A note on information theoretic characterizations of physical theories

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Abstract

Clifton, Bub, and Halvorson (CBH) have recently argued that quantum theory is characterized by its satisfaction of three fundamental information-theoretic constraints. However, it is not difficult to construct apparent counterexamples to the CBH characterization theorem. In this paper, we discuss the limits of the characterization theorem, and we provide some technical tools for checking whether a theory (specified in terms of the convex structure of its state space) falls within these limits.

1 Introduction

Some would like to argue that quantum information theory (QIT) has revolutionary implications for the philosophical foundations of quantum mechanics. Whether or not there is any real substance to this claim, it is undoubtedly true that QIT provides us with new perspectives from which we can approach traditional questions about the interpretation of quantum mechanics. One such question asks whether there are natural physical postulates that capture the essence of quantum mechanics — postulates that tell us what sets quantum mechanics apart from other physical theories, and in particular from its predecessor theories. The advent of QIT suggests that we look for *information-theoretic* postulates that characterize quantum mechanics.

A positive answer to this question has been supplied by Clifton, Bub, and Halvorson [CBH03]. CBH show that, within the C^*-algebraic framework

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for physical theories, quantum theories are singled out by their satisfaction of three information-theoretic postulates: 1. no superluminal information transfer via measurement; 2. no cloning; and 3. no unconditionally secure bit commitment. Nonetheless, the creative thinker will have little trouble concocting a "theory" that satisfies these three constraints, but which does not entail quantum mechanics (see [Spe03, Smo03]). Such toy theories might be thought to show that the CBH characterization theorem does not isolate the essential information-theoretic features of quantum mechanics.

Since the CBH characterization theorem is a valid mathematical result, there is a problem of application here — these apparent counterexamples must not satisfy the premises of the theorem. However, in specific cases, this can be difficult to see. In particular, the CBH theorem requires that a theory be specified in terms of its algebra of observables, which is required to be the set of self-adjoint operators in some C^*-algebra. Since the axioms for C^*-algebras are rather intricate, and since some of these axioms have no direct physical interpretation, it can be difficult to determine whether a physically described theory can be formalized within this framework.

In this note, we provide a partial solution to this difficulty. In particular, in physical applications, it may be more natural (and easier) to describe a theory in terms of the convex structure of its state space. Due to the deep mathematical results of Alfsen et al. (detailed in [AS03]), specifying the convex structure of the state space of a theory is sufficient to determine whether that theory can be formulated within the JB algebraic framework. (Additional structure — viz., an "orientation" on the state space — must be specified in order to determine if a theory falls within the C^*-algebraic framework). However, in the general case, it will also be difficult to determine if Alfsen et al.'s axioms on convex sets are satisfied by a physically described theory. Thus, we derive a couple of easily checked necessary conditions for a theory to admit a formulation within the JB algebraic framework. We then show that some interesting toy theories do not satisfy these necessary conditions. (These investigations will also help to clarify the limits of the JB algebraic framework, and might suggests ways of generalizing the CBH theorem.)

In Section 2, we review the basics of the theory of JB algebras, and of the dual (but more general) theory of compact convex sets. We also distill a "Root Theorem" from the results of Alfsen et al., and we show that a certain class of toy theories — viz., those with ambiguous mixtures, but with only finitely many pure states — falls outside of the JB algebraic framework. Then, in Section 3, we consider a class of theories that are locally quantum-mechanical, but which, unlike quantum mechanics, do not have non-locally entangled states. We show that the simplest of these theories falls outside of the JB algebraic framework; and we adduce considerations which indicate that no such theory falls within the JB algebraic framework.

2 The JB algebraic framework for physical theories

2.1 Jordan-Banach algebras

The CBH characterization theorem shows that among the theories within the C^*-algebraic framework, quantum theories are characterized by their satisfaction of the three information-theoretic postulates. One limitation of the CBH theorem is that it excludes from consideration those theories that employ real or quaternionic Hilbert spaces (and so the theorem cannot shed any light on the physical significance of the choice of the underlying field for a Hilbert space). This limitation could be avoided by expanding the class of theories under consideration to include all those theories that admit a Jordan-Banach (JB) algebraic formulation.

Definition. A *Jordan algebra* over $\mathbb{R}$ is a vector space $\mathfrak{A}$ equipped with a commutative (not necessarily associative) bilinear product $\circ$ that satisfies the identity

$$(A^2 \circ B) \circ A = A^2 \circ (B \circ A), \qquad (A, B \in \mathfrak{A}). \tag{1}$$

A *Jordan-Banach (JB) algebra* is a Jordan algebra $\mathfrak{A}$ over $\mathbb{R}$ with identity element I, and equipped with a complete norm $\| \cdot \|$ satisfying the following requirements:

$$\|A \circ B\| \leq \|A\|\|B\|, \quad \|A^2\| = \|A\|^2, \quad \|A^2\| \leq \|A^2 + B^2\|, \tag{2}$$

for all $A, B \in \mathfrak{A}$.

An element A in a JB algebra $\mathfrak{A}$ is said to be *positive*, written $A \geq 0$, just in case there is a $B \in \mathfrak{A}$ such that $A = B^2$. Let $\mathfrak{A}^*$ denote the set of norm-continuous linear mappings of $\mathfrak{A}$ into $\mathbb{R}$; elements of $\mathfrak{A}^*$ are called *linear functionals* on $\mathfrak{A}$. A linear functional ω on $\mathfrak{A}$ is said to be *positive* just in case $\omega(A) \geq 0$ for all positive $A \in \mathfrak{A}$. A positive linear functional ω on $\mathfrak{A}$ is said to be a *state* if $\omega(I) = 1$. The set K of states is clearly a convex subset of $\mathfrak{A}^*$.

There are two standard topologies on the state space of a JB algebra. First, a net $\{\omega_a\}$ of states converges in the *weak* topology* to a state ω just in case the numbers $\{\omega_a(A) : A \in \mathfrak{A}\}$ converge pointwise to the numbers $\{\omega(A) : A \in \mathfrak{A}\}$. Since K is a weak* closed subset of the unit ball of $\mathfrak{A}^*$, the Alaoglu-Bourbaki theorem [KR97, Thm. 1.6.5] entails that K is weak* compact. Second, since K is a subset of the Banach space dual $\mathfrak{A}^*$, K inherits the standard norm topology from $\mathfrak{A}^*$. A net $\{\omega_a\}$ converges in norm to ω just in case the numbers $\{\omega_a(A) : A \in \mathfrak{A}\}$ converge uniformly to the numbers $\{\omega(A) : A \in \mathfrak{A}\}$. Thus, the norm topology on K is always finer that the weak* topology. When K has finite affine dimension, the two topologies are equivalent.

Roughly speaking, all C^*-algebras are JB algebras. More accurately, there is a canonical mapping from the category of C^*-algebras into the category of JB algebras. Indeed, suppose that $\mathfrak{A}$ is a C^*-algebra, and let $\mathfrak{A}_{\mathrm{sa}}$ be the real vector space of self-adjoint operators in $\mathfrak{A}$. If we define $A \circ B = \frac{1}{2}(AB+BA)$, for $A, B \in \mathfrak{A}_{\mathrm{sa}}$, then $\mathfrak{A}_{\mathrm{sa}}$ is a JB algebra (see [Lan98a, Thm. 1.1.9]). Moreover, the state space of $\mathfrak{A}$ is affinely isomorphic to the state space of $\mathfrak{A}_{\mathrm{sa}}$. More specifically, if $\mathcal{H}$ is a complex Hilbert space, then $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ is a JB algebra; and when $\mathcal{H}$ is finite-dimensional, the state space of $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ is affinely isomorphic to the convex set of positive, trace-1 operators on $\mathcal{H}$. In contrast, it is well-known that there is a JB algebra that is not the self-adjoint part of a C^*-algebra. Thus, the JB algebraic framework is genuinely broader than the C^*-algebraic framework.

2.2 Convex sets

All JB algebra state spaces are compact convex sets. But the converse is not true — not all compact convex sets are JB algebra state spaces. We now briefly recall some of the main definitions in the theory of convex sets.

A point x in a convex set K is *extreme* just in case for any $y, z \in K$ and $\lambda \in (0,1)$, if $x = \lambda y + (1-\lambda)z$, then $x = y = z$. We let $\partial_e K$ denote the set of extreme points in K. If K is the state space of an algebra, we also call extreme points *pure states*. A subset F of a convex set K is said to be a *face* just in case F is convex, and for any $x \in F$, if $x = \lambda y + (1-\lambda)z$ with $\lambda \in (0,1)$, then $y \in F$. Clearly the intersection of an arbitrary family of faces is again a face. For $x, y \in K$, we let $\mathrm{face}(x, y)$ denote the intersection of all faces containing $\{x, y\}$. A pair of faces F, G in K are said to be *split* if every point in K can be expressed uniquely as a convex combination of points in F and G. A convex set K is said to be a *simplex* if mixed states have unique decompositions into pure states. More precisely, K is a simplex if for all $w, x, y, z \in \partial_e K$, when

$$\lambda w + (1-\lambda)x = \mu y + (1-\mu)z, \tag{3}$$

with $\lambda, \mu \in (0,1)$, then either $w = y$ or $w = z$. (This definition differs slightly from the standard definition in the theory of infinite dimensional compact convex sets.) If K and L are convex sets, a mapping $\phi : K \to L$ is an *affine isomorphism* just in case ϕ is bijective, and

$$\phi(\lambda x + (1-\lambda)y) = \lambda\phi(x) + (1-\lambda)\phi(y), \tag{4}$$

for all $x, y \in K$ and $\lambda \in [0,1]$. If there is an affine isomorphism ϕ from K onto L, then K and L are said to be *affinely isomorphic*.

2.3 Toy theories and the JB algebraic framework

Which toy theories admit a JB algebraic formulation? First, since the state space of a JB algebra is convex, the theory must allow arbitrary mixtures of its states. (This immediately disqualifies Spekkens' [Spe03] toy theory.) However, if a state space $S \subseteq \mathbb{R}^n$ is not already convex, we can simply take its closed convex hull $\mathrm{co}(S)^-$, and ask whether the resulting theory admits a JB algebraic formulation — i.e., whether $\mathrm{co}(S)^-$ is affinely isomorphic to the state space of a JB algebra.

Drawing on the results of Alfsen et al., we now derive some easily checked necessary conditions for a theory to admit a JB algebraic formulation. (In this theorem and subsequently, we let B^n denote the closed unit ball in $\mathbb{R}^n$.)

Root Theorem. *Let K be a compact convex subset of a topological vector space V over $\mathbb{R}$, and suppose that K is affinely isomorphic to the state space of a JB algebra. Then for any distinct $x, y \in \partial_e K$, $\mathrm{face}(x,y)$ is affinely isomorphic to B^n, for some $n \geq 1$. Furthermore:*

1. *If $\mathrm{face}(x,y) = B^1$ for all distinct $x, y \in \partial_e K$, then K is a simplex.*

2. *If $\partial_e K$ is connected in the norm topology, then for any distinct $x, y \in \partial_e K$, $\mathrm{face}(x,y) = B^n$ for some $n \geq 2$.*

Proof. The first statement is the content of Corollary 5.56 in [AS03].

(1.) We prove the contrapositive. Suppose that K is not a simplex. Then there are $w, x, y, z \in \partial_e K$ such that

$$\lambda w + (1-\lambda)x = \mu y + (1-\mu)z, \tag{5}$$

where $\lambda, \mu \in (0,1)$, $w \neq y$ and $w \neq z$. But then w is an extreme point in $\mathrm{face}(y,z)$. We know from the first statement that $\mathrm{face}(y,z) = B^n$, for some $n \geq 1$. Since there are three distinct extreme points of $\mathrm{face}(y,z)$, it follows that $n \geq 2$.

(2.) Again we prove the contrapositive. Suppose that there are $x, y \in \partial_e K$ such that $\text{face}(x, y) = B^1$. Then there are split faces F, G of K such that $x \in F$ and $y \in G$ [AS03, Lemma 5.54]. Let $U = F \cap \partial_e K$ and let $V = G \cap \partial_e K$. Since F and G are closed in the norm topology [AS01, Prop. 1.29], U and V are closed in the relativized norm topology on $\partial_e K$. Since $\partial_e K \subseteq F \cup G$, it follows that $\partial_e K = U \cup V$. Thus, U and V are open, and $\partial_e K$ is disconnected. □

It is well-known that quantum mechanical state spaces have *ambiguous* mixtures — i.e., mixed states with more than one decomposition into pure states. Thus, in order to find theories with some interesting information-theoretic properties, it is natural to look at non-simplexes in low dimensional spaces — e.g., the square in $\mathbb{R}^2$, or the octahedron in $\mathbb{R}^3$. However, if a convex set has both ambiguous mixtures and a finite number of pure states, then it is not the state space of a JB algebra.

Theorem 1. *Let K be a compact convex subset of a topological vector space V over $\mathbb{R}$. Suppose that K is affinely isomorphic to the state space of a JB-algebra. If K is not a simplex, then $|\partial_e K| \geq |\mathbb{R}|$.*

Proof. Suppose that K is not a simplex. Then part 1 of the Root Theorem entails that there are $x, y \in \partial_e K$ such that $\text{face}(x, y) = B^n$, with $n \geq 2$. Since every extreme point in $\text{face}(x, y)$ is an extreme point in K, we have $|\partial_e K| \geq |\partial_e B^n| = |\mathbb{R}|$. □

Since the octahedron has finitely many extreme points, Theorem 1 shows that the convex version of Spekkens' [Spe03] toy theory is not the state space of a JB algebra.

3 On the possibility of a local quantum theory

In attempting to characterize quantum theory in terms of information-theoretic constraints, it might seem that there are essentially two type of theories: classical theories, whose algebras of observables are commutative, and quantum theories, whose algebras of observables are non-commutative. As long as we work within the C^*-algebraic framework, this dichotomy is essentially valid. In particular, the Gelfand representation theorem [KR97, Thm. 4.4.3] entails that commutative C^*-algebras have a phase space representation, and the GNS representation theorem [KR97, Thm. 4.5.2] entails that non-commutative C^*-algebras carry copies of the qubit. (More precisely, if the algebra is non-commutative, then there are a pair of states that generate a face that is affinely isomorphic to the Bloch sphere.) But it would be surprising to find that this dichotomy is *logically* necessary — i.e., that our world is necessarily either classical or quantum-mechanical. Motivated by a comment of Schrödinger's, we now consider the possibility of a theory that is locally like quantum mechanics, but which lacks non-locally entangled states.

In his seminal discussion of entanglement, Schrödinger (1936) points out the paradoxical nature of non-locality in quantum mechanics. He ends by noting his hope that quantum mechanics will be replaced by a theory in which there are no non-locally entangled states. He says,

> Indubitably, the situation described here [in which there are non-locally entangled states] is, in present quantum mechanics, a necessary and indispensable feature. The question arises, whether it is so in Nature too. I am not satisfied about there being sufficient experimental evidence for that...
>
> It seems worth noticing that the paradox could be avoided by a very simple assumption, namely if the situation after [two systems] separating were described by the expansion
>
> $$\sum_{i=1}^{n} c_i(x_i \otimes y_i), \tag{6}$$
>
> but with the additional statement that the knowledge of the *phase relations* between the complex constants c_i has been entirely lost in consequence of the process of separation. This would mean that not only the parts, but the whole system, would be in the situation of a mixture, not of a pure state. ... it would utterly eliminate the experimenter's influence on the state of that system which he does not touch.
>
> This is a very incomplete description and I would not stand for its adequateness. But I would call it a possible one, until I am told, either why it is devoid of meaning or with which experiments it disagrees.
>
> ([Sch36, pp. 451–452]. Eqn. 6 has been adapted to the present discussion.)

When Schrödinger speaks of the state in Eqn. 6, but with "the knowledge of the phase relations" lost, he presumably means the mixed state

$$\sum_{i=1}^{n} |c_i|^2 \left(P_{x_i} \otimes P_{y_i}\right). \tag{7}$$

Thus, Schrödinger's hope (as of 1936) is that the true theory will turn out to be locally quantum mechanical, but with some sort of selection rule that prohibits superposition of product states for systems that are spacelike separated.

Schrödinger's hoped-for theory would also be information-theoretically interesting, because it would satisfy two, but not three, of the information-theoretic postulates that characterize quantum mechanics. In particular, Schrödinger's theory would not allow information transfer via measurement, or cloning, but it would allow an unconditionally secure bit commitment protocol.

We now know — due to experimental verification of the violation of Bell's inequality — that Schrödinger's hoped-for theory disagrees with experiment. But, there is also reason to suspect that Schrödinger's hoped-for theory is "devoid of meaning." In fact, the CBH characterization theorem shows that, within the C^*-algebraic framework, if the no cloning and no information transfer via measurement postulates are satisfied, then there are non-locally entangled states. To see this, suppose that there are two physical systems with C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Suppose also that the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies the no information transfer via measurement and no cloning postulates. On the one hand, the no information transfer via measurement postulate entails that observables in $\mathfrak{A}$ commute with observables in $\mathfrak{B}$ [CBH03, Theorem 1]. On the other hand, the no cloning postulate entails that $\mathfrak{A}$ and $\mathfrak{B}$ are non-commutative [CBH03, Theorem 2]. It then follows by mathematical necessity that there are non-locally entangled states across $(\mathfrak{A}, \mathfrak{B})$ [Lan87]. That is, within the C^*-algebraic framework, if a theory is locally quantum mechanical, then it has non-locally entangled states.

We claim that the C^*-algebraic representation of physical theories is broad enough to cover any successful theory from the mature period of physical science (roughly from the time of Newton). Whether this breadth of applicability indicates that there is something *a priori* about the C^*-algebraic representation of theories is a deep question, which we will not attempt to answer here. (Landsman [Lan98b], for one, suggests that JB algebraic structure is not contingent.) Nonetheless, to clarify the limits of the C^*-algebraic framework, we construct a theory that is locally quantum mechanical, but which has no non-locally entangled states.

3.1 The Schr*dinger theory

Consider the simplest composite quantum system, a pair of qubits. Of course, we cannot simply throw away the non-locally entangled states without doing violence to the linear structure of $\mathbb{C}^2 \otimes \mathbb{C}^2$. But since the complement of the set of non-locally entangled states is a compact convex set, we can throw away the entangled states of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and still end up with a theory with a convex state space. More precisely, recall that a density operator $D \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is a *pure product state* just in case $D = P_x \otimes P_y$, where P_x, P_y are one-dimensional projections on $\mathbb{C}^2$. The set of pure product states is a closed subset of the pure state space of $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$; as such, it is a closed subset in the surface of the unit sphere in $\mathbb{R}^{15}$. A density operator $D \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is said to be *separable* just in case it is a convex combination of pure product states. Alternatively, we can define the set K of separable states of $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ as the closed convex hull of the pure product states.

Since K is compact and convex, it gives a genuine theory in the convex sets approach; we call this theory the *Schr*dinger theory.* (This ad hoc construction is for conceptual purposes only; we do not think that Schrödinger really had this theory in mind when he expressed his hope for an alternative to quantum mechanics.) The "observables" of the Schr*dinger theory are the elements of the real vector space $A(K)$ of affine functions from K to $\mathbb{R}$. The "expectation value" of observable $f \in A(K)$ in state $\rho \in K$ is $f(\rho)$. Clearly, each self-adjoint operator $A \in \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ gives an observable for the Schr*dinger theory via the mapping $\rho \mapsto \mathrm{Tr}(\rho A)$.

Since the components systems in the Schr*dinger theory are qubits, the Schr*dinger theory does not allow states to be cloned. (The no-cloning theorem only needs the fact that the affine ratio on local systems takes values strictly between 0 and 1.) However, since the Schr*dinger theory has no entangled states, the original Bennett-Brassard [BB84] bit commitment protocol can be securely implemented.

Clearly, the Schr*dinger theory is not a C^*-algebraic theory. We devote the next section to establishing a stronger claim: the Schr*dinger theory is not a JB algebraic theory.

3.2 Superposability as an equivalence relation

Since all classical phase space theories are JB algebraic theories, it is not the case that states of a JB algebra can always be "superposed." However, in the JB algebraic framework, superposability is an equivalence relation. (The resulting equivalence classes are called *superselection sectors.*) In contrast, we will show that in the Schr*dinger theory, superposability is not transitive. In particular, while $P_x \otimes P_y$ is superposable with $P_y \otimes P_y$, and $P_y \otimes P_y$ is superposable with $P_y \otimes P_x$, $P_x \otimes P_y$ is not superposable with $P_y \otimes P_x$, since a superposition of the latter two states would have to be an entangled state.

To make these claims precise, we need first to define the notion of the *affine ratio* of extreme points in a convex set (see [Mie69], [Lan98a, Prop. 2.8.1]).

Definition. Let K be a compact convex set, and let $A(K)$ be the set of continuous affine functions from K into $\mathbb{R}$. If $x, y \in \partial_e K$, then the *affine ratio* of x and y relative to K is given by

$$\mathbf{p}_K(x/y) \ =_{\text{def}} \ \inf\{f(y); f \in A(K), \ \text{range}(f) \subseteq [0,1], \ \text{and} \ f(x) = 1\}.$$

When K is standardly embedded in a vector space V, the affine ratio has a natural geometrical interpretation. In particular, each affine function $f : V \to \mathbb{R}$ foliates V into a family of hyperplanes $\{f^{-1}(t)\}_{t\in\mathbb{R}}$. Now consider those f's where K falls between the 0 and 1 hyperplanes, and where x lies in the intersection of K with the 1 hyperplane. Then any $y \in \partial_e K$ falls within a unique $t \in [0,1]$ hyperplane. Finally, consider all such foliations, and take the infimum of the t such that $f(y) = t$. In some cases, there is a unique affine function f such that $f(x) = 1$ and $\min\{f(z) : z \in K\} = 0$. (e.g., if $K = B^n$, then $f(z) = \frac{1}{2} + \frac{1}{2}(x \cdot z)$.) In this case, $\mathbf{p}_K(x/y) = f(y)$.

The following Lemma is completely obvious; but, since we use it repeatedly in our subsequent proofs, it is worth stating explicitly.

Lemma 1. *Let K, L be compact convex sets such that $K \subseteq L$. If x, y are extreme points in both K and L, then $\mathbf{p}_K(x/y) \leq \mathbf{p}_L(x/y)$.*

We say that $x, y \in \partial_e K$ are *orthogonal* just in case $\mathbf{p}_K(x/y) = 0$.

Definition. Two orthogonal states $x, y \in \partial_e K$ are *superposable* just in case for each $\lambda \in [0,1]$, there is a state $z \in \partial_e K$ such that $\mathbf{p}_K(x/z) = \lambda$ and $\mathbf{p}_K(y/z) = 1 - \lambda$.

This usage is justified by the fact that z is pure, but for any measurement designed to distinguish x from y, z assigns probability $\mathbf{p}_K(x/z)$ to x, and probability $\mathbf{p}_K(y/z)$ to y.

When K is a JB algebra state space, the superselection sectors in $\partial_e K$ coincide with the path-components in the norm topology on $\partial_e K$. That is, for any $x, y \in \partial_e K$, there is a norm continuous path connecting x and y iff x and y are superposable. Since we will not need the full strength of this result, and since its proof is fairly complicated, we establish a weaker analogue.

Theorem 2. *Let K be the state space of a JB algebra, and suppose that $\partial_e K$ is connected in the norm topology. Then for any orthogonal $x, y \in \partial_e K$, there is a $z \in \partial_e K$ such that $\mathbf{p}_K(x/z) \geq \frac{1}{2}$ and $\mathbf{p}_K(y/z) \geq \frac{1}{2}$.*

Proof. Suppose that $x, y \in \partial_e K$ are orthogonal, and let $F = \text{face}(x, y)$. By part 2 of the Root Theorem, there is an affine isomorphism ϕ from F onto B^n, with $n \geq 2$. Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical orthonormal basis for $\mathbb{R}^n$. Since there is an affine automorphism of B^n that maps $\phi(x)$ to e_1, we may suppose that $\phi(x) = e_1$. An exercise in elementary geometry shows that $\phi(y) = -e_1$. (ϕ preserves affine ratios, and $-e_1$ is the unique $r \in B^n$ such that $\mathbf{p}_{B^n}(e_1/r) = 0$.) Furthermore, $\mathbf{p}_{B^n}(e_1/e_2) = \mathbf{p}_{B^n}(-e_1/e_2) = \frac{1}{2}$. Thus, if we choose $z = \phi^{-1}(e_2)$, then $\mathbf{p}_K(x/z) \geq \mathbf{p}_F(x/z) = \frac{1}{2}$ and $\mathbf{p}_K(y/z) \geq \mathbf{p}_F(y/z) = \frac{1}{2}$. □

Now, it is easy to see that there is a continuous path between any two pure states in the Schr*dinger theory.

Lemma 2. *Let K be the compact convex set of separable states of $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Then $\partial_e K$ is path-connected in the Hilbert-Schmidt norm topology.*

Proof. The extreme points of K are pure product states. So, let $P_x \otimes P_y$ and $P_z \otimes P_w$ be arbitrary elements of $\partial_e K$. By symmetry, and since path-connectedness of points is transitive, it will suffice to consider the case where $w = y$. Let $\|\cdot\|_2$ denote the Hilbert-Schmidt norm on $\mathcal{B}(\mathbb{C}^n)$; i.e., $\|A\|_2 = \text{Tr}(A^*A)^{1/2}$. There is a $\|\cdot\|_2$-continuous function f from $[0, 1]$ into the set of projection operators in $\mathcal{B}(\mathbb{C}^2)$ such that $f(0) = P_x$ and $f(1) = P_z$. Define a function g from $[0, 1]$ into the set of projection operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by setting $f(t) = g(t) \otimes P_y$. Since $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$ for all $A, B \in \mathcal{B}(\mathbb{C}^2)$, it follows that

$$\|g(t) - g(t')\|_2 = \|(f(t) - f(t')) \otimes P_y\|_2 = \|f(t) - f(t')\|_2, \tag{8}$$

for all $t, t' \in [0, 1]$. Thus, f is $\|\cdot\|_2$-continuous as a mapping into $\partial_e L$, and therefore as a mapping into $\partial_e K$ with the relative topology. □

For the set of density operators on a complex Hilbert space, the affine ratio is the transition probability.

Lemma 3. *If K is the state space of $\mathcal{B}(\mathbb{C}^n)$, then*

$$\mathbf{p}_K(P_x/P_y) = \text{Tr}(P_x P_y) = |\langle x, y\rangle|^2, \tag{9}$$

for any unit vectors $x, y \in \mathbb{C}^n$.

Proof. Consider the affine function $f : K \to [0,1]$ given by $f(D) = \mathrm{Tr}(P_x D)$, for all $D \in K$. We claim that $f(P_y) = \mathbf{p}_K(P_x/P_y)$, for all $P_y \in \partial_e K$. For this it will suffice to show that for any $g \in A(K)$, if range$(g) \subseteq [0,1]$, and if $g(P_x) = 1$, then $g \geq f$. Let g be such a function. Since $A(K)$ is order-isomorphic to $\mathcal{B}(\mathbb{C}^n)_{\mathrm{sa}}$, there is a self-adjoint operator A on $\mathbb{C}^n$ with eigenvalues restricted to $[0,1]$, and such that $g(P_y) = \mathrm{Tr}(AP_y)$, for all $P_y \in \partial_e K$. Since $\langle x, Ax\rangle = g(P_x) = 1$, it follows that $Ax = x$; and it then follows from the spectral theorem that $P_x A = AP_x = P_x$. Since A is positive, $(I - P_x)A$ is positive, and therefore

$$g(P_y) = \langle y, Ay\rangle = f(P_y) + \langle y, (I - P_x)Ay\rangle \geq f(P_y), \tag{10}$$

for any $P_y \in \partial_e K$. □

It seems intuitively clear that no pure state in the Schr*dinger theory could count as a non-trivial superposition of $P_x \otimes P_y$ and $P_y \otimes P_x$. We now prove this fact.

Lemma 4. *Suppose that $\{x, y\}$ is an orthonormal basis for $\mathbb{C}^2$. If*

$$|\langle u \otimes v, x \otimes y\rangle|^2 + |\langle u \otimes v, y \otimes x\rangle|^2 = 1, \tag{11}$$

where $\|u\| = \|v\| = 1$, then either $|\langle u \otimes v, x \otimes y\rangle|^2 = 0$ or $|\langle u \otimes v, y \otimes x\rangle|^2 = 0$.

This Lemma follows immediately from the uniqueness of the Schmidt decomposition of $u \otimes v$. However, for completeness' sake, we include an elementary proof.

Proof. Suppose that Eqn. 11 holds. Let

$$a = |\langle u, x\rangle|^2, \quad b = |\langle v, y\rangle|^2, \quad c = |\langle v, x\rangle|^2, \quad d = |\langle u, y\rangle|^2. \tag{12}$$

Thus, Eqn. 11 becomes $ab + cd = 1$. Since $\{x, y\}$ is an orthonormal basis, we also have $b = 1 - c$ and $d = 1 - a$. Hence, $a + c - 2ac = 1$. The functions $[0,1] \ni a \mapsto a + c - 2ac$ (for fixed $c \in [0,1]$) and $[0,1] \ni c \mapsto a + c - 2ac$ (for fixed $a \in [0,1]$) are affine. Thus, $a + c - 2ac$ achieves its maximum value only at extreme points of the convex set $[0,1] \times [0,1]$. Checking these points, we find that $a + c - 2ac \leq 1$, with equality achieved only when $(a, c) = (1, 0)$ or $(a, c) = (0, 1)$. If $a = 1$ and $c = 0$, then $|\langle u \otimes v, y \otimes x\rangle|^2 = cd = 0$. Similarly, if $a = 0$ and $c = 1$, then $|\langle u \otimes v, x \otimes y\rangle|^2 = ab = 0$. □

Finally, we combine Lemma 2 with Theorem 2 to show that the Schr*dinger theory does not fall within the JB algebraic framework.

Theorem 3. *The compact convex set of separable states of* $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ *is not affinely isomorphic to the state space of a JB algebra.*

Proof. Let K be the set of separable states. Suppose for *reductio ad absurdum* that K is affinely isomorphic to the state space of a JB algebra. Since K has finite affine dimension, there is only one topology on K that is compatible with its convex structure. Since $\partial_e K$ is connected in the Hilbert-Schmidt norm topology (Lemma 2), it follows that $\partial_e K$ is connected in the norm topology (as the state space of a JB algebra). Let $\sigma = P_x \otimes P_y$ and let $\tau = P_y \otimes P_x$, where $\{x, y\}$ is an orthonormal basis for $\mathbb{C}^2$. Let L be the state space of $\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Then

$$\mathbf{p}_K(\sigma/\tau) \leq \mathbf{p}_L(P_x \otimes P_y / P_y \otimes P_x) = |\langle x \otimes y, y \otimes x \rangle|^2 = 0. \tag{13}$$

By Theorem 2, there is a $\rho \in \partial_e K$ such that $\mathbf{p}_L(\sigma/\rho) \geq \frac{1}{2}$ and $\mathbf{p}_L(\tau/\rho) \geq \frac{1}{2}$. Since $\rho = P_u \otimes P_v$, with u, v unit vectors in $\mathbb{C}^2$, we have

$$\begin{aligned} 1 &\leq \mathbf{p}_K(\sigma/\rho) + \mathbf{p}_K(\tau/\rho) & (14)\\ &\leq \mathbf{p}_L(P_x \otimes P_y / P_u \otimes P_v) + \mathbf{p}_L(P_y \otimes P_x / P_u \otimes P_v) & (15)\\ &= |\langle x \otimes y, u \otimes v \rangle|^2 + |\langle y \otimes x, u \otimes v \rangle|^2 \ \leq \ 1. & (16) \end{aligned}$$

Thus, Lemma 4 entails that either

$$\mathbf{p}_K(\sigma/\rho) \leq \mathbf{p}_L(P_x \otimes P_y / P_u \otimes P_v) = |\langle x \otimes y, u \otimes v \rangle|^2 = 0, \tag{17}$$

or

$$\mathbf{p}_K(\tau/\rho) \leq \mathbf{p}_L(P_y \otimes P_x / P_u \otimes P_v) = |\langle y \otimes x, u \otimes v \rangle|^2 = 0. \tag{18}$$

In either case, we have a contradiction. Therefore, K is not affinely isomorphic to the state space of a JB algebra. $\square$

This result shows that the simplest Schrödinger-like theory — viz., the Schr*dinger theory — does not admit a JB algebraic formulation. We conjecture that the this result can be generalized to show that there is no Schrödinger-like theory in the JB algebraic framework.

4 Conclusion

This note attempts to clarify the limits of recent information-theoretic characterizations of quantum mechanics. However, in doing so, it has raised a number of further questions, both of a technical and a philosophical nature.

First, although we have worked in the JB algebraic framework, the CBH characterization theorem is formulated in the narrower C^*-algebraic framework. Thus, it is of deep interest to determine whether the CBH characterization theorem generalizes to the JB algebraic framework, and whether there are information-theoretic postulates that pick out the subclass of C^*-algebraic theories within the class of JB algebraic theories.

Second, the considerations in this paper suggest that we take a closer look at different ways of putting together composite systems, where all systems are assumed to have convex state spaces. It is known that there are several different notions of the "tensor product" of compact convex sets (see, e.g., [NP69]). Thus, it would be interesting to see which of these products preserve which information-theoretic properties of the component systems. More specifically, suppose that $\otimes$ is a tensor product of compact convex sets that preserves the defining properties of JB algebra state spaces. Then does it follow that $K \otimes L$ has non-locally entangled states whenever K and L are not simplexes? Or does the JB algebraic framework permit the existence of a Schrödinger-like theory?

Finally, our discussion has raised the question of the role of constraints (either *a priori* or operational) on theory construction. On the one hand, if there are no constraints on theory construction — i.e., if there is no minimum amount of mathematical structure shared by all theories, and if any fairy tale can count as a legitimate "toy theory" — then it would be hopeless to try to *derive* quantum mechanics from information theoretic principles, or from any other sort of principles for that matter. (e.g., why assume that the results of measurements are real numbers? Why assume that measurements have single outcomes? Why assume that the laws of physics are the same from one moment to the next?) On the other hand, the idea that it is legitimate to assume a fixed background framework for physical theories seems to come into tension with the empiricist attitude that drove the two major revolutions in physics in the 20th century; and the last thing we want is to impede the search for a future theory that would generalize quantum mechanics.

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