have? Absent some rule to specify *which* of these people is me, the only options seem to be (1) that I'm all of them, in which case I will presumably have all of their experiences, or (2) that I'm none of them, which seems to suggest that branching events are fatal to me.

As Saunders (1998) has forcefully argued, this is a false dilemma. Even in classical physics, it is a commonplace to suppose that transtemporal identity claims, far from being in some sense primitive, supervene on structural and causal relations between momentary regions of spacetime. Furthermore, the work of Parfit (1984) and others on fission, teletransportation and the like has given us reason to doubt that identity *per se* is what is important to us: rather, it is the survival of people who are appropriately (causally/structurally) related to me that is important, not *my* survival *per se*. In a branching event, then, what is important is that the post-branching people do indeed bear these relations to me. That I care about their future well-being is no more mysterious than — indeed, is precisely analogous to — my caring about my own future well-being.

If this resolves the paradox, still it leaves a choice of ways for Everettians to think about splitting (which is the reason for this section). The choice has not been discussed much in print (though see Greaves 2004 and Wallace 2005a), but it often arises in informal discussions and is of some relevance to the rest of this paper. The two options are:

- The *Subjective-uncertainty (SU) viewpoint:* Given that what it is to have a future self is to be appropriately related to a certain future person, and that in normal circumstances I expect to become my future self, so also in Everettian splittings I should expect to become *one* of my future selves. If there is more than one of them I should be uncertain as to which I will become; furthermore, this subjective uncertainty is compatible with my *total knowledge* of the wavefunction and its dynamics. This is Saunders' own view, argued for at length in Saunders 1998; I discuss his argument for SU in Wallace (2005a), and give my own defence of it in Wallace (2005b).

  ('Subjective' should not be taken too literally here. The subjectivity lies in the essential role of a particular location in the quantum universe (uncertainty isn't visible from a God's-eye view. But it need not be linked to first-person expectations: 'there will be a sea battle tomorrow' might be as uncertain as 'I will see spin up'.)

- The *Objective-determinism (OD) viewpoint:* Branching leads, deterministically, to my having multiple future descendants. Rationally speaking, I should act to benefit my future descendants, for exactly the same reason that people in non-branching possible worlds would act to benefit their single descendant. Situations of conflict may arise between the interests of my descendants (such as when I bet on one possible outcome of a measurement), in which case I will have to weigh up how much I wish to prioritise each descendant's interests. This is perhaps the most literal translation of Parfit's own ideas into a quantum-mechanical context; it appears to be the view espoused by Bostrom (2002) in his discussion of the Everett interpretation, and is defended explicitly by Greaves (2004).

These views should not be taken as automatically in conflict.[2] It is almost impossible *not* to accept the OD viewpoint as valid, since it is just a literal reading of the physics. The conflict is rather between those (such as Saunders and myself) who regard the SU viewpoint as a valid alternative, and those (such as Greaves (2004)) who regard it as incoherent. There is a further question as to whether anything important depends on the validity or otherwise of SU — in my view it is of central, albeit rather philosophical, importance to the epistemology of the Everett interpretation (see Wallace (2005a) for the argument) but others regard it as a purely linguistic distinction.

One of the claims of this paper is that the Born rule can be defended from both the OD and the SU viewpoints, albeit in slightly different ways. I will return to these matters in sections 7 and 9. For now, however, let us move from the question of identity to the question of probability.

[2] Although I erroneously took them as such in an earlier draft of this paper.

3 Weight and probability

The paradigm of a quantum measurement is something like this: prepare a system (represented by a Hilbert space $\mathcal{H}$) in some state (represented by a normalised vector $|\psi\rangle$ in $\mathcal{H}$). Carry out some measurement process (represented by a discrete-spectrum self-adjoint operator $\widehat{X}$ over $\mathcal{H}$) on the system, and look to see what result (represented by some element of the spectrum of $\widehat{X}$) is obtained.

Suppose some such measurement process is denoted by M, and suppose that associated with M is some set $\mathcal{S}_M$ of possible outcomes of the process (for instance, states of the apparatus with pointers pointing in certain directions.) Let us call this the *state space* of the measurement process; and let us call $\mathcal{E}_M$, the set of all subsets of $\mathcal{S}_M$, the *event space* for M. We define the total event space for a set of measurements as the union of all their event spaces.

If the observable being measured is represented by operator $\widehat{X}$, then specifying the measurement process requires us to specify some convention by which elements of $\mathcal{S}_M$ are associated with elements of the spectrum $\sigma(\widehat{X})$ of eigenvalues of $\widehat{X}$. In effect, the convention is a function $\mathcal{C}$ from $\mathcal{S}_M$ onto $\sigma(\widehat{X})$: $\mathcal{C}(s)$ is the outcome which we associate with state s (and $\mathcal{C}(E)$ is shorthand for $\{\mathcal{C}(s)|s \in E\}$).

Now, suppose that the system being measured is in state $|\psi\rangle$. We use it, and $\mathcal{C}$, to define a *weight function* on $\mathcal{E}_M$, as follows:

$$\mathcal{W}_M(E) = \sum_{x \in \mathcal{C}(E)} \langle\psi| \widehat{P}_X(x) |\psi\rangle \quad \forall E \subseteq M, \tag{1}$$

where $\widehat{P}_X(x)$ projects onto the eigenspace of $\widehat{X}$ with eigenvalue x.

All this should be both familiar and essentially interpretation-neutral; familiar, too, should be the

> **Quantum probability rule:** If M is a quantum measurement and $E \in \mathcal{E}_M$, then the probability of E given that M is performed is equal to $\mathcal{W}_M(E)$.

Familiar though it may be, do we actually understand it? Compare it with the

> **Quantum sqwerdleflunkicity rule:** If M is a quantum measurement and $E \in \mathcal{E}_M$, then the sqwerdleflunkicity of E given that M is performed is equal to $\mathcal{W}_M(E)$.

We *don't* understand the quantum sqwerdleflunkicity rule, and for an obvious reason: "sqwerdleflunkicity" is meaningless, or at any rate we have no idea what it is supposed to mean. So if we *do* understand the quantum probability rule, presumably this requires that we understand what "probability" means.

But in fact, the meaning of "probability" is pretty subtle, and pretty controversial. In practice, physicists tend to test "probability" statements by relating them to observed frequencies, and as a result most physicists tend to *define* probability as relative frequency in the limit. This is also the notion of probability used by most investigators of the Everett interpretation, beginning with Everett's own analysis of memory traces under repeated measurements and leading to the elegant relative-frequency theorems established by Hartle (1968) and Farhi, Goldstone, and Gutmann (1989). The general strategy of these investigations is to show that as we approach the limiting case of infinitely many measurements, the weights of those branches in which relative frequencies are anomalous approach zero (or alternatively that in mathematical models appropriate to an *actual* infinity of measurements, there are *no* branches with anomalous frequencies.)

Such arguments have in general failed to convince sceptics. Arguably they either invoke unphysical situations such as an actual infinity of experiments, or they court circularity: if we wish to *prove* that the quantum weight function is something to do with probability, we aren't entitled to *assume* that anomalous branches can be neglected just because they have very low weight.

Now, anyone familiar with the chequered history of frequentist theories of probability should feel uneasy at this point: virtually identical criticisms could easily be levelled at the frequentist definition of probability itself. For that definition, too, either makes use of infinite ensembles or runs into the problem that anomalous frequencies are just *very improbable*, not actually impossible. It is most unclear that Everettian frequentists are any worse off than other species of frequentist.

Nevertheless, if Everettians have an incoherent theory of probability it is cold comfort for them if other people do too. Is there a more positive step that they can make? They could simply take probability as primitive, and declare it to be an interpretative posit that probability=weight; at one point this was Simon Saunders' strategy (see Saunders (1998)), and he has defended (successfully, in my view) the position that non-Everettian theories of physics do no better. Again, though, this seems unsatisfactory: we would like to *understand* Everettian probability, not just observe that non-Everettians also have problems with probability.[3]

To the best of my knowledge, Deutsch (1999) made the first concrete non-frequentist proposal for how probabilities could be derived in the Everett interpretation . His strategy follows in the footsteps[4] of the subjectivist tradition in foundations of probability, originally developed by Ramsey (1931), de Finetti (1974), Savage (1972) and others: instead of reducing probability to frequencies, *operationalise* it by reducing it to the preferences of rational agents. We might

[3]Having said which, if *no-one* has a good theory of probability then it would be unreasonable to dismiss the Everett interpretation in favour of other interpretations purely on the grounds of the probability problem; see Papineau (1996) for a more detailed presentation of this argument.

[4]I should note for the record that Deutsch himself dislikes this description of his project (private conversation), essentially because of his deep skepticism about the coherence of the subjective notion of probability. Nonetheless, at least from a mathematical perspective the description is hard to challenge: Deutsch appeals explicitly to the axioms of decision theory, as originated by Savage et al.

then say that one such agent judges E more likely than F iff that agent would prefer a bet on E to a bet on F (for the same stake); we could similarly say that E is more likely than F *simpliciter* if *all* agents are rationally compelled to prefer bets on E to bets on F.

In principle, we might go further. An agent judges E and F to be *equally likely* iff he is indifferent between a bet on E and one on F; if he judges disjoint events E and F to be equally likely, and judges G to be equally likely as $E \cup F$, then we say that he judges G to be *twice as likely* as E. By this sort of strategy, we can justify not just qualitative but quantitative comparisons of likelihood, and — perhaps — ultimately work up towards a numerical measure of likelihood: that is, a numerical probability.

The strategy of the subjectivists, then, was this: to state intuitively reasonable axioms for rational preference, such that if an agent's preferences conform to those axioms then they are provably required to be given by some probability function. Deutsch essentially takes this strategy over to quantum mechanics, but with a crucial difference: he uses the operationalist notion of probability not only to *make sense* of probabilistic talk within the Everett interpretation, but also to *prove* that the rational agents must use the weight function $\mathcal{W}$ of equation (1) to determine probabilities.

(It might appear from the above that probability in the Everett interpretation is somehow "not objective". This is certainly not the case: the weights of quantum branches are as objective as any other physical property. In fact, the best reading of the decision-theoretic proofs, in my view, is not that they tell us that there are no objective probabilities, but rather that they teach us that objective probability *is* quantum weight. See Saunders (2005) or (Wallace 2005a) for a more detailed analysis of this point.)

4 A rudimentary decision theory

In this section I will develop some of the formal details of the subjectivist program, in a context which will allow ready application to quantum theory. Our starting point is the following: define a *likelihood ordering* as some two-place relation holding between ordered pairs $\langle E, M\rangle$, where M is a quantum measurement and E is an event in $\mathcal{E}_M$ (that is, E is a subset of the possible outcomes of the measurement). We write the relation as $\succeq$:

$$E|M \succeq F|N \tag{2}$$

is then to be read as "It's at least as likely that some outcome in E will obtain (given that measurement M is carried out) as it is that some outcome in F will obtain (given that measurement N is carried out)". We define $\simeq$ and $\succ$ as follows: $E|M \simeq F|N$ if $E|M \succeq F|N$ and $F|N \succeq E|M$; $E|M \succ F|N$ if $E|M \succeq F|N$ but $E|M \not\simeq F|N$. We define $\preceq$ and $\prec$ in the obvious way, as the inverses of $\succeq$ and $\succ$ respectively.

We will say that such an ordering is *represented* by a function Pr from pairs $\langle E, M\rangle$ to the reals if

1. $\Pr(\emptyset|M) = 0$, and $\Pr(\mathcal{S}_M|M) = 1$, for each M.

2. If E and F are disjoint then $\Pr(E \cup F|M) = \Pr(E|M) + \Pr(F|M)$.

3. $\Pr(E|M) \geq \Pr(F|N)$ iff $E|M \succeq F|N$.

The ordering is *uniquely represented* iff there is only one such Pr.

The subjectivist program then seeks to find axioms for $\succeq$ so that any agent's preferences are uniquely represented. Literally dozens of sets of such axioms have been proposed over the years (see Fishburn (1981) for a review). Their forms vary widely, but as a rule there is an inverse correlation between the complexity of the axioms and their individual plausibility. Deutsch, for instance, uses a fairly simple but implausibly strong set of axioms (which I reconstruct in Wallace 2003b).

However, in a *quantum-mechanical* context we can manage with a set of axioms which is both extremely weak — far weaker than Deutsch's set — and fairly simple. To state them, it will be convenient to define a *null event*: an event E is *null with respect to M* (or, equivalently, $E|M$ is null) iff $E|M \simeq \emptyset|M$. (That is: E is certain not to happen, given M). If it is clear which M we're referring to, we will sometimes drop the M and refer to E as null *simpliciter*.

We can then say that a likelihood ordering is *minimally rational* if it satisfies the following axioms:

Transitivity $\succeq$ is transitive: if $E|M \succeq F|N$ and $F|N \succeq G|O$, then $E|M \succeq G|O$.

Separation There exists some E and M such that $E|M$ is not null.

Dominance If $E \subseteq F$, then $F|M \succeq E|M$ for any M, with $F|M \simeq E|M$ iff $E - F$ is null.

This is an extremely weak set of axioms for qualitative likelihood (far weaker, for instance, than the standard de Finetti axioms — see de Finetti 1974). Each, translated into words, should be immediately intuitive:

1. Transitivity: 'If A is at least as likely than B and B is at least as likely than C, then A is at least as likely than C.'

2. Separation: 'There is some outcome that is not impossible.'

3. Dominance: 'An event doesn't get less likely just because more outcomes are added to it; it gets more likely iff the outcomes which are added are not themselves certain not to happen.'

5 A quantum representation theorem

It goes without saying that this set of axioms alone is insufficient to derive the quantum probability rule: absolutely no connection has yet been made between the decision-theoretic axioms and quantum theory. We can make this connection, however, via two further posits. Firstly, we need to assume that we have a fairly rich set of quantum measurements available to us: rich, in fact, in the sense of the following definition.

> **Weight richness:** A set $\mathcal{M}$ of quantum measurements is *rich* provided that, for any positive real numbers $w_1, \ldots w_n$ with $\sum_{i=1}^{n} = 1$, $\mathcal{M}$ includes a quantum measurement with n outcomes having weights $w_1, \ldots, w_n$.

The richness of a set of measurements is an easy consequence of the following (slightly informal) principle: that for any n there exists at least one system whose Hilbert space has dimension n such that that system can be prepared in any state and that at least one non-degenerate observable can be measured on that system. Clearly, this is an idealisation: with sufficiently high computational power we can prepare states with arbitrary accuracy, but presumably there is some limit to that power in a finite universe. However, it seems a reasonable idealisation (just as it is reasonable to idealise the theory of computations slightly, abstracting away the fact that in practice the finitude of the universe puts an upper limit on a computer's memory) and I will assume it without further discussion.

Far more contentious is the second principle which we must assume:

Equivalence If E and F are events and $\mathcal{W}_M(E) = \mathcal{W}_N(F)$, then $E|M \simeq F|N$.

Much of the rest of the paper will be devoted to an analysis of whether **equivalence** is a legitimate requirement for rational agents. But the point of such an analysis, of course, is that in the decision-theoretic context which we are analysing, a great deal can be proved from it. In fact, we are now in a position to prove the

Quantum Representation Theorem: Suppose that $\succeq$ is a minimally rational likelihood order for a rich set of quantum measurements, and suppose that $\succeq$ satisfies **equivalence**. Then $\succeq$ is uniquely represented by the probability measure $\Pr(E|M) = \mathcal{W}_M(E)$.

Proof: We proceed via a series of lemmas.

Lemma 1 *If $\mathcal{W}_M(E) \geq \mathcal{W}_N(F)$, then $E|M \succeq F|N$.*

Since the set of quantum measurements is rich, there exists a quantum measurement O with disjoint events G, H such that $\mathcal{W}_O(G) = \mathcal{W}_N(F)$ and $\mathcal{W}_O(H) = \mathcal{W}_M(E) - \mathcal{W}_N(F)$. By **equivalence** $E|M \simeq G \cup H_O$ and $G|O \simeq F|N$; by **dominance** $G \cup H|O \succeq G|O$; by **transitivity** it then follows that $E|M \succeq F|N$.

Lemma 2 *A quantum event is null iff it has weight zero.*

That weight-zero events are null follows from **equivalence** and $\mathcal{W}_M(\emptyset) = 0$. For the converse, suppose for contradiction that some event of weight $w > 0$ is null; then there must exist n such that $1/n < w$. By lemma 1, any event of weight $1/n$ must also be null. Since the set of measurements is rich, there exists a quantum measurement M with n outcomes all of weight $1/n$: all must be null, and so by repeated use of **dominance** so must $\mathcal{S}_M$. By lemma 1 we conclude that all events are null, in contradiction with **separation**.

Lemma 3 *If* $E|M \succeq F|N$*, then* $\mathcal{W}_M(E) \geq \mathcal{W}_N(F)$.

Suppose that $\mathcal{W}_M(E) < \mathcal{W}_N(F)$. The set of measurements is rich, so there exists a quantum measurement O with disjoint events G, H such that $\mathcal{W}_O(G) = \mathcal{W}_M(E)$ and $\mathcal{W}_O(H) = \mathcal{W}_N(F) - \mathcal{W}_M(E)$. If $E|M \succeq F|N$ then $G|O \succeq G \cup H|O$, which by **dominance** is possible only if H is null. But since $\mathcal{W}_O(H) > 0$, by lemma 2 $H \not\simeq \emptyset$.

Lemmas 1 and 3 jointly prove that $\mathcal{W}$ represents $\succeq$. To show that the representation is unique, suppose M is any quantum measurement with outcomes having weights $k_1/K, k_2/K, \ldots k_N/K$, where k_1 through k_n are positive integers whose sum is K. Since the set of measurements is rich, there exists another measurement M' with K possible outcomes each having weight $1/K$. Since each is equiprobable, any probability function representing $\succeq$ must assign probability $1/K$ to each outcome. Let E_1 be the union of the first k_1 outcomes of M', E_2 be the union of the next k_2 outcomes, and so on; then any probability function must assign probability k_i/K to $E_i|M'$, and hence (since they have equal weight) to the ith outcome of M.

So: any probability function must agree with the weight function on rational-weight events. But since any irrational-weight event is more likely than all rational weight events with lower weight and less likely than all rational weight events with higher weight, agreement on rational values is enough to force agreement on all values.□

6 Equivalence and the single universe

So far, so good. The "decision-theoretic turn" suggested by Deutsch not only allows us to *make sense* of probability in an Everettian universe, but it also allows us to derive the quantitative form of the probability rule from assumptions — **equivalence** and the richness of the set of measurements — which *prima facie* are substantially weaker than the rule itself.

Nonetheless the situation remains unsatisfactory. **Equivalence** has the form of a principle of pure rationality: it dictates that any agent who does not regard equally-weighted events as equally likely is in some sense being irrational. Whether this principle is simple, or "*prima facie* weak", is not the point: the point is whether it is defensible purely from considerations of rationality. Simplicity might be a virtue when we are considering which axioms of fundamental physics to adopt, but can our "axioms of fundamental physics" include statements which speak directly of rationality? Surely not.

In fact, I believe that **equivalence** can be defended on grounds of pure rationality, and need not be regarded as a new physical axiom; I shall spend sections 7–9 providing such a defence. However, before doing so I wish to argue that the Everett interpretation necessarily plays a central role in any such defence: in other interpretations, **equivalence** is not only unmotivated as a rationality principle but is actually absurd.[5]

Why? Observe what **equivalence** actually claims: that if we know that two events have the same weight, then we must regard them as equally likely *regardless of any other information we may have about them*. Put another way, if we wish to determine which event to bet on and we are told that they have the same weight, we will be uninterested in any other information about them.

But in any interpretation which does not involve branching — that is, in any non-Everettian interpretation — there is a further piece of information that cannot but be relevant to our choice: namely, *which event is actually going to happen*? If in fact we know that E rather than F will actually occur, *of course* we will bet on E, regardless of the relative weights of the events.

This objection can be made precise in one of two ways. The first might be called the argument from *fatalism*, and presumes a B-theoretic (or indexical, or 'block-universe') view of time: that from God's perspective there is no difference between past, present and future. If this is the case, then there is a *fact of the matter* (regardless of whether the world is deterministic) as to which future event occurs. Maybe this fact is epistemically inaccessible to us even in principle; nonetheless *if* we knew it, of course it would influence our bets.

Maybe you don't like the B-theoretic view of time; or maybe you see something objectionable in appeal to in-principle-inaccessible facts. Then I commend to your attention the second way of making the objection precise: the argument from *determinism*. Suppose, for the moment, that quantum theory is deterministic and non-branching: for instance, it might be the de Broglie-Bohm pilot-wave theory, or some other deterministic hidden-variables theory). Then a sufficiently complete description of the microstate will determine exactly which event will occur, regardless of weights; again, if we had this complete description it would certainly influence (indeed, fix) our preferences between bets.

(Maybe certain details of the microstate are "in principle inaccessible", as is arguably the case in the pilot-wave theory. This doesn't improve matters: for the theory to have any predictive power at all we need to say *something* about the hidden variables, and in practice we need to give a probability distribution over them. Well, "hidden variables are randomly distributed in such-and-such a way" might be a reasonable *law of physics*, but absent such a law, there seems no justification at all for adopting "it is rational to assume such-and-such distribution of hidden variables".)

What if quantum theory is *in*deterministic? Again, if the theory is to have

[5]In this section I confine my observations to those interpretations of quantum mechanics which are in some sense "realist" and observer-independent (such as collapse theories or hidden-variable theories). I will not consider interpretations (such as the Copenhagen interpretation, or the recent variant defended by Fuchs and Peres 2000) which take a more 'operationalist' approach to the quantum formalism. It is entirely possible, as has been argued recently by Saunders (2003), that an approach based on Deutsch's proof may be useful in these interpretations.

predictive power then we must replace deterministic evolution with something else: a stochastic dynamics. In this case, it only makes sense to adopt **equivalence** if the stochastic dynamics makes equal-weighted events equiprobable — and as Barnum *et al* (2000) have pointed out in their discussion of Deutsch's work, there is no difficulty in constructing a stochastic dynamics for quantum mechanics which does no such thing.

The reason that the Everett interpretation is not troubled by these problems is simple. Regardless of our theories of time, and notwithstanding the determinism of her theory, for the Everettian there is simply *no fact of the matter* which measurement outcome will occur, and so it is not just impossible, but incoherent for an agent to know this fact. This coexistence of determinism with the in-principle-unknowability of the future is from a philosophical point of view perhaps the Everett interpretation's most intriguing feature.

7 Equivalence via measurement neutrality

If we have shown that **equivalence** is implausible except in the Everett interpretation, still we have not shown that it is plausible for Everettians; that is our next task. An obvious starting point is Deutsch's original work; but Deutsch makes no direct use of **equivalence**. Instead, he uses — implicitly, but extensively — a principle which I have elsewhere (Wallace 2003b) called **measurement neutrality**: the principle that once we have specified which system is being measured, which state that system is being prepared in, and which observable is being measured on it, then we have specified everything that we need to know for decision-making purposes. In this section I will show how **measurement neutrality** is sufficient to establish **equivalence**, and discuss whether **measurement neutrality** is itself justifiable; in the next section I will look at more direct arguments for **equivalence**.

In effect, assuming **measurement neutrality** allows us to replace the abstract set $\mathcal{M}$ of measurements with the set of triples $\langle \mathcal{H}, |\psi\rangle, \widehat{X}\rangle$, and the abstract set $\mathcal{S}_M$ of outcomes of the measurement with the spectrum $\sigma(\widehat{X})$ of $\widehat{X}$. The relation $\succeq$ is similarly transferred from ordered pairs $\langle E, M\rangle$ to ordered quadruples $\langle E, \mathcal{H}, |\psi\rangle, \widehat{X}\rangle$ (where $E \subseteq \sigma(\widehat{X})$ is now just a subset of $\Re$).

From a purely mathematical point of view, this move clearly gets us no closer to deriving **equivalence** (and, thus, the quantum representation theorem), and consequently most commentators on Deutsch's work (e. g. Barnum *et al* 2000; Gill 2003; Lewis 2003) have claimed that he is guilty of a *non sequitur* at various points in his derivation. In fact (as I argued in Wallace 2003b) he is guilty of no such thing.

The central insight of Deutsch's argument is that when measurement processes are represented physically there can be no unambiguous way to assign triples $\langle \mathcal{H}, |\psi\rangle, \widehat{X}\rangle$ to these physical processes. Hence, one and the same process may be validly represented by two triples — and if so, of course, the relation $\succeq$ should not distinguish between those triples.

How does the ambiguity arise? For one thing, the physical process of preparing a state and then measuring an observable on it consists of a large number of unitary transformations applied to the state and to various auxiliary systems and there is no privileged moment at which the preparation phase can be said to have finished and the measurement phase begun; nor is there any privileged way of saying which system is the one being measured and which is the 'auxiliary' system.

Consider, for instance, the Stern-Gerlach experiment (see, for instance, Feynman, Leighton, and Sands (1965) for an elementary discussion). A beam of particles prepared in some spin state is placed in an inhomogeneous magnetic field in the $+z$ direction, with the result that particles with spin up (in the z direction) are deflected in one direction and particles of spin down in another. The outgoing beams are incident on some detector which records the location of each particle.

The Stern-Gerlach device is generally treated as a paradigmatic measurement of particle spin — in this case along the z axis. Now suppose that we wish to apply that measurement to particles in spin state $|+_x\rangle = \frac{1}{\sqrt{2}}(|+_z\rangle + |-_z\rangle)$, but our preparation device only outputs spin state $|+_z\rangle$. There is an easy remedy: after the particles emerge from the preparation device, expose them to a homogenous magnetic field which causes their spins to precess, taking $|+_z\rangle$ to the desired state $|+_x\rangle$. Effectively, the new magnetic field is just an extra part of the preparation device.

Conversely, suppose that we want to measure the spin along the $-x$ axis but that it is too difficult physically to rotate the Stern-Gerlach apparatus. Again there is a simple solution: instead of rotating the apparatus, rotate the particles by exposing them to the same magnetic field. That magnetic field is effectively just part of the measurement device.

However, from a purely physical viewpoint there is no difference at all between these two processes. In each case, we prepare particles in state $|+_z\rangle$, expose them to a magnetic field, and then insert them into a Stern-Gerlach device aligned along the $+z$ axis. In the one case we have regarded the magnetic field as part of the preparation, in the other case as part of the measurement — but since this difference is purely a matter of convention and does not correspond to any *physical* difference, it should be regarded as irrelevant to the subjective likelihood of detecting given results.

In fact, we can go further than this: it is also only a matter of convention that we take the process to be a measurement of spin and not of position. For although we have described the splitting of the beam into two as part of the measurement process, it could equally well be regarded as part of a state preparation — in this case, to prepare a particle in a coherent superposition of two positions.[6]

[6]Technically the particles are actually in an entangled state, since their spin remains correlated with their position; the important point remains that the observable being measured could be taken to be a joint spin-position observable, rather than a pure spin observable. (In any case, removing this entanglement in the Stern-Gerlach measurement process would be technically possible, albeit rather difficult in practice.)

The moral of this example is as follows: any process which we describe as 'prepare a state, then measure it' actually consists of a long sequence of unitary transformations on a wide variety of quantum systems (for instance, the magnet performs a unitary transformation on the spin space of the particle, and then the Stern-Gerlach apparatus performs another on the joint spin-position space of the particle). As it is purely conventional when these transformations stop being part of the state preparation and start being part of the measurement process, probabilistic statements about that process cannot depend on that convention. Symbolically, this is to say that there will be some processes which are equally well described by $\langle \mathcal{H}, |\psi\rangle, \widehat{X}\rangle$ and by $\langle \mathcal{H}', \widehat{U}|\psi\rangle, \widehat{X}'\rangle$ where $\widehat{U}: \mathcal{H} \rightarrow \mathcal{H}'$ is a unitary transformation and $\widehat{U}\widehat{X} = \widehat{X}'\widehat{U}$. We can write this as

$$\langle E, \mathcal{H}, |\psi\rangle, \widehat{X}\rangle \sim \langle E, \mathcal{H}', |\psi'\rangle, \widehat{X}'\rangle, \tag{3}$$

which is to be read as 'there is some physical process represented both as a measurement of $\widehat{X}$ on $|\psi\rangle$ followed by a bet on E, and as a measurement of $\widehat{X}'$ on $|\psi'\rangle$ followed by a bet on E.'

Similarly, the physical process of reading off the result of a measurement from the apparatus involves both the physical state of the apparatus *and* some convention as to which number is associated with which physical state — even if the device has a needle pointing to the symbol "**6**" then it is a human convention and not a law of physics that associates that symbol with the sixth positive integer. But if so, then it is purely a matter of convention whether a measurement of $\widehat{X}$ which obtained result x should actually be regarded as a measurement of $f(\widehat{X})$ which obtained result $f(x)$.

Again, conventions should not affect our judgements about likelihood, if they do not correspond to anything physical. As such, it follows that

$$\langle E, \mathcal{H}, |\psi\rangle, \widehat{X}\rangle \sim \langle f(E), \mathcal{H}, |\psi\rangle, f(\widehat{X})\rangle \tag{4}$$

for arbitrary f.

From (3) and (4) we can establish **equivalence**. For if we take $f(x) = 0$ whenever $x \in E$ and $f(x) = 1$ otherwise, it follows from (4) it follows that if M is represented by the quadruple $\langle E, \mathcal{H}, |\psi\rangle, \widehat{X}\rangle$ then that quadruple is $\sim$-equivalent to one of form $\langle \{0\}, \mathcal{H}, |\psi\rangle, \widehat{X}_0, \rangle$, where $\widehat{X}_0$ has only 0 and 1 as eigenvalues.

Now let $\mathcal{H}_0$ be a fixed two-dimensional Hilbert space spanned by vectors $|0\rangle$ and $|1\rangle$. There will exist some unitary operator $\widehat{U}$ from $\mathcal{H}_0$ to $\mathcal{H}$ which takes $|0\rangle$ to the 0-eigenspace of $\widehat{X}$ and $|0\rangle$ to the $1-$eigenspace of $\widehat{X}$. This operator satisfies $\widehat{U}|1\rangle\langle 1| = \widehat{X}\widehat{U}$; from (3) it then follows that $\langle \{0\}, \mathcal{H}, |\psi\rangle, \widehat{X}_0, \rangle$ is $\sim$-equivalent to

$$\langle \{0\}, \mathcal{H}_0, c|0\rangle + d|1\rangle, |1\rangle\langle 1|\rangle, \tag{5}$$

where $c^2 = \mathcal{W}_M(E)$ and $c, d > 0$. So $\sim$-equivalence classes are characterised entirely by the weights they give to events.[7]

[7] For a slightly more detailed version of this argument (in a mildly different notation) see Wallace (2003b).

So: **measurement neutrality** is sufficient to establish the quantum probability theorem, given our physical definition of measurement.[8] Furthermore, the assumption is innocuous on its face: it is part of the basic structure of quantum mechanics that measurement processes are specified up to irrelevant details once we know what observable and state are to be measured.

Unfortunately, that innocuousness is somewhat misleading. The reasons why we treat the state/observable description as complete are not independent of the quantum probability rule. On the contrary, a standard argument might go: "the probability of any given outcome from a measurement process is specified completely by state and observable, so two different systems each described by the same state/observable pair will give the same statistics on measurement." Of course, such a justification is in danger of circularity in the present context.

In fact, it is possible to argue that from an Everettian (or indeed a collapse-theoretic) viewpoint, "measurements" are a misnamed and even unnatural category of processes. For traditionally the point of "measurement" is to learn something, whereas all that happens when we "measure" an already-known state in Everettian quantum mechanics is that we induce a certain decoherent process (and all that happens in collapse-theoretic quantum mechanics is that we trigger the collapse mechanism). In the betting scenarios that are the topic of this paper, in fact, there is in a sense nothing to learn: *ex hypothesi* the state is already known with certainty.

If 'measurements' are an unnatural category, **measurement neutrality** seems to be unmotivated: it's only decision-theoretically relevant that two processes fall under the same description if we have reason to believe that that description captures everything decision-theoretically relevant. As such, it is probably advisable for us to abandon **measurement neutrality** altogether, and look for a more direct justification of **equivalence**; that will be my strategy in the next section.

However, I remain unconvinced that **measurement neutrality** is so unmotivated even for an Everettian. For one thing, something seems wrong about the previous paragraph's attack on 'measurements'. It certainly fails to be an accurate description of real physicists' *modus operandi*: the experimentalist building (say) a cloud chamber believes himself designing a device that will detect particles, not one which induces decoherence, and the design principles he employs are selected accordingly.

This is, I think, one point where the OD and SU viewpoints on branching (discussed in section 2) lead to different results. Someone who regards the SU viewpoint as incoherent is already committed to the falsehood of much of our pre-theoretic discourse about quantum mechanics; as such they will probably be prepared to bite the bullet and say that 'real physicists' are profoundly mistaken about what 'measurement devices' are (just as all of us are profoundly mistaken in regarding measurement results as uncertain.) Without the SU viewpoint, I

[8]It is perhaps worth noting that Deutsch himself does not derive **equivalence** directly from **measurement neutrality**, but rather uses the latter to derive a number of special cases of **equivalence** which are sufficient to prove his own form of the quantum representation theorem. See Wallace (2003b) for more details.

think, **measurement neutrality** has no available defence and it is necessary to look directly for justifications of **equivalence**. (This is the conclusion drawn by Greaves (2004).)

Is it defensible for those who accept the SU viewpoint? As I discuss in Wallace 2003b, they will agree with Dirac and von Neumann that a measurement device performs two functions: it induces wave-function collapse of the state being measured into some eigenstate of the observable being measured, and it then evolves deterministically into a state indicating the eigenvalue of that state. But, *unlike* Dirac and von Neumann, they will not find this dual function mysterious. For an Everettian, the wave-function collapse is only "effective" and "phenomenological". More precisely, the branching structure of the universe (which in turn defines "effective collapse") is defined pragmatically, in terms of which structure(s) are explanatorily and predictively most useful (I defend this view in Wallace 2003a). The very fact that the measurement process is going to occur, and that it will rapidly lead to decoherence between the various output states of the measurement device, guarantees that the explanatorily and predictively natural branching structure to choose is one in which each branch finds the system being measured in an eigenstate.

Thus, from the SU viewpoint there *seems* (I go no further than this) to be an important sense in which "measuring devices" really do deserve that name, and hence in which **measurement neutrality** is indeed innocuous.

8 Equivalence, directly

Perhaps the reader is not prepared to accept the SU viewpoint; or perhaps s/he is unconvinced by the SU-dependent defence of **measurement neutrality** which I offered above. Either way, it seems worth looking directly at **equivalence**, to see if it can be justified without recourse to **measurement neutrality**. This is also of interest because it allows a direct reply to those critics of Deutsch (such as Lewis 2003) who argue that there can be no decision-theoretic reason to be indifferent between choices that lead to very different branching structures, even if they give the same quantum-mechanical weight to a given outcome.

In fact, a very simple and direct justification of **equivalence** is available. Consider, for simplicity, a Stern-Gerlach experiment of the sort discussed in section 7: an atom is prepared in a superposition $|+_x\rangle = \frac{1}{\sqrt{2}}(|+_z\rangle + |-_z\rangle)$ and then measured along the z axis. Accordiing to the result of the measurement, an agent receives some payoff. *Ex hypothesi* the agent is indifferent *per se* to what goes on during the measurement process and to what the actual outcome of the experiment is; all he cares about is the payoff.

We now consider two possible games (that is, associations of payoffs with outcomes):

Game 1: The agent receives the payoff iff the result is spin up.

Game 2: The agent receives the payoff iff the result is spin down.

In each game, the weight of the branch where the agent receives the payoff is 0.5; **equivalence**, in this context, is then the claim that the agent is indifferent between games 1 and 2.

To see that this is indeed the case, we need to model the games explicitly. Let $|\text{‘up’;reward}\rangle$ and $|\text{‘down’; no reward}\rangle$ be the quantum states of the two branches on the assumption that game 1 was played: that is, let the post-game global state[9] if game 1 is played be

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\text{‘up’;reward}\rangle + |\text{‘down’; no reward}\rangle). \tag{6}$$

Similarly, if game 2 is played then the quantum state is

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\text{‘down’;reward}\rangle + |\text{‘up’; no reward}\rangle). \tag{7}$$

Why should an agent be indifferent between a physical process which produces $|\psi_1\rangle$ and one which produces $|\psi_2\rangle$?

Well, recall that the agent is indifferent *per se* to the result of the experiment. This being the case, he will not object if we erase that result. Let $|\text{‘erased’,reward}\rangle$ indicate the state of the branch in which the reward was given post-erasure and $|\text{‘erased’,no reward}\rangle$ the post-erasure state of the no-reward branch. Then (game 1+erasure) leads to the state

$$|\psi_{1;e}\rangle = \frac{1}{\sqrt{2}}(|\text{‘erased’,reward}\rangle + |\text{‘erased’,no reward}\rangle) \tag{8}$$

and (game 2+erasure) to the state

$$|\psi_{2;e}\rangle = \frac{1}{\sqrt{2}}(|\text{‘erased’,reward}\rangle + |\text{‘erased’,no reward}\rangle) \tag{9}$$

— that is, (game 1+erasure) and (game 2+erasure) lead to the same state. If (game 1+erasure) and (game 2+erasure) are just different ways of producing the same physical state — different ways, moreover, which can be made to differ only over a period of a fraction of a second, in which the agent has no interest — then the agent should be indifferent between the two. Since he is also indifferent to erasure, he is indifferent between games 1 and 2, as required by **equivalence**.

Actually, as it stands this is too quick. It is implausible that the erasure process will lead to *precisely* the same state in both cases; in fact, microscopic differences (which the agent is unaware of and indifferent to) are bound to persist, on pain of a failure of decoherence. But this problem can be rectified as follows. Let $|\text{‘erased(i)’, reward}\rangle$ denote one of the vastly many possible quantum states which could be reached by erasure. Each of the $|\text{‘erased(i)’, reward}\rangle$ is indistinguisable to the agent; furthermore, since he is indifferent to the erasure (let alone to its details) he does not care which $|\text{‘erased(i)’, reward}\rangle$ results from the erasure process. Let $|\text{‘erased(j)’, no reward}\rangle$ have its obvious meaning.

[9] More precisely: the global state relative to the pre-game agent: there are of course all manner of other branches which are already effectively disconnected from the agent's branch.

The global state following (game 1+erasure) is then some

$$|\psi_1; i, j\rangle = |\text{'erased(i)', reward}\rangle + |\text{'erased(j)', no reward}\rangle \quad (10)$$

and the agent does not care which one; that is, he is indifferent between processes which produce any state in

$$\mathcal{S} = \{\frac{1}{\sqrt{2}}(|\text{'erased(i)', reward}\rangle + |\text{'erased(j)', no reward}\rangle)|i, j\}, \quad (11)$$

and furthermore he is indifferent between $|\psi_1\rangle$ and any element of $\mathcal{S}$. But of course, exactly the same argument tells us that he is indifferent between $|\psi_2\rangle$ and any element of $\mathcal{S}$; hence that he is indifferent between $|\psi_1\rangle$ and $|\psi_2\rangle$; hence that he is indifferent between games 1 and 2.

(This argument actually gives some insight into why an agent should care about the quantum weight. For suppose that we use an *un*equal superposition:

$$|p\rangle = \sqrt{p}\,|+_z\rangle + \sqrt{1-p}\,|-_z\rangle\,. \quad (12)$$

The result of (game 1+erasure) will be some element of

$$\mathcal{S}_p = \{\sqrt{p}\,|\text{'erased(i)', reward}\rangle + \sqrt{1-p}\,|\text{'erased(j)', no reward}\rangle\,|i, j\}; \quad (13)$$

the result of (game 2+erasure) will be some element of

$$\mathcal{S}_{1-p} = \{\sqrt{1-p}\,|\text{'erased(i)', reward}\rangle + \sqrt{p}\,|\text{'erased(j)', no reward}\rangle\,|i, j\}. \quad (14)$$

$\mathcal{S}_{1-p} = \mathcal{S}_p$ only when $p = 0.5$.)

The generalisation to other weights is straightforward: just add a third possible state ($|0_z\rangle$, say) of the system being measured, and define games 1 and 2 as before. If the system is prepared in state

$$\sqrt{w}\,|+_z\rangle + \sqrt{w}\,|-_z\rangle + \sqrt{1-2w}\,|0_z\rangle \quad (15)$$

then the global state after (game x+erasure) is some

$$|\psi; i, j, k\rangle = |\text{'erased(i)', reward}\rangle + |\text{'erased(j)', no reward}\rangle + |\text{'erased(k)', no reward}\rangle \quad (16)$$

whether we are playing game 1 or game 2. As for phase, this can be incorporated by allowing phase changes in the erasure process: if $|\text{'erased(i)', reward}\rangle$ is a valid erasure state, so is $\exp(i\theta)\,|\text{'erased(i)', reward}\rangle$. More directly, it can be incorporated by observing that a phase transformation of an entire branch is completely unobservable, so an agent should be indifferent to it.

Finally, recall that **equivalence** must hold even when the two weight-equivalent rewards occur in different chance setups. This too can be handled via erasure: simply erase all details of which particular setup is under consideration, except for the weights of the payoff and non-payoff branches.

9 Branching indifference

There are two closely related lacunae in the erasure proof of **equivalence**. Firstly, erasure may lead to branching: realistic erasure processes will usually lead not to a single $|\text{'erased(i)'}, \text{reward}\rangle$ but to a superposition of them. Secondly, **equivalence** must hold not just when we have two equally weighted branches, but when we have one branch whose weight equals the combined weights of several other branches.

Both of these lacunae would be resolved if we could establish

Branching indifference: An agent is rationally compelled to be indifferent about processes whose only consequence is to cause the world to branch, with no rewards or punishments being given to any of his descendants.

This principle is not at all obvious: why should I not care about whether there is one of me or a thousand ten minutes from now? More generally, why should the 'branching microstructure' of constantly dividing worlds which underlies the macroscopic ascription of weights to coarse-grained outcomes be decision-theoretically irrelvant? But it is in fact irrelevant, for two distinct reasons; my last task in this paper will be to establish this.

The first argument for branching indifference works only from the SU viewpoint. From that viewpoint, recall, branching events are to be understood as cases of *subjective uncertainty*: the agent should expect to experience one or other outcome, but does not (and cannot) know which.

But in this case, it is easy to see that the agent should be indifferent to branching *per se*. Suppose that someone proposes to increase a million-fold the number of the agent's descendants who see heads: say, by hiding within the measurement device a randomizer that generates and displays a number from one to 1 million, but whose output the agent doesn't care about and probably never sees. Then from the SU viewpoint, this just corresponds to introducing some completely irrelevant extra uncertainty. For it is the central premise of the SU viewpoint that process which from an objective standpoint involves branching, may be described subjectively as simply one with uncertain outcomes. In this case the objective description is "the agent branches into a million copies who see heads, and one copy who sees tails"; the correct description *for the agent himself* is "I will either see heads or tails, and I am uncertain as to which; if I see heads then I am further uncertain about the result of the randomiser reading — but I don't care about that reading".

But it is a (trivially) provable result of decision theory that introducing "irrelevant" uncertainty of this kind is indeed irrelevant (it is essentially the statement that if we divide one possible outcome into equally-valuable suboutcomes, that division is not decision-theoretically relevant). As such, from the SU viewpoint branching indifference follows trivially.

(Of course, a critic may deny that the observer's description really is correct — but this is simply another way to reject the SU viewpoint itself.)

Everettians who reject the SU viewpoint cannot resort to this strategy, but all Everettians can resort to the other argument: that it is not in fact possible to pursue a non-branch-indifferent strategy in a quantum universe. In part this is due to the fact that branching is going on all the time, which leads to two objections to any proposed violation of branch indifference:

1. The epistemic objection: to take decisions in such a universe, an agent who was not branch indifferent would have to be keeping microscopically detailed track of all manner of branch-inducing events (such as quantum decays) despite the fact that none of these events have any detectable effect on him. This is beyond the plausible capabilities of any agent.

2. The small-world objection: it has long been recognised (see, e. g. , Savage 1972) that decision-making will be impossibly complicated unless it is possible to identify (in at least a rough-and-ready manner) a point after which the dust has settled and the value to an agent of consequences can actually be assessed. But if an agent is not branch indifferent, then such a point will never occur, and he will be faced with the impossible task of calculating how much branching will occur across the entire lifetime of the Universe (contingent on his choice of action) in order to weigh up the value, now, to him of carrying out a certain act.

Both of these objections rely on the assumption that a rational strategy must actually be realisable in at least some idealised sense. In earlier versions of this work I took this as self-evident, but to my surprise this has not generally been accepted (mostly this has emerged in conversation; however, see also Lewis 2003). I therefore offer a brief defence:

1. Decision theory is something of a hybrid. It is to some extent normative (that is, it tells us what we *should* do, and exposes us to rational criticism if we violate its precepts); it is to some extent descriptive (that is, it provides an idealised account of actual decision-making). Both of these are impossible if decision theory instructs us to do something wildly beyond even our idealised abilities.

2. If we are prepared to be even slightly instrumentalist in our criteria for belief ascription, it may not even *make sense* to suppose that an agent genuinely wants to do something that is ridiculously beyond even their idealised capabilities. For instance, suppose I say that I desire (*ceteris paribus*) to date someone with a prime number of atoms in their body. It is not even remotely possible for me to take any action which even slightly moves me towards that goal. In practice my actual dating strategy will have to fall back on "secondary" principles which have no connection at all to my "primary" goal — and since those secondary principles are actually what underwrites my entire dating behaviour, arguably it makes more sense to say that *they* are my actual desires, and that my 'primary' desire is at best an impossible dream, at worst an empty utterance.

However, even if this is not persuasive, then there is a stronger reason why non-branch indifferent strategies cannot be pursued. Namely: non-branch-indifferent strategies require us to know the number of branches, and there is no such thing.

Why? Because the models of splitting often considered in discussions of Everett — usually involving two or three discrete splitting events, each producing in turn a smallish number of branches — bear little or no resemblance to the true complexity of realistic, macroscopic quantum systems. In reality:

- Realistic models of macroscopic systems are invariably infinite-dimensional, ruling out any possibility of counting the number of discrete descendants.
- In such models the decoherence basis is usually a continuous, over-complete basis (such as a coherent-state basis[10] rather than a discrete one, and the very idea of a discretely-branching tree may be inappropriate. (I am grateful to Simon Saunders for these observations).
- Similarly, the process of decoherence is ongoing: branching does not occur at discrete loci, rather it is a continual process of divergence.
- Even setting aside infinite-dimensional problems, the only available method of 'counting' descendants is to look at the time-evolved state vector's overlap with the subspaces that make up the (decoherence-) preferred basis: when there is non-zero overlap with one of these subspaces, I have a descendant in the macrostate corresponding to that subspace. But the decoherence basis is far from being precisely determined, and in particular exactly how coarse-grained it is depends sensitively on exactly how much interference we are prepared to tolerate between 'decohered' branches. If I decide that an overlap of $10^{-10^{10}}$ is too much and change my basis so as to get it down to $0.9 \times 10^{-10^{10}}$, my decision will have dramatic effects on the "head-count" of my descendants.
- Just as the coarse-graining of the decoherence basis is not precisely fixed, nor is its position in Hilbert space. Rotating it by an angle of 10 degrees will of course completely destroy decoherence, but rotating it by an angle of $10^{-10^{10}}$ degrees assuredly will not. Yet the number of my descendants is a discontinuous function of that angle; a judiciously chosen rotation may have dramatic effects on it.
- Branching is not something confined to measurement processes. The interaction of decoherence with classical chaos guarantees that it is completely ubiquitous: even if I don't bother to turn on the device, I will still undergo myriad branching while I sit in front of it. (See Wallace (2001, section 4) for a more detailed discussion of this point.)

The point here is not that there is no *precise* way to define the number of descendants; the entire decoherence-based approach to the preferred-basis problem turns (as I argue in Wallace (2003a)) upon the assumption that exact precision is not required. Rather, the point is that there is *not even an approximate way* to make such a definition.

[10]See, for instance, (Zurek, Habib, and Paz 1993).

In the terminology of Wallace (2003a), the arbitrariness of any proposed definition of number of descendants makes such a definition neither predictive nor explanatory of any detail of the quantum state's evolution, and so no such definition should be treated as part of macroscopic reality. (By contrast, the macroscopic structure defined by decoherence (which can be specified by how the weights of a family of coarse-grained projectors in the decoherence basis change over time) *is* fairly robust, and so should be treated as real. It is only when we start probing that structure to ridiculously high precision — as we must do in order to count descendants — that it breaks down.)

So: whether or not the SU viewpoint is coherence, we are in a position to argue that rational agents do not care about branching *per se*. From this, the erasure argument yields **equivalence**, and with it the quantum representation theorem.

A quick way of understanding why these arguments work goes as follows. Some physical difference between games might be:

1. A change to a given branch which an agent cares about;
2. A change to a given branch which an agent doesn't care about;
3. A change to the relative weights of branches; or
4. a splitting of one branch into many, all of which are qualitatively identical for the agent's descendants in that branch.

By **branching indifference**, (4) may be discounted. Any change of type (1) may be incorporated into an agent's utility function without affecting the probabilities. Changes of type (2) can be erased — by definition the agent doesn't care about the erasure. This only leaves changes of type (3), which cannot be erased on pain of unitarity violation.

10 Conclusion

In Wallace (2003b), I identified four assumptions which I claimed (and still claim!) are required for *Deutsch's* proof of the probability rule to go through: the Everett interpretation, the subjective-uncertainty viewpoint on that interpretation, **measurement neutrality**, and a "fairly strong set of decision-theoretic axioms". I also argued that **measurement neutrality** was at least plausibly a consequence of the SU viewpoint (using roughly the argument of (the current paper's) section 7.)

The present paper may be seen as an exploration of the extent to which these assumptions can or cannot be weakened. We have found the following:

- Despite the substantial reformulation of Deutsch's proof described above, the Everett interpretation remains crucial: as section 6 argued, the central assumption of my reformulation (**equivalence**) is just as dependent on the Everettian assumption as is Deutsch's own proof. Furthermore, there is no realistic prospect of any decision-theoretic proof which applies to (realist) interpretations other than Everett's: all such proofs will inevitably end up requiring us to be indifferent to which outcome is *actually* going to occur, which is absurd in any single-universe interpretation of quantum mechanics.

- **Measurement neutrality** *per se* need not be taken as a premise of the argument. As was argued in section 8, it is possible to defend **equivalence** directly, without recourse to **measurement neutrality**.
- The SU viewpoint is required in Deutsch's proof (I have argued) both to defend **measurement neutrality** and to justify the applicability of decision theory. However, the decision-theoretic axioms which I have advanced have extremely natural justifications from the perspective of the OD viewpoint, and sections 8–9 show how someone who accepts only that viewpoint can defend **equivalence** directly. So it seems that we are not forced to adopt the SU viewpoint, at least not in order to prove the probability rule.
- Deutsch's decision-theoretic axioms are far stronger than is strictly necessary. The notion of a minimally rational preference ordering discussed in section 4 is much weaker, yet still sufficiently strong to derive the quantum representation theorem. Hence, criticisms of Deutsch's program based on the specifics of his decision theory appear to be beside the point.

Given the enormous contribution that decoherence theory has made to the problem of defining a preferred basis, quantitative probability is arguably the last major obstacle confronting the Everett interpretation. In my view the evidence is now quite strong that the decision-theoretic strategy which Deutsch suggested is able to solve the problem; if so, the significance for Everett's program is hard to overstate.

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