Some Aspects of Modality in Analytical Mechanics

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Dedicated to the memory of David Lewis

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Abstract

This paper discusses some of the modal involvements of analytical mechanics. I first review the elementary aspects of the Lagrangian, Hamiltonian and Hamilton-Jacobi approaches. I then discuss two modal involvements; both are related to David Lewis' work on modality, especially on counterfactuals.

The first is the way Hamilton-Jacobi theory uses ensembles, i.e. sets of possible initial conditions. The structure of this set of ensembles remains to be explored by philosophers.

The second is the way the Lagrangian and Hamiltonian approaches' variational principles state the law of motion by mentioning contralegal dynamical evolutions. This threatens to contravene the principle that any actual truth, in particular an actual law, is made true by actual facts. Though this threat can be avoided, at least for simple mechanical systems, it repays scrutiny; not least because it leads to some open questions.

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1 Introduction

Ever since its beginnings, analytical mechanics has been a rich field for philosophical exploration. In particular, the principle of least action—with its various forms, and its strong suggestion of teleology—has been a focus of discussion from the time of Maupertuis to now: as witness some of the essays in this volume, and other excellent recent work such as Stöltzner (2003). However, so far as I can tell, philosophers have not explored the modal involvements of analytical mechanics. So I propose in this paper to make a first foray into this territory.

More specifically, I will discuss two modal involvements. Both are related to David Lewis' work on modality, especially his theory of counterfactuals (1973). (The first concerns Hamilton-Jacobi theory; the second Lagrangian and Hamiltonian mechanics.) So I dedicate the paper to his memory. Although analytical mechanics was not a topic central to his interests, the discussion will illustrate a view central to his metaphysical system, and to his influence on analytical philosophy: that science, indeed all our knowledge and belief, is steeped in modality. Besides, any philosopher who knew Lewis the man as well as the work, knows not only that he was a great philosopher—with transcendent creativity and craftsmanship, and enormous intellectual generosity—but also that he had wide intellectual interests in the sciences. So I like to think my illustrations of Lewisian themes in mechanics would have pleased him.

The modal involvements of analytical mechanics turn out to be rich and subtle. There is much to explore here: as so often in the philosophy of physics, one can mine from a little physics, a lot of philosophy—at least, a lot more than one paper! To be brief enough, I shall have to be selective in various ways. The two main ones are:–

1) I shall consider only a limited class of classical mechanical systems, and give a technically elementary presentation of how the Lagrangian, Hamiltonian and Hamilton-Jacobi approaches treat them (Section 2). To be a bit more specific: I shall consider only systems with finitely many degrees of freedom, for which any constraints can be solved; and my presentation will eschew modern geometry. This limitation is largely a matter of brevity and expository convenience: most of the philosophical discussion in Sections 3 et seq. applies more widely.

2) These modal involvements are entangled, technically and philosophically, with the fact that these three approaches provide general schemes for solving problems, or for representing their solutions. I believe these general schemes hold philosophical morals. But here I will set them aside. (My (2003) takes them up.)

The plan of the paper is as follows. In Section 2, I review elements of analytical mechanics. Since philosophers are often familiar with elementary Lagrangian and Hamiltonian mechanics, but rarely with Hamilton-Jacobi theory, I will give more detail about the latter. In Section 3, I begin my discussion of modality. I distinguish three grades of modal involvement, according to which kind of actual matters of fact they allow to vary counterfactually. The first grade considers counterfactual initial and/or final conditions, but keeps fixed the forces on the system and the laws of motion. It is most strikingly illustrated by Hamilton-Jacobi theory's S-function, which represents a structured ensemble of such conditions. The theory considers many different S-functions, and so ensembles: so I discuss the structure of this set in Section 4. In particular, there is an analogy with Lewis' spheres of worlds. The third grade of modal involvement, which considers counterfactual laws of motion, is illustrated by the way variational principles, such as Hamilton's Principle, invoke evolutions that violate the actual law. This prompts a discussion (Section 5) whether variational principles violate the philosophical principle that any actual truth is made true by actual facts. I argue that they do not, at least for simple mechanical systems. But the topic brings out various points, including another analogy with Lewis' account of counterfactuals. It also raises some open questions.

2 Technical preliminaries

I will first review the mathematics and physics I need; (without proofs, but with a few references). Much of what follows is pure mathematics, though I will use a notation and jargon suggestive of mechanics. Each of the three approaches—Lagrangian, Hamiltonian and Hamilton-Jacobi—has a Subsection.

2.1 Simple systems and Lagrangian mechanics

I begin with the simplest problem of the calculus of variations. This is the variational problem (in a notation suggestive of mechanics)

$$\delta I := \delta I[q_i] = \delta \int_{t_0}^{t_1} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) dt = 0 \quad , \tag{2.1}$$

where [ ] indicates that I is a functional, the dot denotes differentiation with respect to t, and L is to be a C^2 (twice continuously differentiable) function in all $2n+1$ arguments. L is the *Lagrangian* or *fundamental function*; and $\int L \; dt$ is the *action* or *fundamental integral*. I will discuss this only locally; i.e. I will consider a fixed simply connected region G of $(n+1)$-dimensional real space $\mathbb{R}^{n+1}$, on which there are coordinates $(q_1, \dots, q_n, t) =: (q_i, t)$. I will often suppress the subscripts i, j etc. running from 1 to n, and write (q, t) etc.

The singling out of a coordinate t (called the *parameter* of the problem), to give a parametric representation of curves $q(t) := q_i(t)$, is partly a matter of notational clarity. But it is of course suggestive of the application to mechanics, where t is time, q represents the system's configuration and (q_i, t)-space is often called 'extended configuration space' or 'event space'. Besides, the singling out of t reflects the fact that we do not require the fundamental integral to be independent of the choice of t; indeed we shall note in Section 2.2 that allowing this dependence is necessary for making Legendre transformations.[2]

A necessary condition for I to be stationary at the C^2 curve $q(t) := q_i(t)$—i.e. for $\delta I = 0$ in comparison with other C^2 curves that (i) share with $q(t)$ the end-points $q(t_0), q(t_1)$ and (ii) are close to $q(t)$ in both value and derivative throughout $t_0 < t < t_1$—is that: $q(t)$ satisfies for $t_0 < t < t_1$ the n second-order *Euler-Lagrange* (also known as: *Lagrange*) equations

$$\frac{d}{dt}L_{\dot{q}_i} - L_{q_i} = 0 \quad i = 1, \dots, n, \tag{2.2}$$

where as usual subscripts indicate partial differentiation; i.e. $L_{\dot{q}_i} := \frac{\partial L}{\partial \dot{q}_i}$ etc. The proof is elementary. Under certain conditions, the converse also holds: that is, eq. 2.2 is sufficient for eq. 2.1, i.e. for I to be stationary . A curve satisfying eq. 2.2 is called an *extremal*.

We apply these ideas to mechanics, getting *Lagrangian mechanics*. We consider a mechanical system with n configurational degrees of freedom. Note that if the system consists of N point-particles (or bodies small enough to be treated as point-particles), so that a configuration is fixed by $3N$ cartesian coordinates, we may yet have $n < 3N$; for the system may be subject to constraints and the q_i are to be independently variable in the region G.

I shall assume that the system is *simple*, in the sense that it has the following five features, (i) to (v). Note: (1): My discussion of the Hamiltonian and Hamilton-Jacobi approaches will retain this restriction to simple systems; and each will also assume other restrictions. (2): Some of these features, e.g. (iii), evidently involve modal notions; but I will postpone discussion of these aspects till Section 3 et seq..

(i): Any constraints on the system are *holonomic*; i.e. each is expressible as an equation $f(r_1, \dots, r_m) = 0$ among the coordinates r_k of the system's component parts; (here the r_k could be the $3N$ cartesian coordinates of N point-particles, so that $m := 3N$). A set of c such constraints can in principle be solved, defining a $(m - c)$-dimensional hypersurface Q in the m-dimensional space of the rs; so that on the *configuration space* Q we can define $n := m - c$ independent coordinates $q_i, i = 1, \dots, n$.

(ii): Any constraints on the system are *scleronomic*, i.e. independent of time. So the configuration space Q is identified once and for all; and we can take the region $G \subset \mathbb{R}^{n+1}$ as a cartesian product of Q with a time-interval $[t_-, t_+] \subset \mathbb{R}$ (where we allow $t_- = -\infty, t_+ = +\infty$).

(iii): Any constraints on the system are *ideal*; i.e. the forces that maintain the constraints would do no work in any possible displacement consistent with the constraints and applied forces (called a *virtual displacement*). This allows us to deduce the principle of virtual work, and thereby d'Alembert's principle.

[2] Of course, the calculus of variations *can* be developed on the assumption that the fundamental integral is to be parameter-independent—if it could not be, so much the worse for relativistic theories! But the details, in particular of how to set up a canonical formalism, are different from what follows in this Section, and I set them aside; (cf. e.g. Rund (1966, Chapter 3)). Suffice it to say that the philosophical morals of Sections 3 et seq. hold good for parameter-independent problems.

D'Alembert's principle implies that for a holonomic system (i.e. obeying (i)), the *kinetic energy* T (defined in cartesian coordinates, with k now labelling particles, by: $T := \Sigma_k \frac{1}{2} m_k \mathbf{v}_k^2$) and *generalized forces* Q_i (which are defined for $i = 1, \ldots, n$, in terms of the vector applied force $\mathbf{F}_k$ on particle k, and position vector $\mathbf{r}_k$ of particle k, by: $Q_i := \Sigma_k \mathbf{F}_k \cdot \left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right)$) obey, for all i

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \quad \equiv \quad \frac{d}{dt}(T_{\dot{q}_i}) - T_{q_i} \quad = \quad Q_i \quad ; \tag{2.3}$$

which are also sometimes called Lagrange's equations.

(iv): The applied forces are *monogenic*; i.e. the total work δw done in an infinitesimal virtual displacement is integrable; its integral is the *work function* U. (The term 'monogenic' is due to Lanczos (1986, p.30), but followed by others e.g. Goldstein et al. (2002, p. 34).)

(v): Furthermore, the system is *conservative*; i.e. the work function U is independent of both the time and the generalized velocities $\dot{q}_i$, and depends only on the q_i: $U = U(q_1, \ldots, q_n)$. We interpret $V := -U$ as *potential energy*. Then (ii) and (v) together imply the conservation of energy, i.e. the constancy in time of $T + V$.

Besides, (v) and the definition of Q_i in (iii) implies that $Q_i = -V_{q_i}$; so that, defining the Lagrangian $L := T - V$, eq. 2.3 take on the form of the Euler-Lagrange equations, i.e. eq. 2.2. With this $L \equiv T - V$, eq. 2.2 are called Lagrange's equations.

For a simple system, Lagrange's equations are (not just necessary but also) *sufficient* for the action integral $I = \int L \, dt$ to be stationary (Whittaker (1959, Section 99)). So we infer *Hamilton's Principle*: that the motion in configuration space of a simple system, between prescribed configurations at times t_0 and t_1, makes stationary $\int L \, dt$, with the Lagrangian $L(q_i, \dot{q}_i, t) \equiv L(q_i, \dot{q}_i) := T - V$ now having no explicit time-dependence:

$$\delta I = \delta \int_{t_0}^{t_1} L(q_i, \dot{q}_i) dt = 0 \quad . \tag{2.4}$$

As I mentioned in Section 1, my restriction to simple systems is largely a matter of brevity and expository convenience, not of substance. Most of both the formalism below, and the philosophical morals of later Sections, apply much more widely. For example, in the last paragraph's deduction of eq. 2.2, the assumption of conservativity, (v), could be weakened so as to allow V to have explicit time-dependence and even some forms of velocity-dependence; (cf. e.g. Goldstein et al. (2002, p. 22); hence eq. 2.1's allowance of t as an argument of L.).

But beware: some points in what follows *are* restricted. The most important example concerns the deduction of Hamilton's Principle from Lagrange's equations eq. 2.2; (cf. the last paragraph but one). This deduction depends on the system being simple; (more specifically, on the constraints being holonomic, cf. Papastavridis (2002, pp. 960-973)). We shall see in Section 5 that this leaves us open questions about the modal involvements of Lagrangian and Hamiltonian mechanics for non-simple systems.

Finally, I note that the power of Lagrangian mechanics as a scheme for solving problems arises in large part from its equations being invariant under arbitrary transformations, with non-vanishing Jacobian, of the q_i (called *point transformations*). Thus we are free to use coordinates q_i to suit the problem at hand: the equations of motion will retain the form eq. 2.2.

2.2 Canonical equations and Hamiltonian mechanics

Under certain conditions, the variational problem eq. 2.1 has an equivalent form, *the canonical form*, for which the Euler-Lagrange equations are $2n$ first order equations, rather than n second order equations; as follows. Starting from eq. 2.1, we define new variables

$$p_i := L_{\dot{q}_i} \ , \tag{2.5}$$

called *(canonical) momenta*, since in mechanics examples they often coincide with momenta. Recalling that L is C^2 in all its arguments, we now assume that the Hessian with respect to the $\dot{q}$s does not vanish in the domain G considered, i.e. the determinant

$$\mid L_{\dot{q}_i \dot{q}_j} \mid \neq 0 \ ; \tag{2.6}$$

so that eq. 2.5 can be solved for the $\dot{q}_i$ as functions of $q_i, p_i, t : \dot{q}_i = \dot{q}_i(q_j, p_j, t)$. Then the equations

$$p_i = L_{\dot{q}_i} \quad \dot{q}_i = H_{p_i} \quad L(q_i, \dot{q}_i, t) + H(q_i, p_i, t) = \Sigma_i \dot{q}_i p_i \tag{2.7}$$

represent a *Legendre transformation* and its inverse; where in the third equation $\dot{q}_i$ are understood as functions of (q_j, p_j, t) according to the inversion of eq. 2.5. The function $H(q_i, p_i, t)$ is called the *Legendre* (or: *Hamiltonian*) *function* of the variational problem, and the qs and ps are called *canonically conjugate*. It follows that H is C^2 in all its arguments, $H_t = -L_t$, and $\mid L_{\dot{q}_i \dot{q}_j} \mid = \mid H_{p_i p_j} \mid^{-1}$. Besides, any function $H(q_i, p_i, t)$ that is C^2 in all its arguments, and has a non-vanishing Hessian with respect to the ps, $\mid H_{p_i p_j} \mid \neq 0$, is the Legendre function of a C^2 Lagrangian L that is given in terms of H by eq. 2.7.

Applying this Legendre transformation, the Euler-Lagrange equations eq. 2.2 go over to the *canonical system* of equations (also known as: *Hamilton's equations*)

$$\dot{q}_i = H_{p_i} \quad \dot{p}_i = -H_{q_i} \ (= L_{q_i}) \quad . \tag{2.8}$$

(A curve satisfying these equations is also called an extremal.)

Furthermore, these are the Euler-Lagrange equations of a variational problem equivalent to the original one, in which both qs and ps are varied independently, namely the problem

$$\delta \int \left(\Sigma_i \dot{q}_i p_i - H(q_i, p_i, t) \right) \, dt \quad = \quad 0 \ . \tag{2.9}$$

The reason for the equivalence, in brief, is:– The variation of $L = \Sigma_i \dot{q}_i p_i - H$ with respect to p_i gives $\delta L = \Sigma_i(\dot{q}_i - \frac{\partial H}{\partial p_i})\delta p_i$. Since the term in brackets vanishes by Hamilton's equations, an arbitrary variation of the p_i has no influence on the variation of L; so the Euler-Lagrange equations got by varying the qs and ps independently are eq. 2.8, i.e. the Legendre transform of the originals, eq. 2.2.[3]

Applying these ideas to the Lagrangian mechanics of a simple system, understood as in Section 2.1, we get *Hamiltonian mechanics*. Thus we now assume not only that the mechanical system is simple, but also that eq. 2.6 holds. And we think of the system's state-space as, not Q, but the $2n$-dimensional *phase space* Γ coordinatized by the ps and qs; (technically it is the cotangent bundle of Q—but as announced in Section 1, I eschew modern geometry!). The system's motion is given by the new variational principle, sometimes called the *modified Hamilton's Principle*, eq. 2.9; or more explicitly, by Hamilton's equations, eq. 2.8.

The Hamiltonian mechanics of a simple system is equivalent to Section 2.1's Lagrangian mechanics, together with eq. 2.6. But it has several advantages over Lagrangian mechanics, as regards both problem-solving and general theory; though I only mention two.[4]

(i): Its use of first-order ordinary differential equations. In particular, the initial value problem is straightforward, in that through a given point $(q_0, p_0) := (q_{0_1}, \ldots, q_{0_n}; p_{0_1}, \ldots, p_{0_n}) \in \Gamma$, there passes a unique solution of eq. 2.8, i.e. a unique extremal with $q_i(0) = q_{0_i}, p_i(0) = p_{0_i}$.

(ii): Its replacement of the group of point transformations on Q by what is in effect a larger group of transformations on Γ, the canonical transformations. There is a rich and multi-faceted theory of canonical transformations; (to which there are three main approaches—generating functions, symplectic geometry and integral invariants). But I will not need any details about this.

2.3 Hamilton-Jacobi theory

I shall discuss Hamilton-Jacobi theory in more detail than Lagrangian and Hamiltonian mechanics; both because it is less familiar to philosophers and because we need the detail in order to explore its modal involvements. In Section 2.3.1, I follow in Hamilton's (1833, 1834) footsteps, introducing the Hamilton-Jacobi equation via Hamilton's

[3] For more discussion of the Legendre transformation, cf. e.g.: Arnold (1989, Chap.s 3.14, 9.45.C), Courant and Hilbert (1953, Chap. IV.9.3; 1962, Chap. I.6), Lanczos (1986, Chap VI.1-4).) I stress that in the theory of the Legendre transformation, the assumption of a non-vanishing Hessian, eq. 2.6 (equivalently: $\mid H_{p_i p_j} \mid \neq 0$), is crucial; if it fails, we need a different theory (called *constrained dynamics*). Incidentally, it also implies that the fundamental integral cannot be parameter-independent; cf. e.g. Rund (1966, pp. 16, 141-144).

[4] Other advantages of the Hamiltonian approach, from a physical perspective, include: (a) it can be applied to systems to which the Lagrangian approach does not apply, i.e. in modern terms, whose phase space is not a cotangent bundle; (b) it connects analytical mechanics with other fields of physics, especially statistical mechanics and optics.

characteristic function (as do many mechanics textbooks); and then in Section 2.3.2 I discuss hypersurfaces, congruences and fields. Even so, these details will give only a limited view of a rich theory. In particular:

(1) I will ignore aspects to do with problem-solving (especially the use of separation of variables, leading on to action-angle variables and Liouville's theorem) since—though obviously crucial for physics, and so rightly emphasised in textbooks—they are not illuminating about modality.

(2) I will ignore the integration theory of the Hamilton-Jacobi equation, which involves the theory of generating functions and complete integrals. This deep (and beautifully geometric) theory *does* illuminate Hamilton-Jacobi theory's modal involvements; but space prevents me discussing it here.

(3) Both Sections 2.3.1 and 2.3.2 will emphasise the description of motion in the extended configuration space of Section 2.1, i.e. the region $G \subset \mathbb{R}^{n+1}$; while it is equally illuminating to consider Hamilton-Jacobi theory in phase space. But this emphasis on $G \subset \mathbb{R}^{n+1}$ will suffice for our purposes—to reveal some distinctive modal involvements.

2.3.1 The characteristic function and the Hamilton-Jacobi equation

We now assume that our region $G \subset \mathbb{R}^{n+1}$ is sufficiently small that between any two "event" points $E_1 = (q_{1i}, t_1), E_2 = (q_{2i}, t_2)$ there is a unique extremal curve C. To avoid double subscripts, I will in this Section sometimes suppress the i, writing $E_1 = (q_1, t_1), E_2 = (q_2, t_2)$ etc. Then the value of the fundamental integral along C is a function of the coordinates of the end-points; which we call the *characteristic function* and write as

$$S(q_1, t_1; q_2, t_2) = \int_{t_1}^{t_2} L \; dt = \int_{t_1}^{t_2} (\Sigma_i p_i \dot{q}_i - H) \; dt = \int \Sigma_i p_i dq_i - H dt \qquad (2.10)$$

where the integral is taken along the unique extremal C between the end-points, and we have used the Legendre transformation eq. 2.7.

Making arbitrary small displacements $(\delta q_1, \delta t_1), (\delta q_2, \delta t_2)$ at E_1, E_2 respectively, and using the fact that the integral is taken along an extremal, we get for the variation δS of S

$$\delta S := S(q_1 + \delta q_1, t_1 + \delta t_1; q_2 + \delta q_2, t_2 + \delta t_2) - S(q_1, t_1; q_2, t_2) =$$
$$\frac{\partial S}{\partial t_1}\delta t_1 + \frac{\partial S}{\partial t_2}\delta t_2 + \Sigma_i \; \frac{\partial S}{\partial q_{1i}}\delta q_{1i} + \Sigma_i \; \frac{\partial S}{\partial q_{2i}}\delta q_{2i} = [\Sigma_i \; p_i \delta q_i - H(q_j, p_j, t)\delta t]_{t_1}^{t_2} \quad . \qquad (2.11)$$

Since the displacements are independent, we can identify each of the coefficients on the two sides of the last equation in eq. 2.11, getting

$$\frac{\partial S}{\partial t_2} = -[H(q_i, p_i, t)]_{t=t_2} \quad , \quad \frac{\partial S}{\partial q_{2i}} = [p_i]_{t=t_2} \qquad (2.12)$$

$$\frac{\partial S}{\partial t_1} = [H(q_i, p_i, t)]_{t=t_1} \quad , \quad \frac{\partial S}{\partial q_{1i}} = -[p_i]_{t=t_1} \qquad (2.13)$$

in which the p_i refer to the extremal C at E_1 and E_2.

These equations are remarkable, since they enable us, if we know the function $S(q_1, t_1, q_2, t_2)$ to determine all the motions of the system that are possible with the given S—without solving any differential equations! For suppose we are given the initial conditions (q_1, p_1, t_1), i.e. the configuration and canonical momenta at time t_1, and also the function S. The n equations $\frac{\partial S}{\partial q_1} = -p_1$ in eq. 2.13 relate the $n+1$ quantities (q_2, t_2) to the given constants q_1, p_1, t_1. So in principle, we can solve these equations by a purely algebraic process, to get q_2 as a function of t_2 and the constants q_1, p_1, t_1; i.e. to get the system's motion. Finally, we can get the momentum p_2 from the n equations $p_2 = \frac{\partial S}{\partial q_2}$ in eq. 2.12. So indeed the problem is solved without performing integrations, i.e. just by differentiation and elimination: a very remarkable technique.[5]

In fact we can from now on ignore the initial time equations eq. 2.13 and study only the $n+1$ final time equations eq. 2.12. Roughly speaking, the reason is that eq. 2.12 contains enough information for us to analyse fully the system's motion.[6] So we will often write t rather than t_2 and q rather than q_2.

Substituting the second set of equations of eq. 2.12 in the first, and rewriting t_2, q_2 as t, q, yields

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0 \ . \tag{2.14}$$

This first order partial differential equation is the *Hamilton-Jacobi equation*; it is non-linear since the contribution of the kinetic energy T to the Hamiltonian will contain p^2 terms. Throughout this Subsection and the next, we will focus on this equation.

By fixing $E_1 = (q_1, t_1)$ and considering different values for S, we also see that S defines a family of hypersurfaces, which we can call 'spheres' with centre $E_1 = (q_1, t_1)$. Thus the sphere with radius R is given by the equation

$$S(q_1, t_1; q_2, t_2) = R \tag{2.15}$$

with (q_1, t_1) considered fixed. Every point $E_2 = (q_2, t_2)$ on this sphere is connected to the centre $E_1 = (q_1, t_1)$ by a unique extremal along which the fundamental integral has value R. This is amusingly reminiscent of Lewis' spheres of worlds (1973, Chapter 1.3; 1986, Chapter 1.3): and more than amusingly—we will see in Section 4 that the analogy is deeper.

[5] As Hamilton realized. He writes, in the impersonal style of the day, that 'Mr Hamilton's function S ... must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations [i.e. L]. Lagrange's function *states*, Mr Hamilton's function would *solve* the problem. The one serves to form the differential equations of motion, the other would give their integrals' (1834, p. 514).

[6] This insight is essentially due to Jacobi. For discussion, cf. Dugas (1988, p. 401), Lanczos (1986, pp. 225, 231-34, 254-57).

2.3.2 Hypersurfaces, congruences and fields

Of course, partial differential equations have many solutions: (the main contrast with ordinary differential equations being that typically, the solution contains an arbitrary function (or functions) rather than an arbitrary constant (or constants)). So Hamilton-Jacobi theory studies the whole space of solutions of the Hamilton-Jacobi equation. I need to report the main classical result of this study. (For details, a good reference is Rund (1966, Chap. 2), who cites various masters of the last two centuries, especially Carathéodory. I also give more details in (2003a).)

The result connects three diverse notions:—

(a): *Families of hypersurfaces* in our region G of $\mathbb{R}^{n+1}$

$$S(q_i, t) = \sigma \tag{2.16}$$

with $\sigma \in \mathbb{R}$ the parameter labelling the family; where we assume that S is a C^2 function in all $n+1$ arguments, and that the family foliates the region G *simply* in the sense that through each point of G there passes a unique hypersurface in the family.

(b): *Congruences of curves* that: (i) cross the hypersurfaces and fill G simply in the corresponding sense that through each point of G there passes a unique curve in the congruence; and (ii) may be parametrically represented by n equations giving q_i as C^2 functions of n parameters u_α and t

$$q_i = q_i(u_\alpha, t) \ \ , \ \ i = 1, \dots, n \ ; \tag{2.17}$$

where each set of n $u_\alpha = (u_1, \dots, u_n)$ labels a unique curve in the congruence. Thus there is a one-to-one correspondence $(q_i, t) \leftrightarrow (u_\alpha, t)$ in appropriate domains of the variables, with non-vanishing Jacobian

$$\mid \frac{\partial q_i}{\partial u_\alpha} \mid \ \neq \ 0 \ . \tag{2.18}$$

Such a congruence determines tangent vectors $(\dot{q}_i, 1)$ at each (q_i, t); and thereby also values of the Lagrangian $L(q_i(u_\alpha, t), \dot{q}_i(u_\alpha, t), t)$ and of the momentum

$$p_i = p_i(u_\alpha, t) = \frac{\partial L}{\partial \dot{q}_i} \ . \tag{2.19}$$

(c): *Fields*, defined to be a set of $2n$ C^2 functions q_i, p_i of (u_α, t) as in eqs 2.17 and 2.19, i.e. with the qs and ps related by $p_i = \frac{\partial L}{\partial \dot{q}_i}$. So a congruence determines a field, and a field determines (by a Legendre transformation, using eq. 2.6) a set of tangent vectors, and so a congruence.

Some jargon: (i) If all the curves of the congruence determined by a field are extremals, the field is called a *field of extremals.* (ii) We say a field (or its congruence) *belongs to* a family of hypersurfaces given by eq. 2.16 iff throughout the region G the

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ of the field obey the last n equations of eq. 2.12, i.e. iff we have

$$p_i = \frac{\partial}{\partial q_i} S(q_i, t) = \frac{\partial}{\partial q_i} S(q_i(u_\alpha, t), t) \ .^7 \tag{2.21}$$

(iii) Finally, we say that a field $q_i = q_i(u_\alpha, t), p_i = p_i(u_\alpha, t)$ is *canonical* if the q_i, p_i satisfy Hamilton's equations eq. 2.8: equivalently, if the curves of the congruence determined by the field are extremals.

So much for definitions; now the result. The following three conditions on a C^2 function $S : G \rightarrow \mathbb{R}$ are equivalent:

(1): S is a C^2 solution (throughout G) of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0 \ . \tag{2.22}$$

(2): The field belonging to the C^2 function $S : G \rightarrow \mathbb{R}$, i.e. the field defined at each point in G by $p_i = \frac{\partial S}{\partial q_i}$, is canonical. Equivalently: the curves of the congruence belonging to S (the congruence defined from the field by the Legendre transformation) are extremals.

(3): The value of the fundamental integral $\int L\ dt$ along the curve C of the congruence belonging to S, from any point P_1 on the surface $S(q_i, t) = \sigma_1$ to that point P_2 on the surface $S(q_i, t) = \sigma_2$ that lies on C, is the *same* for whatever point P_1 we choose; and the value is just $\sigma_2 - \sigma_1$. That is:

$$\int_{P_1}^{P_2} L\ dt = \sigma_2 - \sigma_1 \ .^8 \tag{2.23}$$

In the light of eq. 2.23, we call a family of hypersurfaces $S = \sigma$ satisfying any, and so all, of these three conditions *geodesically* (or: *geodetically*) *equidistant* (with respect to the Lagrangian L). So the concentric spheres centred on $E_1 = (q_1, t_1)$ introduced above (eq. 2.15) are an example of a geodesically equidistant family. (In fact they are a "fundamental" example, in that other families can be "built" from them in ways studied under the names of 'Green's functions', and 'Huygens' principle'.)

To sum up: Any solution S of the Hamilton-Jacobi equation that is smooth and local (specifically: C^2 and defined in a simply connected region G) has level surfaces

[7] One can show that a field belongs to a family of hypersurfaces iff for all indices $\alpha, \beta = 1, \ldots, n$, the Lagrange brackets of the parameters of the field, i.e.

$$[u_\alpha, u_\beta] := \Sigma_i \left(\frac{\partial q_i}{\partial u_\alpha} \frac{\partial p_i}{\partial u_\beta} - \frac{\partial q_i}{\partial u_\beta} \frac{\partial p_i}{\partial u_\alpha} \right) \tag{2.21}$$

vanish identically. Warning: the role of Lagrange brackets in this theory is sometimes omitted even in excellent accounts, e.g. Courant and Hilbert (1962, Chap. II.9.4).

[8] More precisely: (1) and (3) are exactly equivalent, (1) implies (2); and (2) implies that $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t)$ is a function of t only, and this function can be absorbed into H. For proofs, cf. e.g. Rund (1966, Chap. 2.2-3), Courant and Hilbert (1962, Chapter II.9.2-4).

$S =$ constant which are traversed by a congruence of extremals so as to make the level surfaces a geodesically equidistant family.

But so far it is an open question which n-dimensional surfaces M in G are level surfaces of some smooth, say C^2, solution S (in G). In fact it can be shown, subject to some mild conditions about non-vanishing determinants etc., that:

(1): any n-dimensional surface M is a level surface of a solution, and this solution is uniquely defined throughout G by its value on M (say $S = 0$ on M); and

(2): for any such surface M and any suitably smooth function $S : M \to \mathbb{R}$, there is a uniquely defined solution on all of G which restricts on M to the given S. (So (2) generalizes (1) by M not having to be a level surface.)

In the jargon: the initial value problem for the Hamilton-Jacobi equation has locally a solution, that is unique given suitably smooth prescribed values of S. But I shall not go into details about this. It suffices to state the intuitive idea for the case where M is a level surface: the solution is "grown" from the given surface by erecting a congruence of curves, transverse to the surface, and passing along them to mark off a given value of the fundamental integral $\int L \; dt$; by varying the value, one defines a geodesically equidistant family and so a solution S. (For details, cf. Rund (1966, Chap. 2.7-8), Benton (1977, Chap. 1) or my (2003a, Sections 5,6); or more heuristically, Courant and Hilbert (1962, Chap. II.9.2-5) and Born and Wolf (1999, Appendix I.2-4).)

Returning finally to mechanics: it is clear that each solution S of the Hamilton-Jacobi equation represents a kind of *ensemble*, i.e. a fictitious population of systems (maybe including the actual system). Thus each solution S represents an ensemble with the feature that at all times t, there is a strict configuration—momentum, i.e. $q - p$, correlation given by the gradient of S. That is, S prescribes for any given (q, t), a unique $(p, t) := (\frac{\partial S}{\partial q}, t)$.

So much by way of expounding Hamilton-Jacobi theory. We will return, after Section 3's introduction of modality, to discuss the structure of this set of ensembles (set of S-functions)—and so the modal involvements of Hamilton-Jacobi theory. Finally, I emphasise a point mentioned in Section 1: that this Section's restrictive assumptions about the systems to be considered (that the systems be simple, that eq. 2.6 hold, that any two points in G be connected by a unique extremal etc.) are largely a matter of brevity and expository convenience, not of substance. Much of the formalism above, and the philosophical morals below, apply more widely.

3 Grades of modal involvement

I turn to discussing the modal involvements of our three approaches to analytical mechanics: Lagrangian, Hamiltonian and Hamilton-Jacobi. In this Section, I begin with the obvious fact that postulating a space of possible states (a state-space: Q or Γ or G in Section 2's notation) brings in modality. This leads to a suggested distinction between three grades of modal involvement, which are all illustrated by analytical mechanics. In this Section, I just briefly mention some illustrations: the next two Sections discuss some more striking illustrations of the first and third grades.

At first sight, the philosophical import of postulating a state-space would seem to be at most some uncontroversial version of the idea that laws support counterfactuals. That is: whether or not one believes in a firm distinction between laws of nature and accidental generalizations, and whatever one's preferred account of counterfactuals, a theory that states 'All As are Bs' surely in some sense warrants counterfactuals like 'If any object were an A, it would be a B'. And so when analytical mechanics postulates state-space and then specifies e.g. laws of motion on it, it seems at first that this just corresponds to the passage from 'All actual systems of this kind (having such-and-such initial states—usually a "small" proper subset of state-space) evolve thus-and-so' to 'If any system of this kind were in any of its possible initial states, it would evolve thus-and-so'.[9]

But this first impression is deceptive. The structures with which state-space is equipped by analytical mechanics, and the constructions in which it is involved, make for more varied and nuanced involvements with modality than is suggested by just the idea that laws support counterfactuals. In the light of Section 2, I think it is natural to distinguish (in Quinean fashion!) three grades of modal involvement; so I shall write (Modality;1st) etc. Like Quine's three grades, the first is intuitively the mildest grade, and the third the strongest. But this order will not correspond to Section 2's (and the historical) order of the three approaches to analytical mechanics. In particular, the first and arguably most intuitive approach, Lagrangian mechanics, exhibits the *third* grade of modal involvement.[10]

The grades are defined in terms of the kind of actual matters of fact they allow to vary counterfactually. The first kind is, roughly, the state of the system. The second kind is the physical problem: which we can take as specified by the number of degrees of freedom, and the Lagrangian or Hamiltonian which encodes the forces on the system. The third kind is the laws of motion, as specified by e.g. Hamilton's Principle or Lagrange's or Hamilton's equations. Thus we have the following grades.[11]

(Modality;1st): The first and mildest grade keeps fixed the actual physical problem

[9]Here, and in all that follows, I of course set aside the (apparent!) fact that the actual world is quantum, not classical; so that I can talk about e.g. an actual system obeying Hamilton's Principle. Since my business throughout is the philosophy of classical mechanics, it is unnecessary to encumber my argument with antecedents like 'If the world were not quantum': I leave you to take them in your stride.

[10]Also, my three grades (like Quine's) are often combined: for example, a mechanical theory (or even a small part of one, such as a theorem) might involve the first and third grades. But I will not need to discuss such combinations.

[11]I don't claim that these three grades are the best way to classify the modal involvements of analytical mechanics. For example, it might be at least as fruitful to consider how the various kinds of constraint (holonomic, scleronomic etc. and their contraries) classify the notion of virtual displacement: this classification would cut right across the trichotomy that follows. But at least what follows has the merits of being: obvious, suggested by Section 2's review, and showing at least some of the variety of modal involvement that occurs.

and laws of motion. But it considers different initial or final conditions than the actual given ones. And so it also considers counterfactual histories of the system; (since under determinism, a different initial or final state implies a different history, i.e. trajectory in state-space).

So this grade includes the familiar idea above, that laws support counterfactuals. But analytical mechanics also provides more distinctive illustrations. Some arise from the postulation of a state-space, be it Q or Γ or G. Thus recall from Section 2.1's definition of a simple system: (a) the configuration space Q is to have independently variable coordinates q_i; and (b) to define ideal constraints, one needs to define a virtual displacement (as one that the system could undergo compatibly with the constraints and applied forces). But the most striking illustration of (Modality;1st) is Hamilton-Jacobi theory. As we saw in Section 2.3, Hamilton-Jacobi theory enables you to solve a problem, as it might be an actual one, by introducing an ensemble of systems; i.e. a set of possible systems, of which the actual system is just one member. Besides, the ensemble can be chosen in such a way that given S the problem is solved without performing integrations, i.e. just by differentiation and elimination. For more discussion, cf. Section 4.

(Modality;2nd): The second grade keeps fixed the laws of motion, but considers different problems than the actual one (and thereby different initial states). For example, it considers a counterfactual number of degrees of freedom, or a counterfactual potential function. Maybe no actual system is a simple system with 5,217 coordinates (nor even is well modelled as one); or with a potential given (in certain units) by the polynomial $13x^7 + 5x^3 + 42$. But analytical mechanics continually considers such counterfactual cases: in Section 2, we generalized from the outset about the number n of degrees of freedom, and about what the Lagrangian or Hamiltonian was (subject of course to conditions like eq. 2.6). Such generality of course pays off in countless general theorems.

(Modality;3rd): The third grade considers different laws of motion, even for a given problem. Again, this can happen even in Lagrangian mechanics; namely in its use of Hamilton's Principle eq. 2.1, and eq. 2.4 for simple systems. It also happens in Hamiltonian mechanics with its modified Hamilton's Principle, eq. 2.9. (And Section 2.3 showed that these variational principles are also involved in the Hamilton-Jacobi approach.) In all three approaches, the use of variational principles means—not that one explicitly states non-actual laws, much less calculates with them—but that one states the actual law as a condition that compares the actual history of the system with counterfactual histories of it that do not obey the law (in philosophers' jargon: are *contralegal*). That is, the counterfactual histories share the initial and final conditions, but do not obey the given deterministic laws of motion, with the given forces.[12] This

[12]Besides, one does not require that there *could* be forces which in conjunction with the actual laws and initial and final conditions, would yield the counterfactual history. But I agree that in general for each suitably smooth counterfactual history, there will be some recipe of (in general time-dependent) internal and external forces that would yield the history: (thanks to Michael Dickson for pointing this out). So I also agree that my distinction between (Modality;3rd) and (Modality;2nd), between

is at first sight surprising, even mysterious. How can it be possible to state the actual law by a comparison of the actual history with possible histories that do not obey it? I take up this question in Section 5.

To sum up, analytical mechanics gives many illustrations of all three grades: indeed, the subject is upto its ears in modality. But rather than multiplying examples, the remainder of this paper undertakes two projects suggested by my trichotomy. The first (Section 4) concerns Hamilton-Jacobi theory's use of (Modality;1st). There is no special philosophical *difficulty* here; rather the situation represents an invitation to philosophers to study a new sort of modal structure. The second project (Section 5) concerns variational principles, especially in Lagrangian and Hamiltonian mechanics. Here there is a philosophical difficulty: the variational principles threaten a plausible philosophical principle, and the threat needs to be answered. It can be answered, at least for simple systems; but doing so pays dividends.

4 On the set of ensembles

Since the S-function, representing an ensemble of systems whose q and p are correlated by $p = \frac{\partial S}{\partial q}$, stands at the centre of Hamilton-Jacobi theory, it is clear that the theory illustrates (Modality;1st) in spades. As discussed at the end of Section 2.3, the structure of the set of ensembles is essentially given by the structure of the set of suitably smooth (say C^2) real functions on a n-dimensional manifold M; (M needs to "lie across" the region G so as to be transverse to a congruence of extremals). For since there is a locally unique solution to the Hamilton-Jacobi initial value problem, each such function determines—as well as is determined by—a solution throughout G of the Hamilton-Jacobi equation. So one infers that the set of solutions (ensembles) is some kind of infinite-dimensional set.

This set has various kinds of structure, and a full discussion would have to take account of the aspects listed at the start of Section 2.3, that I am setting aside. In particular, Hamilton-Jacobi integration theory (especially the notions of complete integral, and Jacobi's theorem) picks out subsets of the solution space which are significant, both theoretically and for problem-solving. But even with just the results of Section 2.3, we can discern two kinds of structure—which bear on Lewis' account of modality, especially counterfactuals. These two kinds of structure arise from two different choices about what to take as the analogue, in Hamilton-Jacobi theory, of a Lewisian possible world.

varying the laws and varying the forces, is not as hard-and-fast as it first seems: one can in this sense avoid (Modality;3rd) by accepting (Modality;2nd). But this point will not affect my discussion.

4.1 Configurations as worlds

Let us think of an event (i.e. instantaneous configuration) $(q_i, t) \in G$ as like a possible world. Then Hamilton's characteristic function eq. 2.10, and the geodesic spheres it defines eq. 2.15, yield a neat analogy with Lewis' theory of counterfactuals.

For recall Lewis' proposed truth-conditions for a counterfactual 'If A were the case, then C would be the case', which I will write as $A \to C$ (1973, Chap. 1.3). Roughly speaking, he proposes that $A \to C$ is true at the actual world @ iff: the possible world (or worlds) most similar (for short: "closest") to @ that makes A true, also makes C true. But he wants to avoid the assumption that there is a set of A-worlds tied for first equal as regards similarity to @ (the Limit Assumption). He also allows the counterfactual to be vacuously true: namely iff no world in the union of nested spheres around @, $\cup$ $\$_{@}$, makes A true. That is, Lewis proposes that the counterfactual $A \to C$ is true at @ iff:–

1) no A-world belongs to any sphere S in the system $\$_{@}$ of spheres around @;
or
2) some sphere S in the system $\$_{@}$ contains at least one A-world, and $A \supset C$ is true at every world in S; (i.e. C is true at every A-world in S).

We can easily transplant this kind of truth-condition to geodesic spheres; i.e. taking points $(q_i, t) \in G$ as worlds and $\int L\ dt$ as the measure of distance (dissimilarity) between such worlds. However, the resulting conditionals arguably do not deserve the name 'counterfactual', since both the "base-world" (q_1, t_1) and the "closest A-world", say (q, t), that the evaluation of the conditional carries us to, could be actual configurations of the system.

For simplicity I will ignore the vacuous case, 1) above. This yields the following truth-condition, relative to a given configuration (q_1, t_1):

> a) A is true at a possible configuration (q, t), to which the given configuration (q_1, t_1) could evolve (i.e. would evolve for some p_1 at t_1) with $t > t_1$;
> and
> b) for every possible configuration (q', t') to which (q_1, t_1) could evolve with $t > t_1$, and such that $\int_{q_1,t_1}^{q',t'} L\ dt \leq \int_{q_1,t_1}^{q,t} L\ dt$:
> $A \supset C$ is true at (q', t'); (i.e. if A is true at (q', t'), so is C).

In the abstract, this truth-condition seems a mouthful. But in fact mechanics provides countless examples of such conditional propositions, though of course in a much less formal guise! A very simple example is given by a bead sliding on a wire that lies in a vertical plane; (to be a simple system in Section 2's sense, the bead must slide frictionlessly). We can take A to say that the bead is at the lowest point of the wire, and C to say that its potential energy is at a minumum. Then $A \to C$ can be expressed informally as 'Whenever the bead is next at the lowest point of the wire, its potential energy will then be at a minimum'. Similarly, with C saying instead that the kinetic energy is at a maximum; and so on.

Finally, the results in Section 2.3.2 (especially condition (3)) implies that this discussion of counterfactuals can be generalized so as to define similarity of worlds in terms of level surfaces of any solution S of the Hamilton-Jacobi equation. For example, we could take a n-dimensional surface M that is topologically a sphere surrounding some given point $(q_1, t_1) \in G$, define M to be a surface of constant S, say $S = 0$, and consider the (locally unique) solution of the Hamilton-Jacobi equation thereby defined outside M. That is, we could define the dissimilarity of our worlds (q, t) from the base-world (q_1, t_1), and so the truth-conditions of counterfactuals, in terms of the value of $S(q, t)$.

4.2 States as worlds

On the other hand, let us take as the analogue of a Lewisian world an instantaneous state in the sense of a $2n + 1$-tuple (q_i, p_i, t). This is perhaps a more natural choice than Section 4.1's instantaneous configurations (events), since it determines a history, i.e. a phase space trajectory, of the system, our "toy-universe". (Indeed, an even closer analogue to a Lewisian world would be such a trajectory, which is equivalent to a one-parameter family of tuples (q_i, p_i, t) parameterized by time; but I will not separately discuss this.)

As in Section 4.1, there are various constructions one could make with this concept of world. In particular, one could define conditionals $A \to C$ by using any solution S of the Hamilton-Jacobi equation to define dissimilarity. These conditionals would in general be counterfactual, since the "base-world" (q_1, p_1, t_1) will be on a different phase space trajectory than the (q_2, p_2, t_2) that evaluation of the conditional carries us to. But I shall not pursue this; (partly for the sake of variety—for I will anyway return to counterfactuals in Section 5.1). I shall instead describe how an S-function enables us to define various sets of possible worlds which are "preferred" relative to our choice of S; in fact, the last of these definitions is important for physics.

Here again, the S-function can be any solution of the Hamilton-Jacobi equation. Given such an S, every point $(q, t) \in G$ has an associated canonical momentum, viz. $p := \frac{\partial}{\partial q} S(q, t)$, and so an associated world in our sense, viz. $(q, p \equiv \frac{\partial S}{\partial q}, t)$. If we wish, we can also pick out subsets so that not every event (q, t) is included in a world "preferred" by our S. For example, we could do this by picking out a sub-manifold M of G, and defining the associated worlds $(q, p \equiv \frac{\partial S}{\partial q}, t)$ only for $(q, t) \in M$.

There are two obvious ways to specify such an M; both make M n-dimensional. First, we can define M as the level surface of S that passes through some given $(q, t) \in G$. This definition will connect M with Section 2.3.2's discussion of geodesically equidistant hypersurfaces. And thinking of (q, t) as the system's actual present configuration, M defines a preferred set of counterfactual events, i.e. instantaneous configurations (which are in general not simultaneous with (q, t)).

Secondly, we can fix t. (This will mean that our worlds are given in effect by just (q, p) not (q, p, t).) Each value of t defines M as $Q \times \{t\}$, i.e. the copy of the configuration space Q at time t; (cf. Section 2.1's assumption (ii), of scleronomic constraints). Now writing this copy simply as Q, we consider the gradient $\frac{\partial}{\partial q}S(q,t)$ as a function on Q. The preferred worlds are then given by all $(q, p(q) \equiv \frac{\partial}{\partial q}S(q,t))$ for $q \in Q$. So the worlds are given as before, except that the fixed value of t is now implicit in the definition of p.

In fact this second definition is crucially important for the mathematics and physics of Hamilton-Jacobi theory in phase space. For consider the graph of the function $q \mapsto p(q) := \frac{\partial}{\partial q}S(q,t)$ in the usual logician's sense of the set of ordered pairs of arguments and values; that is, consider the set of pairs $(q, p(q))$. It is a n-dimensional surface in the $2n$-dimensional phase space Γ. It turns out that it is an example of a special kind of surface, called Lagrangian submanifolds. I shall not define this notion: here it suffices to remark that it is crucial for understanding:

(i) the general (symplectic) structure of Hamiltonian mechanics and Hamilton-Jacobi theory;

(ii) physical phenomena like focussing and caustics; these arise when the assumption we made at the start of Section 2.3.1, that any two events $(q_1, t_1), (q_2, t_2) \in G$ are connected by a unique extremal, breaks down;

(iii) the relation of classical and quantum mechanics, since a Lagrangian submanifold is in effect the classical analogue of a pure quantum state (taken as an assignment of values to a complete set of observables).[13]

For us, the main point is, as before, about the structure of the modal involvement. Namely: the graph of $q \mapsto p(q)$ gives us a preferred set of worlds, i.e. alternative states in phase space. Besides, we can analyse the structure of the set of possible preferred sets by studying the set of all Lagrangian manifolds; (or instead, its quotient by the time-evolution under the Hamiltonian H).

So much by way of surveying the structure of Hamilton-Jacobi theory's set of ensembles—surveying the riches of (Modality;1st). I close this Section with a philosophical remark, which looks ahead to Section 5. There I will deny that merely possible facts (or states of affairs or other "truthmakers") could be what make true an actually true proposition; (for, I will claim, only actual facts could do that). But for all I have so far said about Hamilton-Jacobi theory, one might think that it involves precisely this idea—which Lewis once jokingly called "possibilia power" (1986a, p. 158). After all, what else might the use of an S-function i.e. an ensemble of possible systems (for example, to solve a problem) come to?

But in fact, there is no conflict. Agreed, Hamilton and Jacobi teach us to use an S-function to solve problems; and for a single problem there are many S-functions we can consider (which do not all differ just by the time-parameter). But there is no strange influence (whether causal or constitutive) of the S-function, or the ensemble it represents, on the actual system (or propositions about it). In particular, the evolution of a system (its trajectory in configuration space or phase space) *is* fixed by, for example, the initial conditions—$q, \dot{q}, t$ in Lagrangian mechanics and q, p, t in Hamiltonian

[13] For more details about Lagrangian submanifolds, cf. e.g. Arnold (1989, Chap.s 7,8), Littlejohn (1992, Sections 1-3).

mechanics—irrespective of which if any S-function we care to use.[14]

5 Truths without truthmakers?

As I said at the end of Section 3, it seems odd, even mysterious, to state an actual dynamical law by a comparison of the actual history with possible histories that do not obey it—yet variational principles do just this. I shall argue that in fact there is no problem here. But the topic repays scrutiny: it yields insights, both philosophical (Section 5.1) and technical (Section 5.2); and it raises some open questions.[15]

I shall concentrate on Lagrangian mechanics, and specifically on Hamilton's Principle. Recall that for the simple systems we are concerned with, this states that the motion between prescribed configurations at time t_0 and time t_1 makes stationary the action integral:

$$\delta I = \delta \int_{t_0}^{t_1} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)\; dt = 0 \; . \tag{5.1}$$

This involves (Modality;3rd): not because it formulates non-actual laws, but because of the kind of variation it uses to state the actual law.

I say 'shall concentrate' for two reasons, the first "positive" and the second "negative". (1): I shall also mention the modified Hamilton's Principle of Hamiltonian mechanics. Of course, for our simple systems with non-vanishing Hessian, eq. 2.6, these are equivalent; and so the discussion applies equally to Lagrangian and Hamiltonian mechanics. But there will also be a distinction between the Lagrangian and Hamiltonian approaches which is worth noting.

(2): There are, even within Lagrangian mechanics, several other variational principles (e.g. principles of least action, least constraint and least curvature) which I will *not* discuss. My reason for ignoring them is not just lack of space. Also, (i): Broadly speaking, Hamilton's Principle is more important than them, since in almost all developments of Lagrangian mechanics it acts as the main postulate, the other variational principles being deduced from it under various conditions. (ii): Broadly speaking, the philosophical discussion in Section 5.1 carries over to these other principles. Or so I contend, without having the space to prove it![16]

[14]Incidentally, the situation is different in quantum theory. There, S has a close mathematical cousin (also written S) whose values do influence the motion of the system. But again, this does not represent any weird "possibilia power". For this influence is regarded as a strong, indeed the strongest, reason to take the quantum S-function as part of the actual physical state of the individual system; i.e. not as in classical mechanics, as "just" a description of an ensemble.

[15]The topic seems wholly ignored in the philosophical literature about variational principles. But thanks to the rise of modal metaphysics in analytical philosophy—over which Lewis presided so magnificently—the topic is nowadays plainly visible. Incidentally: the literature has instead focussed almost entirely on the way (i) specifying final conditions and (ii) referring to *least* action, suggests teleology. Indeed, this focus has been dominant ever since Maupertuis (cf. e.g. Yourgrau and Mandelstam (1979, Chap. 14), Dugas (1988)). But I set it aside entirely.

[16]I admit that there are many philosophically important differences between the various principles,

5.1 The threat and the answer

In Section 5.1.1, I shall state the threat that a variational principle like Hamilton's Principle poses; this Section will be wholly philosophical, involving no technicalities of physics. Then in Section 5.1.2, I shall argue that fortunately, the threat can be answered: the answer will involve technicalities.

5.1.1 The threat

The threat is that a formulation of an actual law (in this case, the law of motion of a classical mechanical system)[17] that mentions other possible evolutions of the system apparently violates the principle that *any* actually true proposition (not only: any law of nature) should be made true by actual facts, i.e. goings-on in the actual world. (So the threat does not depend on the evolutions mentioned by the law being contralegal: what matters is that they are not actual.)

I admit that this principle, often called the *truthmaker principle*, is both vague and disputable. Indeed, this is so, even apart from disputes about the nature of modality (in particular, the status of possible worlds). For the terms 'true', 'proposition' and 'fact' are vague and disputable. In particular, philosophers disagree about whether (*contra* Frege) we need a notion of fact distinct from (especially: more substantial or "thicker" than) the notion of a true proposition; and even those who accept such facts disagree about the truthmaker principle thus understood, i.e. about whether every true proposition is made true by such a fact.

But I think the principle *sounds* right when one first hears it: (witness the fact that it has been articulated by philosophers working in different philosophical traditions—cf. Mulligan et al. (1984)). I also find that non-philosophers endorse the principle. In particular, it surely underlies the point often stressed in physics that a system's history, for given initial conditions, cannot depend on what ensemble it is considered to be a member of: (cf. the discussion at the end of Section 4.2 of the quantum-classical contrast concerning whether S is physically real, as shown by its influence on the system's trajectory).

So I endorse various philosophers' efforts to articulate a precise and true form of the principle; where precision and truth will presumably require the notion of fact to be not *too* "thick". Of course, controversy continues about how to do this. Here are two examples. (1): Assuming that each of a certain collection of propositions $A, B, \ldots$ is made true by such a fact, should we say that the same goes for their Boolean compounds such as $\neg A$, $A \wedge B$ and $A \vee B$; which would amount to admitting negative,

including about their modal involvements. Consider for example the different definitions of variation used in Hamilton's Principle, Gauss' principle of least constraint and Euler-Lagrange-Jacobi's principle of least action (cf. e.g. Lanczos 1986, Chap.s IV.8 and V.4-8). But these differences do not affect the philosophical discussion of (Modality;3rd) that follows.

[17] Recall from footnote 9 that my saying 'actual' here and elsewhere is not meant to deny the quantum!

conjunctive and disjunctive facts? (2): Are true propositions made true not by, or not just by, a fact—but by an object (i.e. individual, particular) or objects? Most authors would say that this is at most true of some true propositions, not all. For there is a mis-match between the Boolean algebra of propositions, and objects—which do not carry a corresponding Boolean algebra. Thus suppose we say that A and B are made true by a and b respectively. If we also believe that any such objects a, b have a mereological fusion $a + b$, we might say $A \wedge B$ is made true by $a + b$; but there seems to be no object to make true a disjunction such as $A \vee \neg B$.

Of course, I do not have the space to enter into disputes like those mentioned: (for recent discussions cf. e.g. Armstrong (1997, Chap. 8), Mellor (2003)). But fortunately, I do not need to. I can leave the truthmaker principle vague, mainly because I will need only the fact that various authors advocate some suitably weak form of it. In fact Lewis himself is one such author: (so that what follows has a further *ad hominem* interest). The reason I will need only this fact is that the threat posed to the truthmaker principle by variational principles is different from the threats and putative counterexamples mentioned above; and so far as I know, different from those discussed in the literature—with one exception.

In fact, the literature discusses two broad kinds of threat:—

(A): Threats that, like the problems about Boolean compounds I mentioned, fall squarely in the tradition of modern analytic metaphysics. These threats are often broadly logical, and largely independent of the subject-matter of the propositions concerned; e.g. the problem of what, if anything, are the truthmakers of universal generalizations.

(B): Threats based on rejecting the initial idea of a substantive ("thick") notion of a truthmaking fact. These may arise either from holding a general "minimalism" about truth, or from holding that some specific subject-matter, such as ethics or mathematics, has true propositions without any corresponding "thick" facts. (Of course, a position that went further, and denied that the subject-matter has true propositions, would be more radical as an "anti-realism" or scepticism; but it would not pose a threat to the truthmaker principle.)

As we shall see, variational principles will differ from both (A) and (B). And this difference will mean that I can make do without a precise truthmaker principle—at least in a paper that is a first foray into analytical mechanics' modal involvements!

I said there was one discussion in the literature of a threat to the truthmaker principle similar to that posed by variational principles. In fact it is by Lewis himself! He briefly discusses how counterfactuals, analysed in terms of possible worlds as he proposes, pose such a threat—to which he then replies. In fact, we will see in Section 5.1.2 that a variational principle such as Hamilton's Principle can be regarded as a long conjunction of Lewisian counterfactuals—and this makes the threats to the truthmaker principle posed by variational principles and by counterfactuals closely analogous.

So in the rest of this Section, I propose to report Lewis' discussion of the threat by counterfactuals, and his reply. But it will help set the stage for that discussion, to first state two precise forms of the truthmaker principle, as formulated by Lewis. The first illustrates how the principle can be formulated so weakly as not be threatened (by variational principles, counterfactuals or indeed any proposition). The second formulation is stronger, and *is* threatened by variational principles and counterfactuals: (it will also clarify how the threat posed by these is different from those in the literature).[18]

(1): *Truth supervenes on being*: Bigelow (1988, 132-3, 158-9) makes the idea of truthmakers more precise along the following lines: that how things are determines which propositions are true—which he expresses with the slogan 'truth supervenes on being'. Lewis incorporates this in his framework of possible worlds, in such a way that it becomes *a priori*. Accordingly, Lewisian counterfactuals and variational principles—as well as the other cases considered in the literature—pose no threat to it. (As I said, some forms of the truthmaker principle are weak!) Thus Lewis uses his ideas (1988) that:

(a): a proposition is a set of worlds, viz. the worlds where the proposition is true;

(b): a subject-matter is a partition of the set of all worlds, with any two worlds in a cell of the partition matching as regards the subject-matter; and

(c): a proposition is wholly about a subject-matter if it (i.e. its set of worlds) is a union of the cells of the subject-matter's partition.

Lewis then construes Bigelow's 'being' as the largest subject-matter, i.e. the maximal partition. So truth's supervening on being becomes an *a priori* truth. Every proposition is a union of the cells of the maximal partition; and which of those cells contains the actual world trivially determines which propositions are actually true (1992, Section 6; 1994, Section 1; 2003, sections 1-2).[19]

(2): *Counterparts as truthmakers*: Lewis has recently proposed that some propositions have truthmakers that are objects—in his jargon: individuals; (2003, Section 3 et seq., overcoming previous scepticism in e.g. his (1992, Section 5)). More precisely, he defines a possible individual a to be a truthmaker for a proposition A iff every world where a exists is a world where A is true.[20] Here 'every world where a exists' must be understood, in the light of Lewis' denial of strict identity across possible worlds, in terms of his counterpart theory. Lewis goes on to show that counterpart theory yields a truthmaker in his precise sense for many propositions, in particular for predications. Besides, a postscript (co-authored with G. Rosen) argues that the proposal

[18] Both formulations also illustrate how in some of his work, Lewis incorporates current positions and insights into his own philosophical system—and in doing so, often makes them more vivid and precise. Indeed, this ability to incorporate ideas that are "in the air" into his system, and to make them vivid and precise, is one of his great strengths as a philosopher. His writings provide many examples: e.g. his treatment of indexicality as attitudes *de se* (1979), and his treatment of chance (1980, 1994).

[19] Cf. Lewis (2003) for discussion of further aspects. In particular: (a) this notion of aboutness does not suit necessary or impossible propositions—it is intensional but not hyperintensional; (b) cells of the maximal partition might not be singleton sets of worlds, since there might be indiscernible worlds and we might ban non-qualitative propositions.

[20] Others give verbally the same definition, though in their own metaphysical frameworks: e.g. Armstrong (1997, p.115), Bigelow (1988, p.122,126).

can be extended to other propositions, even negative existentials like 'There are no unicorns'. (Lewis also compares his proposal with proposals for facts as truthmakers made by Armstrong and Mellor.) Again, I cannot go into details. For this paper's purposes, it suffices to note the contrast with (1). That is to say: for objects as truthmakers, the threat that concerns us arises again: variational principles and Lewisian counterfactuals, with their transworld comparisons, apparently do not have this sort of truthmaker.

These items (1) and (2) set the stage for Lewis' discussion of how counterfactuals threaten the idea of truthmakers. For that discussion falls between (1) and (2), in the sense that there *is* a threat (like (2) and unlike (1)), but one that (he maintains) can be answered (unlike (2)). I turn to reporting that discussion.

Lewis of course recognizes that his proposed truth-conditions for counterfactuals in terms of similarity between possible worlds threaten the the idea of truthmakers; (although his discussion does not use the term 'truthmaker', the connection will be clear). After all, Lewis proposes for an actually true counterfactual, truth-conditions in terms of other worlds! Thus recall that, roughly speaking, $A \rightarrow C$ is actually true if some $(A\&C)$-world is closer (i.e. more similar) to the actual world than any $(A\&\neg C)$-world is. So he writes:

> Here is our world, which has a certain qualitative character. (In as broad a sense of 'qualitative' as may be required—include irreducible causal relations, laws, chances, and whatnot if you believe in them.)[21] There are all the various A-worlds, with their various characters. Some of them are closer to our world than others. If some $(A\&C)$-world is closer to our world than any $(A\&\neg C)$-world is, that's what makes the counterfactual true at our world. Now ... it's the character of our world that makes some A-worlds be closer to it than others. So, after all, it's the character of our world that makes the counterfactual true—in which case why bring the other worlds into the story at all?
>
> To which I reply that it is indeed the character of our world that makes the counterfactual true. But it is only by bringing the other worlds into the story that we can say in any concise way what character it takes to make what counterfactuals true. The other worlds provide a frame of reference whereby we can characterize our world. By placing our world within this frame, we can say just as much about its character as is relevant to the truth of a counterfactual (1986, p. 22).

This passage makes two main claims, one in each paragraph:
(Actual): although Lewis' truth-conditions mention other worlds, it is the character of the actual world that makes the counterfactual actually true;
(Concise): mentioning other worlds is the only concise way to state what in the

[21] [By JNB]: Thus this threat is independent of Lewis' neo-Humean analyses of causation, law and chance; as also of his more speculative additional doctrine, Humean supervenience.

actual world's character is relevant to the counterfactual's truth.
Of these two claims, (Actual) is the more important for us—it summarizes both the threat to truthmakers and Lewis' reply. But I shall also briefly discuss (Concise).

We can better understand (Actual) by recalling Lewis' (1986, p. 62) distinction between (a) relations that supervene on the intrinsic properties of their *relata*, which Lewis calls 'internal', and (b) relations that do not thus supervene, which I will call 'external'. (I will not need Lewis' doctrines about which properties are intrinsic, and can make do with some intuitive if disputable examples of intrinsic properties. Nor will I need Lewis' allowance that a relation might supervene on the composite of the *relata* taken together: his main example of this category being spatiotemporal relations.) Thus relations of similarity or difference in intrinsic respects are internal; so that if an object's mass is an intrinsic property of it, the relation 'is more massive than' is internal. An example of an external relation would be 'has the same owner as': a and a' could match in all their intrinsic properties and yet a person might own a and some other object b, but not a'; so that 'has the same owner as' holds of $\langle a, b\rangle$ but not $\langle a', b\rangle$.

Lewis applies this distinction not just to relations between objects in a single world, but to objects in different worlds. Thus a sentence such as 'He is slimmer than he would have been without the diet' reports an internal relation between objects in different worlds (a man and one of his counterparts). A sentence reporting a transworld external relation seems harder to construct; I suppose because our thought and language has little use for them. But Lewis' own counterpart theory gives examples. For counterparthood, though it sometimes emphasises intrinsic similarity, often emphasises extrinsic similarity, especially as regards the object's origins (Lewis 1986, p.88). Thus two objects a and a' (in the same world, or in two different worlds) might be duplicates, while only a is a counterpart of some object b in another world—say an actual object b.[22]

Furthermore, Lewis also takes worlds to be objects (in short: the mereological fusion of their parts) and so allows them as *relata*; and therefore applies this distinction to relations between worlds. And he says explicitly (1986, p. 62,177) that since the relation of closeness between possible worlds used in his analysis of counterfactuals is a relation of similarity, it is internal. Hence his claim in (Actual) that the truth-values of counterfactuals are determined by the character of our world. For the character of our world determines which worlds are similar to it. (Though it is a vague and controversial matter which respects of similarity are relevant to the truth-conditions of counterfactuals (Lewis 1979a), any resolution of those issues will render the overall

[22]Here is an example with a, b both actual, and indeed identical: 'an atom-for-atom replica of Humphrey (as he actually was at, say, noon 4 July 1968), who had been born of different parents than the actual Humphrey (in say Latvia, never setting foot in the USA etc.), would not have been [folk-language, according to Lewis, for: would not have been a counterpart of] Humphrey'. Here, $a = b =$ the actual Humphrey, and $a' =$ the replica. Another example, with a and b in different worlds: 'Each of two people might be atom-for-atom replicas of Humphrey as he actually was at noon, 4 July 1968; but only the person whose origin matched (at least: sufficiently closely) that of the actual Humphrey, would be Humphrey'. Here, a, a' are the replicas, b is the actual Humphrey.

similarity relation internal.)

The connection of Lewis' (Actual) with the idea of truthmakers is clear. Though 'truthmaker' is a philosophical term-of-art awaiting strict definition, the way that Lewis' truth-conditions mention other worlds makes one think that—whether one takes truthmakers to be facts or objects—a counterfactual has truthmakers "scattered across the worlds": apparently violating the principle that actual truths have actual truthmakers. To which threat, Lewis replies: 'No worries: *which* facts, objects etc. in other worlds get mentioned in the truth-conditions is wholly determined by the character of the actual world—and that is sufficient for satisfying the idea that actual truths have actual truthmakers.' And Lewis might well go on: 'If you want, you can call the facts, objects etc. in the other worlds that get mentioned in the truth-conditions 'truth-makers'. But the point remains that their being scattered across the worlds is innocuous. The fact that the character of the actual world determines them (and thereby the truth-value of the counterfactual) is sufficient to satisfy the spirit, if not the letter, of the truthmaker principle that 'actual truths have actual truthmakers'.'

So much by way or explicating Lewis' claim (Actual), i.e. his reply to the threat posed by counterfactuals. I think that within the framework of Lewis' metaphysics, it faces no objections. But of course, my main purpose is not to report or defend Lewis. Rather, the point of my endorsement of (Actual) is that, as we shall see in Section 5.1.2, variational principles can be similarly reconciled with the spirit, if not the letter, of the principle 'actual truths have actual truthmakers'—just because we can read such principles as long conjunctions of Lewisian counterfactuals.

I turn to briefly discuss Lewis' claim (Concise): that mentioning other worlds is the only concise way to state what in the actual world's character is relevant to a counterfactual's truth. I would have liked Lewis to say more about this, especially in view of (a) the importance in his philosophical system of the threatening counterfactuals, and (b) the importance in his late work of the threatened idea of truthmakers. One naturally asks: *why* is mention of other worlds the only concise way to describe the relevant part of the the actual world's character? But so far as I know, this passage is all Lewis says on the topic.[23]

In any case, we will see in Sections 5.1.2 and 5.2 a contrast between Lewis' philosophical system and our concern: mechanics and the calculus of variations. In our simpler and more technical framework, one *can* say more about why mentioning possi-

[23]Here is an analogy that I use in explaining Lewis' reply. To describe Buenos Aires concisely to a friend who is unfamiliar with it, you might forego listing its intrinsic properties, and instead use a comparison with something familiar to your friend; thinking of the harbour and summer in January, you might say, for example, 'It's like a Spanish-speaking Sydney'. I confess I believed Lewis invented this (Australophile!) analogy; but I cannot find it in his published work—maybe he just told me it. Alan Hajek tells me that he, Hajek, invented this same analogy for the same purpose, except that Hajek's example was that Perth is the Australian city most similar to San Diego. Hajek also reports that Lewis endorsed the analogy; (so maybe Lewis got the analogy from Hajek, and passed it to me). But more important than who invented it: according to Hajek, Lewis insisted that it was a fact *about San Diego* that the Australian city most similar to it is Perth—cf. Lewis' claim (Actual) above.

ble histories is useful. Indeed, one can also say more about the analogue of Lewis' claim (Actual): about the character of the actual world (i.e. the this-worldly truthmakers) that makes a variational principle true.

Finally, before turning to this—i.e. answering the threat that variational principles pose to the idea of truthmakers—I should set aside two ways in which a philosopher might dismiss this threat, so that there would be no "case to answer".

(1): The first dismissal echoes (B) above: i.e. the rejection of the idea of substantive, truthmaking, facts, either generally or for a particular subject matter. Thus a philosopher might think variational principles are not "in the market" for truthmakers, for a variety of reasons: ranging from

(i): "minimalism" about truth (either generally or for variational principles); through

(ii): some kind of instrumentalism (i.e. denial that variational principles are true, and even that they purport to be true, yet acceptance of their usefulness); to

(iii) some kind of eliminativist "anti-realism" (i.e. denial of usefulness as well as truth).

Needless to say, I will not try to reply at length to all these positions! Suffice it to make one reply to each of (i)-(iii).

(i'): I am unconvinced of minimalism in general, and see no special motivation for holding it for variational principles.

(ii'): I have two replies to instrumentalism: the first general and firm, the second special and yielding. (a): First, I see no motivation for instrumentalism about all variational principles, except as an instance of a general instrumentalism about all theoretical claims: a general instrumentalism which I have no truck with, and which is anyway nowadays unpopular, displaced in large part by van Fraassen's constructive empiricism.[24] (b): However, I will concede at the end of Section 5.1.2 that instrumentalism about variational principles *is* tenable for non-simple mechanical systems.

(iii'): I wholly reject the idea that variational principles are not useful: I shall develop this theme in Section 5.2.

(2): Finally, a philosopher might say that the variations mentioned in variational principles have nothing to do with possibilities of the sort discussed in the literature about truthmakers (or in modal metaphysics generally). Again, I have no truck with this. As I said in Section 3, mechanics is up to its ears in modality, of *some* kind or kinds. And no sign is ever given that modal locutions like 'could', with which the notions and mathematics of virtual displacements, variations etc. are introduced, are to be understood differently from elsewhere in science or everyday life. So why should they be?

[24]But there is a rich subject here. Stöltzner (2003) is a fascinating study of the logical empiricists' treatment—and mistreatment!—of variational principles in mechanics.

5.1.2 The answer

To answer the threat, I shall adapt a two-stage strategy which is straightforward, and traditional in philosophy. According to this strategy, when one is confronted with apparently problematic entities, one has to consider two tasks. The first takes one of two forms; but it is compulsory, in the sense that one must undertake either the first form or the second. On the other hand, one faces the second task only if one undertakes the second form of the first task; and even then, the second task is optional, not compulsory—though succeeding in it would be a significant merit of one's philosophical position.

Thus the first task is as follows. One must either show that the problem is an illusion—the entities are not really problematic, after all: they can be vindicated. I shall call this task (Vindicate). Or, accepting that the entities really are problematic, one must show how to do without them: one must eliminate them. I call this (Eliminate). If one undertakes (Eliminate), one should, if possible, undertake the second task: to show how it is useful or convenient to speak as if the entities exist (so as to explain, perhaps even justify, "how the folk speak"). I shall call this task (Useful).

For variational principles in mechanics, the entities at issue—the possible histories of the system—are not themselves problematic; (I of course set aside the debate about the nature of possibilities, i.e. Lewis' modal realism vs. various ersatzisms and fictionalisms). Rather, what seems problematic is the role these entities are assigned: viz. being, when taken together with the actual history, truthmakers of actual truths (indeed the actual laws). But clearly, we can adapt the two-stage strategy to apparently problematic roles rather than entities. Thus we envisage arguing that:

(Vindicate): This role of possible (indeed, contralegal) histories can be vindicated—it is not problematic, after all; or instead that

(Eliminate): This role of possible histories can be eliminated—the laws can be formulated without invoking it; in which case we should also try to argue that

(Useful): The variational formulation of the laws is nevertheless useful, or even advantageous compared with formulations that do not mention possible histories. (Cf. (iii') at the end of Section 5.1.1.)

In this Section, I will discuss (Vindicate) and (Eliminate), in order. But I postpone (Useful) to Section 5.2.

5.1.2.A Vindicating possible histories One can argue for (Vindicate) on strict analogy with Lewis' own answer to the threat that counterfactuals apparently pose to the truthmaker principle: i.e. Lewis' claim (Actual), reported in Section 5.1.1. For there is a striking analogy between what a variational principle says and Lewis' proposed truth-conditions for counterfactuals $A \to C$. Roughly speaking, a history of the mechanical system corresponds to a Lewisian possible world (cf. Section 4.2's suggestion); and similarity of histories is a matter of first fixing the configurations at the end-points, and secondly closeness of the values of q and $\dot{q}$. Given this correspondence, a variational principle turns out to be an infinite conjunction of Lewisian counterfactuals.

To spell this out, let us first recall Lewis' proposal (cf. Section 4.1). To avoid the Limit Assumption (that there is a set of A-worlds all tied for first equal as regards similarity to the actual world @), and setting aside the case of vacuous truth, Lewis proposes that a counterfactual $A \rightarrow C$ is true at @ iff: some sphere S in the system $\$_{@}$ contains at least one A-world, and $A \supset C$ is true at every world in S; (i.e. C is true at every A-world in S).

Turning to variational principles, I shall take Hamilton's Principle; though essentially the same analogy could be drawn with any number of principles. The principle says that the actual history is a stationary point of the action. Here, 'stationary point' rather than 'minimum' allows that:

(i): the actual history could be a maximum of the action $\int L\ dt$, not a minimum;

(ii): the minimum or maximum need only be local;

(iii): the actual history could be a point of inflexion (associated with the vanishing of second derivative of the action, not just the first).

But I now need to spell this out in more detail than I did in Section 2.1 (paragraph 3). Roughly speaking, Hamilton's Principle says that:

> For any one-parameter family, parametrized by α say, of kinematically possible histories of the mechanical system, that may deviate from the actual history between t_0 and t_1, but must match the actual history as regards the configurations q_0, q_1 at times t_0, t_1:
> the action as a function of α, $I(\alpha) = \int_{t_0}^{t_1} L\ dt$ with the integral taken along the history labelled by parameter-value α, has zero gradient at the value of α corresponding to the actual history.

To be precise, we consider any one-parameter family, parametrized by α, of curves from q_0, t_0 to q_1, t_1; so we write $q_i = q_i(t, \alpha)$. We also suppose that the curve (let us call it @!) that makes stationary the integral $I(\alpha) = \int_{t_0}^{t_1} L\ dt$ (taken along the curve labelled by α) has parameter-value $\alpha = 0$; which we write as $q_i(t) := q_i(t, 0)$. So the family of curves is given by $q_i(t, \alpha) = q_i(t) + \alpha\eta_i(t)$. Then for the action integral to be stationary at $\alpha = 0$ means that in the Taylor expansion about $\alpha = 0$, i.e.

$$I(\alpha) \equiv \int_{t_0}^{t_1} L(q_i(t) + \alpha\eta_i(t), \dot{q}_i(t) + \alpha\dot{\eta}_i(t), t)\ dt = I(0) + \alpha \left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} + \mathrm{O}(\alpha^2)\ , \quad (5.2)$$

we have:

$$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0\ . \quad (5.3)$$

That is, by the elementary definition of the derivative as a limit of a quotient:

$$\forall \varepsilon > 0\ \ \exists \delta > 0\ \ \forall\, 0 < \alpha \leq \delta\ , \quad \frac{I(\alpha) - I(0)}{\alpha} < \varepsilon\ . \quad (5.4)$$

Now I can state the analogy between Lewis' truth-condition and Hamilton's Principle (for our given one-parameter family of curves). We take the parameter α to define a system of nested spheres (sets of curves) $\$_@$, centred on the curve @ which itself has parameter $\alpha = 0$: the spheres are defined by inequalities $\alpha \leq r \in \mathbb{R}$, so that a smaller value of α represents greater similarity to @.

With this understanding, we can read Hamilton's Principle, in the form eq. 5.4, as a battery of Lewisian counterfactuals; indeed as a infinitely long conjunction of them (there are at least continuously many conjuncts). This plethora of counterfactuals arises from two sources:

(i): the continuously large range of values of ε; and

(ii): for given ε, the at-least-continuously large range of antecedents A that are false at the actual curve @ but true at some curve in the family with a parameter-value $0 < \alpha \leq \delta$. (Here we think of δ as determined using eq. 5.4 from the given ε.)

But on the other hand, the counterfactuals are similar as regards their consequents C: they are all given by the inequality in eq. 5.4.[25]

That is to say: we can read eq. 5.4 as saying that for each value of ε, and any A that is false at the actual curve @ but true at some curve in the family with a parameter-value $0 < \alpha \leq \delta \equiv \delta(\varepsilon)$: that curve labelled by α makes the quotient $\frac{I(\alpha)-I(0)}{\alpha}$ less than ε, as do all curves with an α in the same range.

In other words, now using the Lewisian spheres defined by inequalities $\alpha \leq r \in \mathbb{R}$, so as to talk about counterfactuals:—

> for all ε, and all such A (so that ε fixes a range of α, viz. $0 < \alpha \leq \delta \equiv \delta(\varepsilon)$, and A is true at such an α, but false at $\alpha = 0$): the Lewisian counterfactual
>
> $$A \to [\text{the quotient } \frac{I(\alpha)-I(0)}{\alpha} \text{ is less than } \varepsilon]$$
>
> is true at @.

(Incidentally: we need A to be false at @, i.e. the counterfactual $A \to C$ to be contrary to fact, so as to force $0 < \alpha$, so that the quotient is well-defined.) To sum up: Hamilton's Principle is an infinitely long conjunction of Lewisian counterfactuals.

Returning (at last!) to philosophy: this discussion makes clear the analogy with Lewis' claim (Actual), reported in Section 5.1.1, and thereby our argument for (Vindicate). For all the ingredients in the above transworld comparisons involve only internal relations, either between objects in possible worlds (i.e. components of the system in possible histories, possible curves) or between worlds (histories, curves) themselves. Thus similarity between worlds is a matter of: first, shared initial and final configurations; and second, closeness of values of q and so $\dot{q}$, where closeness is encoded by the parameter α. So the character of the actual world @ (i.e. the course of values of the functions $q, \dot{q}$ along the actual history) determines, for any antecedent proposition A, at which if any of the histories in the system of nested spheres (sets of histories) $\$_@$, A

[25] Besides, there is another source of yet more counterfactuals, viz. the various choices for our one-parameter family of curves: which I have for the sake of clarity suppressed, by fixing on a single family.

is true. The case is even clearer as regards the consequent C, which as noted is similar for all the counterfactuals involved. The inequality $\frac{I(\alpha)-I(0)}{\alpha} < \varepsilon$ compares the values of I on different histories. This is an internal relation between worlds, i.e. histories, since whether or not this relation holds is determined by the values $I(\alpha)$ and $I(0)$: which are part of the intrinsic natures of the worlds. To sum up, by echoing Lewis' claim (Actual): you can say if you like that the truthmakers of Hamilton's Principle are "scattered across the worlds"; but the spirit, if not the letter, of the truthmaker principle is satisfied.[26]

So much for arguing for (Vindicate). But notwithstanding this success, some will still worry! Some philosophers are very wary about modality. And even if one relishes modality, it may seem risky to rely on satisfying just the spirit, and not the letter, of the truthmaker principle: especially while a precise and correct formulation of the principle remains controversial—for the formulation eventually agreed on might have a more demanding spirit than one now guesses! So I ought also to consider how one might argue for the other option—(Eliminate).

5.1.2.B Eliminating possible histories Focussing again on Hamilton's Principle, I shall first consider whether there is a statement (or statements) equivalent to Hamilton's Principle, that does not mention possible histories of the system. Here 'equivalence' means logical equivalence; or perhaps mathematical equivalence, understood in the usual way as logical equivalence, once given the assumption of appropriate pure mathematical propositions. However, the idea of (Eliminate) does not require that there be such an equivalence: it would surely be enough that there are alternatives to Hamilton's Principle—i.e. statement(s) that do not mention possible histories, and yet function as laws of motion. So after considering equivalence, I shall discuss this alternative.

For the simple mechanical systems we have focussed on since Section 2.1, there *are* equivalent statements. For as noted in Section 2.1, the Lagrange equations eq. 2.2 are, for simple systems, not only necessary but also *sufficient* for Hamilton's Principle eq. 2.4. And these equations do not mention possible histories. Agreed, they are modally involved; at least if we take them as putative laws of analytical mechanics—as we no doubt should! For then we will take them as applying to possible as well as actual initial conditions (given by the qs and $\dot{q}$s), and to possible as well as actual problems. That is, they will illustrate (Modality;1st) and (Modality;2nd), in Section 3's classification; but not (Modality;3rd). In short: Hamilton's Principle can be regarded for simple systems

[26] To make the argument clearer, I have suppressed a wrinkle about spatiotemporal relations. According to Lewis, these relations are not internal: they supervene on the intrinsic properties of the composite of the relata, not on the properties of the relata individually. This might seem an obstacle to my argument, since spatiotemporal relations are surely involved in assessing similarity of worlds in our sense, viz. courses of values of q and $\dot{q}$. But no worries. It is as relations between objects in a single world that spatiotemporal relations are not internal; but for variational principles, the assessment of similarity makes a comparison of the spatiotemporal structures of entire worlds—and the ensuing similarity is obviously an internal relation between the worlds.

as a corollary of the "kosher" this-worldly laws, Lagrange's equations eq. 2.2.

The same point applies to Hamiltonian mechanics for simple systems with non-zero Hessian, eq. 2.6. In this context, Hamilton's equations eq. 2.8 are equivalent, by the Legendre transformation eq. 2.7, to Lagrange's equations eq. 2.2. So again, taking Hamilton's equations as laws of analytical mechanics—as we no doubt should—they illustrate (Modality;1st) and (Modality;2nd) but not (Modality;3rd).

Incidentally, Hamiltonian mechanics raises another point, concerning the free variation of the ps in the modified Hamilton's Principle eq. 2.9. This gives another illustration of (Modality;3rd). But it is a more "extreme" illustration than that given by the original Hamilton's Principle eq. 2.4. For the latter, we contralegally vary q and so $\dot{q}$. But for the modified Hamilton's Principle, once we are given such a variation of q (and so $\dot{q}$), we independently vary the ps (violating $p = \partial L/\partial \dot{q}$). So our variations are "doubly contralegal".

But what about arguing for (Eliminate) for mechanical systems that are *not* simple: where the above equivalence breaks down? That is (cf. discussion after eq. 2.4): for systems for which the this-worldly Lagrange equations are only necessary but not sufficient for Hamilton's Principle?

In fact, I believe (Eliminate) can be defended for such systems; (and more generally, for systems for which the this-worldly dynamical equations are only necessary but not sufficient for a variational principle). I must postpone this topic to another occasion; not least because it has technical aspects (cf. Papastavridis 2002, pp. 960-973), as well as philosophical ones. But I should admit here that this defence opens the door to instrumentalism about variational principles. Thus suppose one says that the laws of motion are given by the (true and this-worldly!) Lagrange equations, not by Hamilton's principle. Then it seems one can turn instrumentalist about the latter: since these principles are sufficient but not necessary for the laws, one need not accept them as true, in order to agree that they have various uses. And they certainly *do* have uses numerous and important enough to earn them their central place in expositions of mechanics, even if they are "merely" instruments. I will return to this in Section 5.2's defence of (Useful). For the moment, I just mention one main use: a variational principle is often used as a way of guessing or deriving the laws of motion, since it is often easier to guess a Lagrangian that obeys a required symmetry than a set of laws of motion that obeys it.

Finally, a philosophical point that bring us back to Lewis. There is a temptation to say that a mystery remains, even *after* the argument for (Eliminate) for simple systems. It is tempting to ask: how can one of two *equivalent* formulations of a law (or theory)—Hamilton's Principle, on the one hand, and Lagrange's equations or Hamilton's equations, on the other—have a modal involvement that the other lacks? Indeed, more generally: How can there be a logical equivalence between a proposition with "this-worldly truth-conditions" and one making "transworld comparisons"?[27]

[27]Most philosophers agree that there may well be a notion of theoretical, not merely empirical,

I think Section 5.1.1's discussion, especially Lewis' claim (Actual), gives most of the reply to this question; and I will not rehearse it again. But another point, *independent* of (Actual) and indeed of the general idea of truthmakers, is worth stressing. Namely, there is no logical or semantical problem about evaluating as true at a single world a proposition making a transworld comparison. After all, this is exactly what is proposed by analyses of counterfactuals like those of Lewis and Stalnaker; and proposed by these analyses as semantical doctrines, independently of Lewis' metaphysical thesis (Actual). So suppose someone thought some propositions make transworld comparisons, in the strong sense that their truth-conditions (or if you like: truthmakers) are scattered across the worlds in ways *not* determined by internal relations of those worlds to the actual world. Such a person could nevertheless accept, as a matter of logic or semantics, that such a proposition be assigned a truth-value relative to the actual world.[28]

To sum up this Section:— I first argued for (Vindicate). Variational principles' mention of possible histories can be vindicated by an argument parallel to Lewis' argument that counterfactuals are made true by the character of the actual world— since their mention of other worlds reflects only internal relations between worlds. This parallel was based on showing that a variational principle is itself an infinite conjunction of Lewisian counterfactuals. Then I argued for (Eliminate), at least for simple systems. That is: we can identify the this-worldly truthmakers of Hamilton's Principle, namely *via* Lagrange's (or equivalently: Hamilton's) equations.

5.2 The uses of variational principles

I turn to the claim that at the start of Section 5.1.2, I dubbed (Useful): that formulating classical dynamical laws as variational principles is useful, or even advantageous compared with other formulations.

I admit that I shall duck out of giving a general argument for (Useful). Rather like Lewis with his claim (Concise) about counterfactuals (cf. Section 5.1.1), I offer no single general advantage of variational formulations. My reason is that the advantages are many, diverse and sometimes very technical. The calculus of variations remains an active research area, with deep connections to various branches of mathematics in addition to mechanics. (For a taster, cf. e.g. Courant and Hilbert (1953, Chap IV); for a banquet, cf. Giaquinta and Hildebrandt (1996).) So it would be well-nigh impossible

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equivalence such that laws or even whole theories that are theoretically equivalent could yet have heuristic, and even ontological, differences. Still, there can seem to be a mystery about our argument for (Eliminate); since both the equivalence of the formulations, and their having different modal involvements, seem matters of logic.

[28]Incidentally, the use of truth-assignments relative to two or more worlds in many-dimensional modal logic (and similarly: relative to two or more times in temporal logic) is, so far as I can tell, no evidence against the truthmaker principle. For these logics invoke multiple worlds or times to keep track of rigidified uses of 'actually' or temporal indexicals: not to keep track of truthmakers scattered among the worlds—indeed, not even 'scattered among the worlds' in the innocuous sense allowed by Lewis' (Actual), i.e. in the innocuous sense of determined by internal relations among the worlds.

even to list, let alone discuss, the advantages gained by adopting the notions, and general perspective, of the calculus of variations; not just for mechanics, but for any field that uses variational principles. So I shall just mention two examples of advantages of variational principles in analytical mechanics that would appear on any such list.

(i): The role of variational principles in understanding symmetry; especially the way that symmetries of the Lagrangian give Noether's theorems.

(ii): I choose my second example to illustrate how a piece of formalism within a theory can be advantageous not only as regards that theory, but also in illuminating another theory; (and maybe even heuristically useful in constructing that other theory). I have in mind how Hamilton's Principle illuminates the path integral formulation of quantum theory; both by providing a classical limit of it, and by heuristically suggesting it.

Finally, I should note an important topic related to (Useful): the topic, not of the *advantages* of a variational formulation of laws, but of the *conditions* under which such a formulation can be given. This is a large topic, which has been investigated since the nineteeth century, mostly in the more precise form: what are the necessary and sufficient conditions for a set of dynamical (differential) equations governing variables q_i to be the Euler-Lagrange equations of a variational principle? For example, the first major result was by Helmholtz in 1887. This topic also has philosophical aspects—not least the question I raised in Section 5.1.2.B, about how to argue for (Eliminate) for non-simple systems.

Though I cannot here develop this topic (cf. e.g. Santilli (1979), Lopuszanski (1999)), I should end by considering a small aspect of it: viz. a general correspondence between sets of canonical equations and variational principles—variational principles that even allow higher-order derivatives in the Lagrangian. I say 'I should consider this', because in effect, the question which has been the focus of this whole Section—'How can it be that the actual laws of motion admit a variational formulation?'—gets from this correspondence a technical interpretation—and an answer.

The key idea is that the modified Hamilton's Principle provides a correspondence between a general class of variational problems and systems of ordinary differential equations arranged in conjugate pairs.[29] The class of variational problems is given by the extremization of an integral

$$\int L(q_i, \dot{q}_i, \ddot{q}_i, \ldots, q_i^{(m)}, t)\, dt\ ; \tag{5.5}$$

where (m) indicates the mth derivative with respect to t; and where L is of course an arbitrary function (it need not have mechanical significance); with the extremization subject to not only the q_i, but also their derivatives upto the $(m-1)$th, being prescribed at the end-points.

[29] What follows is "well-known": it was discovered by Ostrogradskii writing in 1850! My summary is based on Lanczos (1986, pp. 170-72). For details and references about Ostrogradskii, cf. Kolmogorov and Yushkevich (1998, p. 201-207).

I shall describe the correspondence for the simplest case beyond the already familiar one, i.e. $m = 1$, $\delta \int L(q_i, \dot{q}_i, t)\ dt = 0$. That is, I shall allow at most a second time derivative as an argument of L. I shall also assume just one degree of freedom. It will be clear enough how the correspondence generalizes to more than one q, and to yet higher derivatives.

Consider then the extremization of

$$\int L(q, \dot{q}, \ddot{q}, t)\ dt \tag{5.6}$$

subject to q and $\dot{q}$ being prescribed at the initial and final times. One easily adapts the usual calculus of variations argument to this case. The boundary conditions now require the arbitrary function representing the variation of q, say η, not only to vanish at the end-points, but also to have vanishing first derivative there. The deduction proceeds much as usual, but now includes an integration by parts of $\frac{\partial L}{\partial \ddot{q}}\ddot{\eta}$, as well as integrations by parts of $\frac{\partial L}{\partial \dot{q}}\dot{\eta}$. We get:

$$\delta \int L(q, \dot{q}, \ddot{q}, t)\ dt \ = \ 0 \quad \text{iff} \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) \ = \ 0. \tag{5.7}$$

We proceed to find corresponding canonical equations. First we define a "momentum" $u := \frac{\partial L}{\partial \ddot{q}}$, and then perform a Legendre transformation, defining $H \equiv H(q, \dot{q}, u, t) := u\ddot{q} - L$; so that $L = u\ddot{q} - H(q, \dot{q}, u, t)$. So our variational problem $\delta \int L\ dt = 0$ is modified to $\delta \int [u\ddot{q} - H(q, \dot{q}, u, t)]\ dt = 0$. An integration by parts of the first term reduces this to a variational problem of the familiar kind, in q, u and their first derivatives: i.e. $\delta \int [-H(q, \dot{q}, u, t) - \dot{u}\dot{q}]\ dt = 0$. Now given *this* problem, we can in the familiar way define conjugate momenta, p_1, p_2 say, of q, u, and get two pairs of canonical equations. These are *equivalent* to the differential equation eq. 5.7.

This method easily generalizes to any number of degrees of freedom, q_i; and it generalizes to higher derivatives than the second. In the general case of mth derivatives, we first reduce them to $(m-1)$th derivatives by an integration by parts, and then repeat the process until eventually only first derivatives appear in the integrand, and we can pass to the corresponding canonical equations.

This result also gives a characterization of the differential equations corresponding to a variational principle (of the above class). Though an arbitrary system of differential equations can be given the form

$$\dot{q}_i = f_i(q_1, \ldots, q_n, t) \tag{5.8}$$

by introducing suitable independent variables $q_1, \ldots, q_n$, in general the functions f_i will of course be different for different i. On the other hand: differential equations obtained from a variational principle are derivable from a *single* function H by differentiation. In short: Hamilton's canonical equations are a normal form for the differential equations arising from a variational principle.

To sum up: we have shown how to pass from an arbitrary variational principle (of our class) to a system of canonical equations, all first-order in time and with all variables' time-derivatives given by differentiating a single function H. In effect, this result takes this Section's over-arching question—'how can it be that the actual laws of motion admit a variational formulation?'—as a technical question (instead of as a philosophical question, as in Section 5.1). And the result, i.e. the correspondence between a large class of variational problems and sets of canonical equations, answers as follows:—

'This is possible because the actual laws of motion, i.e. Hamilton's equations, have the very special feature that their right-hand sides, that specify the time derivatives of all the variables, are all derivatives of one and the same function H. If that were not so, one could *not* pass by a Legendre transformation to a variational formulation $\delta \int L dt = 0$.'

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