A Topos Perspective on the Kochen-Specker Theorem:
I. Quantum States as Generalized Valuations

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Abstract

Any attempt to construct a realist interpretation of quantum theory founders on the Kochen-Specker theorem, which asserts the impossibility of assigning values to quantum quantities in a way that preserves functional relations between them. We construct a new type of valuation which is defined on all operators, and which respects an appropriate version of the functional composition principle. The truth-values assigned to propositions are (i) contextual; and (ii) multi-valued, where the space of contexts and the multi-valued logic for each context come naturally from the topos theory of presheaves.

The first step in our theory is to demonstrate that the Kochen-Specker theorem is equivalent to the statement that a certain presheaf defined on the category of self-adjoint operators has no global elements. We then show how the use of ideas drawn from the theory of presheaves leads to the definition of a generalized valuation in quantum theory whose values are sieves of operators. In particular, we show how each quantum state leads to such a generalized valuation.

A key ingredient throughout is the idea that, in a situation where no normal truth-value can be given to a proposition asserting that the value of a physical quantity A lies in a subset $\Delta \subset \mathbb{R}$, it is nevertheless possible to ascribe a partial

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truth-value which is determined by the set of all coarse-grained propositions that assert that some function $f(A)$ lies in $f(\Delta)$, and that *are* true in a normal sense. The set of all such coarse-grainings forms a sieve on the category of self-adjoint operators, and is hence fundamentally related to the theory of presheaves.

1 Introduction

1.1 Preliminary Remarks

Anyone who has taught an introductory course on quantum theory will have encountered the anguish that can accompany a student's first engagement with the problematic status of beliefs previously deemed to be self-evidently true. In particular, it is difficult to remove the compelling conviction that, at any given time, any physical quantity must have a value.

In classical physics, there is no problem with this belief since the underlying mathematical structure is geared precisely to express it. Specifically, if $\mathcal{S}$ is the state space of some classical system, a physical quantity A is represented by a real-valued function $\bar{A} : \mathcal{S} \to \mathbb{R}$; and then the value $V_s(A)$ of A in any state $s \in \mathcal{S}$ is simply

$$V^s(A) = \bar{A}(s). \tag{1.1}$$

Thus all physical quantities possess a value in any state. Furthermore, if $h : \mathbb{R} \to \mathbb{R}$ is a real-valued function, a new physical quantity $h(A)$ can be defined by requiring the associated function $\overline{h(A)}$ to be

$$\overline{h(A)}(s) := h(\bar{A}(s)) \tag{1.2}$$

for all $s \in \mathcal{S}$; *i.e.*, $\overline{h(A)} := h \circ \bar{A} : \mathcal{S} \to \mathbb{R}$. Thus the physical quantity $h(A)$ is *defined* by saying that its value in any state s is the result of applying the function h to the value of A; hence, by definition, the values of the physical quantities $h(A)$ and A satisfy the 'functional composition principle'

$$V^s(h(A)) = h(V^s(A)) \tag{1.3}$$

for all states $s \in \mathcal{S}$.

However, to the distress of angst-ridden students, standard quantum theory precludes any such naive realist interpretation of the relation between formalism and physical world. And this is not just because of some wilfully obdurate philosophical interpretation of the theory: rather, the obstruction comes from the mathematical formalism itself, in the guise of the famous Kochen-Specker theorem which asserts the impossibility of assigning values to all physical quantities whilst, at the same time, preserving the functional relations between them [1].[4]

In a quantum theory, a physical quantity A is represented by a self-adjoint operator $\hat{A}$ on the Hilbert space of the system, and the first thing one has to decide is whether to regard a valuation as a function of the physical quantities themselves, or on the operators that represent them. From a mathematical perspective, the latter strategy is preferable, and we shall therefore define a (global) valuation to be a real-valued function V on the set of all bounded, self-adjoint operators, with the properties that : (i) the value $V(\hat{A})$ of

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[4]As has been emphasized by Brown [2], the essential result is already contained in Bell's seminal first paper on hidden variables [3].

the physical quantity A represented by the operator $\hat{A}$ belongs to the spectrum of $\hat{A}$ (the so-called 'value rule'); and (ii) the functional composition principle (or FUNC for short) holds:

$$V(\hat{B}) = h(V(\hat{A})) \tag{1.4}$$

for any pair of self-adjoint operators $\hat{A}$, $\hat{B}$ such that $\hat{B} = h(\hat{A})$ for some real-valued function h. If they existed, such valuations could be used to embed the set of self-adjoint operators in the commutative ring of real-valued functions on an underlying space $\mathcal{S}$ of microstates, thereby laying the foundations for a hidden-variable interpretation of quantum theory.

Several important results follow from the definition of a valuation. For example, if $\hat{A}_1$ and $\hat{A}_2$ commute, there exists an operator $\hat{C}$ and functions h_1 and h_2 such that $\hat{A}_1 = h_1(\hat{C})$ and $\hat{A}_2 = h_2(\hat{C})$; it then follows from FUNC that

$$V(\hat{A}_1 + \hat{A}_2) = V(\hat{A}_1) + V(\hat{A}_2) \tag{1.5}$$

and

$$V(\hat{A}_1\hat{A}_2) = V(\hat{A}_1)V(\hat{A}_2). \tag{1.6}$$

The defining equation Eq. (1.4) for a valuation makes sense whatever the nature of the spectrum $\sigma(\hat{A})$ of the operator $\hat{A}$. However, if $\sigma(\hat{A})$ contains a continuous part, one might doubt the physical meaning of assigning one of its elements as a value; indeed, in the present paper, we shall consider valuations in this sense as being defined only on the subset of operators whose spectrum is purely discrete. To handle the more general case, we shall reconceive a valuation as primarily giving *truth-values* to *propositions* about the values of a physical quantity, rather than assigning a specific value to the quantity itself.

The propositions concerned are of the type '$A \in \Delta$', which asserts that the value of the physical quantity A lies in the Borel subset Δ of the spectrum $\sigma(\hat{A})$ of the associated operator $\hat{A}$. Of course, such assertions are meaningful for both discrete and continuous spectra: which motivates studying the general mathematical problem of assigning truth-values to projection operators.

If $\hat{P}$ is a projection operator, the identity $\hat{P} = \hat{P}^2$ implies that $V(\hat{P}) = V(\hat{P}^2) = (V(\hat{P}))^2$ (from Eq. (1.6)); and hence, necessarily, $V(\hat{P}) = 0$ or 1. Thus V defines a homomorphism from the Boolean algebra $\{\hat{0}, \hat{1}, \hat{P}, \neg\hat{P} \equiv (\hat{1} - \hat{P})\}$ to the 'false(0)-true(1)' Boolean algebra $\{0, 1\}$. More generally, a valuation V induces a homomorphism $\chi^V : W \to \{0, 1\}$ where W is any Boolean subalgebra of the lattice $\mathcal{P}$ of projectors on $\mathcal{H}$. In particular,

$$\hat{\alpha} \leq \hat{\beta} \text{ implies } \chi^V(\hat{\alpha}) \leq \chi^V(\hat{\beta}) \tag{1.7}$$

where '$\hat{\alpha} \leq \hat{\beta}$' refers to the partial ordering in the lattice $\mathcal{P}$, and '$\chi^V(\hat{\alpha}) \leq \chi^V(\hat{\beta})$' is the ordering in the Boolean algebra $\{0, 1\}$. This result has an important implication for us, to which we shall return shortly.

The Kochen-Specker theorem asserts that no global valuations exist if the dimension of the Hilbert space $\mathcal{H}$ is greater than two. The obstructions to the existence of such valuations typically arise when trying to assign a single value to an operator $\hat{C}$ that can be written as $\hat{C} = g(\hat{A})$ and as $\hat{C} = h(\hat{B})$ with $[\hat{A}, \hat{B}] \neq 0$.

One response to this result is to note that the theorem does not preclude the existence of 'partial', or 'local', valuations—*i.e.*, valuations that are defined only on some subset of the set of self-adjoint operators; a typical example would be any complete set of commuting operators on the Hilbert space. However, if partial valuations are to form part of a proper interpretative framework, the question immediately arises as to how the domain of any such valuation is to be chosen.

The extant interpretations of quantum theory that aspire to use 'beables', rather than 'observables', are all concerned in one way or another with addressing this issue. One well-known approach is that of Bohm, where certain physical quantities—for example, the position of a particle—are declared by fiat to be those that always have a value. In other, so-called 'modal' approaches, the domain of a partial valuation depends on the quantum state; as, for example, in the works of van Fraassen [4, 5], Kochen [6], Healey [7], Clifton [8], Dieks [9], Vermaas and Dieks [10], Bacciagaluppi and Hemmo [11], and Bub [12].

Inherent in such schemes is a type of 'contextuality' in which a value ascribed to a physical quantity C cannot be part of a global assignment of values but must, instead, depend on some *context* in which C is to be considered. In practice, contextuality is endemic in any attempt to ascribe properties to quantities in a quantum theory. For example, as emphasized by Bell [3], in the situation where $\hat{C} = g(\hat{A}) = h(\hat{B})$, if the value of C is construed counterfactually as referring to what would be obtained *if* a measurement of A or of B is made—and with the value of C then being *defined* by applying the relation $C = g(A)$, or $C = h(B)$, to the result of the measurement—then one can claim that the actual value obtained depends on whether the value of C is determined by measuring A, or by measuring B.

In the programme to be discussed here, the idea of a contextual valuation will be developed in a different direction from that of the existing modal interpretations. In particular, rather than accepting only a limited domain of beables we shall propose a theory of 'generalized' valuations that are defined globally on *all* propositions about values of physical quantities. However, the price of global existence is that any given proposition may have only a '*partial*' truth-value. More precisely, (i) the truth-value of a proposition '$A \in \Delta$' belongs to a logical structure that is larger than $\{0, 1\}$; and (ii) these target-logics are context-dependent.

It is clear that the main task is to formulate mathematically the idea of a contextual, 'partial' truth-value in such a way that the assignment of generalized truth-values is consistent with an appropriate analogue of the functional composition principle FUNC. The scheme also has to have some meaningful physical interpretation; in particular, we want the set of all possible partial truth-values for any given context to form some sort of *distributive* logic, in order to facilitate a proper semantics for this 'neo-realist' view of quantum theory.

1.2   Generalized Logic in Quantum Physics

Our central idea is that, although in a given situation in quantum theory it may not be possible to declare a particular proposition '$A \in \Delta$' to be true (nor false), nevertheless there may be (Borel) functions f such that the associated propositions '$f(A) \in f(\Delta)$' *can* be said to be true. This possibility arises for the following reason.

Let W_A denote the spectral algebra of the operator $\hat{A}$ that represents a physical quantity A: thus W_A is the Boolean algebra of projectors $\hat{E}[A \in \Delta]$ that project onto the eigenspaces associated with the Borel subsets Δ of the spectrum $\sigma(\hat{A})$ of $\hat{A}$; physically speaking, $\hat{E}[A \in \Delta]$ represents the proposition '$A \in \Delta$'. It follows from the spectral theorem that, for all Borel subsets J of the spectrum of $f(\hat{A})$, the spectral projector $\hat{E}[f(A) \in J]$ for the operator $f(\hat{A})$ is equal to the spectral projector $\hat{E}[A \in f^{-1}(J)]$ for $\hat{A}$. In particular, if $f(\Delta)$ is a Borel subset of $\sigma(f(\hat{A}))$ (which is automatically true if the spectrum of $\hat{A}$ is discrete; we shall discuss the non-discrete case later) then, since $\Delta \subseteq f^{-1}(f(\Delta))$, we have $\hat{E}[A \in \Delta] \leq \hat{E}[A \in f^{-1}(f(\Delta))]$; and hence

$$\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in f(\Delta)]. \tag{1.8}$$

Physically, the inequality in Eq. (1.8) reflects the fact that the proposition '$f(A) \in f(\Delta)$' is generally weaker than the proposition '$A \in \Delta$' in the sense that the latter implies the former, but not necessarily vice versa. For example, the proposition '$f(A) = f(a)$' is weaker than the original proposition '$A = a$' if the function f is many-to-one and such that more than one eigenvalue of $\hat{A}$ is mapped to the same eigenvalue of $f(\hat{A})$. In general, we shall say that '$f(A) \in f(\Delta)$' is a *coarse-graining* of '$A \in \Delta$'.

Now if the proposition '$A \in \Delta$' is evaluated as 'true' by, for example, a partial valuation V of the type mentioned at the end of Section 1.1—so that $V(\hat{E}[A \in \Delta]) = 1$—then, from Eq. (1.7) and Eq. (1.8), it follows that the weaker proposition '$f(A) \in f(\Delta)$' is also evaluated as 'true'.

This remark provokes the following observation. There may be situations in which, although the proposition '$A \in \Delta$' cannot be said to be either true or false, the weaker proposition '$f(A) \in f(\Delta)$' can be. In particular, if the latter *can* be given the value 'true' in a total sense, then—by virtue of the remark above—it is natural to suppose that any further coarse-graining to give an operator $g(f(\hat{A}))$ will yield a proposition '$g(f(A)) \in g(f(\Delta))$' that also is to be evaluated as 'true'. Note that there may be more than one possible choice for the 'initial' function f, each of which can then be further coarse-grained in this way. This multi-branched picture of coarse-graining is one of the main justifications for our invocation of the topos-theoretic idea of a presheaf.

In fact, guided by the remarks above, the procedure we shall adopt in Section 3 is first to consider partial valuations—which assign truth-values 0 or 1 in a standard way, but are defined on less than all the operators—and then to go on to *define* the partial truth-value (associated with each partial valuation V) of *any* proposition '$A \in \Delta$' to be the set of all operators $\hat{B}$ of the form $\hat{B} = f(\hat{A})$ that are in the domain of V, and which are such that the weaker proposition '$f(A) \in f(\Delta)$' is 'totally true'—*i.e.*, it is assigned the unit in the logic of partial truth-values.

We shall then generalize this idea in Section 4 where we extract the key properties of these partial truth-values and use them to formulate a definition of a 'generalized valuation', the semantic interpretation of which is that the truth-value of a proposition '$A \in \Delta$' is a set of coarse-grained propositions '$f(A) \in f(\Delta)$' each of which can be regarded as being totally true. As we shall show, any quantum state gives rise to such a generalized valuation.

The key property of such a generalized truth-value is that it is a *sieve* in a certain category formed from the self-adjoint operators on the Hilbert space of the system—and it is a fundamental property of sieves that they form a *Heyting* algebra, and hence have the structure of a distributive logic; albeit one that is *intuitionistic*, not classical, in the sense that the logical law of excluded middle is replaced with the weaker condition $\alpha \vee \neg \alpha \leq 1$. These sieves are associated with a certain presheaf—the 'spectral presheaf'— that is naturally associated with any quantum theory: this is how ideas from topos theory enter our scheme.

This procedure was partly motivated by an earlier paper in which topos ideas were applied to the consistent histories approach to quantum theory [13]; in particular, it was shown there how a topos framework fits naturally with the multi-branched, coarse-graining operations that play a central role in the construction of consistent sets of propositions. Contextuality arises explicitly there as the need to choose a particular consistent set of histories; and—in fact—topos-theoretic ideas can be expected to arise naturally in any physical theory where contextuality plays a central role. Presheaves are particularly important in this respect since they are naturally associated with contextual, generalized truth-values given by the so-called 'subobject classifier'.

Another motivation for our procedure is more general and conceptual. In short, it represents a *via media* between two extremes in the semantics, or interpretation, of quantum theory. For, on the one hand, the Kochen-Specker theorem shows the impossibility of sustaining any naive realist interpretation of quantum theory in which propositions about the values of physical quantities are handled with the simple type of Boolean logic which is characteristic of, for example, the set of subsets of a classical state space. And, on the other hand, we believe that the 'logical' structure inherent in the lattice of projection operators that represent quantum propositions mathematically is too non-classical—in particular, it is non-distributive—to fulfill any genuine semantic role. (This is a well-known viewpoint; for example, see Dummett's [14] critique of Putnam's proposal to "read the logic off Hilbert space" [15].) Our aim is to find a middle path between these extremes with the aid of logical structures that are certainly not just simple Boolean algebras—our logics are contextual and intuitionistic—but which retain the semantically crucial property of distributivity. We hope that this intermediate position will extend a little our encompassing of 'quantum reality'.

1.3 Some Expected Properties of Generalized Truth Values

To further motivate the detailed constructions that will be made in this paper it is helpful at this stage to consider what can be said *ab initio* about the assignment of partial truth

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values. For example, presumably the minimum that should be satisfied by the analogue of FUNC is that if $\hat{B} = h(\hat{A})$, and if the proposition '$A \in \Delta$' is assigned the value 'totally true', then the proposition '$B \in h(\Delta)$' should also be 'totally true'. As we shall see, this requirement is implemented in a simple way in the presheaf framework which we employ.

A central problem in handling multi-valued truth-values is to understand how the internal mathematical operations of the 'target' logic are to be related to the logical structure of the propositions being evaluated. More precisely, let L denote the Boolean algebra of all propositions of the type '$A \in \Delta$' for some fixed physical quantity A—and suppose we have some assignment of partial truth-values, $\nu : L \to T(L)$, where $T(L)$ is the target logic in the context of L. Then how should the structure of L be reflected in the properties of ν and the logical structure of $T(L)$? For example, is ν some type of algebraic homomorphism? The minimum that can be said in this direction would seem to be the following.

Firstly, the null proposition corresponding to the zero element $0_L \in L$ should presumably always be valued as totally false; and hence we expect $\nu(0_L) = 0_{T(L)}$ in all contexts.

Secondly, if $\alpha, \beta \in L$ are such that $\alpha \leq \beta$, then the physical interpretation is that the proposition α implies the proposition β; an example is '$A \in \Delta_1$' and '$A \in \Delta_2$' respectively, with $\Delta_1 \subseteq \Delta_2$. Under these circumstances, the analogy with Eq. (1.7) suggests that the generalized truth-values should satisfy $\nu(\alpha) \leq \nu(\beta)$ in the target logic $T(L)$. In what follows, we shall refer to this central requirement as the 'monotonicity' condition.

Now, for any $\alpha, \beta \in L$, we have $\alpha \leq \alpha \vee \beta$ and $\beta \leq \alpha \vee \beta$; hence it follows from monotonicity that $\nu(\alpha) \leq \nu(\alpha \vee \beta)$ and $\nu(\beta) \leq \nu(\alpha \vee \beta)$. This implies that, in the logic $T(L)$,

$$\nu(\alpha) \vee \nu(\beta) \leq \nu(\alpha \vee \beta) \tag{1.9}$$

if we assume that the 'or' operation in the target logic $T(L)$ behaves as expected, *i.e.*, it is the least upper bound for the partial ordering.

One might wonder if the stronger disjunctive rule $\nu(\alpha \vee \beta) = \nu(\alpha) \vee \nu(\beta)$ holds but, on reflection, this is at variance with certain key ideas of quantum theory. For example, suppose that α and β are the propositions '$A = a_1$' and '$A = a_2$' respectively, with $a_1 \neq a_2$. Then the projection operators that represent these propositions project onto the eigenstates of $\hat{A}$ corresponding to the eigenvalues a_1 and a_2 respectively. However, in the lattice of projectors, the disjunction of these operators projects onto the two-dimensional space spanned by these eigenvectors, which is strictly *bigger* than the union of the pair of one-dimensional spaces (which, indeed, is not a linear subspace at all). Hence a generalized truth-value $\nu(\alpha \vee \beta)$ of $\alpha \vee \beta$ might be greater (in the logical sense) than the disjunction of the generalized truth-values of α and β separately. We shall see in several concrete examples that this is indeed the case.

Similarly, for any $\alpha, \beta \in L$, we have $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$, so that, by monotonicity, $\nu(\alpha \wedge \beta) \leq \nu(\alpha)$ and $\nu(\alpha \wedge \beta) \leq \nu(\beta)$. Assuming that the 'and' operation, '$\wedge$', in the target logic $T(L)$ behaves as expected—*i.e.*, is the greatest lower bound for the partial ordering—it follows that

$$\nu(\alpha \wedge \beta) \leq \nu(\alpha) \wedge \nu(\beta). \tag{1.10}$$

Here also, one might wonder if a stronger conjunctive rule $\nu(\alpha \wedge \beta) = \nu(\alpha) \wedge \nu(\beta)$ holds; but we can see at once that it cannot do so in any scheme in which 'blurred' truth-values occur. For example, suppose once more that α and β are the propositions '$A = a_1$' and '$A = a_2$' respectively, with $a_1 \neq a_2$. Then, as explained earlier, our key idea is to assign a partial truth-value to a proposition like '$A = a$' by finding a 'coarse-grained' operator $\hat{B} = f(\hat{A})$ such that the weaker proposition '$f(A) = f(a)$' *is* totally true. One consequence is that, even though the propositions '$A = a_1$' and '$A = a_2$' are disjoint—so that $\alpha \wedge \beta = 0$—this does not imply that $\nu(\alpha) \wedge \nu(\beta)$ is totally false: all that is needed is an operator $\hat{B} = f(\hat{A})$ with $f(a_1) = f(a_2)$ and such that '$f(A) = f(a_1)$' is unequivocally true. In this circumstance, the strict inequality holds in Eq. (1.10).

The monotonicity rule requires supplementing in one respect. Consider again the propositions '$A = a_1$' and '$A = a_2$' with $a_1 \neq a_2$, and suppose the generalized valuation is such that $\nu(A = a_1) = 1_{T(L)}$—i.e., the proposition '$A = a_1$' is totally true in the logic $T(L)$. Then it seems natural to require that the disjoint proposition '$A = a_2$' cannot also be *totally* true, even though it need not be totally false either. However, for the following reason, this restriction—which we shall refer to as 'exclusivity'—cannot be deduced from the monotonicity condition.

The disjointness condition $\alpha \wedge \beta = 0$ in the Boolean algebra L, implies that $\beta \leq \neg\alpha$; and hence, using monotonicity,

$$\nu(\beta) \leq \nu(\neg\alpha). \tag{1.11}$$

Now, *if* we assumed that $\nu : L \rightarrow T(L)$ commutes with the negation operation, in the sense that

$$\nu(\neg\alpha) = \neg\nu(\alpha), \tag{1.12}$$

then Eq. (1.11) plus the hypothesis $\nu(\alpha) = 1_{T(L)}$, would imply that $\nu(\beta) \leq \neg\nu(\alpha) = \neg 1_{T(L)} = 0_{T(L)}$; hence $\nu(\beta) = 0_{T(L)}$, which certainly satisfies exclusivity. However, it turns out that the equality Eq. (1.12) is precisely what *cannot* be assumed in our theory since, as we shall see later, the target logic for the generalized truth-values is a Heyting algebra, and the negation operation in an intuitionistic logic of this type behaves differently from that in a Boolean algebra. As a result, the exclusivity condition cannot be derived from monotonicity, and it must therefore be added as an extra requirement.

Putting together all these remarks, we arrive at the following tentative, minimal list of algebraic properties that we expect to be satisfied by a generalized valuation $\nu : L \rightarrow T(L)$ of a Boolean logic L:

Null condition:	$\nu(0_L) = 0_{T(L)}$	(1.13)
Monotonicity:	$\alpha \leq \beta$ implies $\nu(\alpha) \leq \nu(\beta)$	(1.14)
Exclusivity:	If $\alpha \wedge \beta = 0_L$ and $\nu(\alpha) = 1_{T(L)}$, then $\nu(\beta) < 1_{T(L)}$	(1.15)

As we shall see, the examples of generalized valuations in quantum theory given in this paper satisfy these requirements. Another condition that we might want to add is

Unit condition:	$\nu(1_L) = 1_{T(L)}$	(1.16)

which, as we shall see, is also satisfied by the valuations associated with quantum states. On the other hand, it can be violated by the generalized valuations, which we mentioned

in Section 1.2 that are associated with partial valuations (mentioned in Section 1.2). We shall see this explicitly in Section 3.4.

1.4 Prospectus

The plan of the paper is as follows. In Section 2, we show how the Kochen-Specker theorem can be viewed as asserting the non-existence of global sections of certain presheaves that are naturally associated with any quantum theory. A key ingredient here is the idea that the set of all bounded, self-adjoint operators forms an appropriate category on which to form presheaves, as does the set of all Boolean subalgebras of projectors. Readers unfamiliar with topos theory may find it helpful to read the Appendix before embarking on this section.

By rewriting the Kochen-Specker theorem in terms of presheaves, several ways of generalizing the idea of a valuation present themselves. In this paper we pursue *one* particular scheme: to motivate the definition we finally arrive at, we show in Section 3 how a partial valuation (of the type used in the extant modal interpretations of quantum theory) gives rise to a generalized valuation whose truth-values lie in the Heyting algebra of sieves on an object in the category of self-adjoint operators. By these means, we arrive naturally at contextualized, multi-valued truth-value assignments.

Then, in Section 4 we use these results to motivate the formal definition of a generalized valuation, and we show how any state in a quantum theory gives rise to one such. In Section 5, we extend these ideas to the case where the space of contexts is taken as the category of all Boolean subalgebras of projectors, rather than the category of self-adjoint operators.

This paper is intended to be the first in a series devoted to an extensive analysis of the possible uses of topos ideas in quantum theory. Our main aim in the present paper is to present the basic mathematical tools and some general ideas about using quantum states to produce generalized valuations, but this leaves much work to be done: in particular, an analysis of the philosophical implications of generalized truth-values will be given in a future paper, as will the way in which similar ideas can arise in classical physics [16]. For this reason, the present paper concludes with a short summary of what has been achieved so far, and a list of some of the more significant topics for further research.

2 The Kochen-Specker Theorem in the Language of Topos Theory

2.1 The Categories of Boolean Algebras and Self-adjoint Operators

A key step in formulating the Kochen-Specker theorem in the language of topos theory is the construction of several categories that will form the domains of the presheaf functors

we shall be using later. Readers unfamiliar with topos theory may find it helpful to read the Appendix first. This contains a short introduction to the relevant parts of topos theory, particularly the theory of presheaves and the associated use of sieves as generalized truth-values.

We start with the set $\mathcal{W}$ of all Boolean subalgebras of the lattice $\mathcal{P}(\mathcal{H})$ of projection operators on the Hilbert space $\mathcal{H}$ of the quantum system. This forms a poset under subalgebra inclusion, $W_2 \subseteq W_1$. As with any poset, $\mathcal{W}$ can be regarded as a category in which (i) the objects are defined to be the elements $W \in \mathcal{W}$ of the poset; and (ii) a morphism is defined to exist from W_2 to W_1 if $W_2 \subseteq W_1$; we shall write this morphism as $i_{W_2 W_1} : W_2 \to W_1$. Thus there is at most one morphism between any two objects.

The next step is to introduce the set $\mathcal{O}$ of all bounded, self-adjoint operators on the Hilbert space $\mathcal{H}$. First, recall that any such operator $\hat{A}$ has the spectral representation[5]

$$\hat{A} = \int_{\sigma(\hat{A})} \lambda \, d\hat{E}^A_\lambda \tag{2.1}$$

where $\sigma(\hat{A}) \subset \mathbb{R}$ is the spectrum of $\hat{A}$, and $\{\hat{E}^A_\lambda \mid \lambda \in \sigma(\hat{A})\}$ is the spectral family of $\hat{A}$. The spectral projection operators $\hat{E}[A \in \Delta]$ are determined by the spectral family according to

$$\hat{E}[A \in \Delta] = \int_\Delta d\hat{E}^A_\lambda \tag{2.2}$$

where Δ is any Borel subset of the spectrum of $\hat{A}$. In particular, if a belongs to the discrete spectrum of $\hat{A}$, the projector onto the eigenspace with eigenvalue a is

$$\hat{E}[A = a] := \hat{E}[A \in \{a\}]. \tag{2.3}$$

Then, if $f : \mathbb{R} \to \mathbb{R}$ is any bounded Borel function, the operator $f(\hat{A})$ is defined by

$$f(\hat{A}) := \int_{\sigma(\hat{A})} f(\lambda) \, d\hat{E}^A_\lambda. \tag{2.4}$$

Note that if functions f and g exist such that $\hat{B} = f(\hat{A})$ and $\hat{B} = g(\hat{A})$, this does not imply that f and g are equal: in the discrete case it means only that their restrictions to $\sigma(\hat{A})$ are equal; more generally, measure-theoretic issues arise, and we shall define two bounded Borel functions $f, g : \sigma(\hat{A}) \to \mathbb{R}$ to be *equivalent* if $f(\hat{A}) = g(\hat{A})$.

We are now ready to turn $\mathcal{O}$ into a category. We define the objects to be the elements of $\mathcal{O}$, and we say that there is a 'morphism' from $\hat{B}$ to $\hat{A}$ if there exists a Borel function (more precisely, an equivalence class of Borel functions) $f : \sigma(\hat{A}) \to \mathbb{R}$ such that $\hat{B} = f(\hat{A})$. As implied above, any such function on $\sigma(\hat{A})$ is unique (up to the equivalence relation), and hence there is at most one morphism between any two operators; if such exists—*i.e.*, if $\hat{B} = f(\hat{A})$, for some $f : \sigma(\hat{A}) \to \mathbb{R}$—the corresponding morphism in the category $\mathcal{O}$ will be denoted $f_\mathcal{O} : \hat{B} \to \hat{A}$. Note that we could make the corresponding definitions for any

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[5]As usual, the expression in Eq. (2.1) is shorthand for the equation $\langle \psi, \hat{A}\phi \rangle = \int \lambda \, d\langle \psi, \hat{E}_\lambda \phi \rangle$ for all $\psi, \phi \in \mathcal{H}$, whose right hand side is to be interpreted as a Stieltjes integral. A similar remark applies to the integrals in Eq. (2.2) and Eq. (2.4).

subset of $\mathcal{O}$ that it is closed under the action of constructing functions of its members. In what follows, we shall be especially concerned with the category $\mathcal{O}_d$ of all bounded self-adjoint operators whose spectra are discrete.

The categories $\mathcal{W}$ and $\mathcal{O}$ are closely related[6] via a certain covariant functor $\mathbf{W} : \mathcal{O} \to \mathcal{W}$:

Definition 2.1 *The* spectral algebra functor *is the covariant functor* $\mathbf{W} : \mathcal{O} \to \mathcal{W}$ *defined as follows:*

- *On objects:* $\mathbf{W}(\hat{A}) := W_A$, *where* W_A *is the spectral algebra of the operator* $\hat{A}$ *(i.e., the collection of all projectors onto the subspaces of* $\mathcal{H}$ *associated with Borel subsets of* $\sigma(\hat{A})$*).*

- *On morphisms: If* $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, *then* $\mathbf{W}(f_{\mathcal{O}}) : W_B \to W_A$ *is defined as the subset inclusion* $i_{W_B W_A} : W_B \to W_A$.

In defining the map $\mathbf{W}(f_{\mathcal{O}}) : W_B \to W_A$ we have exploited the fact that the spectral algebra for $\hat{B} = f(\hat{A})$ is naturally embedded in the spectral algebra for $\hat{A}$ according to the result $\hat{E}[f(A) \in J] = \hat{E}[A \in f^{-1}(J)]$, for all Borel subsets $J \subseteq \sigma(\hat{B})$. Rigorously speaking, we could write $i_{W_{f(A)} W}(\hat{E}[f(A) \in J]) = \hat{E}[A \in f^{-1}(J)]$.

Note that we have defined $f_{\mathcal{O}}$ to be a morphism from $\hat{B}$ to $\hat{A}$—rather than from $\hat{A}$ to $\hat{B}$—so as to ensure that the categories $\mathcal{O}$ and $\mathcal{W}$ match up in this way. One consequence of this choice is the reversal of arrows in the equation

$$f_{\mathcal{O}} \circ g_{\mathcal{O}} = (g \circ f)_{\mathcal{O}} \tag{2.5}$$

where the left hand side denotes composition in the category $\mathcal{O}$, and the right hand side denotes normal composition of functions, so that if $\hat{B} = f(\hat{A})$ and $\hat{C} = g(\hat{B})$, the functional relation $\hat{C} = g(f(\hat{A})) \equiv g \circ f(\hat{A})$ translates to the morphism $f_{\mathcal{O}} \circ g_{\mathcal{O}} : \hat{C} \to \hat{A}$ in the category $\mathcal{O}$.

It should be noted that pairs of operators $\hat{A} \neq \hat{B}$ exist such that $\hat{B} = f(\hat{A})$ and $\hat{A} = g(\hat{B})$ for suitable functions f and g. In the category $\mathcal{O}$, these relations become $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ and $g_{\mathcal{O}} : \hat{A} \to \hat{B}$ with

$$g_{\mathcal{O}} \circ f_{\mathcal{O}} = \mathrm{id}_B; \quad f_{\mathcal{O}} \circ g_{\mathcal{O}} = \mathrm{id}_A. \tag{2.6}$$

One consequence of the existence of such pairs is that $\mathcal{O}$ is only a pre-ordered space since it lacks the antisymmetry property[7] of a true poset (which $\mathcal{W}$ is). However, it follows from Eq. (2.6) that two such operators are *isomorphic* objects in the category $\mathcal{O}$, and it is therefore possible to construct a new category $[\mathcal{O}]$ whose objects are the equivalence classes of operators, where two operators are regarded as being equivalent if they are isomorphic as objects in $\mathcal{O}$. Finally, we note that if $\hat{A}$ and $\hat{B}$ are related as in Eq. (2.6) then they have the same spectral algebras; *i.e.*, $W_A = W_B$, and hence $[\mathcal{O}]$ is closely related to the category $\mathcal{W}$.

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[6]Another, closely related, category has as its objects the abelian subalgebras of the algebra of bounded, self-adjoint operators. The fact that this can be regarded as a category was mentioned in the original paper of Kochen and Specker [1].

[7]A pre-ordered set X is said to have the *antisymmetry* property if $x \preceq y$ and $y \preceq x$ implies $x = y$.

2.2 The Spectral Presheaf on $\mathcal{O}_d$ and the Kochen-Specker Theorem

A central step in developing our use of topos theory is the observation that the spectra of the self-adjoint operators on a Hilbert space can be used to form a presheaf on the category $\mathcal{O}_d$ of self-adjoint operators whose spectra are discrete. Specifically:

Definition 2.2 *The* spectral presheaf *on $\mathcal{O}_d$ is the contravariant functor $\boldsymbol{\Sigma} : \mathcal{O}_d \to \text{Set}$ defined as follows:*

1. *On objects: $\boldsymbol{\Sigma}(\hat{A}) := \sigma(\hat{A})$—the spectrum of the self-adjoint operator $\hat{A}$.*

2. *On morphisms: If $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$, so that $\hat{B} = f(\hat{A})$, then $\boldsymbol{\Sigma}(f_{\mathcal{O}_d}) : \sigma(\hat{A}) \to \sigma(\hat{B})$ is defined by $\boldsymbol{\Sigma}(f_{\mathcal{O}_d})(\lambda) := f(\lambda)$ for all $\lambda \in \sigma(\hat{A})$.*

Note that $\boldsymbol{\Sigma}(f_{\mathcal{O}_d})$ is well-defined since, if $\lambda \in \sigma(\hat{A})$, then $f(\lambda)$ is indeed an element of the spectrum of $\hat{B}$; indeed, for these discrete-spectrum operators we have $\sigma(f(\hat{A})) = f(\sigma(\hat{A}))$.

It is straightforward to see that $\boldsymbol{\Sigma}$ is a genuine functor. It clearly respects domains and codomains of a morphism in $\mathcal{O}_d$ in the desired way, and $\boldsymbol{\Sigma}(\text{id}_A) = \text{id}_{\sigma(A)}$. The key step is to show that $\boldsymbol{\Sigma}(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \boldsymbol{\Sigma}(g_{\mathcal{O}_d}) \circ \boldsymbol{\Sigma}(f_{\mathcal{O}_d})$. So, suppose that $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$ and $g_{\mathcal{O}_d} : \hat{C} \to \hat{B}$, so that $\hat{B} = f(\hat{A})$ and $\hat{C} = g(\hat{B})$. Then $f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d} : \hat{C} \to \hat{A}$ with $\hat{C} = g(f(\hat{A})) = g \circ f(\hat{A})$. Hence, for all $\lambda \in \sigma(\hat{A})$, we have

$$\boldsymbol{\Sigma}(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d})(\lambda) = g(f(\lambda)) = \boldsymbol{\Sigma}(g_{\mathcal{O}_d})(f(\lambda)) = \boldsymbol{\Sigma}(g_{\mathcal{O}_d})(\boldsymbol{\Sigma}(f_{\mathcal{O}_d})(\lambda)) = \boldsymbol{\Sigma}(g_{\mathcal{O}_d}) \circ \boldsymbol{\Sigma}(f_{\mathcal{O}_d})(\lambda) \tag{2.7}$$

so that

$$\boldsymbol{\Sigma}(f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d}) = \boldsymbol{\Sigma}(g_{\mathcal{O}_d}) \circ \boldsymbol{\Sigma}(f_{\mathcal{O}_d}) \tag{2.8}$$

as required. Thus $\boldsymbol{\Sigma}$ is a contravariant functor from $\mathcal{O}_d$ to Set, and hence a presheaf on $\mathcal{O}_d$.

The key remark now is the following. As discussed in the Appendix, a *global section*, or *global element*, of a contravariant functor $\mathbf{X} : \mathcal{C} \to \text{Set}$ is defined to be a function γ that assigns to each object A in the category $\mathcal{C}$ an element $\gamma_A \in \mathbf{X}(A)$ in such a way that if $f : B \to A$ then $\mathbf{X}(f)(\gamma_A) = \gamma_B$, as in Eq. (A.22).

In the case of the spectral functor $\boldsymbol{\Sigma} : \mathcal{O}_d^{\text{op}} \to \text{Set}$, a global section/element is therefore a function γ that assigns to each self-adjoint operator $\hat{A}$ with a purely discrete spectrum, a real number $\gamma_A \in \sigma(\hat{A})$ such that if $\hat{B} = f(\hat{A})$ then $f(\gamma_A) = \gamma_B$. But this is precisely the condition FUNC in Eq. (1.4) for a valuation! Thus, for operators with a discrete spectrum, the Kochen-Specker theorem is equivalent to the statement that, if $\dim \mathcal{H} > 2$, there are no global sections of the spectral presheaf $\boldsymbol{\Sigma} : \mathcal{O}_d^{\text{op}} \to \text{Set}$.

The situation for operators whose spectra contains continuous parts is more complex since it is no longer necessarily true that $\sigma(f(\hat{A})) = f(\sigma(\hat{A}))$. Indeed, the most that can be proved in general[8] is that

$$\sigma(f(\hat{A})) = \bigcap_{\Delta} \{\overline{f(\Delta)} \mid \hat{E}[A \in \Delta] = \hat{1}\} \tag{2.9}$$

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[8]For details see page 900 of [19].

where $\overline{f(\Delta)}$ is the topological closure of $f(\Delta) \subset \mathbb{R}$, and Δ denotes Borel subsets of $\mathbb{R}$. The idea of the spectral presheaf can be extended to this case by using a more sophisticated approach that involves the spectral theorem for commutative von Neumann algebras. However, we shall not develop this particular approach further in the present paper because of the problematic physical meaning of assigning an exact value to a quantity whose range of values is continuous. Of much greater relevance is the assignment of truth-false values to propositions of the type '$A \in \Delta$', as discussed in Section 1 and in the original Kochen-Specker paper: as we shall see shortly in Section 2.3, the relevant presheaf in this case can be defined for the category $\mathcal{O}$ of all bounded self-adjoint operators on the Hilbert space of the quantum system.

Note that, in the form above, the Kochen-Specker theorem looks remarkably like the theorem in fibre-bundle theory which says that there are no global cross-sections of a non-trivial principal bundle [20]. Thus, *cum grano salis*, one might be tempted to say that the Kochen-Specker theorem in quantum theory is analogous to the 'Gribov effect' in Yang-Mills gauge theories (which arises from the non-triviality of the gauge bundle)!

More seriously, the non-triviality of a principal fibre bundle is related to the existence of certain non-vanishing cohomology classes that arise as obstructions to the step-wise construction of a cross-section on the simplices of a locally-trivializing triangulation of the base manifold. It would be intriguing to see if the non-existence of global valuations in the quantum theory can be related to the non-vanishing of some topos-based cohomology structure. If so, this would open an perspective on the Kochen-Specker theorem that would be extremely interesting; not least because most of the existing literature on the theorem is concerned with finding concrete counter-examples to the existence of a global valuation rather than studying the phenomenon in a general sense.

However, from our immediate perspective the most important reason for presenting this topos-theoretic restatement of the Kochen-Specker theorem is that, as we shall see, it suggests specific ways of implementing the idea of constructing generalized valuations; particularly in regard to using the contextual logic that forms the heart of the theory of presheaves.

2.3   The Kochen-Specker Theorem in Terms of the Dual Presheaves on $\mathcal{W}$ and on $\mathcal{O}$.

1.   The Dual Presheaf on $\mathcal{W}$:   The Kochen-Specker theorem is usually stated in terms of the features of a valuation on the Boolean subalgebras of the lattice $\mathcal{P}(\mathcal{H})$ of projectors on the Hilbert space $\mathcal{H}$. This is useful for handling operators whose spectra contain continuous parts; it is also the starting point for most constructions of explicit counter-examples to the existence of global valuations. For these reasons, it is very useful to restate, and extend, the results above using the category $\mathcal{W}$ rather than $\mathcal{O}_d$. This will enable us in Section 2.3.2 to state the Kochen-Specker theorem in terms of the category $\mathcal{O}$ of all bounded self-adjoint operators.

Once again we start with the definition of an appropriate presheaf; this time on the category $\mathcal{W}$.

Definition 2.3 *The* dual presheaf *on* $\mathcal{W}$ *is the contravariant functor* $\mathbf{D} : \mathcal{W} \to$ Set *defined as follows:*

1. *On objects:* $\mathbf{D}(W)$ *is the* dual *of* W; *thus it is the set* $\mathrm{Hom}(W, \{0, 1\})$ *of all homomorphisms from the Boolean algebra* W *to the Boolean algebra* $\{0, 1\}$.

2. *On morphisms: If* $i_{W_2 W_1} : W_2 \to W_1$ *then* $\mathbf{D}(i_{W_2 W_1}) : \mathbf{D}(W_1) \to \mathbf{D}(W_2)$ *is defined by* $\mathbf{D}(i_{W_2 W_1})(\chi) := \chi|_{W_2}$ *where* $\chi|_{W_2}$ *denotes the restriction of* $\chi \in \mathbf{D}(W_1)$ *to the subalgebra* $W_2 \subseteq W_1$.

A global section of the functor $\mathbf{D} : \mathcal{W} \to$ Set is then a function γ that associates to each $W \in \mathcal{W}$ an element γ_W of the dual of W such that if $i_{W_2 W_1} : W_2 \to W_1$ then $\gamma_{W_1}|_{W_2} = \gamma_{W_2}$; thus, for all $\hat{\alpha} \in W_2$,

$$\gamma_{W_2}(\hat{\alpha}) = \gamma_{W_1}((i_{W_2 W_1}(\hat{\alpha}))). \tag{2.10}$$

Since each projection operator belongs to at least one Boolean algebra (for example, the algebra $\{\hat{0}, \hat{1}, \hat{\alpha}, \neg\hat{\alpha}\}$) it follows that a global section of $\mathbf{D} : \mathcal{W}^{\mathrm{op}} \to$ Set associates to each projection operator $\hat{\alpha}$ a number $V(\hat{\alpha})$ which is either 0 or 1, and is such that, if $\hat{\alpha} \wedge \hat{\beta} = \hat{0}$, then $V(\hat{\alpha} \vee \hat{\beta}) = V(\hat{\alpha}) + V(\hat{\beta})$. These are precisely the types of valuation considered in the discussions of the Kochen-Specker theorem that focus on the construction of specific counter-examples. Thus an alternative way of expressing the Kochen-Specker theorem is that, if $\dim \mathcal{H} > 2$, the dual presheaf $\mathbf{D} : \mathcal{W}^{\mathrm{op}} \to$ Set has no global sections.

2. The Dual Presheaf on $\mathcal{O}$: The covariant functor $\mathbf{W} : \mathcal{O} \to \mathcal{W}$ of Definition 2.1 and the contravariant functor $\mathbf{D} : \mathcal{W} \to$ Set, can be composed to give a contravariant functor $\mathbf{D} \circ \mathbf{W} : \mathcal{O} \to$ Set, which we shall call the *dual presheaf* on $\mathcal{O}$. It has the following properties:

1. On objects: $\mathbf{D} \circ \mathbf{W}(\hat{A})$ is the dual of the spectral Boolean algebra W_A; thus it is the set $\mathrm{Hom}(W_A, \{0, 1\})$ of all homomorphisms from W_A to the Boolean algebra $\{0, 1\}$.

2. On morphisms: If $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ then $\mathbf{D} \circ \mathbf{W}(f_{\mathcal{O}}) : D(W_A) \to D(W_B)$ is defined by $\mathbf{D} \circ \mathbf{W}(f_{\mathcal{O}})(\chi) := \chi|_{W_{f(A)}}$ where $\chi|_{W_{f(A)}}$ denotes the restriction of $\chi \in D(W_A)$ to the subalgebra $W_{f(A)} \subseteq W_A$.

Note that a global section γ of the presheaf $\mathbf{D} \circ \mathbf{W} : \mathcal{O} \to$ Set would correspond to a consistent association of each physical quantity A with an element $\gamma_A \in \mathrm{Hom}(W_A, \{0, 1\})$, and hence with a statement of which propositions of the form '$A \in \Delta$' are true, and which are false. The non-existence of such global sections is perhaps the most physically transparent statement of the Kochen-Specker theorem in the language of presheaves.

Finally, we note that, as might be expected, there is a close relationship between the spectral presheaf $\mathbf{\Sigma}$ on $\mathcal{O}_d$ and the corresponding dual presheaf $\mathbf{D} \circ \mathbf{W}$ on $\mathcal{O}_d$. Specifically, there is a natural transformation $T : \mathbf{\Sigma} \to \mathbf{D} \circ \mathbf{W}$ between these presheaves, whose component $T_A : \mathbf{\Sigma}(\hat{A}) \to \mathbf{D} \circ \mathbf{W}(\hat{A})$ at each stage $\hat{A} \in \mathcal{O}_d$,

$$T_A : \sigma(\hat{A}) \to \mathrm{Hom}(W_A, \{0, 1\}) \tag{2.11}$$

is defined by (where $\lambda \in \sigma(\hat{A})$)

$$T_A(\lambda)(\hat{E}[A \in \Delta]) := \begin{cases} 1 & \text{if } \lambda \in \Delta \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

for all projection operators $\hat{E}[A \in \Delta] \in W_A$.

3 From Partial Valuations to Generalized Valuations

3.1 Some Implications of the Presheaf Version of the Kochen-Specker Theorem

We are now ready to begin the presentation of our theory of generalized valuations. From a pedagogical perspective, this could be done in several ways. One possibility would be to start with the formal definition and then to exhibit some physically relevant examples. However, although the definition of a generalized valuation is partly motivated by the conclusions of our earlier discussion in Section 1.3, one of the central components—the presheaf analogue of the functional composition principle FUNC—is best justified by seeing how it arises in a particular case. Therefore, we shall devote this section to a fairly extensive discussion of a concrete example of a particular class of generalized valuation that will serve to illustrate the ideas that lie behind our later, more abstract, constructions in Sections 4 and 5.

As we have seen, the Kochen-Specker theorem asserts that, if $\dim \mathcal{H} > 2$, there do not exist valuations that are globally defined in the sense that FUNC is satisfied for all pairs of operators $\hat{A}$, $\hat{B}$ (with discrete spectra) in the Hilbert space with $\hat{B} = f(\hat{A})$ for some f; or, in the language of topos theory, the spectral presheaf $\boldsymbol{\Sigma} : \mathcal{O}_d{}^{\text{op}} \to \text{Set}$ has no global sections. More generally, the theorem asserts that there are no global sections of the dual presheaf $\mathbf{D} \circ \mathbf{W}$ on $\mathcal{O}$; and hence there is no consistent way of assigning the values true or false to propositions of the type '$A \in \Delta$' for all bounded physical quantities A.

Rewriting the Kochen-Specker theorem in the language of topos theory suggests several ways in which the idea of a valuation might be generalized so that globally-defined entities *do* exist. For example, one possibility is to embed the spectral presheaf $\boldsymbol{\Sigma}$ in a larger presheaf that does have global elements. The existence of at least one such presheaf follows from some general considerations in topos theory[9]: in the present case, a relevant example is the presheaf on $\mathcal{O}$ whose objects are *subsets* of $\sigma(\hat{A})$ at each stage of truth $\hat{A}$. A global section of this presheaf would comprise a consistent assignment of a *range* of values for each physical quantity. This option sounds physically plausible, and is something to which we may return in a later paper.

Another possibility is to replace the dual presheaf $\mathbf{D} \circ \mathbf{W}$ on $\mathcal{O}$ with a presheaf $\mathbf{H}$ in which $\mathbf{H}(\hat{A})$ is defined to be the set of homomorphisms from W_A into some larger algebra than the $\{0, 1\}$ used by $\mathbf{D} \circ \mathbf{W}$, thus building in the idea of multi-valued truth in a rather

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[9]The existence of injective resolutions of a presheaf.

direct way. Of course, guided by our remarks in the Introduction, this target logic could itself depend on the stage of truth $\hat{A}$ (*i.e.*, it could be contextual), and it is not clear that we would want to use genuine homomorphisms; for example, if the target algebra was an intuitionistic logic, then the negation operation would behave differently from that in W_A, as was mentioned briefly in the Introduction in the context of the (incorrect!) equation Eq. (1.12). We shall see an example of this type of structure in Section 5 in the form of the 'valuation presheaf' of Definition 5.2.

However, in this Section, we will take our departure from the property of presheaves that even if a global section/element does not exist, typically there will be plenty of *local* sections (just as there are in a non-trivial principal bundle), which are defined to be morphisms of a subobject of the terminal object into the presheaf. In the case of the spectral presheaf Σ, any such local element corresponds to what we shall call a 'partial' valuation, and the main thrust of this section of the paper is to show how each such locally-defined normal valuation ('normal' in the sense that assigned values lie in the minimal Boolean algebra $\{0, 1\}$) gives rise to a *globally* defined 'generalized' valuation with truth-values in the Heyting algebra of sieves on $\mathcal{O}$. This also allows comparison to be made with the various modal approaches to the interpretation of quantum theory, all of which use local valuations of one type or another; however, we shall not pursue that comparison in this paper.

3.2 The Idea of a Partial Valuation

The precise definition of a 'partial valuation' is that it is a local section of the spectral presheaf Σ on the category $\mathcal{O}_d$ of bounded self-adjoint operators with discrete spectra. This translates into the following explicit set of properties:

Definition 3.1 *A* partial valuation *on the set of bounded, self-adjoint operators with discrete spectra is a map $V : \operatorname{dom} V \to \mathbb{R}$ defined on a subset $\operatorname{dom} V$ of such operators (called the* domain *of V) such that:*

  1. If $\hat{A} \in \operatorname{dom} V$, then $V(\hat{A}) \in \sigma(\hat{A})$.

  2. If $\hat{A} \in \operatorname{dom} V$ and $\hat{B} = f(\hat{A})$ then (i) $\hat{B} \in \operatorname{dom} V$; and (ii) $V(\hat{B}) = f(V(\hat{A}))$.

One consequence of this definition is that if $\hat{A}$ belongs to the domain of V, then so do all its spectral projectors. This is because any such projector $\hat{E}[A \in \Delta]$ can be written as

$$\hat{E}[A \in \Delta] = \chi_\Delta(\hat{A}) \tag{3.1}$$

where $\chi_\Delta : \sigma(\hat{A}) \to \mathbb{R}$ is the characteristic function of $\Delta \subseteq \sigma(\hat{A})$. It follows that

$$V(\hat{E}[A \in \Delta]) = \chi_\Delta(V(\hat{A})) = \begin{cases} 1 & \text{if } V(\hat{A}) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Note that, provided $\operatorname{dom} V \neq \emptyset$, real multiples of the unit operator $\hat{1}$ belong to the domain of any partial valuation V. This is because if $\hat{A}$ is any operator in $\operatorname{dom} V$, then

$r\hat{1} = c_r(\hat{A})$ where $c_r : \sigma(\hat{A}) \to \mathbb{R}$, $r \in \mathbb{R}$, is the constant map $c_r(a) := r$ for all $a \in \sigma(\hat{A})$. This also shows that $V(\hat{1}) = 1$.

The definition of a partial valuation is not empty since there clearly exists a 'trivial' example V_0 whose domain is defined as $\mathrm{dom}\, V_0 := \{r\hat{1} \mid r \in \mathbb{R}\}$, and with $V_0(r\hat{1}) := r$. However, non-trivial partial valuations also exist. For example, we have the following definition:

Definition 3.2 *Let $\hat{M}$ be any bounded, self-adjoint operator with a purely discrete spectrum, and let $m \in \sigma(\hat{M})$ be one its eigenvalues. Then the associated partial valuation $V^{M,m}$ is defined as follows:*

1. *The domain of $V^{M,m}$ is*

$$\mathrm{dom}\, V^{M,m} :=\downarrow \hat{M} := \{f_{\mathcal{O}_d} : \hat{A} \to \hat{M}\} = \{\hat{A} \mid \exists f \text{ s.t. } \hat{A} = f(\hat{M})\}, \qquad (3.3)$$

   where the last equality holds since there is at most one morphism between two objects in $\mathcal{O}_d$.

2. *If $\hat{A} \in \mathrm{dom}\, V^{M,m}$ with $\hat{A} = f(\hat{M})$, then the value of $V^{M,m}(\hat{A})$ is*

$$V^{M,m}(\hat{A}) := f(m). \qquad (3.4)$$

It is straightforward to check that this satisfies the requirements for a partial valuation.

Note that, generally speaking, a partial valuation of this type can be extended 'upwards' in the sense that if there is a morphism $h_{\mathcal{O}_d} : \hat{M} \to \hat{N}$, so that $\hat{M} = h(\hat{N})$, then $V^{M,m}$ can be extended to $\hat{N}$ by defining $V^{M,m}(\hat{N})$ to be any eigenvalue n of $\hat{N}$ such that $h(n) = m$ (there must be at least one such eigenvalue since $\sigma(\hat{M}) = h(\sigma(\hat{N}))$). Therefore, one might as well suppose in the first place that $\hat{M}$ is a *maximal* operator[10].

The domain of a valuation $V^{M,m}$ forms a commutative set of operators; however, there is no reason in general why the domain of a partial valuation should be commutative. For example, the use of a non-abelian domain forms an integral part of the modal interpretation of Clifton and Bub [8, 21, 12].

We note that a partial valuation V gives a simple 'false-true' assignment to propositions of the type '$A \in \Delta$' provided that A lies in the domain of V; specifically:

$$V(A \in \Delta) := \begin{cases} \text{'true' if } V(A) \in \Delta; \\ \text{'false' otherwise.} \end{cases} \qquad (3.5)$$

Thus the proposition '$A \in \Delta$' is true if A lies in the domain of V, and if the value of A lies in the range Δ; it is false, if A lies in the domain of V and the value of A does not lie in Δ. If A is *not* in the domain of V, no truth-value at all is assigned to propositions about the value of A. Of course, these assignments are consistent with the assignment in Eq. (3.2) of a $0 - 1$ value to the projection operator $\hat{E}[A \in \Delta]$.

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[10]In the present context, we could define an operator $\hat{M}$ to be *maximal* if, for any operator $\hat{N}$ and function $h : \sigma(\hat{N}) \to \mathbb{R}$ such that $\hat{M} = h(\hat{N})$, there exists $g : \sigma(\hat{M}) \to \mathbb{R}$ such that $\hat{N} = g(\hat{M})$; *i.e.*, $h_{\mathcal{O}} : \hat{M} \to \hat{N}$ implies that $\hat{M}$ and $\hat{N}$ are isomorphic objects in the category $\mathcal{O}$.

3.3 The Construction of a Generalized Valuation from a Partial Valuation

1. The Basic Idea: Let V be any partial valuation, and consider a proposition of the form '$A = a$', where a is an eigenvalue of $\hat{A}$ and where $\hat{A}$ does *not* lie in the domain of V. The implication of the Kochen-Specker theorem is that it may not be possible to extend the domain of V to include $\hat{A}$. If this is indeed the case, then the proposition '$A = a$' cannot be given a value true or false in a way that is consistent with the values already given by V to the operators in its domain.

However, consider a proposition of the form '$f(A) = f(a)$'. As was emphasized earlier, this will generally be weaker than '$A = a$'; both in a conceptual sense—knowing that the quantity $f(A)$ has the value $f(a)$ gives only limited information on the value of A itself (it could be any number b such that $f(b) = f(a)$)—and in the mathematical sense that, in the lattice of projection operators, (*cf.*, Eq. (1.8)),

$$\hat{E}[A = a] \leq \hat{E}[f(A) = f(a)] \tag{3.6}$$

where $\hat{E}[A = a]$ projects onto the eigenspace of $\hat{A}$ with eigenvalue a, and $\hat{E}[f(A) = f(a)]$ projects onto the eigenspace of $f(\hat{A})$ with eigenvalue $f(a)$. More precisely

$$\hat{E}[f(A) = f(a)] = \sum_{b \in \sigma(A), f(b) = f(a)} \hat{E}[A = b] \quad = \quad \hat{E}[A \in f^{-1}(f(\{a\}))]. \tag{3.7}$$

In other words, $\hat{E}[f(A) = f(a)]$ is the sum of the (orthogonal) set of those projectors in the spectral decomposition of $\hat{A}$ whose corresponding eigenvalues are mapped into the number $f(a)$ by the function $f : \sigma(\hat{A}) \to \mathbb{R}$.

The key remark is then the following. It is possible that, for one (or more) function f, (i) the coarse-grained operator $f(\hat{A})$ *does* lies in the domain of V (*i.e.*, at least part of the spectral algebra of $\hat{A}$ lies in dom V); and (ii) $V(f(\hat{A})) = f(a)$. Under these circumstances, we can assign a *true* value to the *weaker* proposition '$f(A) = f(a)$', and thereby assign a *partial* truth-value to the original proposition '$A \in \Delta$'. We note that if $g : \sigma(f(\hat{A})) \to \mathbb{R}$ then $V(g(f(\hat{A}))) = g(f(a))$—*i.e.*, $V(g \circ f(\hat{A})) = g \circ f(a)$—and hence if the function f satisfies the above conditions, so does $g \circ f$ for any g; in other words, the set of such functions determines a *sieve* on $\hat{A}$ in the category $\mathcal{O}_d$.

Motivated by these remarks, we propose the following definition of a *generalized valuation associated* with a partial valuation.

Definition 3.3 *Given a partial valuation V on the set of bounded self-adjoint operators with discrete spectra, the* associated generalized valuation *is defined on a proposition '$A = a$' as*

$$\nu^V(A = a) := \begin{cases} \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \text{dom}\,V,\ V(\hat{B}) = f(a)\} & \textit{if } a \in \sigma(\hat{A}); \\ \emptyset & \textit{otherwise.} \end{cases} \tag{3.8}$$

A crucial consequence of this definition is that, as indicated above, $\nu^V(A = a)$ is a *sieve* on $\hat{A}$ in the category $\mathcal{O}_d$. Indeed, suppose $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$ belongs to $\nu^V(A = a)$, and consider

any morphism $g_{\mathcal{O}_d} : \hat{C} \to \hat{B}$. Then, since $\hat{B} \in \mathrm{dom}\, V$, and $\hat{C} = g(\hat{B})$, the definition of a partial valuation shows that (i) $\hat{C} \in \mathrm{dom}\, V$, and (ii) $V(\hat{C}) = g(V(\hat{B}))$. However, $g(V(\hat{B})) = g(f(a)) = g \circ f(a)$; and hence $V(\hat{C}) = g \circ f(a)$. Thus $f_{\mathcal{O}_d} \circ g_{\mathcal{O}_d} : \hat{C} \to \hat{A}$ is in the set $\nu^V(A = a)$, which is therefore a sieve.

Thus the partial truth-value $\nu^V(A = a)$ of the proposition '$A = a$' is defined to be the sieve on $\hat{A}$ of coarse-grainings $f(\hat{A})$ of $\hat{A}$, at which the proposition '$f(A) = f(a)$' is 'totally' true according to the partial valuation V.

2. The Origin of Contextuality: The fact that $\nu^V(A = a)$ is a sieve is of considerable importance since it shows that the target space of the valuation $\nu^V(A = a)$ is a genuine mathematical logic: namely, the Heyting algebra $\mathbf{\Omega}(\hat{A})$ of sieves on the object $\hat{A}$ in the category $\mathcal{O}_d$.

In more general terms, the sieve-like nature of the generalized valuation gives strong support to our claim that topos theory is the appropriate mathematical framework in which to develop these ideas. This is particularly so in regard to the presheaf idea of 'contextual' logic. From the defining property of a generalized valuation in Eq. (3.8), it is clear that if the propositions '$A = a$' and '$C = c$' happen to correspond to the *same* projection operator $\hat{P}$, so that $\hat{E}[A = a] = \hat{E}[C = c] = \hat{P}$, this does *not* mean that $\nu^V(A = a)$ is equal to $\nu^V(C = c)$; indeed, the former is a sieve on $\hat{A}$, whilst the latter is a sieve on $\hat{C}$. Furthermore, if the projection operator $\hat{P}$ is thought of as representing some physical quantity P directly, then the proposition '$P = 1$' can also be assigned a partial truth-value $\nu^V(P = 1)$; which, as a sieve on $\hat{P}$, is different from both $\nu^V(A = a)$ and $\nu^V(C = c)$.

The situation can be summarized by saying that if we think of ourselves as assigning partial truth-values to projection operators, then the actual value assigned to any specific projector $\hat{P}$ will depend on the *context* chosen—*i.e.*, we have to choose a particular self-adjoint operator $\hat{A}$ from the set of all operators whose associated sets of spectral projectors include $\hat{P}$; hence each context corresponds to a 'stage of truth' for the presheaf.

Thus we see that, in the notation '$\nu^V(A = a)$', the argument '$A = a$' serves two purposes: (i) it specifies the associated projection operator $\hat{E}[A = a]$; and (ii) it indicates the context (*i.e.*, $\hat{A}$) in which a partial truth-value is to be ascribed to this projector. This manifest contextuality is one of the crucial features that distinguishes our scheme from a naive one in which one tries simply to assign to each projector the value 1 or 0—an attempt that immediately falls foul of the Kochen-Specker theorem.

If desired, this situation can be reflected in the notation by rewriting $\nu^V(A = a)$ as $\nu_A^V(\hat{P})$ to emphasize that the former can be construed as the partial truth-value assigned to the projection operator $\hat{P}$ ($= \hat{E}[A = a]$) in the context/stage of truth of the self-adjoint operator $\hat{A}$. Notice that, as is characteristic of presheaf logic, the Heyting algebra to which $\nu_A^V(\hat{P})$ belongs itself depends on the context $\hat{A}$; namely, it is the algebra $\mathbf{\Omega}(\hat{A})$ of sieves on $\hat{A}$.

3. Extending to Propositions '$A \in \Delta$': The construction above of a generalized valuation can be extended in an obvious way to include more general propositions of the form '$A \in \Delta$', where Δ is any Borel[11] subset of the spectrum $\sigma(\hat{A})$. Note that the set of these propositions is naturally equipped with the logical structure of the Boolean algebra of all Borel subsets of $\sigma(\hat{A})$; in the quantum theory, this algebra is represented by the spectral algebra W_A of projectors onto the eigenspaces associated with these Borel subsets.

Specifically, we define:

Definition 3.4 *Given a partial valuation V, the associated* generalized valuation *is defined on a proposition '$A \in \Delta$' as*

$$\nu^V(A \in \Delta) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\Delta)\} \tag{3.9}$$

It is a straightforward exercise to show that the right hand side is a sieve on $\hat{A}$ in the category $\mathcal{O}_d$.

4. 'Totally true' and 'Totally false': This is a convenient point at which to give a precise meaning to the concepts 'totally true' and 'totally false' that have been employed up to now in a rather heuristic way. These concepts, too, are contextual in nature.

The formal definition is as follows:

Definition 3.5

1. *The proposition '$A \in \Delta$' is* totally true *at the stage of truth $\hat{A}$ if*

$$\nu^V(A \in \Delta) = \mathrm{true}_A := \downarrow\hat{A} = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A}\}. \tag{3.10}$$

2. *The proposition '$A \in \Delta$' is* totally false *at the stage of truth $\hat{A}$ if*

$$\nu^V(A \in \Delta) = \mathrm{false}_A := \emptyset. \tag{3.11}$$

Thus a proposition is totally true in the context $\hat{A}$ if its partial truth-value is equal to the principal sieve on $\hat{A}$, which is the unit element in the Heyting algebra $\mathbf{\Omega}(\hat{A})$; the proposition is totally false if it is equal to the empty sieve, which is the zero element in $\mathbf{\Omega}(\hat{A})$.

Note that if $\nu^V(A = a) = \downarrow\hat{A}$, then, in particular, the identity morphism $\mathrm{id}_A : \hat{A} \to \hat{A}$ belongs to the sieve $\nu^V(A = a)$. According to the Definition 3.3 this means that (i) $\hat{A} \in \mathrm{dom}\, V$, and (ii) $V(\hat{A}) = a$. Conversely, if $\hat{A} \in \mathrm{dom}\, V$ and $V(\hat{A}) = a$ then $\nu^V(A = a) = \downarrow\hat{A}$. Thus the proposition '$A = a$' is totally true if, and only if, $\nu^V(A = a) = \mathrm{true}_A$. Hence the notion of total truth of the proposition '$A = a$' captures precisely the idea that the quantity A does indeed have a value, and that value is a.

More generally, $\nu^V(A \in \Delta) = \mathrm{true}_A$ if, and only if, A lies in the domain of V, and the value of A assigned by V lies in the subset $\Delta \subseteq \sigma(\hat{A})$.

---

[11]Note that any subset of the spectrum of an operator in $\mathcal{O}_d$ is Borel, and hence the qualification is unnecessary. However, we shall leave in references to 'Borel' subsets as this *is* of importance for operators whose spectra is not just discrete.

5. Possible Modification of $\mathcal{O}_d$ to Remove Minimal Truth-Values: As things stand, if $\Delta \neq \emptyset$, the proposition '$A \in \Delta$' is never totally false since, as mentioned in Section 3.2, real multiples of the unit operator $\hat{1}$ belongs to the domain of any partial valuation, and so $c_{r\,\mathcal{O}_d} : r\hat{1} \to \hat{A}$ with $V(r\hat{1}) = r = c_r(a)$ for all $a \in \sigma(\hat{A})$, is bound to be in $\nu^V(A \in \Delta)$ if $\Delta \neq \emptyset$. Thus the morphism $c_{r\,\mathcal{O}_d} : r\hat{1} \to \hat{A}$ always belongs to the sieve $\nu^V(A \in \Delta)$ provided only that Δ is not the empty set.

If $\nu(A \in \Delta) = \{c_{r\,\mathcal{O}_d} : r\hat{1} \to A \mid r \in \mathbb{R}\}$ then we will say that the proposition '$A \in \Delta$' is *minimally true*; that is, it really provides no interesting information about the value of A. If desired, the existence of such minimal truth-values can be removed by the simple expedient of replacing the category $\mathcal{O}_d$ with the category $\mathcal{O}_{d*}$, which is defined to be $\mathcal{O}_d$ minus (i) the objects $r\hat{1}$, $r \in \mathbb{R}$, and (ii) all morphisms that have these objects as domains. Clearly there is a precise analogue of this construction for the category $\mathcal{O}$ of all bounded, self-adjoint operators on $\mathcal{H}$. The analogous modification of the category $\mathcal{W}$ consists in removing the trivial Boolean algebra $\{0, 1\}$ as a possible context/stage of truth; we shall denote the resulting category by $\mathcal{W}_*$.

Whether or not one wants to make the change from $\mathcal{O}$ to $\mathcal{O}_*$ is not totally clear, and for the moment we prefer to keep the two options open as two slightly different schemes. Most of the material that follows is valid irrespective of whether $\mathcal{O}$ or $\mathcal{O}_*$ is used; where there is a significant difference, we shall point it out.

6. The Analogue of FUNC: Let us turn now to the crucial question of the analogue of the functional composition condition FUNC; in particular, we must check that if the proposition '$A \in \Delta$' is given the value 'totally true' then, in an appropriate sense, this is also the case for the proposition '$h(A) \in h(\Delta)$' for any bounded Borel function $h : \sigma(\hat{A}) \to \mathbb{R}$. The following theorem provides the key to seeing that this is so.

Theorem 3.1 *If $h_{\mathcal{O}_d} : \hat{C} \to \hat{A}$, so that $\hat{C} = h(\hat{A})$, then*

$$\nu^V(C \in h(\Delta)) = h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta)) \tag{3.12}$$

where the pull-back $h^*_{\mathcal{O}_d}(S)$ of $S \in \mathbf{\Omega}(\hat{A})$ by $h_{\mathcal{O}_d} : \hat{C} \to \hat{A}$ is the sieve on $\hat{C}$ defined as (cf., Eq. (A.11))

$$h^*_{\mathcal{O}_d}(S) := \{k_{\mathcal{O}_d} : \hat{D} \to \hat{C} \mid h_{\mathcal{O}_d} \circ k_{\mathcal{O}_d} \in S\}. \tag{3.13}$$

Proof

We have

$$\nu^V(C \in h(\Delta)) := \{k_{\mathcal{O}_d} : \hat{D} \to \hat{C} \mid \hat{D} \in \mathrm{dom}\, V,\ V(\hat{D}) \in k(h(\Delta))\} \tag{3.14}$$

and so, since $\hat{C} = h(\hat{A})$, if $k_{\mathcal{O}_d} \in \nu^V(C \in h(\Delta))$ then $h_{\mathcal{O}_d} \circ k_{\mathcal{O}_d} : \hat{D} \to \hat{A}$ with $\hat{D} \in \mathrm{dom}\, V$ and $V(\hat{D}) \in k \circ h(\Delta)$; hence $h_{\mathcal{O}_d} \circ k_{\mathcal{O}_d} \in \nu^V(A \in \Delta)$, so that $k_{\mathcal{O}_d} \in h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta))$. Thus $\nu^V(C \in h(\Delta)) \subseteq h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta))$.

Conversely, let $k_{\mathcal{O}_d} : \hat{D} \to \hat{C}$ belong to $h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta))$; thus $h_{\mathcal{O}_d} \circ k_{\mathcal{O}_d} \in \nu^V(A \in \Delta)$. Then $\hat{D} \in \mathrm{dom}\, V$, and $V(\hat{D}) \in k(h(\Delta))$, and so $k_{\mathcal{O}_d} \in \nu^V(C \in h(\Delta))$. Hence $h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta)) \subseteq \nu^V(C \in h(\Delta))$. **Q.E.D.**

In particular, suppose that $\nu^V(A \in \Delta)$ has the value 'totally true', *i.e.*, it is equal to the unit $1_A = \downarrow\hat{A}$ (or 'true$_A$') of the Heyting algebra $\mathbf{\Omega}(\hat{A})$ of sieves on $\hat{A}$. Then

$$h^*_{\mathcal{O}_d}(\nu^V(A \in \Delta)) = h^*_{\mathcal{O}_d}(\downarrow\hat{A}) = \downarrow\hat{C} \tag{3.15}$$

and so, by Eq. (3.12), we get $\nu^V(C \in h(\Delta)) = \downarrow C = 1_C$; hence the proposition '$C \in h(\Delta)$' has the value 'totally true' in the Heyting algebra of sieves on $\hat{C}$.

In summary: if the proposition '$A \in \Delta$' is totally true at the stage of truth $\hat{A}$, then the weaker proposition '$h(A) \in h(\Delta)$' is also totally true at the stage of truth $h(\hat{A})$. This result is precisely the type of thing we wanted, and justifies our taking Eq. (3.12) to be the presheaf analogue of the functional composition rule.

Furthermore, the pull-back of a sieve by a morphism that is itself a member of the sieve, is the principal sieve (see the discussion around Eq. (A.12) in the Appendix). Thus Eq. (3.12) implies that the partial truth-value of a proposition '$A \in \Delta$' is the set of coarse-grainings of $\hat{A}$ which are such that the associated propositions are totally true at their own 'stages of truth'.

3.4 Algebraic Properties of the Generalized Valuation ν^V

Let us consider now the extent to which the generalized valuation Eq. (3.9) satisfies the conditions listed in Eqs.(1.13–1.15) in the Introduction. We shall also consider explicitly the possibility that the generalized valuation might satisfy strong disjunctive or conjunctive conditions.

1. The Null Proposition Condition: The null proposition regarding the value of the physical quantity A is '$A \in \emptyset$', and then $\nu^V(A \in \emptyset) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\,V,\ V(\hat{B}) \in f(\emptyset)\}$. But $f(\emptyset) = \emptyset$, and hence $\nu^V(A \in \emptyset) = \emptyset$, which is the zero element of the Heyting algebra $\mathbf{\Omega}(\hat{A})$. Hence, as required, $\nu^V(0) = 0_A$; or, to indicate the context in a more precise way,

$$\nu^V_A(\hat{0}) = 0_A. \tag{3.16}$$

2. The Monotonicity Condition: To check monotonicity we consider propositions '$A \in \Delta_1$' and '$A \in \Delta_2$' with $\Delta_1 \subseteq \Delta_2$, which is equivalent to the propositional relation '$A \in \Delta_1 \leq A \in \Delta_2$'.

Then if $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$ belongs to $\nu^V(A \in \Delta_1)$, we have $\hat{B} \in \mathrm{dom}\,V$ and $V(\hat{B}) \in f(\Delta_1)$. However, $\Delta_1 \subseteq \Delta_2$ implies $f(\Delta_1) \subseteq f(\Delta_2)$; and hence $V(\hat{B}) \in f(\Delta_2)$. Thus $f_{\mathcal{O}_d}$ also belongs to $\nu^V(A \in \Delta_2)$. This proves the monotonicity condition

$$\text{'}A \in \Delta_1 \leq A \in \Delta_2\text{' implies } \nu^V(A \in \Delta_1) \leq \nu^V(A \in \Delta_2). \tag{3.17}$$

2.1 A Strong Disjunctive Condition: As noted in Section 1.3, the monotonicity condition implies the weak disjunctive and conjunctive conditions

$$\nu^V(A \in \Delta_1) \vee \nu^V(A \in \Delta_2) \leq \nu^V(A \in \Delta_1 \vee A \in \Delta_2) \tag{3.18}$$

and

$$\nu^V(A \in \Delta_1 \wedge A \in \Delta_2) \leq \nu^V(A \in \Delta_1) \wedge \nu^V(A \in \Delta_2) \tag{3.19}$$

respectively.

However, it turns out that ν^V satisfies a strong disjunctive condition in which the inequality in Eq. (3.18) is replaced by an equality.

To see this, consider propositions '$A \in \Delta_1$' and '$A \in \Delta_2$', so that '$A \in \Delta_1 \vee A \in \Delta_2$' is the equivalent to the proposition '$A \in \Delta_1 \cup \Delta_2$', (*i.e.*, the logical '$\vee$' operation is taken in the Boolean algebra of propositions about the value of A lying in Borel subsets of $\sigma(\hat{A})$). Then

$$\nu^V(A \in \Delta_1 \vee A \in \Delta_2) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\Delta_1 \cup \Delta_2)\} \tag{3.20}$$

which, since $f(\Delta_1 \cup \Delta_2) = f(\Delta_1) \cup f(\Delta_2)$, gives

$$\nu^V(A \in \Delta_1 \vee A \in \Delta_2) = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\Delta_1),\ \text{or}\ V(\hat{B}) \in f(\Delta_2)\}. \tag{3.21}$$

However, the right hand side of this expression is just $\nu^V(A \in \Delta_1) \cup \nu^V(A \in \Delta_2)$. Thus we see that

$$\nu^V(A \in \Delta_1 \vee A \in \Delta_2) = \nu^V(A \in \Delta_1) \vee \nu^V(A \in \Delta_2) \tag{3.22}$$

where the '$\vee$'-operation on the right hand side is taken in the Heyting algebra $\mathbf{\Omega}(\hat{A})$, and where we recall from Eq. (A.16) that if S_1 and S_2 are sieves on the same object, then $S_1 \vee S_2 := S_1 \cup S_2$. Thus, the generalized valuation ν^V satisfies a disjunctive condition in the strong sense that the equality holds. As we shall see later , this is not the case for other types of generalized valuation (see the discussion around Eqs. (4.48–4.50) in Section 4.4).

2.2 No Strong Conjunctive Condition: One might wonder if there is not a strong version of the conjunctive condition too, in which the inequality in Eq. (3.19)—which comes purely from monotonicity—is replaced by an equality.

To check this, we note that the conjunction '$A \in \Delta_1 \wedge A \in \Delta_2$' = '$A \in \Delta_1 \cap \Delta_2$', receives the truth-value

$$\nu^V(A \in \Delta_1 \wedge A \in \Delta_2) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\Delta_1 \cap \Delta_2)\} \tag{3.23}$$

whereas

$$\begin{aligned}
\nu^V(A \in \Delta_1) \wedge \nu^V(A \in \Delta_2) &= \nu^V(A \in \Delta_1) \cap \nu^V(A \in \Delta_2) := \\
&\{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\Delta_1)\ \text{and}\ \ V(\hat{B}) \in f(\Delta_2)\}
\end{aligned} \tag{3.24}$$

where we have used the definition in Eq. (A.15) that if S_1 and S_2 are sieves on the same object, then $S_1 \wedge S_2 := S_1 \cap S_2$.

However $f(\Delta_1 \cap \Delta_2) \subseteq f(\Delta_1) \cap f(\Delta_2)$, and the equality may not hold if f is many-to-one. Thus the most that can be deduced from Eqs. (3.23–3.24) is that $\nu^V(A \in \Delta_1 \cap \Delta_2) \subseteq \nu^V(A \in \Delta_1) \cap \nu^V(A \in \Delta_2)$, which gives only the inequality

$$\nu^V(A \in \Delta_1 \wedge A \in \Delta_2) \leq \nu^V(A \in \Delta_1) \wedge \nu^V(A \in \Delta_2) \tag{3.25}$$

that could have been derived directly from the monotonicity result in Eq. (3.17).

As anticipated in the Introduction (the discussion in Section 1.3), there are good reasons for expecting the strict equality not to hold. For example, consider the propositions '$A \in \{a_1\}$' and '$A \in \{a_2\}$' with $a_1 \neq a_2$. Then

$$\nu^V(A \in \{a_1\} \cap \{a_2\}) = \nu^V(A \in \emptyset) = \emptyset \tag{3.26}$$

whereas

$$\nu^V(A \in \{a_1\}) \wedge \nu^V(A \in \{a_2\}) = \{f_{\mathcal{O}_d} : \hat{B} \to A \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) = f(a_1) = f(a_2)\} \tag{3.27}$$

and there is no reason for this to be the empty set, or even to be just minimally true: all that is necessary is that there is some nontrivial function $f : \sigma(\hat{A}) \to \mathbb{R}$ such that $f(\hat{A}) \in \mathrm{dom}\, V$ and $f(a_1) = f(a_2)$. Thus, in this special 'topos' sense, a physical quantity can have more than one partial value at once!

3. The Exclusivity Condition: It is necessary to check the exclusivity condition since this cannot be derived directly from the monotonicity result in Eq. (3.17).

So, suppose that $\nu^V(A \in \Delta_1) = \mathrm{true}_A = \downarrow \hat{A}$, and that Δ_2 is such that $\Delta_1 \cap \Delta_2 = \emptyset$. Then, from the definition of ν^V, it follows that id_A belongs to the sieve $\nu^V(A \in \Delta_1)$, and hence $\hat{A} \in \mathrm{dom}\, V$ and $V(\hat{A}) \in \Delta_1$. Therefore, since $\Delta_1 \cap \Delta_2 = \emptyset$, we have $V(\hat{A}) \notin \Delta_2$, and hence id_A is not a member of the sieve $\nu^V(A \in \Delta_2)$. This does not mean that $\nu^V(A \in \Delta_2)$ is equal to $\mathrm{false}_A\ (= \emptyset)$, but it does make it strictly less than true_A. Thus we have shown that if $A \in \Delta_1$ and $A \in \Delta_2$ are disjoint propositions, and if $\nu^V(A \in \Delta) = \mathrm{true}_A$, then $\nu^A(A \in \Delta_2) < \mathrm{true}_A$; hence exclusivity is satisfied.

4. No Unit Proposition Condition: The unit proposition in the Boolean algebra of propositions about A is simply '$A \in \sigma(\hat{A})$', and *a priori* one might expect that this is always given the value 'true_A', so that there is a unit analogue of the null condition Eq. (3.16). We shall refer to this as the 'unit proposition condition', and state it formally as:

Unit Proposition Condition: For all stages of truth $\hat{A}$

$$\nu(A \in \sigma(\hat{A})) = \mathrm{true}_A \tag{3.28}$$

or, in the alternative notation for valuations on projection operators,

$$\nu_A(\hat{1}) = \mathrm{true}_A. \tag{3.29}$$

However, in fact, this is not necessarily satisfied by the generalized valuation ν^V. Indeed, from the definition of ν^V we see at once that

$$
\begin{aligned}
\nu^V(A \in \sigma(\hat{A})) &:= \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V,\ V(\hat{B}) \in f(\sigma(\hat{A}))\} \\
&= \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{B} \in \mathrm{dom}\, V\}
\end{aligned} \tag{3.30}
$$

where the last equality holds since, for these discrete-spectra operators, $f(\sigma(\hat{A})) = \sigma(\hat{B})$, which means that $V(\hat{B})$ is always an element of the set $f(\sigma(\hat{A}))$. Thus

$$\nu^V(A \in \sigma(\hat{A})) = \operatorname{dom} V \cap \downarrow\hat{A} \tag{3.31}$$

which could well be a proper subset of the sieve $\text{true}_A := \downarrow\hat{A}$. Thus, in this situation, even the proposition 'A has *some* value' is not totally true! Rather, the partial truth-value of this proposition is a measure of the 'proximity' of the observable A to the domain of the partial valuation. Borrowing a standard piece of nomenclature from topos theory, one could say that the physical quantity A only 'partially exists' in this situation. As we shall see in Section 4.3, the unit proposition condition *is* satisfied by the generalized valuation associated with a quantum state.

4   Generalized Valuations and Quantum States

Motivated by Definition 3.3 as an example of a sieve-valued generalized valuation, and by the properties of these valuations, we turn now in Section 4.1 to the formal definition of a generalized valuation that is not based on the existence of any partial valuation. In Section 4.2 we discuss the precise way in which this fits into a topos framework; finally in Section 4.3 we show how any quantum state gives rise to a generalized valuation.

4.1   The Definition of a Generalized Valuation

Since we wish to apply these methods to the category $\mathcal{O}$ of all bounded, self-adjoint operators, the first step is to give meaning to the projector $\hat{E}[f(A) \in f(\Delta)]$ in those cases in which $f(\Delta)$ is not a Borel subset of $\sigma(f(\hat{A}))$. The main ingredient is the following theorem (which is also used in Section 5.3):

Theorem 4.1 *If Δ is a Borel subset of $\sigma(\hat{A})$, and if $f : \sigma(\hat{A}) \to \mathbb{R}$ is a Borel function such that $f(\Delta)$ is a Borel subset of $\sigma(f(\hat{A}))$, then if $W_{f(A)}$ is viewed as a subalgebra of the Boolean algebra W_A we have*

$$\hat{E}[f(A) \in f(\Delta)] = \inf\{\hat{Q} \in W_{f(A)} \subseteq W_A \mid \hat{E}[A \in \Delta] \le \hat{Q}\} \tag{4.1}$$

where the infimum of projectors is taken in the (complete) lattice structure of $\mathcal{P}$.

Proof

Let $\hat{I} := \inf\{\hat{Q} \in W_{f(A)} \subseteq W_A \mid \hat{E}[A \in \Delta] \le \hat{Q}\}$; then, since $E[A \in \Delta] \le \hat{E}[f(A) \in f(\Delta)]$, we clearly have $\hat{I} \le \hat{E}[f(A) \in f(\Delta)]$.

Conversely, suppose $\hat{Q} \in W_{f(A)} \subseteq W_A$ is such that $\hat{E}[A \in \Delta] \le \hat{Q}$. There is some Borel subset $K \subseteq \sigma(f(\hat{A}))$ such that $\hat{Q} = \hat{E}[f(A) \in K]$, and so $\hat{E}[A \in \Delta] \le \hat{E}[f(A) \in K]$. However, $\hat{E}[f(A) \in K] = \hat{E}[A \in f^{-1}(K)]$; and hence the inequality reads $\hat{E}[A \in \Delta] \le \hat{E}[A \in f^{-1}(K)]$, which implies $\Delta \subseteq f^{-1}(K)$ (up to sets of spectral-measure zero), and

hence that $f(\Delta) \subseteq f(f^{-1}(K)) \subseteq J$. In turn, this implies that $\hat{Q} = \hat{E}[f(A) \in K] \geq \hat{E}[f(A) \in f(\Delta)]$. In summary: $\hat{E}[A \in \Delta] \leq \hat{Q}$ implies that $\hat{E}[f(A) \in f(\Delta)] \leq \hat{Q}$, and hence $\hat{E}[f(A) \in f(\Delta)] \leq \hat{I}$. Thus $\hat{E}[f(A) \in f(\Delta)] = \hat{I}$. **Q.E.D.**

The key idea now is to use the right hand side of Eq. (4.1) as the *definition* of the symbol $\hat{E}[f(A) \in f(\Delta)]$ in those cases in which $f(\Delta)$ is not a Borel subset of $\sigma(f(\hat{A}))$. In this context, we note that since $W_{f(A)}$ is a complete sublattice of $\mathcal{P}$, the right hand side of Eq. (4.1) is always of the form $E[f(A) \in J]$ for *some* Borel subset J of $\sigma(f(\hat{A}))$. Note also that, considered as a definition of $\hat{E}[f(A) \in f(\Delta)]$, the expression Eq. (4.1) can be usefully rewritten as

$$\hat{E}[f(A) \in f(\Delta)] \quad := \quad \inf_{K \subseteq \sigma(f(\hat{A}))} \{\hat{E}[f(A) \in K] \mid \hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in K]\} \quad (4.2)$$

$$= \quad \inf_{K \subseteq \sigma(f(\hat{A}))} \{\hat{E}[f(A) \in K] \mid \hat{E}[A \in \Delta] \leq \hat{E}[A \in f^{-1}(K)]\} \quad (4.3)$$

$$= \quad \inf_{K \subseteq \sigma(f(\hat{A}))} \{\hat{E}[f(A) \in K] \mid \Delta \subseteq f^{-1}(J)\} \quad (4.4)$$

where the *infinum* is taken over all Borel subsets J of $\sigma(f(\hat{A}))$. From now on we shall use Eq. (4.2) as the definition of $\hat{E}[f(A) \in f(\Delta)]$ for the category of operators $\mathcal{O}$.

Equipped with this idea, we can give the definition of a generalized valuation on propositions about the values of any physical quantity represented by a bounded self-adjoint operator $\hat{A}$ in $\mathcal{O}$:

Definition 4.1 *A* generalized valuation *on the propositions in a quantum theory is a map ν that associates to each proposition of the form '$A \in \Delta$' (where Δ is a Borel subset of $\sigma(\hat{A})$) a sieve $\nu(A \in \Delta)$ on $\hat{A}$ in $\mathcal{O}$. These sieves must satisfy the following properties:*

(i) Functional composition:

For any Borel function $h : \sigma(\hat{A}) \to \mathbb{R}$ we have

$$\nu(h(A) \in h(\Delta)) = h_{\mathcal{O}}^*(\nu(A \in \Delta)). \quad (4.5)$$

(ii) Null proposition condition:

$$\nu(A \in \emptyset) = 0_A \quad (4.6)$$

(iii) Monotonicity:

If $\Delta_1 \subseteq \Delta_2$ then $\nu(A \in \Delta_1) \leq \nu(A \in \Delta_2)$. $\quad (4.7)$

(iv) Exclusivity:

If $\Delta_1 \cap \Delta_2 = \emptyset$ and $\nu(A \in \Delta_1) = \text{true}_A$, then $\nu(A \in \Delta_2) < \text{true}_A$. $\quad (4.8)$

We may also wish to add the 'unit proposition condition':

(v) Unit proposition condition:

$$\nu(A \in \sigma(\hat{A})) = \mathrm{true}_A. \tag{4.9}$$

Note that this definition of a generalized valuation makes sense for operators whose spectra contains continuous parts, as well as for those whose spectra is purely discrete. However, in order to give meaning to the proposition '$h(A) \in h(\Delta)$' in Eq. (4.5) if $h(\Delta)$ is not a Borel subset of $\sigma(h(\hat{A}))$, it is more appropriate to think of a generalized valuation as being defined on the projectors $\hat{E}[A \in \Delta]$, rather than on the more abstract propositions '$A \in \Delta$' themselves; for this enables the definition in Eq. (4.2) to be used.

The physical interpretation of a generalized valuation is motivated by the special case of the valuations ν^V discussed in the last section. Namely, the partial truth-value $\nu(A \in \Delta)$ of the proposition '$A \in \Delta$' is a sieve of coarse-grainings $f(\hat{A})$ of $\hat{A}$, at which each associated proposition '$f(A) \in f(\Delta)$' is totally true—reflecting the fact that if $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ belongs to the sieve $\nu(A \in \Delta)$ on $\hat{A}$ then, by the definition of a sieve, the pull-back $f_{\mathcal{O}}^* \nu(A \in \Delta)$ to $f(\hat{A})$ of this sieve is necessarily the principal sieve on $f(\hat{A})$ (see Eq. (A.12)). In general terms, we can say that the 'size' of the sieve $\nu(A \in \Delta)$ determines the degree of the partial truth of the proposition '$A \in \Delta$'.

We note that, as in the earlier discussion of the generalized valuation ν^V, the phrase '$A \in \Delta$' in $\nu(A \in \Delta)$ performs the dual function of specifying (i) the projection operator whose partial truth-value is to be given; and (ii) the context—the operator $\hat{A}$—in which this valuation takes place.

As in the previous section, this contextuality can be made more explicit by shifting the emphasis to think of valuations as being defined on projection operators in the explicit context of a specific physical quantity. Then, the truth-value associated with a specific $\hat{P} \in \mathcal{P}$ depends on the context of a particular self-adjoint operator $\hat{A}$ whose set of spectral projectors W_A includes $\hat{P}$. In this manifestly contextual form, the definition of a generalized valuation would read as follows:

Definition 4.2 *A generalized valuation on the lattice of projection operators $\mathcal{P}$ in a quantum theory is a collection of maps $\nu_A : W_A \to \Omega(\hat{A})$, one for each 'stage of truth' $\hat{A}$ in the category $\mathcal{O}$, with the following properties:*

(i) Functional composition:

For any Borel function $h : \sigma(\hat{A}) \to \mathbb{R}$,
$$\nu_{h(A)}(\hat{E}[h(A) \in h(\Delta)]) = h_{\mathcal{O}}^*(\nu_A(\hat{E}[A \in \Delta]). \tag{4.10}$$

(ii) Null proposition condition:
$$\nu_A(\hat{0}) = 0_A \tag{4.11}$$

(iii) Monotonicity:

If $\hat{\alpha}, \hat{\beta} \in W_A$ with $\hat{\alpha} \le \hat{\beta}$, then $\nu_A(\hat{\alpha}) \le \nu_A(\hat{\beta})$. $\tag{4.12}$

(iv) Exclusivity:

$$\text{If } \hat{\alpha}, \hat{\beta} \in W_A \text{ with } \hat{\alpha}\hat{\beta} = \hat{0} \text{ and } \nu_A(\hat{\alpha}) = \text{true}_A, \text{ then } \nu_A(\hat{\beta}) < \text{true}_A. \quad (4.13)$$

We may wish to supplement this list with:

(v) Unit proposition condition:

$$\nu_A(\hat{1}) = \text{true}_A. \quad (4.14)$$

Note that, in writing Eq. (4.10) we have employed the specific 'coarse-graining' function from the Boolean algebra W_A to the Boolean algebra $W_{h(A)}$, defined by the map

$$\hat{E}[A \in \Delta] \mapsto \hat{E}[h(A) \in h(\Delta)] \quad (4.15)$$

where, if necessary, the right hand side is to be understood in the sense of Eq. (4.2). In Section 5.3 we shall consider a more general way of understanding this operation.

4.2 The Topos Interpretation of Generalized Valuations

1. The Coarse-graining Presheaf: From what has been said so far it should be clear that ideas of topos theory lie at the heart of our constructions. However, the only explicit feature that has appeared so far is our use of sieves as truth-values, and we wish now to explain more fully how our ideas fit in with the theory of presheaves.

A key ingredient in exhibiting the underlying topos framework of generalized valuations is a certain presheaf on $\mathcal{O}$ that incorporates our central idea of operator coarse-graining. This is contained in the following definition.

Definition 4.3 *The* coarse-graining presheaf *over $\mathcal{O}$ is the covariant functor $\mathbf{G} : \mathcal{O}^{\text{op}} \to$* Set *defined as follows.*

1. *On objects in $\mathcal{O}$: $\mathbf{G}(\hat{A}) := W_A$, where W_A is the spectral algebra of $\hat{A}$.*

2. *On morphisms in $\mathcal{O}$: If $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ (i.e., $\hat{B} = f(\hat{A})$), then $\mathbf{G}(f_{\mathcal{O}}) : W_A \to W_B$ is defined as*

$$\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)] \quad (4.16)$$

   where, if necessary, the right hand side is to be understood in the sense of Eq. (4.2).

Note that $\mathbf{G}(f_{\mathcal{O}}) : W_A \to W_{f(A)}$ is just the coarse-graining operation considered above in Eq. (4.15).

The main step in proving that $\mathbf{G}$ is a contravariant functor from $\mathcal{O}$ to Set is to show that if $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ and $g_{\mathcal{O}} : \hat{C} \to \hat{B}$, then $\mathbf{G}(f_{\mathcal{O}} \circ g_{\mathcal{O}}) = \mathbf{G}(g_{\mathcal{O}}) \circ \mathbf{G}(f_{\mathcal{O}})$, as in Eq. (A.6). However, $\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)]$, and therefore, if $f(\Delta)$ and $g(f(\Delta))$ are Borel subsets of the appropriate spectra, then

$$\mathbf{G}(g_{\mathcal{O}})(\hat{E}[f(A) \in f(\Delta)]) := \hat{E}[g(f(A)) \in g(f(\Delta))] \quad (4.17)$$

while

$$\mathbf{G}(f_{\mathcal{O}} \circ g_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[g(f(A)) \in g(f(\Delta))]. \tag{4.18}$$

Hence $\mathbf{G}(f_{\mathcal{O}} \circ g_{\mathcal{O}}) = \mathbf{G}(g_{\mathcal{O}}) \circ \mathbf{G}(f_{\mathcal{O}})$, as desired. If $f(\Delta)$ or $g(f(\Delta))$ are not Borel subsets then the result follows (using the definition in Eq. (4.2)) as a special case of the result stated after Definition 5.4.

2. The Natural Transformation Between G and Ω: A key technical result in revealing the topos content of our constructions is the following.

Theorem 4.2 *To each generalized valuation ν on $\mathcal{P}$ there corresponds a natural transformation N^{ν} between the contravariant functors $\mathbf{G}$ and $\mathbf{\Omega}$, in which, at each stage of truth $\hat{A}$, the component $N_A^{\nu} : \mathbf{G}(\hat{A}) \to \mathbf{\Omega}(\hat{A})$ is defined by*

$$N_A^{\nu}(\hat{P}) := \nu_A(\hat{P}) \tag{4.19}$$

for all $\hat{P} \in W_A = \mathbf{G}(\hat{A})$.

Proof

We recall that the subobject classifier $\mathbf{\Omega}$ in the topos $\mathrm{Set}^{\mathcal{O}^{\mathrm{op}}}$ is defined (i) on objects by $\mathbf{\Omega}(\hat{A}) := \{S \mid S \text{ is a sieve on } \hat{A} \text{ in } \mathcal{O}\}$; and (ii) on morphisms $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ by $\mathbf{\Omega}(f_{\mathcal{O}}) : \mathbf{\Omega}(\hat{A}) \to \mathbf{\Omega}(\hat{B})$ where $\mathbf{\Omega}(f_{\mathcal{O}})(S) := f_{\mathcal{O}}^*(S)$ for all sieves $S \in \mathbf{\Omega}(\hat{A})$.

As discussed in Section A.2, a natural transformation N between the contravariant functors $\mathbf{G}$ and $\mathbf{\Omega}$ is defined to be a family of functions $N_A : \mathbf{G}(\hat{A}) \to \mathbf{\Omega}(\hat{A})$—one for each stage of truth $\hat{A}$—such that, if $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, the composite map $\mathbf{G}(\hat{A}) \xrightarrow{N_A} \mathbf{\Omega}(\hat{A}) \xrightarrow{\mathbf{\Omega}(f_{\mathcal{O}})} \mathbf{\Omega}(\hat{B})$ is equal to $\mathbf{G}(\hat{A}) \xrightarrow{\mathbf{G}(f_{\mathcal{O}})} \mathbf{G}(\hat{B}) \xrightarrow{N_B} \mathbf{\Omega}(\hat{B})$ (*cf.* the commutative diagram in Eq. (A.9)).

In our case, if ν is a generalized valuation, the associated natural transformation N^{ν} is defined at stage $\hat{A}$ on $\mathbf{G}(\hat{A}) := W_A$ by $N_A^{\nu}(\hat{E}[A \in \Delta]) := \nu_A(\hat{E}[A \in \Delta]) \equiv \nu(A \in \Delta)$. Then

$$\mathbf{\Omega}(f_{\mathcal{O}}) \circ N_A^{\nu}(\hat{E}[A \in \Delta]) = \mathbf{\Omega}(f_{\mathcal{O}})(N_A^{\nu}(\hat{E}[A \in \Delta])) = f_{\mathcal{O}}^*(\nu_A(\hat{E}[A \in \Delta])) \tag{4.20}$$

while

$$\begin{aligned} N_B^{\nu} \circ \mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) &= N_B^{\nu}(\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta])) \\ &= N_B^{\nu}(\hat{E}[f(A) \in f(\Delta)]) = \nu_{f(A)}(\hat{E}[f(A) \in f(\Delta)]) \end{aligned} \tag{4.21}$$

However, the functional composition principle Eq. (4.10) says that the right hand sides of Eq.(4.20) and Eq. (4.21) are identical, which shows that $\mathbf{\Omega}(f_{\mathcal{O}}) \circ N_A^{\nu}(\hat{E}[A \in \Delta]) = N_B^{\nu} \circ \mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta])$. Hence N^{ν} is a natural transformation between the contravariant functors $\mathbf{G}$ and $\mathbf{\Omega}$. **Q.E.D.**

Note that, in the language of Definition 4.1, the components of the natural transformation are $N_A^{\nu}(\hat{E}[A \in \Delta]) := \nu(A \in \Delta)$.

The next three subsections bring out some of the implicit 'topos content' of Theorem 4.2.

3. Another Perspective on the Coarse-Graining Presheaf: There is another way of looking at the coarse-graining presheaf which may help to clarify its place in the theory; at least in the case of operators with purely discrete spectra. Associated with the spectral presheaf $\mathbf{\Sigma} : \mathcal{O}_d{}^{\mathrm{op}} \to$ Set of Definition 2.2 there is another covariant functor $B\mathbf{\Sigma} : \mathcal{O}_d{}^{\mathrm{op}} \to$ Set, defined as follows:

1. *On objects: $B\mathbf{\Sigma}(\hat{A}) := B(\sigma(\hat{A}))$—the Boolean algebra of Borel subsets of the spectrum of $\hat{A}$.*

2. *On morphisms: If $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$, so that $\hat{B} = f(\hat{A})$, then $B\mathbf{\Sigma}(f_{\mathcal{O}_d}) : B(\sigma(\hat{A})) \to B(\sigma(\hat{B}))$ is defined by*

$$B\mathbf{\Sigma}(f_{\mathcal{O}_d})(\Delta) := f(\Delta) \tag{4.22}$$

for all Borel subsets $\Delta \subseteq \sigma(\hat{A})$.

Note that the spectral Boolean algebra W_A is isomorphic to the Boolean algebra $B(\sigma(\hat{A}))$ by the map that associates the projector $\hat{E}[A \in \Delta] \in W_A$ with the Borel subset $\Delta \in B(\sigma(\hat{A}))$. From equations Eq. (4.16) and Eq. (4.22), it is clear therefore that the coarse-graining presheaf $\mathbf{G}$ is essentially the same thing as the 'power-object' $B\mathbf{\Sigma}$.[12]

4. Generalized Valuations as Subobjects of G: We recall that, in a topos of presheaves such as $\mathrm{Set}^{\mathcal{O}^{\mathrm{op}}}$, a morphism between a pair of functors (*i.e.*, a pair of objects in the topos) is defined to be a natural transformation between them. Therefore, Theorem 4.2 implies that to each generalized valuation ν there corresponds a morphism $N^\nu : \mathbf{G} \to \mathbf{\Omega}$ between the coarse-graining object $\mathbf{G}$ and the subobject classifier $\mathbf{\Omega}$ in the topos $\mathrm{Set}^{\mathcal{O}^{\mathrm{op}}}$. However, precisely because $\mathbf{\Omega}$ *is* the subobject classifier in this topos, morphisms $\mathbf{G} \to \mathbf{\Omega}$ are in one-to-one correspondence with subobjects of $\mathbf{G}$ (see the end of Section A.2; especially equations Eq. (A.20) and Eq. (A.21)). Thus, we conclude that to every generalized valuation there corresponds a subobject of the coarse-graining object $\mathbf{G}$; or, equivalently, of the power object $B\mathbf{\Sigma}$.

Conversely, of course, we could turn this around and *define* a generalized valuation to be any subobject of $\mathbf{G}$, or $B\mathbf{\Sigma}$, that is subject to the conditions Eqs. (4.11–4.13), or to the equivalent set Eqs. (4.6–4.8).

One important consequence of looking at a generalized valuation as a certain type of morphism from $\mathbf{G}$ to $\mathbf{\Omega}$, comes from the fact that, in any topos, the collection of all subobjects of a given object has the structure of a Heyting algebra. This is of considerable interest to us since it raises the possibility that the subset of subobjects that satisfy our extra conditions Eqs. (4.11–4.13)—*i.e.*, the set of generalized valuations—may inherit some, or all, of this logical structure. This could be expected to play an important role in exploring the physical implications of these valuations. We shall return in a later paper to discussing the structure of the space of all generalized valuations.

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[12]With any object X in a topos, there is associated another object $PX := \Omega^X$, known as the 'power object', which is the topos analogue of the power set of a set (the set of all subsets of the set). In our case, $B\mathbf{\Sigma}$ is the subobject of the power object $P\mathbf{\Sigma}$ obtained by requiring the elements of $B\mathbf{\Sigma}(\hat{A})$ to be *Borel* subsets of $\mathbf{\Sigma}(\hat{A}) := \sigma(\hat{A})$ only—rather than arbitrary subsets—at each stage $\hat{A}$. Thus the 'coarse-graining' presheaf is closely related to the power object $P\mathbf{\Sigma}$.

5. Generalized Valuations as Global Sections of a Presheaf: We note in passing that there is a bijection between morphisms from **G** to $\mathbf{\Omega}$, and global elements of the 'exponential' object $\mathbf{\Omega}^{\mathbf{G}}$ which, roughly speaking, is the topos analogue of the set Y^X of all maps from X to Y in normal set theory. Thus a generalized valuation *does* turn out to be a global section of a certain presheaf on $\mathcal{O}$, but it is the presheaf $\mathbf{\Omega}^{\mathbf{G}}$, not the simple dual presheaf $\mathbf{D} \circ \mathbf{W}$ to which the Kochen-Specker 'no-go' theorem applies.

6. The Generalized Valuation of a Physical Quantity: Definition 4.1 gives generalized truth-values to propositions of the form '$A \in \Delta$' but this leaves open the question whether in the case of operators with purely discrete spectra, there is some corresponding concept of a 'generalized value' for the physical quantity A.

Clearly this cannot generally be a single real number, unless all the propositions '$A = a$', $a \in \sigma(\hat{A})$, (where $\hat{A}$ has a purely discrete spectrum) are evaluated as false$_A$ except for one, '$A = a_0$' say, which is evaluated as true$_A$; in this case one can say that the value of A is a_0. More generally, however, the quantity A has to be given some sort of 'smeared' value, corresponding to the collection of propositions '$A = a$' that are not evaluated as totally false. In fact, this suggests that, given a generalized valuation ν, we might try *defining* the 'value' of the physical quantity A as $V^{\nu}(A) := \{\langle a, \nu(A = a)\rangle \mid a \in \sigma(\hat{A})\}$, so that we assign to A the collection of the eigenvalues of $\hat{A}$ 'weighted' with the generalized valuations of the associated propositions.

With this preliminary definition, $V^{\nu}(\hat{A})$ is a subset of $\sigma(\hat{A}) \times \mathbf{\Omega}(\hat{A})$, and is hence a relation between $\sigma(\hat{A})$ and $\mathbf{\Omega}(\hat{A})$. However, since each $a \in \sigma(\hat{A})$ is associated with a *unique* element $\nu(A = a) \in \mathbf{\Omega}(\hat{A})$, this relation defines a function from $\sigma(\hat{A})$ to $\mathbf{\Omega}(\hat{A})$, and thus we arrive at the idea that $V^{\nu}(A)$ should be such a function. However, this holds at each stage of truth $\hat{A}$ and, it transpires, these fit together nicely to give a morphism between the presheaves $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ in the category $\text{Set}^{\mathcal{O}^{\text{op}}}$. More precisely, we have the following theorem:

Theorem 4.3 *To each generalized valuation ν in the sense of Definition 4.1 applied to the category $\mathcal{O}_d$, there is associated a natural transformation $V^{\nu} : \mathbf{\Sigma} \to \mathbf{\Omega}$ for which, at each stage of truth $\hat{A}$, the component $V_A^{\nu} : \mathbf{\Sigma}(\hat{A}) \to \mathbf{\Omega}(\hat{A})$ is defined by*

$$V_A^{\nu}(a) := \nu(A = a). \tag{4.23}$$

Proof

To see that this is a natural transformation we have to show that, if $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$, the composite map $\mathbf{\Sigma}(\hat{A}) \xrightarrow{V_A^{\nu}} \mathbf{\Omega}(\hat{A}) \xrightarrow{\mathbf{\Omega}(f_{\mathcal{O}_d})} \mathbf{\Omega}(\hat{B})$ is equal to $\mathbf{\Sigma}(\hat{A}) \xrightarrow{\mathbf{\Sigma}(f_{\mathcal{O}_d})} \mathbf{\Sigma}(\hat{B}) \xrightarrow{V_B^{\nu}} \mathbf{\Omega}(\hat{A})$ (*cf.*, the commutative diagram in Eq. (A.9)).

It is a straightforward task to prove this directly, but in fact this is not necessary since the theorem can be derived at once from the earlier result in Theorem 4.2 that N^{ν} is a natural transformation from **G** (or $B\mathbf{\Sigma}$) to $\mathbf{\Omega}$. The main step is to note the existence of a

natural transformation[13] $\{\}_{\boldsymbol{\Sigma}} : \boldsymbol{\Sigma} \to B\boldsymbol{\Sigma}$ whose components $\{\}_{\boldsymbol{\Sigma}A}$ that map $\boldsymbol{\Sigma}(\hat{A}) = \sigma(\hat{A})$ to $B\boldsymbol{\Sigma}(\hat{A}) = B(\sigma(\hat{A}))$ are

$$\{\}_{\boldsymbol{\Sigma}A}(a) := \{a\}. \tag{4.24}$$

This is well-defined since $\{a\}$ is a Borel subset of the (discrete) spectrum $\sigma(\hat{A})$ of $\hat{A}$; that it satisfies the requirements for a natural transformation is obvious. Then, identifying the coarse-graining presheaf $\mathbf{G}$ with $B\boldsymbol{\Sigma}$, we see that $V^{\nu}_A : \boldsymbol{\Sigma}(\hat{A}) \to \boldsymbol{\Omega}(\hat{A})$ can be written as $V^{\nu}_A = (N^{\nu} \circ \{\}_{\boldsymbol{\Sigma}})_A$ for all stages $\hat{A}$. Thus

$$V^{\nu} = N^{\nu} \circ \{\}_{\boldsymbol{\Sigma}} \tag{4.25}$$

which, as a composition of natural transformations, is itself a natural transformation. **Q.E.D.**

In particular, it follows that each generalized valuation defines a subobject of the spectral presheaf $\boldsymbol{\Sigma}$ (*cf.* the remarks in Subsection 4. above, or Eq. (A.21) for the general definition of the subobject associated with a morphism into $\boldsymbol{\Omega}$). Note that the exclusivity condition means that, in the map $V^{\nu}_A : \boldsymbol{\Sigma}(\hat{A}) \to \boldsymbol{\Omega}(\hat{A})$, at most one element in $\boldsymbol{\Sigma}(\hat{A}) = \sigma(\hat{A})$ is assigned the value 'totally true' (true_A). Thus, the subobject of $\boldsymbol{\Sigma}$ defined by V^{ν} has the property that the associated subset of each $\boldsymbol{\Sigma}(\hat{A})$ is either a singleton or it is empty. In fact, it defines a partial section of the presheaf $\boldsymbol{\Sigma}$, and hence a partial valuation in the sense of Section 3 (*i.e.*, a number-valued valuation with a limited domain)—which we will also denote V^{ν}—with

$$\mathrm{dom}\, V^{\nu} := \{\hat{A} \mid \exists a \in \sigma(\hat{A}), \text{ s.t. } V^{\nu}_A(a) = \mathrm{true}_A\} \tag{4.26}$$

and with the value of any operator $\hat{A}$ in this domain being defined as the associated real number $a \in \sigma(\hat{A})$.

In Definition 3.3 we showed how to go from a partial valuation/section to a generalized valuation; here we have shown how each generalized valuation leads back to a partial valuation. We note in passing that the chain

$$\text{partial valuation} \to \text{generalized valuation} \to \text{partial valuation} \tag{4.27}$$

takes any given partial valuation back to itself. However we do not necessarily return to the starting point if we begin the 'chain' with a generalized valuation; *i.e.*,

$$\text{generalized valuation} \to \text{partial valuation} \to \text{generalized valuation}. \tag{4.28}$$

We shall see an explicit example of this in Section 4.5.

4.3 The Generalized Valuation Associated with a Quantum State Vector

We shall now show that any quantum state gives rise to an associated generalized valuation.

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[13]The notation reflects that fact that $\{\}_{\boldsymbol{\Sigma}} : \boldsymbol{\Sigma} \to B\boldsymbol{\Sigma}$ is a topos analogue of the map $X \to PX$, $x \mapsto \{x\}$, in standard set theory.

Let us start by considering the extent to which a vector $\psi \in \mathcal{H}$ can be regarded as assigning a value to a physical quantity A represented by a self-adjoint operator $\hat{A}$ (whose spectrum may not necessarily be purely discrete). In the standard interpretation of quantum theory, one makes only the minimal claim that a physical quantity A possesses a value a when the state ψ is an eigenstate of $\hat{A}$ with eigenvalue a; *i.e.*, $\hat{A}\psi = a\psi$.

However, the ideas we have been developing in this paper suggest that even when ψ is *not* an eigenvector of $\hat{A}$, it may still be possible to give a *partial* truth-value to the proposition '$A = a$'. Indeed, in the light of our earlier discussion, it is natural to reflect on the possibility that some function $f(\hat{A})$ of $\hat{A}$ may have ψ as an eigenvector, even though $\hat{A}$ itself does not. Thus we are led to define, for each state $\psi \in \mathcal{H}$, an associated generalized valuation ν^{ψ} on propositions '$A = a$' as

$$\nu^{\psi}(A = a) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{B}\psi = f(a)\psi\}. \tag{4.29}$$

The condition $\hat{B}\psi = f(a)\psi$ is equivalent to $\hat{E}[B = f(a)]\psi = \psi$, and this suggests an obvious extension to include propositions of the form '$A \in \Delta$':

Definition 4.4 *The generalized valuation ν^{ψ} associated with a vector $\psi \in \mathcal{H}$ is*

$$\nu^{\psi}(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{E}[B \in f(\Delta)]\psi = \psi\} \tag{4.30}$$

where Δ is a Borel subset of the spectrum $\sigma(\hat{A})$ of $\hat{A}$. If necessary, the right hand side of Eq. (4.30) is to be understood in the sense of Eq. (4.2).

Note that if ψ is actually an eigenstate of $\hat{A}$ with eigenvalue a, then $\nu^{\psi}(A \in \Delta) = \text{true}_A$, if $a \in \Delta$. This is a good illustration of the general rule-of-thumb that if a proposition is evaluated as 'totally true', this is equivalent to saying that it is true in the normal sense; *i.e.* in the sense of simple two-valued logic.

At this point we could check explicitly that the right hand side of Eq. (4.29) is a sieve, and that ν^{ψ} possesses the extra properties Eqs. (4.5–4.8) required for a generalized valuation. However, we shall first give a few simple examples, and then press on to give a substantial extension of the definition to include generalized valuations associated with density matrices, and then prove all the needed results for that.

1. An Example with Spin-$1/2$**:** We take a two-dimensional spin system with $\psi := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$—which is an eigenstate of $\hat{S}_x$—and consider the generalized evaluation of the propositions '$S_z = \frac{1}{2}$' and '$S_z = -\frac{1}{2}$' (we choose units in which $\hbar = 1$).

The physical quantity S_z is represented by the matrix $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the only functions of this for which ψ is an eigenvector are $r(\hat{S}_z)^2$, $r \in \mathbb{R}$. Thus, if we use the category $\mathcal{O}$, the definition Eq. (4.29) of the generalized valuation ν^{ψ}, says that both the propositions '$S_z = \frac{1}{2}$' and '$S_z = -\frac{1}{2}$' are only minimally true. If we use the category $\mathcal{O}_*$ (so that multiples of the unit operator are excluded as stages of truth) then

$$\nu^{\psi}(S_z = \tfrac{1}{2}) = \emptyset; \quad \nu^{\psi}(S_z = -\tfrac{1}{2}) = \emptyset. \tag{4.31}$$

Hence we see that, in this particular example, the physical quantity S_x—which, unequivocally, has the value $1/2$ in the state ψ—is sufficiently 'far' from S_z that propositions assigning a definite value to the latter cannot be evaluated as anything other than (i) totally false, if $\mathcal{O}_*$ is used as the category of contexts; or (ii) minimally true, if $\mathcal{O}$ is used.

On the other hand, the spectral projector corresponding to the proposition $S_z \in \{-1/2, 1/2\} = \sigma(\hat{S}_z)$ is the unit operator $\hat{1}$, and hence

$$\nu^\psi(S_z \in \{-1/2, 1/2\}) = \text{true}_{S_z}. \tag{4.32}$$

This result might be construed as asserting that the quantity S_z 'exists', even if it is not possible to assign a non-trivial truth-value to a proposition that asserts it has any specific value. As we shall see shortly, this unit proposition condition (defined earlier in Eq. (3.28)) is always satisfied by a generalized valuation produced by a quantum state.

We note in passing that the result in Eq. (4.31) means that this particular type of generalized valuation cannot be used by itself to construct a stochastic hidden variable theory. More precisely, the example shows that, given a valuation ν^ψ generated by a normalised state $\psi \in \mathcal{H}$, one cannot expect to find a measure μ_A on the space of sieves on $\hat{A}$, such that $\mu_A[\nu^\psi(A \in \Delta)]$ is equal to the quantum-mechanical value $\langle \psi, \hat{E}[A \in \Delta]\psi \rangle$ for the probability that a measurement of A will yield a result lying in Δ. Thus, in the example, we have $\text{Prob}(S_z = 1/2; \psi) = 1/2$ and $\text{Prob}(S_z = -1/2; \psi) = 1/2$, whereas the generalized truth-values of the propositions '$S_z = \frac{1}{2}$' and '$S_z = -\frac{1}{2}$' are both null (or minimal).

2. An Example with Spin-1: We shall now consider an example where a non-trivial generalized valuation is obtained. This involves a spin-1 system where the physical quantities S_x and S_z are represented by the matrices

$$\hat{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \hat{S}_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.33}$$

respectively.

Let the quantum state ψ be $(0, 1, 0)$—which is an eigenstate of $\hat{S}_z$ with eigenvalue 0—and consider the propositions '$S_x = 1$' and '$S_x = -1$'. Since ψ is not an eigenstate of $\hat{S}_x$, neither of these propositions is totally true at stage $\hat{S}_x$. On the other hand,

$$\hat{S}_x^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{4.34}$$

and we see that $\hat{S}_x^2\psi = \psi$. Furthermore, taking the square of $\hat{S}_x^2$ gives just a multiple of itself, and taking the cube of $\hat{S}_x$ gives just a multiple of $\hat{S}_x$; hence all functions of $\hat{S}_x$ are of the form $t\hat{1} + k\hat{S}_x + r\hat{S}_x^2$. Note that the real numbers t, k, r have to be such that k and r are not both zero if we use the category $\mathcal{O}_*$, since that excludes multiples of $\hat{1}$ as possible contexts/stages of truth.

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It is easy to check that ψ is an eigenstate of $t\hat{1} + k\hat{S}_x + r\hat{S}_x^2$ if, and only if, $k = 0$; hence, if $s_{t,r\,\mathcal{O}} : t\hat{1} + r\hat{S}_x^2 \to \hat{S}_x$ denotes the morphism in $\mathcal{O}$ that corresponds to the function $s_{t,r} : \sigma(\hat{S}_x) \to \mathbb{R}$ defined by $s_{t,r}(\lambda) := t + r\lambda^2$, we see that

$$\nu^\psi(S_x = 1) = \{s_{t,r\,\mathcal{O}} : t\hat{1} + r\hat{S}_x^2 \to \hat{S}_x \mid t, r \in \mathbb{R}\} \tag{4.35}$$

and

$$\nu^\psi(S_x = -1) = \{s_{t,r\,\mathcal{O}} : t\hat{1} + r\hat{S}_x^2 \to \hat{S}_x \mid t, r \in \mathbb{R}\}. \tag{4.36}$$

The conclusion is that the propositions '$S_x = 1$' and '$S_x = -1$' are both assigned a non-trivial partial truth-value: namely the sieve $\{s_{t,r\,\mathcal{O}} : t\hat{1} + r\hat{S}_x^2 \to \hat{S}_x \mid t, r \in \mathbb{R}\}$; if $\mathcal{O}_*$ is used, then the value $r = 0$ is excluded.

On the other hand, we note that the proposition '$S_x \in \{-1, 1\}$' is represented by the projector $\hat{E}[S_x = -1] + \hat{E}[S_x = +1]$, and also

$$\psi := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} - \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \tag{4.37}$$

where the column vectors on the right hand side are eigenvectors of $\hat{S}_x$ with eigenvalues $+1$ and -1 respectively. It follows that $\hat{E}[S_x \in \{-1, 1\}]\psi = \psi$, and hence

$$\nu^\psi(S_x \in \{-1, 1\}]) = \text{true}_{S_x} \tag{4.38}$$

whereas, as shown by Eqs. (4.35–4.36),

$$\nu^\psi(S_x = 1) \vee \nu^\psi(S_x = -1) = \{s_{t,r\,\mathcal{O}} : t\hat{1} + r\hat{S}_x^2 \to \hat{S}_x \mid t, r \in \mathbb{R}\} < \text{true}_{S_x}. \tag{4.39}$$

This failure of a strong disjunctive condition is typical of the generalized valuations produced by quantum states, and we shall return to this feature shortly. As emphasized in the Introduction, it can be regarded as a fundamental consequence of the superposition principle of quantum theory.

4.4 The Generalized Valuation Associated with a Density Matrix

We shall now show that it is possible to associate a generalized valuation to each density matrix state ρ in the quantum theory. To this end, we note that the previous definition Eq. (4.30) for ν^ψ can be re-expressed as

$$\nu^\psi(A \in \Delta) = \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \langle\psi, \hat{E}[B \in f(\Delta)]\psi\rangle = \langle\psi, \psi\rangle\} \tag{4.40}$$

or, in more physical terms,

$$\nu^\psi(A \in \Delta) = \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \text{Prob}(B \in f(\Delta); \psi) = 1\} \tag{4.41}$$

where $\text{Prob}(B \in f(\Delta); \psi)$ denotes the usual quantum mechanical probability that the result of a measurement of B will lie in $f(\Delta) \subseteq \sigma(\hat{B}) \subset \mathbb{R}$, given that the quantum state is ψ.

This way of expressing ν^ψ clarifies a little the physical meaning of the generalized valuation—it is the set of coarse-grainings $f(\hat{A})$ of $\hat{A}$ such that the probability that $f(A)$ lies in $f(\Delta)$ is 1; something that is construed in the standard interpretation as equivalent to saying that $f(A)$ actually has a value in $f(\Delta)$. It also suggests the following definition for a generalized valuation associated with any density matrix:

Definition 4.5 *The* generalized valuation ν^ρ *associated with a density matrix ρ is*

$$\begin{aligned}
\nu^\rho(A \in \Delta) &:= \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{Prob}(B \in f(\Delta); \rho) = 1\} \\
&= \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\Delta)]) = 1\}.
\end{aligned} \tag{4.42}$$

If necessary, the right hand side of Eq. (4.42) is to be understood in the sense of Eq. (4.2).

This class of generalized valuation is clearly of considerable physical interest, and therefore it is important to check that the necessary conditions are satisfied (of course, this will include as a special case the generalized valuations ν^ψ, $\psi \in \mathcal{H}$.)

First we show that $\nu^\rho(A \in \Delta)$ is a sieve on $\hat{A}$ in $\mathcal{O}$. Thus, suppose $f_\mathcal{O} \in \nu^\rho(A \in \Delta)$, and let $h_\mathcal{O} : \hat{C} \to \hat{B}$. Then, in the lattice $\mathcal{P}$ of projection operators, $\hat{E}[B \in f(\Delta)] \leq \hat{E}[C \in h(f(\Delta))]$; and hence $\mathrm{tr}(\rho\,\hat{E}[B \in f(\Delta)]) \leq \mathrm{tr}(\rho\hat{E}[C \in h(f(\Delta))])$. In particular, since $f_\mathcal{O} \in \nu^\rho(A \in \Delta)$, we have $\mathrm{tr}(\rho\,\hat{E}[B \in f(\Delta)]) = 1$, and hence $\mathrm{tr}(\rho\hat{E}[C \in h(f(\Delta))]) = 1$ (since $\mathrm{tr}(\rho\hat{P}) \leq 1$ for all projection operators $\hat{P}$). Thus $h_\mathcal{O} \in \nu^\rho(A \in \Delta)$, which proves that $\nu^\rho(A \in \Delta)$ is a sieve on $\hat{A}$.

1. The Functional Composition Rule: Next we must show that the functional composition rule is satisfied. If $k_\mathcal{O} : \hat{C} \to \hat{A}$, then

$$\begin{aligned}
k_\mathcal{O}^*(\nu^\rho(A \in \Delta)) &:= \{j_\mathcal{O} : \hat{D} \to \hat{C} \mid k_\mathcal{O} \circ j_\mathcal{O} \in \nu^\rho(A \in \Delta)\} \\
&= \{j_\mathcal{O} : \hat{D} \to \hat{C} \mid \mathrm{tr}(\rho\,\hat{E}[D \in j(k(\Delta))]) = 1\}
\end{aligned} \tag{4.43}$$

whereas

$$\nu^\rho(k(A) \in k(\Delta)) := \{h_\mathcal{O} : \hat{D} \to k(\hat{A}) \mid \mathrm{tr}(\rho\,\hat{E}[D \in h(k(\Delta))]) = 1\}. \tag{4.44}$$

Thus $k_\mathcal{O}^*(\nu^\rho(A \in \Delta)) = \nu^\rho(k(A) \in k(\Delta))$, as required.

We shall now consider the extent to which the object ν^ρ defined in Eq. (4.42) satisfies the remaining conditions Eqs. (4.6–4.8) in the formal definition of a generalized valuation.

2. The Null Proposition Condition: To check this, we note that $\nu^\rho(A \in \emptyset) := \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\emptyset)]) = 1\}$. But this is the empty set since $\hat{E}[B \in f(\emptyset)] = \hat{0}$. Hence the null proposition condition is satisfied.

3. The Monotonicity Condition: Suppose $f_{\mathcal{O}} \in \nu^\rho(A \in \Delta_1)$ where

$$\nu^\rho(A \in \Delta_1) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \operatorname{tr}(\rho\,\hat{E}[B \in f(\Delta_1)]) = 1\}. \qquad (4.45)$$

If $\Delta_1 \subseteq \Delta_2$, then $f(\Delta_1) \subseteq f(\Delta_2)$; and in the lattice of projection operators we then have

$$\hat{E}[B \in f(\Delta_1)]) \leq \hat{E}[B \in f(\Delta_2)]). \qquad (4.46)$$

But then $\operatorname{tr}(\rho\,\hat{E}[B \in f(\Delta_1)]) = 1$ implies that $\operatorname{tr}(\rho\,\hat{E}[B \in f(\Delta_2)]) = 1$ (since $\operatorname{tr}(\rho\hat{P}) \leq 1$ for all projection operators $\hat{P}$). Thus $f_{\mathcal{O}}$ also belongs to $\nu(A \in \Delta_2)$, which means that $\nu^\rho(A \in \Delta_1) \subseteq \nu^\rho(A \in \Delta_2)$. However, in the Heyting algebra of sieves on $\hat{A}$, the partial ordering operations is just subset inclusion; hence we have shown that

$$\Delta_1 \subseteq \Delta_2 \text{ implies } \nu^\rho(A \in \Delta_1) \leq \nu^\rho(A \in \Delta_2), \qquad (4.47)$$

as required.

3.1 No Strong Disjunctive Condition: From the monotonicity result in Eq. (4.47) one can immediately derive the weak disjunctive condition

$$\nu^\rho(A \in \Delta_1) \cup \nu^\rho(A \in \Delta_2) \leq \nu^\rho(A \in \Delta_1 \cup \Delta_2). \qquad (4.48)$$

However, in Section 1.3 we remarked, in rather general terms, that the existence of the quantum superposition principle suggests that the reverse inequality may not hold in Eq. (4.48). To see this explicitly, consider the special case when ρ comes from a state vector ψ, and let $\Delta_1 := \{a_1\}$, $\Delta_2 := \{a_2\}$ with $a_1 \neq a_2$—i.e., we are considering the propositions '$A = a_1$' and '$A = a_2$'. Then

$$\nu^\psi(A \in \{a_1\}) \cup \nu^\psi(A \in \{a_2\}) = \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{B}\psi = f(a_1)\psi \text{ or } \hat{B}\psi = f(a_2)\psi\} \quad (4.49)$$

whereas

$$\nu^\psi(A \in \{a_1, a_2\}) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{E}[B \in f(\{a_1, a_2\})]\psi = \psi\}. \qquad (4.50)$$

Now suppose $f : \sigma(\hat{A}) \to \mathbb{R}$ is such that $f(a_1) \neq f(a_2)$. Then satisfaction of the condition in Eq. (4.50) requires only that ψ lies in the direct sum of the eigenspaces of the operator $\hat{B} := f(\hat{A})$ that are associated with the eigenvalues $f(a_1)$ and $f(a_2)$; in particular, if ψ is a non-trivial linear superposition of these eigenstates of $\hat{B}$, then $f_{\mathcal{O}} : \hat{B} \to \hat{A}$ will belong to the sieve $\nu^\psi(A \in \{a_1, a_2\})$, but it will not belong to $\nu^\psi(A \in \{a_1\}) \cup \nu^\psi(A \in \{a_2\})$. Thus, there is a strict inequality in Eq. (4.48); an explicit example is Eqs.(4.38–4.39) in the spin-1 model discussed above, with f chosen to be the identity map on $\hat{S}_x$. This should be contrasted with the generalized valuation ν^V that satisfies the strong disjunctive condition Eq. (3.22).

3.2 No Strong Conjunctive Condition: We can also confirm the absence of any strong conjunctive condition. Indeed, using the same pair of propositions as above, we have '$A \in \{a_1\} \wedge A \in \{a_2\}$' = '$A \in \{a_1\} \cap \{a_2\}$' = '$A \in \emptyset$'; and hence

$$\nu^\rho(A \in \{a_1\} \wedge A \in \{a_2\}) = \emptyset = 0_A. \qquad (4.51)$$

On the other hand

$$\nu^\rho(A \in \{a_1\}) \cap \nu^\rho(A \in \{a_2\}) = \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\{a_1\})]) = 1 \text{ and}$$
$$\mathrm{tr}(\rho\,\hat{E}[B \in f(\{a_2\})]) = 1\}. \quad (4.52)$$

Then, if we chose f such that $f(a_1) = f(a_2)$ it is perfectly possible for the right hand side of Eq. (4.52) to be non-trivial. Thus, in general, there is no strong conjunctive condition.

4. The Exclusivity Condition: Finally, we must check the exclusivity condition. Thus suppose $\Delta_1 \cap \Delta_2 = \emptyset$ and $\nu^\rho(A \in \Delta_1) = \mathrm{true}_A$; then, in particular, $\mathrm{tr}(\rho\,\hat{E}[A \in \Delta_1]) = 1$. Now define the real number $k := \mathrm{tr}(\rho\,\hat{E}[A \in \Delta_2])$; this satisfies $0 \leq k \leq 1$. Then, since $\Delta_1 \cap \Delta_2 = \emptyset$, the projectors $\hat{E}[A \in \Delta_1]$ and $\hat{E}[A \in \Delta_2]$ are orthogonal, and therefore $\hat{E}[A \in \Delta_1 \cup \Delta_2] = \hat{E}[A \in \Delta_1] + \hat{E}[A \in \Delta_2]$. Thus $\mathrm{tr}(\rho\,\hat{E}[A \in \Delta_1 \cup \Delta_2]) = 1 + k$. However, since $\mathrm{tr}(\rho\hat{P}) \leq 1$ for all projection operators $\hat{P}$, and $k \geq 0$, we deduce that $k = 0$, *i.e.*, $\mathrm{tr}(\rho\,\hat{E}[A \in \Delta_2]) = 0$. This means that $\nu^\rho(A \in \Delta_2) < \mathrm{true}_A$; which proves exclusivity.

5. The Unit Proposition Condition: We recall that, in the case of the generalized valuation ν^V, the unit proposition $A \in \sigma(\hat{A})$ is not necessarily given the truth-value true_A but instead satisfies the equation Eq. (3.30).

The situation for ν^ρ is as follows. We have

$$\nu^\rho(A \in \sigma(\hat{A})) := \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\sigma(\hat{A}))]) = 1\}. \quad (4.53)$$

But, according to the definition in Eq. (4.2), $\hat{E}[B \in f(\sigma(\hat{A}))] = \hat{1}$; and thus, for these types of generalized valuation, we do have

$$\nu^\rho(A \in \sigma(\hat{A})) = \mathrm{true}_A \quad (4.54)$$

or, equivalently,

$$\nu^\rho_A(\hat{1}) = \mathrm{true}_A \quad (4.55)$$

for all contexts $\hat{A}$.

The Negation Operation: We have not made any use so far of the negation operation in the Heyting algebra of sieves, which is defined in general in Eq. (A.18). In the case of the sieve $\nu^\rho(A \in \Delta)$, this gives

$$\neg\nu^\rho(A \in \Delta) \;\; := \;\; \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \forall g_\mathcal{O} : \hat{C} \to \hat{B},\; f_\mathcal{O} \circ g_\mathcal{O} \notin \nu^\rho(A \in \Delta)\}$$
$$= \;\; \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \forall g_\mathcal{O} : \hat{C} \to \hat{B},\; \mathrm{tr}(\rho\hat{E}[g(f(A)) \in g(f(\Delta))]) < 1\}. \quad (4.56)$$

This is one point at which there is a real difference between using $\mathcal{O}$ and $\mathcal{O}_*$ as the category of contexts. In the former case, we are allowed the unit operator $\hat{1}$ as an allowed stage of truth, and then the choice of g as the constant map $c_{1,\mathcal{O}} : \hat{1} \to \hat{B}$ gives the spectral projector $\hat{E}[g(f(A)) \in g(f(\Delta)) = \hat{E}[1 \in \{1\}] = \hat{1}$, for which $\mathrm{tr}(\rho\hat{E}) = 1$. Thus the right hand side of Eq. (4.56) would always be the empty set, since this particular g would exist and violate the strict inequality.[14]

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[14]In fact, this is true of presheaves defined over *any* category $\mathcal{C}$ that has an initial object;*i.e.*, an object I such that there is a morphism from I to every object in the category.

Thus, the negation operation is essentially trivial if the category $\mathcal{O}$ is used, and this might suggest employing $\mathcal{O}_*$ instead. On the other hand, if we do keep the unit operator as a possible stage of truth, then the definition of ν^ρ shows that the set of operators appearing as the domains of morphisms in the sieve $\nu^\rho(A \in \Delta)$ form an *abelian algebra* of operators. This is an attractive feature, and might suggest that using $\mathcal{O}$ has certain advantages too. Note that the spin-1 example discussed earlier shows this effect very clearly: the set of operators $\{t\hat{1} + r\hat{S}_x^2 \mid t, r \in \mathbb{R}\}$ that appear in the right hand sides of Eq. (4.35) and Eq. (4.36) are both abelian subalgebras, but cease to be so if the value $r = 0$ is excluded—as would be the case if $\mathcal{O}_*$ is used as the category of contexts.

A Generalization of the Valuations ν^ρ: Finally, we note in passing that there exists a one-parameter family of extensions of our valuations ν^ρ. Namely, we define

$$
\begin{aligned}
\nu^{r,\rho}(A \in \Delta) &:= \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{Prob}(B \in f(\Delta); \rho) \geq r\} \\
&= \{f_\mathcal{O} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\,\hat{E}[B \in f(\Delta)]) \geq r\}
\end{aligned}
\tag{4.57}
$$

where r is a real parameter satisfying $0 < r \leq 1$. It is straightforward to show that for all real numbers r in this range, $\nu^{r,\rho}$ satisfies all our defining conditions for a generalized valuation, with the exception of exclusivity. Exclusivity is also satisfied if the parameter r lies in the range $\frac{1}{2} \leq r \leq 1$. This is an intriguing class of generalized valuation, because it seems to promise a topos perspective on the probabilistic statements of quantum theory.

4.5 From Generalized Valuation to Partial Valuation, and Back Again

As mentioned in the context of Theorem 4.3: in the case of operators with a purely discrete spectrum, each generalized valuation ν on propositions leads to the valuation $V^\nu : \Sigma \to \Omega$ on physical quantities, as defined in Eq. (4.23). In particular, for the generalized valuation ν^ρ associated with a density matrix ρ, we have

$$
V_A^{\nu^\rho}(a) = \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \mathrm{tr}(\rho\hat{E}[B = f(a)]) = 1\}.
\tag{4.58}
$$

Now consider the generalized valuation ν^ψ associated with a quantum state ψ (as in Eq. (4.30)) in a situation where all the operators concerned have a discrete spectrum only. The corresponding generalized valuation Eq. (4.58) on physical quantities gives rise to a partial valuation, which we shall denote V^ψ, whose domain is defined as in Eq. (4.26); thus

$$
\mathrm{dom}\, V^\psi = \{\hat{B} \mid \hat{B}\psi = b\psi \text{ for some } b\}
\tag{4.59}
$$

and, of course, if $\hat{B}$ belongs to the domain of V^ψ, then $V^\psi(\hat{B}) := b$.

We can now apply Definition 3.4 to the partial valuation V^ψ to get an associated generalized valuation ν^{V^ψ} with

$$
\nu^{V^\psi}(A \in \Delta) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \exists b, \hat{B}\psi = b\psi \text{ and } b \in f(\Delta)\},
\tag{4.60}
$$

which should be contrasted with the original definition of ν^ψ:

$$\nu^\psi(A \in \Delta) := \{f_{\mathcal{O}_d} : \hat{B} \to \hat{A} \mid \hat{E}[B \in f(\Delta)]\psi = \psi\}. \tag{4.61}$$

We point we wish to emphasize is that the generalized valuations in Eq. (4.60) and Eq. (4.61) assign the same truth-values to propositions of the type '$A = a$', but they differ in the way they treat more general propositions '$A \in \Delta$'.

Thus, the definition of ν^{V^ψ} in Eq. (4.60) shows that $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$ belongs to the sieve $\nu^{V\psi}(A \in \Delta)$ if, and only if, (i) ψ is an eigenvector of $\hat{B} = f(\hat{A})$; and (ii) the corresponding eigenvalue b belongs to $f(\Delta)$—in other words, the coarse-grained operator $f(\hat{A})$ *has* a value in the state ψ, and this value lies in $f(\Delta)$. On the other hand, for $f_{\mathcal{O}_d} : \hat{B} \to \hat{A}$ to belong to the sieve $\nu^\psi(A \in \Delta)$ requires only that ψ is some *linear combination* of such eigenstates of $f(\hat{A})$. In particular, this proves our earlier remark that the chain in Eq. (4.28) is not the identity transformation on generalized valuations.

4.6 The Generalized Valuation Produced by a Projection Operator

There is apparently another way of constructing generalized valuations using the mathematical ingredients of quantum theory. To see this, we note that the defining condition $\hat{E}[B \in f(\Delta)]\psi = \psi$ for $\nu^\psi(A \in \Delta)$ (see equation Eq. (4.30)) can be written as

$$\hat{E}[B \in f(\Delta)]|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|\hat{E}[B \in f(\Delta)] = |\psi\rangle\langle\psi| \tag{4.62}$$

where $|\psi\rangle\langle\psi|$ denotes the projector onto the vector ψ. The expression Eq. (4.62) suggests an immediate generalization to

$$\hat{E}[B \in f(\Delta)]\,\hat{P} = \hat{P}\,\hat{E}[B \in f(\Delta)] = \hat{P} \tag{4.63}$$

where $\hat{P}$ is now an arbitrary projection operator. In turn, this condition is equivalent to the relation $\hat{P} \leq \hat{E}[B \in f(\Delta)]$ in the lattice of projection operators. Hence we are led to the following definition:

Definition 4.6 *The* generalized valuation ν^P *associated with a projection operator* $\hat{P}$ *is*

$$\nu^P(A \in \Delta) := \{f_{\mathcal{O}} : \hat{B} \to \hat{A} \mid \hat{P} \leq \hat{E}[B \in f(\Delta)]\}. \tag{4.64}$$

It is relatively straightforward to show that the necessary conditions for a generalized valuation are satisfied; and, for reasons of space, we shall not go into the details here. In fact, if $\hat{P}$ is a finite projector (*i.e.*, its range is a finite-dimensional subspace of the Hilbert space $\mathcal{H}$) then ν^P is just a special case of the density-matrix construction given above:

Theorem 4.4 *If* $\hat{P}$ *is a projector such that* $\dim \hat{P} = n < \infty$, *then, for all propositions* '$A \in \Delta$',

$$\nu^P(A \in \Delta) = \nu^{\rho^P}(A \in \Delta) \tag{4.65}$$

where $\rho^P := \frac{1}{n}\hat{P}$ *is the density matrix given by the projection operator* $\hat{P}$.

Proof

If $\hat{P} \leq \hat{E}[B \in f(\Delta)]$ we have $\hat{P}\hat{E}[B \in f(\Delta)] = \hat{P}$, and hence $\mathrm{tr}(\rho^P \hat{E}[B \in f(\Delta)]) = \mathrm{tr}\,\frac{1}{n}\hat{P} = 1$. Thus $\nu^P(A \in \Delta) \subseteq \nu^{\rho^P}(A \in \Delta)$.

Conversely, suppose $\hat{B}$ is such that $\mathrm{tr}(\rho^P \hat{E}[B \in f(\Delta)]) = 1$. Then $\mathrm{tr}\,\frac{1}{n}\hat{P} = 1 = \mathrm{tr}\,(\frac{1}{n}\hat{P}\hat{E}[B \in f(\Delta)])$, which implies at once that $\hat{P} \leq \hat{E}[B \in f(\Delta)]$. Therefore $\nu^{\rho^P}(A \in \Delta) \subseteq \nu^P(A \in \Delta)$. Hence $\nu^{\rho^P}(A \in \Delta) = \nu^P(A \in \Delta)$. **Q.E.D.**

Thus nothing new is gained by introducing the valuations ν^P on a finite-dimensional Hilbert space. However, if $\mathcal{H}$ has an infinite dimension, then ν^P does give a new type of valuation provided that the projection operator $\hat{P}$ has an infinite range.

5 Using the Set of Boolean Sub-Algebras as the Space of Contexts

5.1 Preliminary Definitions

We remarked earlier on the existence of a number of isomorphic pairs of objects in the category $\mathcal{O}$. This occurs whenever operators $\hat{A}$ and $\hat{B}$ are related by $\hat{B} = f(\hat{A})$ and $\hat{A} = g(\hat{B})$ for some functions $f : \sigma(\hat{A}) \to \mathbb{R}$ and $g : \sigma(\hat{B}) \to \mathbb{R}$.

From a physical perspective, if we know the value of one member of such a pair of physical quantities, then we automatically know the value of the other, and vice versa. In this sense, the quantities are 'physically equivalent' and, in some circumstances, it is natural therefore to concentrate on the equivalence classes, rather than on the individual quantities themselves. In particular—since the spectral Boolean algebras W_A and W_B of such pairs of operators are isomorphic—a unique Boolean algebra can be associated with each equivalence class of physical quantities.

Viewed mathematically, this suggests moving towards a formalism in which the space of contexts, or stages of truth, is the category $\mathcal{W}$ of all Boolean subalgebras of the projection lattice, rather than the category $\mathcal{O}$ of self-adjoint operators. Actually, we could have started *ab initio* with $\mathcal{W}$ as the space of contexts, but we elected to use $\mathcal{O}$ instead since the physical motivation for some of the mathematical constructions is more transparent in this case; in particular, this is true of the coarse-graining operation. However, as we shall see in Sections 5.2 and 5.3, the use of $\mathcal{W}$ also suggests generalizations of the idea of coarse-graining which do not arise in such a natural way if the category $\mathcal{O}$ is used. Another significant reason for studying the use of $\mathcal{W}$ is that most of the discussion extends at once to the general quantum logic situation in which all that is said of the basic mathematical structure of a quantum theory is that the propositions are represented by elements in an orthomodular, orthocomplemented lattice; however, we do not take up this generalization here.

We start by constructing several important presheaf objects in the topos $\mathrm{Set}^{\mathcal{W}^{\mathrm{op}}}$. The dual presheaf $\mathbf{D} : \mathcal{W}^{\mathrm{op}} \to \mathrm{Set}$ on $\mathcal{W}$ was introduced in Definition 2.3, with $\mathbf{D}(W)$ defined to be the dual of the Boolean algebra W; *i.e.*, the set of homomorphisms from W to the

Boolean algebra $\{0, 1\}$. In our case, we are interested in a generalization of this situation in which the 'homomorphisms' from W takes their values in the Heyting algebra $\mathbf{\Omega}(W)$ of sieves on W in the category $\mathcal{W}$ rather than in $\{0, 1\}$. Furthermore, we must satisfy the algebraic conditions that specify a generalized valuation. To formalize these ideas we start with the following definition.

Definition 5.1 *A* valuation *of a Boolean algebra B in a Heyting algebra H is a map $\phi : B \to H$ such that the following conditions are satisfied:*

$$\text{Null proposition condition}: \quad \phi(0_B) = 0_H \tag{5.1}$$

$$\text{Monotonicity}: \quad \alpha \le \beta \text{ implies } \phi(\alpha) \le \phi(\beta) \tag{5.2}$$

$$\text{Exclusivity}: \quad \text{If } \alpha \wedge \beta = 0_B \text{ and } \phi(\alpha) = 1_H, \text{ then } \phi(\beta) < 1. \tag{5.3}$$

The set of all valuations from B to H will be denoted $\mathrm{Val}(B, H)$.

These have been chosen to be the analogues of the conditions that we have used a number of times already; and, as before, we may also want to add the 'Unit condition':

$$\text{Unit proposition condition}: \quad \phi(1_B) = 1_H \tag{5.4}$$

In the case when B is a Boolean subalgebra $W \in \mathcal{W}$, and H is $\mathbf{\Omega}(W)$, the elements of $\mathrm{Val}(W, \mathbf{\Omega}(W))$ will be referred to as '*local valuations*'.

We can now define a natural generalization of the dual presheaf $\mathbf{D}$ on $\mathcal{W}$ (see Definition 2.3) in which the standard dual of a Boolean algebra is replaced with an $\mathbf{\Omega}(W)$-valued valuation.

Definition 5.2 *The* valuation presheaf *of $\mathcal{W}$ is the contravariant functor $\mathbf{V} : \mathcal{W} \to \mathrm{Set}$ defined as follows:*

1. On objects in $\mathcal{W}$: $\mathbf{V}(W) := \mathrm{Val}(W, \mathbf{\Omega}(W))$, *the set of local valuations on W.*

2. On morphisms in $\mathcal{W}$: *If $i_{W_2 W_1} : W_2 \to W_1$ (i.e., $W_2 \subseteq W_1$), then $\mathbf{V}(i_{W_2 W_1}) : \mathrm{Val}(W_1, \mathbf{\Omega}(W_1)) \to \mathrm{Val}(W_2, \mathbf{\Omega}(W_2))$ is defined by*

$$[\mathbf{V}(i_{W_2 W_1})(\phi)](\hat{\alpha}) := i_{W_2 W_1}^*(\phi(i_{W_2 W_1}(\hat{\alpha}))) \tag{5.5}$$

where $\phi \in \mathrm{Val}(W_1, \mathbf{\Omega}(W_1))$ and $\hat{\alpha} \in W_2$, and where, in the poset category $\mathcal{W}$, we have $i_{W_2 W_1}^*(S) = \downarrow W_2 \cap S$ for all $S \in \mathbf{\Omega}(W_1)$ (cf., Eq. (A.13)).

It is interesting consider global elements of $\mathbf{V}$ for two reasons: (i) in order to compare with the dual presheaf $\mathbf{D}$, for which—as we saw in Section 2.3—global elements are ruled out by the Kochen-Specker theorem; and (ii) in order to make a contrast with the definition of a generalized valuation in Section 4.

A global element γ of the valuation presheaf corresponds to a family of local valuations $\gamma_W \in \mathrm{Val}(W, \mathbf{\Omega}(W))$, $W \in \mathcal{W}$, such that, if $W_2 \subseteq W_1$ then, for all $\hat{\alpha} \in W_2$,

$$\gamma_{W_2}(\hat{\alpha}) = [\mathbf{V}(i_{W_2 W_1})(\gamma_{W_1})](\hat{\alpha}) = i_{W_2 W_1}^*\{\gamma_{W_1}(i_{W_2 W_1}(\hat{\alpha}))\}. \tag{5.6}$$

In order to see the potential application for such global elements, it is instructive to study these equations in the special case where $W_1 = W_A$ and $W_2 = W_{h(A)}$ for some function $h : \sigma(\hat{A}) \to \mathbb{R}$. Thus, suppose that $\hat{\alpha}$ is the projection operator $\hat{E}[h(A) \in \Lambda]$ for some Borel subset $\Lambda \subseteq \sigma(h(\hat{A}))$. Then

$$i_{W_{h(A)}W_A}(\hat{E}[h(A) \in \Lambda]) = \hat{E}[A \in h^{-1}(\Lambda)] \qquad (5.7)$$

and hence the matching condition in Eq. (5.6) reads

$$\gamma_{W_{h(A)}}(\hat{E}[h(A) \in \Lambda]) = i^*_{W_{h(A)}W_A}\{\gamma_{W_A}(\hat{E}[A \in h^{-1}(\Lambda)])\}. \qquad (5.8)$$

In particular,

$$\gamma_{W_{h(A)}}(\hat{E}[h(A) \in h(\Delta)]) = i^*_{W_{h(A)}W_A}\{\gamma_{W_A}(\hat{E}[A \in h^{-1}(h(\Delta))])\}. \qquad (5.9)$$

The corresponding matching equation in Section 4 for the case of a generalized valuation on $\mathcal{O}$ was (Eq. (4.5))

$$\nu(h(A) \in h(\Delta)) = h^*_{\mathcal{O}}\{\nu(A \in \Delta)\} \qquad (5.10)$$

or, in explicit contextual form,

$$\nu_{h(A)}(\hat{E}[h(A) \in h(\Delta)]) = h^*_{\mathcal{O}}\{\nu_A(\hat{E}[A \in \Delta])\}. \qquad (5.11)$$

Here it is important to contrast equations Eq. (5.9) and Eq. (5.11). In Eq. (5.11) the truth-value of the proposition '$h(A) \in h(\Delta)$' in the context $h(\hat{A})$ is equated with the pull-back of the truth value of the *finer* proposition '$A \in \Delta$' at stage $\hat{A}$; whereas in Eq. (5.9) it is equated with the pull-back of the valuation of the proposition '$A \in h^{-1}(h(\Delta))$'.

However, in the lattice of projectors, the projectors $\hat{E}[A \in h^{-1}(h(\Delta))]$ and $\hat{E}[h(A) \in h(\Delta)]$ are equal: so one is not pulling back a valuation of a finer proposition. Indeed, this equality is reflected in Eq. (5.6) which guarantees the consistency of (i) the sieve valuation of a given projector $\hat{\alpha} \in W_2$, in the context W_2, with (ii) the sieve valuation of $\hat{\alpha}$ if W_2 is embedded in the larger Boolean algebra W_1 and the valuation is then taken in the context of W_1.

To sum up: the equality of $\hat{E}[A \in h^{-1}(h(\Delta))]$ and $\hat{E}[h(A) \in h(\Delta)]$ means that if we were to define a generalized valuation to be a global section of the valuation presheaf $\mathbf{V}$, this would *not* be equivalent to our earlier definition 4.1, or 4.2, using the category $\mathcal{O}$.

Global sections of $\mathbf{V}$ could possibly be used to develop another topos semantics for quantum theory—certainly, we would not wish to claim that the approach adopted in the present paper is necessarily the *only* one. The first step would be to show that global sections of $\mathbf{V}$ actually exist; preferably by finding concrete examples in analogy to, for example, the quantum-state induced general valuations ν^ρ discussed earlier.

We may return in a later paper to the possible use of $\mathbf{V}$ in the semantics of quantum theory. But for the remainder of this section we shall concentrate on showing how the analogue of the coarse-graining operation—which played a central role in our definition of a generalized valuation on $\mathcal{O}$—can be introduced into the mathematical framework based on $\mathcal{W}$.

5.2 The Motivation for the Coarse-Graining Axioms

Motivated by what we did using the category $\mathcal{O}$, we wish to define a coarse-graining operation from W_1 to W_2 where W_1 and W_2 are Boolean subalgebras of projectors with $W_2 \subseteq W_1$. This is intended to play an analogous role to that of the coarse-graining functor $\mathbf{G} : \mathcal{O}^{\mathrm{op}} \rightarrow \mathrm{Set}$, where the map $\mathbf{G}(f_{\mathcal{O}}) : W_A \rightarrow W_B$, with $\hat{B} = f(\hat{A})$, was defined in Eq. (4.16) to map the projector $\hat{E}[A \in \Delta]$ to $\hat{E}[f(A) \in f(\Delta)]$.

The procedure we shall follow is to extract certain key properties of the coarse-graining process in $\mathcal{O}$ in this Section, and then in Section 5.3 use these as the basis for an axiomatization of an analogous procedure for $\mathcal{W}$.

1. Coarse Graining: The first step is to express more precisely the coarse-graining property itself. We start by recalling that, in the lattice of projection operators,

$$\hat{E}[A \in \Delta] \leq \hat{E}[f(A) \in f(\Delta)]. \tag{5.12}$$

However, if we wish to think of the operators on the left and right hand sides of Eq. (5.12) as elements of the Boolean subalgebras W_A and $W_{f(A)}$ respectively, then it pays to be pedantic by rewriting Eq. (5.12) as

$$\hat{E}[A \in \Delta] \leq i_{W_{f(A)}W_A}(\hat{E}[f(A) \in f(\Delta)]) \tag{5.13}$$

where $i_{W_{f(A)}W_A} : W_{f(A)} \rightarrow W_A$ is the embedding of the Boolean algebra $W_{f(A)}$ in W_A. In this sense, the precise statement of the coarse-graining property is

$$\hat{E}[A \in \Delta] \leq i_{W_{f(A)}W_A}(\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta])) \tag{5.14}$$

where the partial ordering '$\leq$' takes place in the Boolean algebra W_A. The analogue of this expression will play a key role in what follows.

2. The Retraction Property: Considered as an element of W_A, the spectral projector $\hat{E}[f(A) \in J]$ is $E[A \in f^{-1}(J)]$; more precisely,

$$i_{W_{f(A)}W_A}(\hat{E}[f(A) \in J]) = \hat{E}[A \in f^{-1}(J))], \tag{5.15}$$

and hence

$$\mathbf{G}(f_{\mathcal{O}}) \circ i_{W_{f(A)}W_A}(\hat{E}[f(A) \in J]) = \hat{E}[f(A) \in f(f^{-1}(J))]. \tag{5.16}$$

However, using the definition in Eq. (4.2), it is easy to show that the right hand side of Eq. (5.16) is equal to $\hat{E}[f(A) \in J]$. Hence Eq. (5.16) becomes, for all Borel subsets $J \subseteq \sigma(f(\hat{A}))$,

$$\mathbf{G}(f_{\mathcal{O}}) \circ i_{W_{f(A)}W_A}(\hat{E}[f(A) \in J]) = \hat{E}[f(A) \in J] \tag{5.17}$$

which can be rewritten succinctly as

$$\mathbf{G}(f_{\mathcal{O}}) \circ i_{W_{f(A)}W_A} = \mathrm{id}_{W_{f(A)}}. \tag{5.18}$$

This is expressed by saying that $\mathbf{G}(f_{\mathcal{O}}) : W_A \rightarrow W_{f(A)}$ is a *retraction*[15] map from W_A onto its embedded subalgebra $W_{f(A)}$.

---

[15]In general, a map $r : Y \rightarrow X$ is a *retraction* of a subset embedding $i : X \subseteq Y$ if $r(x) = x$ for all $x \in X \subseteq Y$; X is then said to be a *retract* of Y. Formally, we can write this as $r \circ i = \mathrm{id}_X$.

3. Composition Conditions: Since $\mathbf{G}$ is a contravariant functor from $\mathcal{O}$ to Set, it follows that if $h_{\mathcal{O}} : \hat{C} \to \hat{B}$ and $f_{\mathcal{O}} : \hat{B} \to \hat{A}$, then $f_{\mathcal{O}} \circ h_{\mathcal{O}} : \hat{C} \to \hat{A}$, and

$$\mathbf{G}(f_{\mathcal{O}} \circ h_{\mathcal{O}}) = \mathbf{G}(h_{\mathcal{O}}) \circ \mathbf{G}(f_{\mathcal{O}}). \tag{5.19}$$

These can be thought of as the 'composition conditions' that must be satisfied by a coarse-graining operation.

4. Monotonicity: If $\Delta_1 \subseteq \Delta_2$, then $f(\Delta_1) \subseteq f(\Delta_2)$, and hence $\hat{E}[f(A) \in f(\Delta_1)] \leq \hat{E}[f(A) \in f(\Delta_2)]$. From this we deduce the monotonicity condition that is satisfied by the coarse-graining presheaf $\mathbf{G}$. Namely, if $\Delta_1 \subseteq \Delta_2$, then

$$\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta_1]) \leq \mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta_2]) \tag{5.20}$$

Note that the partial-ordering operation '$\leq$' in Eq. (5.20) is taken in the Boolean algebra $W_{f(A)}$.

5.3 The Definition of Coarse-Graining on $\mathcal{W}$

1. A Coarse-Graining Presheaf on $\mathcal{W}$: Motivated by the above we can now give our formal definition of a 'coarse-graining' operation on the category $\mathcal{W}$.

Definition 5.3 *A* coarse-graining *on $\mathcal{W}$ is an operation that associates to each pair $W_2 \subseteq W_1$, a 'coarse-graining' map $\theta_{W_1 W_2} : W_1 \to W_2$ with the following properties:*

1. Coarse-graining: *For all $\hat{\alpha} \in W_1$,*

$$\hat{\alpha} \leq i_{W_2 W_1}(\theta_{W_1 W_2}(\hat{\alpha})). \tag{5.21}$$

   If $W_2 = W_1$, then $\theta_{W_1 W_1} = id_{W_1}$.

2. Monotonicity: *If $\hat{\alpha}, \hat{\beta} \in W_1$ are such that $\alpha \leq \beta$, then*

$$\theta_{W_1 W_2}(\hat{\alpha}) \leq \theta_{W_1 W_2}(\hat{\beta}) \tag{5.22}$$

3. Retraction: *For all $\hat{\alpha} \in W_2$,*

$$\theta_{W_1 W_2}(i_{W_2 W_1}(\hat{\alpha})) = \hat{\alpha}. \tag{5.23}$$

   Thus $\theta_{W_1 W_2}$ is a retraction of W_2 onto W_1; i.e., $\theta_{W_1 W_2} \circ i_{W_2 W_1} = \mathrm{id}_{W_2}$.

4. Composition conditions: *If $W_3 \subseteq W_2 \subseteq W_1$ then*

$$\theta_{W_2 W_3} \circ \theta_{W_1 W_2} = \theta_{W_1 W_3}. \tag{5.24}$$

From a topos perspective, the composition conditions show that θ defines a presheaf $\mathbf{\Theta} : \mathcal{W}^{\mathrm{op}} \to$ Set that is defined (i) on objects as $\mathbf{\Theta}(W) := W$; and (ii) on a morphism $i_{W_2 W_1} : W_2 \to W_1$ as $\mathbf{\Theta}(i_{W_2 W_1}) := \theta_{W_1 W_2}$. Conversely, we could *define* a 'coarse-graining presheaf on $\mathcal{W}$' to be a presheaf on W that satisfies the remaining conditions, viz. coarse-graining, monotonicity, and retraction.

2. The Canonical Coarse-Graining Presheaf: It is important to show that there exists at least one coarse-graining presheaf. In the analogous case of contextualizing over $\mathcal{O}$, there was a 'canonical' coarse-graining operation that came from considering the implications of writing one operator $\hat{B}$ as a function $f(\hat{A})$ of another. The key to finding the analogue of this construction for the category $\mathcal{W}$ is contained in Theorem 4.1. This result leads naturally to the following definition:

Definition 5.4 *The* canonical coarse-graining *of $\mathcal{W}$ associates to each pair $W_2 \subseteq W_1$, the coarse-graining map $\phi_{W_1 W_2} : W_1 \to W_2$ defined by*

$$\phi_{W_1 W_2}(\hat{\alpha}) := \inf\{\hat{\beta} \in W_2 \mid \hat{\alpha} \le i_{W_2 W_1}(\hat{\beta})\} \tag{5.25}$$

for all $\hat{\alpha} \in W_1$.

We shall leave as a straightforward exercise the task of showing that the entity thus defined really does satisfy all the requirements for a coarse-graining operation: coarse-graining, monotonicity, retraction, and the composition condition.

3. Generalized Valuations Associated with a Coarse-Graining Presheaf: We shall now show that for any given coarse-graining presheaf, there is an associated definition of a generalized valuation that is constructed as a matching family of local valuations:

Definition 5.5 *A* generalized valuation *on $\mathcal{W}$ associated with a coarse-graining presheaf Θ is a family of local valuations $\phi_W : W \to \Omega(W)$, $W \in \mathcal{W}$, such that if $W_2 \subseteq W_1$ then, for all $\hat{\alpha} \in W_1$,*

$$\phi_{W_2}(\theta_{W_1 W_2}(\hat{\alpha})) = i^*_{W_2 W_1}(\phi_{W_1}(\hat{\alpha})). \tag{5.26}$$

From a physical perspective, the interpretation of a generalized valuation on $\mathcal{W}$ is closely analogous to that of generalized valuation on $\mathcal{O}$ as given by the discussion following Definition 4.1. Specifically: although a particular projector $\hat{\alpha} \in W_1$ may not be assigned the value 'totally true' at a stage of truth W_1, it does have a partial truth-value that is given by the set of coarser Boolean algebras W_2 that belong to the sieve $\phi_{W_1}(\hat{\alpha})$, where, on account of Eq. (5.26), each corresponding coarse-grained projector $\theta_{W_1 W_2}(\hat{\alpha})$ is given the value 'totally true' at the corresponding stage of truth W_2. (This should be compared with the discussion following Eq. (A.12), and after Eq. (3.15).)

4. The Generalized Valuation Produced by a Density Matrix: There is no difficulty in finding examples of generalized valuations associated with any coarse-graining presheaf. In particular, each density-matrix state ρ produces one according to the following definition.

Definition 5.6 *The* generalized valuation ν^ρ on $\mathcal{W}$ associated with a coarse-graining presheaf Θ and a density matrix ρ, is defined at each stage W by

$$\nu^\rho_W(\hat{\alpha}) := \{W' \subseteq W \mid tr(\rho\, \theta_{WW'}(\hat{\alpha})) = 1\} \tag{5.27}$$

for all $\hat{\alpha} \in W$.

To show that this is indeed a generalized valuation it is necessary to show that (i) each $\nu_W^\rho : W \to \mathbf{\Omega}(W)$ is a local valuation; and (ii) the maps ν_W^ρ fit together in the way indicated by the intertwining condition in Eq. (5.26). The proofs are contained in the following theorem.

Theorem 5.1 *The quantity ν^ρ defined in Eq. (5.27) satisfies all the conditions for a generalized valuation on $\mathcal{W}$.*

Proof

A. For each stage $W \in \mathcal{W}$, ν_W^ρ is a local valuation:

1. $\nu_W^\rho(\hat{\alpha})$ is a sieve: The first step is to show that $\nu_W^\rho(\hat{\alpha})$ is a sieve on W in $\mathcal{W}$. Thus suppose that $W' \in \nu_W^\rho(\hat{\alpha})$ and consider any subalgebra $W'' \subseteq W'$. The composition condition Eq. (5.24) applied to the chain $W'' \subseteq W' \subseteq W$ gives

$$\theta_{WW''}(\hat{\alpha}) = \theta_{W'W''}(\theta_{WW'}(\hat{\alpha})) \tag{5.28}$$

for all $\hat{\alpha} \in W$. Then applying the coarse-graining condition Eq. (5.21) to $\theta_{WW'}(\hat{\alpha})$, and using Eq. (5.28), we get

$$\theta_{WW'}(\hat{\alpha}) \leq i_{W''W'}\left(\theta_{W'W''}(\theta_{WW'}(\hat{\alpha}))\right) = i_{W''W'}(\theta_{WW''}(\hat{\alpha})). \tag{5.29}$$

Hence, in the Boolean algebra W', we have $\theta_{WW'}(\hat{\alpha}) \leq \theta_{WW''}(\hat{\alpha})$. Thus, in particular, $\mathrm{tr}(\rho\,\theta_{WW'}(\hat{\alpha})) = 1$ implies $\mathrm{tr}(\rho\,\theta_{WW''}(\hat{\alpha})) = 1$; and hence $\nu_W^\rho(\hat{\alpha})$ is a sieve on W in $\mathcal{W}$.

2. The null proposition condition: The equations Eq. (5.23) and $i_{W'W}(0_{W'}) = 0_W$, imply $\theta_{WW'}(\hat{0}) = \hat{0}$, from which the null proposition condition follows at once. It is also trivial to check that ν^ρ satisfies the unit proposition condition $\nu_W^\rho(\hat{1}) = \mathrm{true}_W$.

3. The monotonicity condition: To show monotonicity, suppose that $\hat{\alpha}, \hat{\beta} \in W$ satisfy $\hat{\alpha} \leq \hat{\beta}$, and that $W' \in \nu_W^\rho(\hat{\alpha})$, so that $\mathrm{tr}(\rho\,\theta_{WW'}(\hat{\alpha})) = 1$. Then the monotonicity condition Eq. (5.22) obeyed by the coarse-graining operation implies that $\theta_{WW'}(\hat{\alpha}) \leq \theta_{WW'}(\hat{\beta})$, and hence that $\mathrm{tr}(\rho\,\theta_{WW'}(\hat{\alpha})) \leq \mathrm{tr}(\rho\,\theta_{WW'}(\hat{\beta}))$. However, $\mathrm{tr}(\rho\hat{P}) \leq 1$ for all projection operators $\hat{P}$, and hence $\mathrm{tr}(\rho\,\theta_{WW'}(\hat{\alpha})) = 1$ implies $\mathrm{tr}(\rho\,\theta_{WW'}(\hat{\beta})) = 1$, which means that $W' \in \nu_W^\rho(\beta)$; hence the monotonicity condition is satisfied.

4. The exclusivity condition: To show exclusivity, suppose that $\hat{\alpha}, \hat{\beta} \in W$ satisfy $\hat{\alpha} \wedge \hat{\beta} = 0$, and that $\nu_W^\rho(\hat{\alpha}) := 1_W$. The latter implies that $W \in \nu_W^\rho(\hat{\alpha})$, and hence, since $\theta_{WW} = \mathrm{id}_W$, we have $\mathrm{tr}(\rho\,\hat{\alpha}) = 1$. However, $\hat{\alpha} \wedge \hat{\beta} = 0$ implies that $\hat{\beta} \leq \neg\hat{\alpha}$ and, since $\neg\hat{\alpha} = \hat{1} - \hat{\alpha}$, we get

$$0 \leq \mathrm{tr}(\rho\hat{\beta}) \leq \mathrm{tr}(\rho(\hat{1} - \hat{\alpha})) = 0. \tag{5.30}$$

Thus $\mathrm{tr}(\rho\hat{\beta}) = 0$, and hence $W \notin \nu_W^\rho(\hat{\beta})$. Therefore, $\nu_W^\rho(\hat{\beta}) < 1_W$, which proves exclusivity.

B. For each stage $W \in \mathcal{W}$, ν^ρ satisfies the intertwining condition Eq. (5.26):

To see that Eq. (5.26) is satisfied, let $W_2, W_1 \in \mathcal{W}$ be such that $W_2 \subseteq W_1$. Then, for all $\hat{\alpha} \in W_1$,

$$
\begin{aligned}
\nu^\rho_{W_2}(\theta_{W_1 W_2}(\hat{\alpha})) &:= \{W' \subseteq W_2 \mid \mathrm{tr}(\rho\, \theta_{W_2 W'}(\theta_{W_1 W_2}(\hat{\alpha}))) = 1\} \\
&= \{W' \subseteq W_2 \mid \mathrm{tr}(\rho\, \theta_{W_1 W'}(\hat{\alpha})) = 1\}
\end{aligned}
\tag{5.31}
$$

where the last line follows from the composition conditions Eq. (5.24). On the other hand,

$$
\begin{aligned}
\{W' \subseteq W_2 \mid \mathrm{tr}(\rho\, \theta_{W_1 W'}(\hat{\alpha})) = 1\} &= \downarrow W_2 \cap \{W' \subseteq W_1 \mid \mathrm{tr}(\rho\, \theta_{W_1 W'}(\hat{\alpha})) = 1\} \\
&= i^*_{W_2 W_1}(\nu^\rho_{W_1}(\hat{\alpha})),
\end{aligned}
\tag{5.32}
$$

so that $\nu_{W_2}(\theta_{W_1 W_2}(\hat{\alpha})) = i^*_{W_2 W_1}(\nu_{W_1}(\hat{\alpha}))$, as required. **Q.E.D.**

5. The Topos-Theoretic Perspective: From a topos-theoretic perspective we note that each generalized valuation ν on $\mathcal{W}$ defines a natural transformation N^ν between the coarse-graining presheaf Θ and the subobject classifier Ω, in which, at each stage of truth W, $N^\nu_W : \Theta(W) \to \Omega(W)$ is defined by $N^\nu_W(\hat{\alpha}) := \nu_W(\hat{\alpha})$. It is a straightforward exercise in diagram chasing to show that N^ν really is a natural transformation.

Thus to each generalized valuation ν on $\mathcal{W}$ there corresponds a morphism in the topos $\mathrm{Set}^{\mathcal{W}^{\mathrm{op}}}$ between the coarse-graining presheaf Θ and the sub-object classifier. In particular, therefore, each generalized valuation on $\mathcal{W}$ corresponds to a subobject of Θ. The overall implications of this are the same as for the analogous result in the case of generalized valuations defined on $\mathcal{O}$.

6 Conclusion

The Kochen-Specker theorem shows the non-existence of global valuations on the self-adjoint operators in a quantum theory if the dimension of the underlying Hilbert space $\mathcal{H}$ is greater than two. We have shown that this theorem is equivalent to the statement that a certain presheaf on the category of bounded self-adjoint operators has no global sections. Then, motivated by the underlying topos structure, we introduced a new type of valuation which *is* globally defined, but whose truth values (i) are contextual; and (ii) lie in a larger Heyting algebra than the minimal $\{0, 1\}$ Boolean algebra of standard logic.

Thus our construction shows clearly how contextual features enter into a 'neo-realist' interpretation of quantum theory. It also shows that the use of multi-valued logic is perfectly feasible. In particular, there is no ambiguity or uncertainty about what the logical connectives are: the Heyting algebra of the sieves at any particular stage of truth, or context, is precisely fixed by the structure of the base category—in our case $\mathcal{O}$ or $\mathcal{W}$—on which the relevant presheaves are defined.

As we generalize at the end of the Introduction, the main aim of the present paper is to provide the main mathematical tools and some of the general ideas involved in the

application of topos ideas in quantum theory. Much remains to be done to develop both the mathematical and the conceptual implications of these ideas; the latter in particular are discussed in a forthcoming paper [16].

At the mathematical level, the work reported in this paper suggests a number of topics for further research. Of particular importance is the study of the space of all generalized valuations which—as mentioned in Section 4.2—might carry an intuitionistic logical structure by virtue of the identification of each generalized valuation with a subobject of the coarse-graining presheaf $\mathbf{G}$. An important part of any such study is likely to involve a closer investigation of the negation operation in the Heyting algebras, which we have not exploited in any significant way so far.

A crucial question regarding the space of all generalized valuations is to understand the mathematical status of the valuations ν^ρ generated by the mixed states ρ in the quantum system. In particular, if we impose the 'unit proposition condition' of Eq. (3.28), is it possible to find a set of extra conditions to be imposed on the generalized valuations that will guarantee that *every* subobject of $\mathbf{G}$ that satisfies these and the original defining conditions Eqs. (4.6—4.8), has the form ν^ρ for some density matrix ρ? In effect, we are asking for a contextualized, Heyting-algebra valued analogue of the Gleason theorem. It seems likely that an important role in such an analysis will be played by the one-parameter family of generalized valuations $\nu^{r,\rho}$ defined in Eq. (4.57).

A number of other questions suggest themselves. For example, is our theory of generalized values of physical quantities and propositions related at all to existing ideas on 'unsharp' values of quantum quantities (as described, for example, in [22])? Another important example is the relation of our constructions to the standard probabilistic statements of quantum theory.

Another important issue is to see how the phenomenon of quantum entanglement is reflected in the truth-values assigned by our generalized valuations. Thus we should study possible relations between a generalized valuation ν^ψ, where ψ is an entangled state in a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, and the generalized valuations associated with vectors in the constituent Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

The discussion in Section 5.3 of coarse-graining in the category $\mathcal{W}$ of Boolean subalgebras implies that there might be coarse-graining functors other than the canonical one given in Definition 5.4. It is clearly important to see if this is indeed the case, since each such functor would give rise to a whole new class of generalized valuations. In particular, this is relevant to the problem mentioned above of classifying generalized valuations. It would also be interesting to study this question in a simple model quantum-logic situation in which the orthoalgebra of propositions is not the projection lattice of a Hilbert space.

Finally, there is the question of the Kochen-Specker theorem itself: in particular, the possibility of finding a new proof based on some theory of obstructions to the construction of global sections of the spectral presheaf, rather as one studies obstructions to the construction of global cross-sections of non-trivial fibre bundles. This is an intriguing mathematical challenge, and one whose solution could generate a deeper insight into the ultimate significance of the Kochen-Specker theorem. It could also suggest ways of using topos ideas in quantum theory other than the coarse-graining scheme employed in the

present paper.

Acknowledgements

A   A Brief Account of the Relevant Parts of Topos Theory

A.1   Presheaves on a Poset

Topos theory is a remarkably rich branch of mathematics which can be approached from a variety of different viewpoints. The relevant general area of mathematics is category theory; where, we recall, a category consists of a collection of *objects* and a collection of *morphisms* (or *arrows*). In the special case of the category of sets, the objects are sets, and a morphism is a function between a pair of sets. In general, each morphism f in a category is associated with a pair of objects, known as its 'domain' and the 'codomain', and is written in the form $f : B \rightarrow A$ where B and A are the domain and codomain respectively. Note that this arrow notation is used even if f is not a function in the normal set-theoretic sense. A key ingredient in the definition of a category is that if $f : B \rightarrow A$ and $g : C \rightarrow B$ (*i.e.*, the codomain of g is equal to the domain of f) then f and g can be 'composed' to give an arrow $f \circ g : C \rightarrow A$; in the case of the category of sets, this is just the usual composition of functions.

In many categories, the objects are sets equipped with some type of additional structure, and the morphisms are functions that preserve this structure; for example, in the category of groups, an object is a group, and a morphism $f : G_1 \rightarrow G_2$ is a map from the group G_1 to the group G_2 that is also a homomorphism. However, not all categories are of this type. For example, any partially-ordered set ('poset') $\mathcal{C}$ can be regarded as a category in which (i) the objects are defined to be the elements of $\mathcal{C}$; and (ii) if $p, q \in \mathcal{C}$, a morphism from p to q is defined to exist if, and only if, $p \leq q$ in the poset structure. Thus, in a poset regarded as a category, there is at most one morphism between any pair of objects $p, q \in \mathcal{C}$; if it exists, we shall write this morphism as $i_{pq} : p \rightarrow q$.

From our perspective, the most relevant feature of a topos is that it is a category in which the subobjects of an object behave in many ways like the subsets of a set in set theory [17, 18]. In particular, the subsets $K \subseteq X$ of a set X are in one-to-one correspondence with functions $\chi^K : X \rightarrow \{0, 1\}$, where $\chi^K(x) = 1$ if $x \in K$, and $\chi^K(x) = 0$ otherwise. Thus the target space $\{0, 1\}$ can be regarded as the simplest 'false-true' Boolean algebra, and the proposition '$x \in K$' is true if $\chi^K(x) = 1$, and false otherwise.

In the case of a topos, the subobjects K of an object X in the topos are in one-to-

one correspondence with morphisms $\chi^K : X \to \Omega$, where the special object Ω in the topos—called the 'subobject classifier', or 'object of truth-values'—plays an analogous role to that of $\{0, 1\}$ in the category of sets. In particular, we are interested in the theory of presheaves where, as we shall see, a morphism $\chi^K : X \to \Omega$ corresponds to a contextualized, multi-valued truth assignment.

To illustrate the main ideas, we will first give a few definitions from the theory of presheaves on a partially ordered set (or 'poset'); physically, this poset will represent the space of 'contexts' in which generalized truth-values are to be assigned. We shall then use these ideas to motivate the definition of a presheaf on a general category. Only the briefest of treatments is given here, and the reader is referred to the standard literature for more information [17, 18].

A *presheaf* (also known as a *varying set*) X on a poset $\mathcal{C}$ is a function that assigns to each $p \in \mathcal{C}$, a set X_p; and to each pair $p \leq q$, a map $X_{qp} : X_q \to X_p$ such that (i) $X_{pp} : X_p \to X_p$ is the identity map id_{X_p} on X_p, and (ii) whenever $p \leq q \leq r$, the composite map $X_r \xrightarrow{X_{rq}} X_q \xrightarrow{X_{qp}} X_p$ is equal to $X_r \xrightarrow{X_{rp}} X_p$, so that[16]

$$X_{rp} = X_{qp} \circ X_{rq}. \tag{A.1}$$

A *morphism* $\eta : X \to Y$ between two presheaves X, Y on $\mathcal{C}$ is a family of maps $\eta_p : X_p \to Y_p$, $p \in \mathcal{C}$, that satisfy the intertwining conditions

$$\eta_p \circ X_{qp} = Y_{qp} \circ \eta_q \tag{A.2}$$

whenever $p \leq q$. This is equivalent to the commutative diagram

$$\begin{array}{ccc} X_q & \xrightarrow{X_{qp}} & X_p \\ \downarrow{\scriptstyle\eta_q} & & \downarrow{\scriptstyle\eta_p} \\ Y_q & \xrightarrow{Y_{qp}} & Y_p \end{array} \tag{A.3}$$

A *subobject* of a presheaf X is a presheaf K, with a morphism $i : K \to X$ such that (i) $K_p \subseteq X_p$ for all $p \in \mathcal{C}$; and (ii) for all $p \leq q$, the map $K_{qp} : K_q \to K_p$ is the restriction of $X_{qp} : X_q \to X_p$ to the subset $K_q \subseteq X_q$. This is shown in the commutative diagram

$$\begin{array}{ccc} K_q & \xrightarrow{K_{qp}} & K_p \\ \downarrow & & \downarrow \\ X_q & \xrightarrow{X_{qp}} & X_p \end{array} \tag{A.4}$$

where the vertical arrows are subset inclusions.

The collection of all presheaves on a poset $\mathcal{C}$ forms a category, denoted $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. The morphisms between presheaves in this category are defined as the morphisms above.

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[16]A matter of convention is involved here. Sometimes a presheaf is defined as above except that, to each $p \leq q$, one associates a function $X_{pq} : X_p \to X_q$ that maps X_p to X_q, rather than the function X_{qp} that maps X_q to X_p. To reflect this, equation Eq. (A.1) is replaced by $X_{pr} = X_{qr} \circ X_{pq}$ for $p \leq q \leq r$. Presheaves in the sense of the main text are in one-to-one correspondence with presheaves in this alternative sense, in which the latter are defined on the *opposite* poset $\mathcal{C}^{\mathrm{op}}$—defined to be the same set as $\mathcal{C}$ but with all the partial ordering relations reversed.

A.2 Presheaves on a General Category

The ideas sketched above admit an immediate generalization to the theory of presheaves on an arbitrary 'small' category $\mathcal{C}$ (the qualification 'small' means that the collection of objects is a genuine set, as is the collection of all morphisms between any pair of objects). To make the necessary definition we first need the idea of a 'functor':

1. The Idea of a Functor: A central concept is that of a 'functor' between a pair of categories $\mathcal{C}$ and $\mathcal{D}$. Broadly speaking, this is a morphism-preserving function from one category to the other. The precise definition is as follows.

Definition A.1

1. A covariant functor $\mathbf{F}$ *from a category* $\mathcal{C}$ *to a category* $\mathcal{D}$ *is a function that assigns*

   (a) *to each* $\mathcal{C}$-*object* A, *a* $\mathcal{D}$-*object* $\mathbf{F}(A)$;

   (b) *to each* $\mathcal{C}$-*morphism* $f : B \to A$, *a* $\mathcal{D}$-*morphism* $\mathbf{F}(f) : \mathbf{F}(B) \to \mathbf{F}(A)$ *such that* $\mathbf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{F}(A)}$; *and, if* $g : C \to B$, *and* $f : B \to A$ *then*

$$\mathbf{F}(f \circ g) = \mathbf{F}(f) \circ \mathbf{F}(g). \tag{A.5}$$

2. A contravariant functor $\mathbf{X}$ *from a category* $\mathcal{C}$ *to a category* $\mathcal{D}$ *is a function that assigns*

   (a) *to each* $\mathcal{C}$-*object* A, *a* $\mathcal{D}$-*object* $\mathbf{X}(A)$;

   (b) *to each* $\mathcal{C}$-*morphism* $f : B \to A$, *a* $\mathcal{D}$-*morphism* $\mathbf{X}(f) : \mathbf{X}(A) \to \mathbf{X}(B)$ *such that* $\mathbf{X}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{X}(A)}$; *and, if* $g : C \to B$, *and* $f : B \to A$ *then*

$$\mathbf{X}(f \circ g) = \mathbf{X}(g) \circ \mathbf{X}(f). \tag{A.6}$$

The connection with the idea of a presheaf on a poset is straightforward. As mentioned above, a poset $\mathcal{C}$ can be regarded as a category in its own right, and it is clear that a presheaf on the poset $\mathcal{C}$ is the same thing as a contravariant functor $\mathbf{X}$ from the category $\mathcal{C}$ to the category 'Set' of normal sets. Equivalently, it is a covariant functor from the 'opposite' category[17] $\mathcal{C}^{\mathrm{op}}$ to Set. More precisely, in terms of the notation used earlier, the sets X_p, $p \in \mathcal{C}$, are defined as

$$X_p := \mathbf{X}(p) \tag{A.7}$$

and, if $p \leq q$ (so that $i_{pq} : p \to q$), the map $X_{qp} : X_q \to X_p$ is defined as

$$X_{qp} := \mathbf{X}(i_{pq}). \tag{A.8}$$

Clearly, Eq. (A.1) corresponds to the contravariant condition Eq. (A.6).

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[17]The 'opposite' of a category $\mathcal{C}$ is a category, denoted $\mathcal{C}^{\mathrm{op}}$, whose objects are the same as those of $\mathcal{C}$, and whose morphisms are defined to be the opposite of those of $\mathcal{C}$; *i.e.*, a morphism $f : A \to B$ in $\mathcal{C}^{\mathrm{op}}$ is said to exist if, and only if, there is a morphism $f : B \to A$ in $\mathcal{C}$.

2. Presheaves on an Arbitrary Category $\mathcal{C}$: These remarks motivate the definition of a presheaf on an arbitrary small category $\mathcal{C}$: namely, a *presheaf* on $\mathcal{C}$ is a covariant functor $\mathbf{X} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ from $\mathcal{C}^{\mathrm{op}}$ to the category of sets. Equivalently, a presheaf is a contravariant functor from $\mathcal{C}$ to the category of sets.

We want to make the collection of presheaves on $\mathcal{C}$ into a category, and therefore we need to define what is meant by a 'morphism' between two presheaves $\mathbf{X}$ and $\mathbf{Y}$. The intuitive idea is that such a morphism from $\mathbf{X}$ to $\mathbf{Y}$ must give a 'picture' of $\mathbf{X}$ within $\mathbf{Y}$. Formally, such a morphism is defined to be a *natural transformation* $N : \mathbf{X} \to \mathbf{Y}$, by which is meant a family of maps (called the *components* of N) $N_A : \mathbf{X}(A) \to \mathbf{Y}(A)$, A in $\mathcal{C}$, such that if $f : B \to A$ is a morphism in $\mathcal{C}$, then the composite map $\mathbf{X}(A) \xrightarrow{N_A} \mathbf{Y}(A) \xrightarrow{\mathbf{Y}(f)} \mathbf{Y}(B)$ is equal to $\mathbf{X}(A) \xrightarrow{\mathbf{X}(f)} \mathbf{X}(B) \xrightarrow{N_B} \mathbf{Y}(B)$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathbf{X}(A) & \xrightarrow{\mathbf{X}(f)} & \mathbf{X}(B) \\ \downarrow{\scriptstyle N_A} & & \downarrow{\scriptstyle N_B} \\ \mathbf{Y}(A) & \xrightarrow{\mathbf{Y}(f)} & \mathbf{Y}(B) \end{array} \qquad (A.9)$$

of which Eq. (A.3) is clearly a special case. The category of presheaves on $\mathcal{C}$ equipped with these morphisms is denoted $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

The idea of a subobject generalizes in an obvious way. Thus we say that $\mathbf{K}$ is a *subobject* of $\mathbf{X}$ if there is a morphism in the category of presheaves (*i.e.*, a natural transformation) $i : \mathbf{K} \to \mathbf{X}$ with the property that, for each A, the component map $i_A : \mathbf{K}(A) \to \mathbf{X}(A)$ is a subset embedding, *i.e.*, $\mathbf{K}(A) \subseteq \mathbf{X}(A)$. Thus, if $f : B \to A$ is any morphism in $\mathcal{C}$, we get the analogue of the commutative diagram Eq. (A.4):

$$\begin{array}{ccc} \mathbf{K}(A) & \xrightarrow{\mathbf{K}(f)} & \mathbf{K}(B) \\ \downarrow & & \downarrow \\ \mathbf{X}(A) & \xrightarrow{\mathbf{X}(f)} & \mathbf{X}(B) \end{array} \qquad (A.10)$$

where, once again, the vertical arrows are subset inclusions.

The category of presheaves on $\mathcal{C}$, $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, forms a topos. We do not need the full definition of a topos; but we do need the idea, mentioned in Section A.1, that a topos has a subobject classifier Ω, to which we now turn.

3. Sieves and The Subobject Classifier Ω: Among the key concepts in presheaf theory, and something of particular importance for this paper, is that of a 'sieve', which plays a central role in the construction of the subobject classifier in the topos of emphasized on a category $\mathcal{C}$.

A *sieve* on an object A in $\mathcal{C}$ is defined to be a collection S of morphisms $f : B \to A$ in $\mathcal{C}$ with the property that if $f : B \to A$ belongs to S, and if $g : C \to B$ is any morphism, then $f \circ g : C \to A$ also belongs to S. [18] In the simple case where $\mathcal{C}$ is a poset, a sieve on

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[18]A *cosieve* on A is defined to be a collection S of morphisms $f : A \to B$ with the property that if $f : A \to B$ belongs to S, and if $g : B \to C$ is any morphism, then $g \circ f : A \to C$ also belongs to S. However, another matter of convention is involved here: some authors interchange our usage of the words 'sieve' and 'cosieve'. Note that, in any event, a sieve in $\mathcal{C}$ is the same thing as a cosieve in $\mathcal{C}^{\mathrm{op}}$, and vice versa.

$p \in \mathcal{C}$ is any subset S of $\mathcal{C}$ such that if $r \in S$ then (i) $r \leq p$, and (ii) $r' \in S$ for all $r' \leq r$; in other words, a sieve is nothing but a *lower* set in the poset.

The presheaf $\mathbf{\Omega} : \mathcal{C} \to \mathrm{Set}$ is now defined as follows. If A is an object in $\mathcal{C}$, then $\mathbf{\Omega}(A)$ is defined to be the set of all sieves on A; and if $f : B \to A$, then $\mathbf{\Omega}(f) : \mathbf{\Omega}(A) \to \mathbf{\Omega}(B)$ is defined as

$$\mathbf{\Omega}(f)(S) := \{h : C \to B \mid f \circ h \in S\} \tag{A.11}$$

for all $S \in \mathbf{\Omega}(A)$; the sieve $\mathbf{\Omega}(f)(S)$ is often written as $f^*(S)$, and is known as the *pull-back* to B of the sieve S on A by the morphism $f : B \to A$.

For our purposes in what follows, it is important to note that if S is a sieve on A, and if $f : B \to A$ belongs to S, then from the defining property of a sieve we have

$$f^*(S) := \{h : C \to B \mid f \circ h \in S\} = \{h : C \to B\} =: \downarrow B \tag{A.12}$$

where $\downarrow B$ denotes the *principal* sieve on B, defined to be the set of all morphisms in $\mathcal{C}$ whose codomain is B. In words: the pull-back of any sieve on A by a morphism from B to A that belongs to the sieve, is the *principal* sieve on B.

If $\mathcal{C}$ is a poset, the pull-back operation corresponds to a family of maps $\Omega_{qp} : \Omega_q \to \Omega_p$ (where Ω_p denotes the set of all sieves on p in the poset) defined by $\Omega_{qp} = \mathbf{\Omega}(i_{pq})$ if $i_{pq} : p \to q$ (*i.e.*, $p \leq q$). It is straightforward to check that if $S \in \Omega_q$, then

$$\Omega_{qp}(S) := \downarrow p \cap S \tag{A.13}$$

where $\downarrow p := \{r \in \mathcal{C} \mid r \leq p\}$.

A crucial property of sieves is that the set $\mathbf{\Omega}(A)$ of sieves on A has the structure of a Heyting algebra.[19] This is defined to be a distributive lattice, with null and unit elements, that is *relatively complemented*, which means that to any pair S_1, S_2 in $\mathbf{\Omega}(A)$, there exists an element $S_1 \Rightarrow S_2$ of $\mathbf{\Omega}(A)$ with the property that, for all $S \in \mathbf{\Omega}(A)$,

$$S \leq (S_1 \Rightarrow S_2) \text{ if and only if } S \wedge S_1 \leq S_2. \tag{A.14}$$

Specifically, $\mathbf{\Omega}(A)$ is a Heyting algebra where the unit element $1_{\mathbf{\Omega}(A)}$ in $\mathbf{\Omega}(A)$ is the principal sieve $\downarrow A$, and the null element $0_{\mathbf{\Omega}(A)}$ is the empty sieve $\emptyset$. The partial ordering in $\mathbf{\Omega}(A)$ is defined by $S_1 \leq S_2$ if, and only if, $S_1 \subseteq S_2$; and the logical connectives are defined as:

$$S_1 \wedge S_2 := S_1 \cap S_2 \tag{A.15}$$
$$S_1 \vee S_2 := S_1 \cup S_2 \tag{A.16}$$
$$S_1 \Rightarrow S_2 := \{f : B \to A \mid \text{ for all } g : C \to B \text{ if } f \circ g \in S_1 \text{ then } f \circ g \in S_2\} \tag{A.17}$$

As in any Heyting algebra, the negation of an element S (called the *pseudo-complement* of S) is defined as $\neg S := S \Rightarrow 0$; so that

$$\neg S := \{f : B \to A \mid \text{for all } g : C \to B, f \circ g \notin S\}. \tag{A.18}$$

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[19]The paradigmatic example of a Heyting algebra is the set of all open sets in a topological space Z. The algebraic operations are defined as $O_1 \wedge O_2 := O_1 \cap O_2$; $O_1 \vee O_2 := O_1 \cup O_2$; and $\neg O := \mathrm{int}(Z - O)$.

The main distinction between a Heyting algebra and a Boolean algebra is that, in the former, the negation operation does not necessarily obey the law of excluded middle: instead, all that be can said is that, for any element S,

$$S \vee \neg S \leq 1. \tag{A.19}$$

It can be shown that the presheaf $\mathbf{\Omega}$ is a subobject classifier for the topos $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. That is to say, subobjects of any object $\mathbf{X}$ in this topos (*i.e.*, any presheaf on $\mathcal{C}$) are in one-to-one correspondence with morphisms $\chi : \mathbf{X} \to \mathbf{\Omega}$. This works as follows. First, let $\mathbf{K}$ be a subobject of $\mathbf{X}$. Then there is an associated *characteristic* morphism $\chi^{\mathbf{K}} : \mathbf{X} \to \mathbf{\Omega}$, whose 'component' $\chi_A^{\mathbf{K}} : \mathbf{X}(A) \to \mathbf{\Omega}(A)$ at each 'stage of truth' A in $\mathcal{C}$ is defined as

$$\chi_A^{\mathbf{K}}(x) := \{f : B \to A \mid \mathbf{X}(f)(x) \in \mathbf{K}(B)\} \tag{A.20}$$

for all $x \in \mathbf{X}(A)$. That the right hand side of Eq. (A.20) actually *is* a sieve on A follows from the defining properties of a subobject.

Thus, in each 'branch' of the category $\mathcal{C}$ going 'down' from the stage A, $\chi_A^{\mathbf{K}}(x)$ picks out the first member B in that branch for which $\mathbf{X}(f)(x)$ lies in the subset $\mathbf{K}(B)$, and the commutative diagram Eq. (A.10) then guarantees that $\mathbf{X}(h \circ f)(x)$ will lie in $\mathbf{K}(C)$ for all $h : C \to B$. Thus each stage of truth A in $\mathcal{C}$ serves as a possible context for an assignment to each $x \in \mathbf{X}(A)$ of a generalized truth-value: which is a sieve, belonging to the Heyting algebra $\mathbf{\Omega}(A)$, rather than an element of the Boolean algebra $\{0, 1\}$ of normal set theory. This is the sense in which contextual, generalized truth-values arise naturally in a topos of presheaves.

There is a converse to Eq. (A.20): namely, each morphism $\chi : \mathbf{X} \to \mathbf{\Omega}$ (*i.e.*, a natural transformation between the presheaves $\mathbf{X}$ and $\mathbf{\Omega}$) defines a subobject $\mathbf{K}^{\chi}$ of $\mathbf{X}$ via

$$\mathbf{K}^{\chi}(A) := \chi_A^{-1}\{1_{\mathbf{\Omega}(A)}\}. \tag{A.21}$$

at each stage of truth A.

For this reason, the presheaf $\mathbf{\Omega}$ is known as *the subobject classifier* in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. As mentioned above, the existence of such an object is one of the defining properties for a category to be a topos, which $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is.

3. Global Sections of a Presheaf: In any category, a *terminal object* is defined to be an object 1 with the property that, for any object X in the category, there is a unique morphism $X \to 1$; it is easy to show that terminal objects are unique up to isomorphism. A *global element* of an object X is then defined to be any morphism $1 \to X$. The motivation for this nomenclature is that, in the case of the category of sets, a terminal object is any singleton set $\{*\}$; and then it is true that there is a one-to-one correspondence between the elements of a set X and functions from $\{*\}$ to X.

For the category of presheaves on $\mathcal{C}$, a terminal object $\mathbf{1} : \mathcal{C} \to \mathrm{Set}$ can be defined by $\mathbf{1}(A) := \{*\}$ at all stages A in $\mathcal{C}$; if $f : B \to A$ is a morphism in $\mathcal{C}$ then $\mathbf{1}(f) : \{*\} \to \{*\}$ is defined to be the map $* \mapsto *$. This is indeed a terminal object since, for any presheaf

X, we can define a unique natural transformation $N : \mathbf{X} \to \mathbf{1}$ whose components $N_A : \mathbf{X}(A) \to \mathbf{1}(A) = \{*\}$ are the constant maps $x \mapsto *$ for all $x \in \mathbf{X}(A)$.

A global element of a presheaf $\mathbf{X}$ is also called a *global section*. As a morphism $\gamma : 1 \to \mathbf{X}$ in the topos $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, a global section corresponds to a choice of an element $\gamma_A \in \mathbf{X}(A)$ for each stage of truth A in $\mathcal{C}$, such that, if $f : B \to A$, the 'matching condition'

$$\mathbf{X}(f)(\gamma_A) = \gamma_B \tag{A.22}$$

is satisfied. As we shall see, the Kochen-Specker theorem can be read as asserting the non-existence of any global sections of certain presheaves that arises naturally in any quantum theory.

4. Local Sections of a Presheaf: One of the important properties of a general topos category is that an object may have 'partial', or 'local', elements even if there are no global ones. In general, a *local element* of an object X in a category with a terminal object is defined to be a morphism $U \to X$, where U is a subobject of the terminal object 1. In the category of sets, there are no-nontrivial subobjects of $1 := \{*\}$, but this is not the case in a general topos.

In particular, in the case of presheaves on $\mathcal{C}$, a subobject $\mathbf{U}$ of $\mathbf{1}$ is a collection of subsets $\mathbf{U}(A) \subseteq \{*\}$, A in $\mathcal{C}$, that satisfy the appropriate form of the commutative diagram Eq. (A.10) that describes a subobject. However, the only subsets of $\{*\}$ are $\{*\}$ itself, and the empty set $\emptyset$. Furthermore, there is a unique function $\emptyset \to \{*\}$ (the 'empty' function) but no function $\{*\} \to \emptyset$. It follows, therefore, that in assigning the sets $\emptyset$ or $\{*\}$ to each stage A for a subobject $\mathbf{U}$ of $\mathbf{1}$, the assignments of the singleton sets $\{*\}$ must be 'closed downwards' in the sense that if $\mathbf{U}(A) = \{*\}$ and if $f : B \to A$ is a morphism in $\mathcal{C}$, then we must have $\mathbf{U}(B) = \{*\}$ also.

We deduce from this that a partial element of a presheaf $\mathbf{X}$ is an assignment γ of an element γ_A to a certain *subset* of objects A in $\mathcal{C}$—what we shall call the *domain* $\mathrm{dom}\,\gamma$ of γ—with the properties that (i) the domain is closed downwards in the sense that if $A \in \mathrm{dom}\,\gamma$ and $f : B \to A$, then $B \in \mathrm{dom}\,\gamma$; and (ii) for objects in this domain, the matching condition Eq. (A.22) is satisfied.

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