Quantum Fields on Noncommutative Spacetimes

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As much in physics as in life...Hasta la victoria, Siempre!

Contents

Introduction: The Quest of Quantum Gravity and Noncommutativity

The 20th century has been defined by many the century of physics. When it started, 111 years back, our understanding of the world was still constrained within the context of what is now called classical physics. The two major revolutions in physics, which later would be called *General Relativity* and *Quantum Mechanics*, were yet to come. Physicists were still thinking in terms of just three flat dimensions and in a complete deterministic manner.

Not more than 30 years later the warm and convenient setting of classical physics every physicist was comfortable with, was completely turned up side down. By the thirties Quantum Mechanics and General Relativity had already become almost universally accepted theories which constituted the new frame where physics speculations could find place. The transition from the three dimensional, ether-filled, static and eternal flat universe where everything happened in a deterministic and uniquely predictable way, to a strange four dimensional, curved, shape changing object, now appropriately called space-time, where notion of simultaneity and present-past-future became much fuzzier concepts and with a totally messy and chaotic behaviour at the microscopic level, was long and painful. At least for the physicists that from the beginning embraced the new ideas of the universe. Only incontrovertible evidences could defeat an extremely conservative old school physics community which for years laughed at the ones presenting the ideas which later on revolutionized completely our understanding of how nature works.

Such new ideas affected not only physics, but the whole approach of human thinking. Countless are the philosophical implications of Special Relativity, but probably none is as deeply disturbing as the ones brought up by quantum mechanics. Once and for all, human ambitions to be able to understand the world as a clock going forward in time, where everything is a uniquely determined effect of the cause who produced it, were killed by the theory of quanta. And it was quite of a brutal murder.

Once the new theories were accepted and the new ideas pushed forward it was time to let them bloom. And the spring of physics was a very pleasant one. For the remaining 60/70 years it was a non-stop flow of wonderful insights which let the seeds of General Relativity and Quantum Mechanics flower in what is today known as the Standard Model of Particle Physics and Cosmology. We now understand physics all the way down to almost 10^{-18} cm and could reconstruct the history of the universe from 10^{-36} s after a yet not understood event which has been called Big Bang but of which we really have little idea about.

Despite how pleasant the status of High Energy physics might appear, very few physicists have chosen to step back and spend the rest of their time proudly celebrating their achievements. The theory is still missing few ingredients (or maybe many more than we now think) which will hopefully lead us to fill up the holes still present in our current understanding. Discontent and desire to better understanding, easily prevailed on pride. Already Einstein in the last few decades of his life felt the urge of a theory which would unify General Relativity and Quantum Mechanics and would push the limits of our understanding, both in time and distance, all the way down to what needed to enlighten the darkness surrounding the Big Bang, the beginning of the whole. Although Einstein was not very successful in his attempts, many have decided to follow his steps and to embark in the challenging trail which might lead to what Einstein liked to call "The theory of everything". Such a theory is now called "Quantum Gravity" and this thesis is aimed to give a modest small contribution in the direction of its development.

Do we have any ideas on what *Quantum Gravity* should look like? Will it be a quantum or a classical theory? And how do we know that such a theory should exist to begin with? Any of these questions has hardly a firm and solid answer but easily many speculative ones. We hope to soon be able to answer all. And as it happens in physics it will only come from experiments. But let's try to unroll a few ideas here to get a taste of how the challenge looks like.

The questions are likely in decreasing order of difficulty to be answered. It is fair to say that we really have little idea on what the features of Quantum Gravity might be. More and more physicists have convinced themselves that the new pillar of physics which will be taken down is the number of spacetime dimensions. Extra-dimensions, whose number varies a lot depending on the specific proposal going from 1, in Randall-Sundrum type of model, all the way up to 26 in the case of bosonic String Theory, have become almost ubiquitous in the realm of Beyond the Standard Model physics proposals. But other physicists prefer to get rid of something else feeling particularly comfortable in a four-dimensional spacetime. Most of these theories are not less deeply disturbing. The notion to be abandoned is our perception of spacetime as a continuous. Noncommutative Geometry, which it is the subject of the present work, falls into this set of theories. As we will have time to explain below, we will get rid of the notion of a point altogether.

Possibly the second question has a more grounded answer. Although we do not have a certainty, nor has anybody come up with a no-go theorem, there is common agreement that quantum effects should have their appearance in Quantum Gravity. There exist arguments which strongly sustain the inconsistency of a unified theory of Gravity and Quantum Mechanics where the former is still treated classically but yet interacts with quantized matter fields. We don't want to get deeper into such arguments. We just want to mention that most of the problems with a non-quantized theory of gravity which interacts with particle described quantum mechanically, arise because of the way a gravity mediated measurement would affect the wave function. It can be easily proven that, under a wide range of assumption, such a possibility opens to exchange of superluminal signal. We will refer to the literature for more details (**?**). In the treatment of Noncommutative Geometry which will be presented here, in fact, Quantum Fields will still represent the mathematical objects which we will be using to construct our formalism. Part of the treatment will be in fact devoted on how to construct quantum fields on a spacetime which does not carry a meaningful notion of points, a noncommutative spacetime.

The answer to the last question is probably the most important one and at the same time likely the most solid. Although we know quite little about what a theory of

"Quantum Gravity" might look like, there are plenty of insights on its existence. Thus far what physicists have exploited to circumvent the absence of a unified theory, is the huge scale separation between the three "quantum forces" and gravity[1]. Thanks to this seemingly lucky coincidence, for the most part we can carry out extremely precise calculations completely neglecting one of the two side of the whole picture. The really problematic side of the story is that there is no sharp distinction in nature between what can be treated classically and what quantum mechanically. And there are cases where there is absolutely no possibility to favor one approach over the other (black holes or the Big Bang are two cases where gravity should be as important as quantum effects). If from the theoretical point of view seems to be obvious that a theory of Quantum Gravity should exist and we should soon abandon our vision of a two-sided world made of either classical or quantum object (and often the same object acquires different connotations depending on what it is interacting with or which one of its features is under study), the experiments have coexisted quite well with such a fuzzy distinction. Despite lack of experimental results that quantum gravity proposals can attempt to explain, there are reasons to believe that things can change drastically in the years to come. We have recently entered an extremely promising age of physics with many running experiments aiming at probing our current frontiers of knowledge. It is therefore not exceedingly optimistic to hope that soon we could also appeal to experimental evidences of a theory of quantum gravity. Let us keep our hopes well alive, further major revolutions in our understanding of physics might be waiting for us around the corner, and their secrets might be unveiled much sooner than we expect.

0.1 Noncommutative Geometry

As we explained above, the central idea in noncommutative geometry is dealing with spaces which do not rely on the notion of points. In fact, we will show below that spacetime coordinates in a theory of quantum gravity, under certain set of assumptions, do not commute, therefore it is forbidden to make an infinite precise measurement of spacetime coordinates. Before entering a more technical discussion, we would like to

[1]Feynman once calculated that including gravitational interaction in the study of the Hydrogen atom would change its wave function phase by a tiny 43 arcseconds in $100T$, where T is the age of the universe! (**?**).

present a more heuristic argument that already shows how different spacetime might look at the Plank scale.

Imagine to set up a measurement which aims to probe the spacetime at the Plank scale, L_P

$$L_P = \sqrt{\frac{G\hbar}{c^3}} \simeq 1.6 \times 10^{-33} \text{ cm}. \tag{1}$$

(Just for this section we will keep track of all the G, $\hbar$ and c factors which we in later sections set equal to 1.) Such a measurement requires very energetic particles, precisely we need a particle whose Compton Wavelength is smaller than L_P

$$\lambda_C = \frac{\hbar}{M_* c} \leq L_P \tag{2}$$

from which we can easily solve for the Compton Mass of the particle

$$M_* \geq \frac{\hbar}{L_P c} \simeq 10^{19} \text{ GeV} \tag{3}$$

which is 15 orders of magnitude higher than what the LHC will be able to produce at its maximum center of mass energy! But the complications of probing the spacetime at the Plank Length go even beyond the enormous amount of energy already required to be able to achieve the goal. One of these complications is the possible collapse of the spacetime. The theory of General Relativity sets an upper limit to the amount of energy density that the spacetime structure can sustain, exceeded which nothing can prevent its collapse into a black hole. Such a limit is given by the Schwarzshild Radius R_S associated to a given massive object (we assume spherical symmetry for simplicity). If the radius of the object is pushed below R_S, keeping its mass fixed, the collapse becomes unavoidable. We can then compute R_S for our particle and ask whether it is bigger or not than the length scale we are trying to probe with it, that is L_P

$$R_S^* = \frac{2GM_*}{c^2} \geq 2L_P \quad . \tag{4}$$

From (**??**) we can conclude that there is an intrinsic limit in how precise a spacetime measure can be. We have in fact shown that such a limit should be bigger than the Plank Scale. Probing the spacetime at L_P requires an amount of energy density which exceeds what the spacetime can sustain. The 4-volume we attempt to probe collapses into a black hole and falls beyond the event horizon and therefore beyond our reach.

0.1.1 FDR argument

Let's now try to carry out a treatment of the measurement process in a slightly more quantitative and formal way (**? ?**). We treat the measurement process semi-classically assuming the gravitational field propagates at the speed of light. This means that the expression for a the gravitational field generated at the point $\vec{r'}$ by a given energy density distribution $\rho(\vec{r}, t)$ is

$$\varphi(\vec{r'}) = -G \int \frac{\rho(\vec{r}, t_r)}{|\vec{r} - \vec{r'}|} d^3 r \tag{5}$$

where t_r is the retarded time which takes into account the finite speed of propagation of the field

$$t_r = \frac{|\vec{r} - \vec{r'}|}{c} \tag{6}$$

In the following we work in natural units so we set $G = c = \hbar = 1$.

Assume that we now want measure the event $\bar{x} = (-t, 0, 0, 0)$ with uncertainties $\Delta\bar{x} = (\Delta x_0, \Delta x_1, \Delta x_2, \Delta x_3)$ and compute how the gravitational field generated by the energy density needed to perform the measurement (for instance the compton mass of the probe)3 affects an observer seating at the origin $x_O = (0, 0, 0, 0)$. E represents the energy of the measurement probe and we also assume uniform spreading of the energy E after the measurement, with all speeds not exceeding the speed of light. The gravitational field at x_O is then:

$$\varphi(x_O) \approx -\int_0^\infty \frac{1}{r} \frac{E}{\Pi_{i=1}^3 (\Delta x_i + r)} r^2 \, dr \quad . \tag{7}$$

An expression for E can be derived from Heisenberg uncertainty principle, $E \approx 1/\Delta x_0$ or $E \approx 1/\Delta x_i$, depending which uncertainty is smaller, we will assume $\Delta x_0 \ll \Delta x_i$ so that

$$\varphi(x_O) \approx -\frac{1}{\Delta x_0} \int_0^\infty \frac{r}{\Pi_{i=1}^3 (\Delta x_i + r)} dr \quad . \tag{8}$$

In order to derive limitations on the uncertainties Δx_μ we should impose that the gravitational field, generated by the performed measurement, does not create a black hole. In our semi-classical framework this can be done by asking that a photon with energy ε at the origin does not get trapped by the gravitational field, that is its energy remains always positive

$$\varepsilon + \varepsilon\varphi(x_O) \gtrsim 0 \quad \Rightarrow \quad -\varphi(x_O) \lesssim 1 \tag{9}$$

We now focus just on the uncertainty $\Delta x_0 \Delta x_1$, the other relations can be derived in a completely analogous manner. There are three different regimes which we should consider

$$\Delta x_1 \sim \Delta x_2 \sim \Delta x_3, \qquad \Delta x_1 \sim \Delta x_2 \gg \Delta x_3, \qquad \Delta x_1 \gg \Delta x_2 \sim \Delta x_3 \tag{10}$$

Consider the latter, $\Delta x_1 \gg \Delta x_2 \sim \Delta x_3$. From (**??**) and (**??**)

$$\Delta x_0 \gtrsim \int_0^\infty \frac{r}{(\Delta x_1 + r)(\Delta x_2 + r)^2} dr \tag{11}$$

We can pull out $1/\Delta x_1$ and change variable $r \to r' = r/\Delta x_1$

$$\Delta x_0 \Delta x_1 \gtrsim \int_0^\infty \frac{r'}{(1+r')\left(\frac{\Delta x_2}{\Delta x_1} + r'\right)^2} dr' \tag{12}$$

performing the integral and neglecting higher powers in $\Delta x_2/\Delta x_1$ we get

$$\Delta x_0 \Delta x_1 \gtrsim \ln\left(\frac{\Delta x_1}{\Delta x_2}\right) \quad . \tag{13}$$

Similarly in the other two regimes in (**??**) we obtain

$$\Delta x_0 \Delta x_1 \gtrsim 1 \tag{14}$$

Since $\Delta x_1 \gg \Delta x_2$ the absolute limitation is $\Delta x_0 \Delta x_1 \gtrsim 1$. Following a similar reasoning, and reintroducing the appropriate powers of G, c and $\hbar$, we can derive the following absolute limitations for Δx_μ

$$\Delta x_0 \sum_{i=1}^{3} \Delta x_i \gtrsim L_P^2 \tag{15}$$

$$\sum_{i<j=1}^{3} \Delta x_i \Delta x_j \gtrsim L_P^2 \tag{16}$$

where L_P is the previously introduced Plank Length. We therefore obtained that including the effect of gravity implies a lower bound on uncertainties on the measurement of space-time coordinates. In the formalism of quantum mechanics this means that the spacetime coordinates operators do not commute. In particular it can be proven that relations (**??**) and (**??**) are implied by the following two conditions:

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \qquad [x_\rho, \theta_{\mu\nu}] = 0 \tag{17}$$

so $\theta_{\mu\nu}$ must be a constant or more precisely it must belong to the center of the algebra. Thus every value of $\theta_{\mu\nu}$ identifies a representation of the "Quantum space-time".

The noncommutation relations introduced in (**??**) will be our starting point for what we are going to present in the next chapter. As we will show, they imply a modification of the product between functions over space-time. This will bring new features in the quantum theories defined over such spaces which is the main topic of the current thesis.

We now present a different motivation for noncommutativity both for completion and for the extreme neatness of the result that Connes and his group were able to achieve.

0.1.2 Connes' proposal

We now present another argument which supports the idea of the space-time as a noncommutative space rather than a differential manifold (we will explain how to deal with noncommutative spaces in the next chapter). It follows somewhat different ideas. In particular our main motivation does not come from measuring the very fine structure of space-time but from the symmetry group of a theory of quantum gravity. So we turn now to discuss such symmetries.

Symmetry groups play an absolutely essential role in physics. In our modern understanding, particles that mediate forces come as connections of principal bundles, or in a less technical sense, as fields with certain properties, to enforce a specific symmetry group to be a gauge group. We no longer write down the Lagrangian and then study its symmetries but we choose the symmetry group first and write down the most general Lagrangian having the chosen group as gauge symmetry. In this sense global symmetries are treated in a completely different manner. They are not imposed but derived and for this reason often called accidental global symmetries. Yet it is very likely that Quantum Gravity breaks any global, that is non gauge, symmetry (**? ?**). We will now concentrate just on gauge symmetries and will not mention global symmetries anymore.

Our current understanding of physics is based on the theory of General Relativity and the Standard Model of Particle Physics. These two theories have very different symmetry groups. Any theory of gravity defined on any space-time manifold $\mathcal{M}$, as we currently understand it, it is invariant under the group of diffeomorphisms of $\mathcal{M}$. We call such a group $\mathcal{D}\mathit{iff}(\mathcal{M})$. On the other side the Standard Model is the theory of

Strong, *Weak* and *Electro-Magnetic* interactions which come from the Gauged $SU(3) \times SU(2) \times U(1)$.

The symmetry group of the two theory is a slightly non trivial combination of the two that takes into account that the action of a diffeomorphism on $\mathcal{M}$ does affect Standard Model gauge transformations which are obviously gauged over the same manifold $\mathcal{M}$. The total symmetry group is precisely the semi-direct product of the two:

$$\mathcal{G}_{GR+SM} = \Big(SU(3) \times SU(2) \times U(1)\Big) \rtimes \mathcal{D}\mathit{iff}\,(\mathcal{M}) \quad . \tag{18}$$

Two remarks are in order. First the total group comes from two different arguments, one based on General Relativity and the other on the Standard Model. We instead would like to be able to obtain (**??**) as a symmetry group of a single theory. Second, more on the technical side, the two groups play quite a different role in terms of the proper mathematical formulation of gauge theories. Specifically $\mathcal{D}\mathit{iff}(\mathcal{M})$ acts also on the base manifold whereas $SU(3) \times SU(2) \times U(1)$ just on the fiber. Connes then asked whether could be possible to find a unified picture which naturally gives (**??**) as symmetry group.

Specifically is it possible to have a theory of gravity which naturally contains the Standard Model? We could call such a theory Quantum Gravity, and such a requirement can be re-phrased as finding a space-time manifold $\mathcal{N}$ whose diffeomorphism group is isomorphic to (**??**)

$$\mathcal{D}\mathit{iff}\,(\mathcal{N}) \cong \Big(SU(3) \times SU(2) \times U(1)\Big) \rtimes \mathcal{D}\mathit{iff}\,(\mathcal{M}) \quad . \tag{19}$$

It is possible to prove that so long as we look for $\mathcal{N}$ as a differential manifold, (**??**) has no solution. The reason being that if $\mathcal{N}$ is a differential manifold (more generically a topological space), $\mathcal{D}\mathit{iff}(\mathcal{N})$ is always simple (that is does not contain normal subgroup) whereas in (**??**), the right hand side contains $SU(3) \times SU(2) \times U(1)$ has a normal subgroup. Things change drastically if $\mathcal{N}$ is a noncommutative space. Moreover in such a case $\mathcal{D}\mathit{iff}(\mathcal{N})$ has a semi-direct product structure

$$\text{if } \mathcal{N} \text{ is noncommutative} \quad \Rightarrow \quad \mathcal{D}\mathit{iff}\,(\mathcal{N}) \cong \mathcal{G}_1 \rtimes \mathcal{G}_2 \tag{20}$$

We will come back to this argument in the next chapter after we introduce how to mathematically describe noncommutative spaces showing that we can in fact find a solution for (**??**) if we are willing to release the commutativity of our space-time.

Connes at al.'s approach, which we just briefly outlined, has achieved even more remarkable results than finding the non-commutative space $\mathcal{N}$ solving (**??**). Using the appropriate formalism to treat noncommutative space-times, which we are going to discuss in the next chapter, they were able to show that such models are highly constrained, that is very few solutions exist. In particular there is a constrained on the number of matter fields (fermions) which the theory can accommodate. It happens that the Standard Model has just the right number of fermions, that is 16. We want to stress that besides cancellation of anomalies, which constrains, roughly speaking, the number of leptonic chiral families to be the same of the quark chiral families, and asymptotic freedom, which through the strong coupling beta function constrains the number of families to be less than 16 (number of families is very different from number of fermionic fields, also the Standard Model only provides an upper bound), there is no analog constrain on the Standard Model. Also using the extremely involved techniques of Spectral Analysis (**? ?**), which are beyond the scope of this thesis, Connes at al. Noncommutative Standard Model constrains the Higgs mass to be around 110 GeV. Although this value has been ruled out by now, it is nevertheless extremely remarkable that a value so close to where we expect the Higgs to be can be derived by solving an equation of the kind of (**??**).

0.1.3 Final Remarks

Before entering a more formal treatment of noncommutative spacetimes, it is helpful to pause on the physical interpretation on what we have presented so far:

- In the noncommutative spaces we have called for in the previous two sections, the notion of continuous does not break down. More precisely the spectrum of each x_μ is still continuous. In other words we can make a measurement of a single spacetime coordinate as precise as we want, at the cost of having complete uncertainty on the others. The situation resembles closely the noncommutativity of Quantum phase space. There are example of noncommutativity in which the space-time becomes a lattice[1] but we will not treat such cases in the present

[1] One of the most famous examples is the "Fuzzy Sphere" (**?**) where the algebra of spacetime coordinates is isomorphic to the $SU(2)$ algebra. It is well-known from angular momentum theory, that representation of such an algebra are finite dimensional and identified by a half-integer. The spectrum of coordinate functions operator becomes discrete and the spacetime as a continuous is lost.

thesis.

Thus it is not appropriate to think at noncommutativity as a spacetime with a lowest length scale $\sqrt{\theta}$ but more as spacetime with a lowest four volume $\sim \theta^2$. Discreteness was the way noncommutativity was initially introduced by Snyder (**?**). A lowest lenghtscale $\sqrt{\theta}$ would introduce a natural cut-off in momentum space $\Lambda \sim 1/\sqrt{\theta}$ which would fix all the ubiquitous UV-divergencies in Quantum Field Theory. In many of the formulations of quantum field theory on noncommutative space-times, the UV-behaviour is instead worsened. In such theories a new phenomenon which entangles together the IR and UV behaviour, arises. The UV-IR mixing makes application of renormalization techniques to noncommutative QFT's difficult and not at all straightforward (**?**).

- The fate of Poincaré symmetry is not a trivial point either. Although not discrete, it is still not easy to come up with a definition of Lorentz symmetry for noncommutative spacetimes. This is potentially a very serious problem. The kind of particles which can be observed, and we do observe, is strongly constrained by Poincaré symmetry (**?**). We will discuss this point in much more depth in the next chapter. The definition of symmetries on a noncommutative spacetime turns out to be not at all an easy task. We will need to introduce the notion of Quantum Groups and Hopf Algebras which allows to restore Poincaré symmetry in a deformed fashion. Quantum Groups allow to circumvent the naively explicit breaking of Poincaré invariance by $\theta_{\mu\nu}$ and recover its essential role in any QFT formulation.

Having enlightened the connections between Quantum Gravity and Noncommmutative Geometry, we hope that by now the reader feels the need to understand noncommutative spaces in a more quantitative and formal sense. And this is exactly what we are turning into in the next chapter.

1

The Math of noncommutativity: noncommutative spaces and Quantum Symmetries

In the previous chapter we motivated in many different ways, why we expect noncommutativity of the spacetime to appear once we get to probe its finer structure. Yet we don't know how to deal with these spaces and how to even treat them mathematically. Differential geometrical or topological methods will definitely not suffice for it: spacetime coordinates do not commute

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1.1}$$

so on noncommutative spaces there is no operational definition of points. Topology and differential geometry are the study of respectively continuous and smooth spaces. We then need to turn our perspective on how to deal with spacetimes quite radically. In this chapter we introduce the tools to do it. The main idea which will allow us to circumvent the problem of lack of points is contained in the first Gel'fand-Naimark theorem which proves that there is an algebraic way of looking at topological spaces. Algebras will be the tool we use to talk about some kind of "noncommutative topology".

Most of the technical details will not be needed for the remaining part of the thesis. Therefore we will try to keep the treatment as simple as possible but still mathematically accurate, focusing mainly on conceptual ideas rather than formal subtleties.

Yet some mathematical notion about algebras will be required. In order to keep the treatment self-contained, we devoted an appendix to "algebraic preliminaries".

The presentation of the material in this chapter, follows quite closely (**?**).

1.1 From topological/geometrical methods to algebraic ones

In this section we will make heavy use of the terminology and definitions introduced in Appendix **??** so we refer to it for most of the definitions not given here. Below we will outline the main idea which allows us to treat mathematically noncommutative spaces, the existence of a duality between topological methods and algebraic ones, starting with the main pillar, a duality between topological spaces and commutative algebras.

1.1.1 Topological spaces $\leftrightarrow$ Commutative C^*-algebras

The first *Gel'fand-Naimark* theorem represents the pillar of the reformulation of topology in terms of algebraic notions: **Every commutative C^*-algebra can be realized as an algebra of continuous function on a certain Hausdorff topological space M.** In other words there is a one to one correspondence between Hausdorff topological spaces and commutative C^*-algebras.

Given a Hausdorff topological space M, we can immediately obtain a commutative C^*-algebra as the algebra of continuous functions over M, $C_0(M)$, with supremum norm:

$$\forall f \in C_0(M) \quad \|f(x)\| = \sup_{x\in M}|f(x)| \quad . \tag{1.2}$$

If M is not compact, $C_0(M)$ is the algebra of continuous functions vanishing at infinity. We then obtain that (non)compact Hausdorff topological spaces are dual to (non)unital C^*-algebra.

The other side of the correspondence is less trivial. So given a commutative (non)unital C^*-algebra $\mathcal{C}$ how do we re-construct M? First we should explain how to construct $\widehat{C}$, the space of all *characters* of $\mathcal{C}$. A character of $\mathcal{C}$ is a one-dimensional irreducible, $*$-linear representation of $\mathcal{C}$, $x : \mathcal{C} \to \mathbb{C}$. x preserves the multiplication map on $\mathcal{C}$

$$x(f \cdot g) = x(f) \cdot x(g) \quad \forall f, g \in \mathcal{C} \tag{1.3}$$

where the RHS is just the multiplication on $\mathbb{C}$. The space $\widehat{C}$ can be made into a topological space endowing it with the topology of pointwise convergence on $\mathcal{C}$. This simply means that a sequence of characters $\{x_n\}$ in $\widehat{C}$ converges to $x \in \widehat{C}$ if and only if the sequence $\{x_n(f)\} \in \mathbb{C}$ converges to $\{x(f)\}$ in the topology of $\mathbb{C}$ for any $f \in \mathcal{C}$.

We then have shown that given a Hausdorff topological space M we can construct a commutative C^*-algebra $C_0(M)$ and that given a commutative C^*-algebra $\mathcal{C}$ we can construct a topological space associated with it, $\widehat{C}$. The last question left is, if we start from M, construct $C_0(M)$, and then its topological space $\widehat{C_0(M)}$, how are $\widehat{C_0(M)}$ and M related? The relation between the two can be constructed by noticing that any $x \in M$ allows us to construct a $*$-linear homomorphism $\phi_x \in \widehat{C_0(M)}$ through the evaluation map:

$$\phi_x : C_0(M) \to \mathbb{C}, \quad \phi_x(f) = f(x), \quad \phi_x(f \cdot g) = (f \cdot g)(x) = f(x) \cdot g(x) \tag{1.4}$$

Then the map ϕ_x is a homeomorphism of M onto $\widehat{C_0(M)}$[1]. We then obtain the result which we state few lines above, there is a one-to-one correspondence between commutative C^*-algebras and the homeomorphism classes of locally compact topological spaces. We can rephrase it also in a fancy, categorial language as the fact that there is a complete duality between the category of (locally) compact Hausdorff spaces and continuous maps and the category of commutative (non) unital C^*-algebras and $*$-homomorphisms.

1.1.2 Vector Bundles $\leftrightarrow$ Finite Projective Modules

The *Serre-Swan* theorem represents the analog of the first *Gel'fand-Naimark* theorem for Vectors Bundles: **Complex vector bundles over an Hausdorff space M are in one-to-one correspondence with finite projective modules over the algebra $\mathcal{A} = C^\infty(M)$.** In this section we will first introduce what projective modules are and then we will proceed to convince the reader that the correspondence is in fact one-to-one. We will again avoid the formal proof of the mathematical theorem.

[1]To avoid confusion we want to clarify this point. ϕ_x is an homomorphism if seen as a map from $C_0(M) \to \mathbb{C}$. In fact (**??**) shows that ϕ_x preserves the multiplication in $C_0(M)$. At the same time ϕ_x can be seen as a map between topological spaces, namely the topological space M and the space of character $\widehat{C_0(M)}$ with the topology of pointwise convergence. In this latter perspective ϕ_x is instead a homeomorphism.

Free Modules

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$ and $\mathcal{E}$ a vector space over $\mathbb{C}$. $\mathcal{E}$ is a (right) left module over $\mathcal{A}$ if it carries a (right) left action such that

$$\mathcal{A} \times \mathcal{E} \ni (a, v) \to av \in \mathcal{E}$$
$$(a_1 a_2)v = a_1(a_2 v), \quad (a_1 + a_2)v = a_1 v + a_2 v, \quad a(v_1 + v_2) = av_1 + av_2 \tag{1.5}$$

$\forall a_1, a_2 \in \mathcal{A}$ & $v_1, v_2 \in \mathcal{E}$, in other words $\mathcal{E}$ carries a (right) left representation of $\mathcal{A}$.

Let $\{e_n\}$ be a set of linearly independent vectors of $\mathcal{E}$. If any element in $\mathcal{E}$ can be uniquely written as linear combination of $\{e_n\}$ with coefficient in $\mathcal{A}$, that is $\forall v \in \mathcal{E}, \exists! \{a_i\} \in \mathcal{A}$ such that $v = \sum_i a_i e_i$, then $\{e_n\}$ is a *basis* for $\mathcal{E}$. A module is *free* if it admits a basis. In other words a module is *free* if it is isomorphic to $\mathcal{A}^N = \mathcal{A} \otimes \mathbb{C}^N$. This looks very much similar to the condition of triviality for a vector bundle, $E = V \times M$. As we just anticipated trivial bundles correspond to *free modules* but in order to accommodate non trivial ones we should also introduce *projective modules*.

Projective Modules

An $\mathcal{A}$-module $\mathcal{E}$ is *projective* if it is a direct summand in a free module, that if there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ such that

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{E}' \tag{1.6}$$

From the definition given, it follows immediately that $\mathcal{E}'$ is then projective as well.

The definition above is quite obscure. We can provide an equivalent and hopefully more transparent definition in terms of a $\mathcal{A}$-modules' homomorphism, that is a linear map ρ which also preserves the $\mathcal{A}$ action

$$\rho : \mathcal{E} \to \mathcal{E}' \qquad \rho(av) = a\rho(v) \quad \forall a \in \mathcal{A} \;\&\; v \in \mathcal{E} \tag{1.7}$$

A module $\mathcal{E}$ is projective if and only if there exists an $\mathcal{A}^N$-endomorphism e, which is *idempotent*, $e^2 = e$, such that $\mathcal{E} = e\mathcal{A}^N$. It then follows that

$$\mathcal{A}^N = e\mathcal{A}^N \oplus (\mathbb{1} - e)\mathcal{A}^N = \mathcal{E} \oplus \mathcal{E}' \tag{1.8}$$

And $\mathcal{A}^N$ is obviously free. The map e can be represented as $N \times N$ matrix with entries in $\mathcal{A}$ acting on $\mathcal{A}^N$. The above definition means that $\mathcal{E}$ is *projective* if and only if it can be written as a non-trivial projection of a free module $\mathcal{A}^N$. In the definition above the projection map is e.

The correspondence between projective modules and bundles goes as follow. A vector bundle $E \to M$ is completely characterized by the space $\mathcal{E} = \Gamma(E, M)$ of its section, that is vector fields. It can be easily shown that $\mathcal{E}$ is a finite projective module over $C^\infty(M)$. The converse requires a bit more care, that is given $\mathcal{E} \cong \Gamma(E, M)$ how do we construct the vector bundle $E \to M$.

The fiber $E_{\tilde{x}}$ of $E \to M$ over the point $\tilde{x} \in M$, is a vector space. It can be obtained from $\Gamma(E, M)$ as a space of equivalence classes where two vector fields are equivalent if they coincide at $\tilde{x}$ or alternatively they differ by a vector field which vanishes at $\tilde{x}$. The space of vector fields which vanish at $\tilde{x}$ is generated by $\mathcal{I}_{\tilde{x}}$, the set of functions which vanish at $\tilde{x}$ (such a set is in fact a ideal of $\mathcal{A} = C^\infty(M)$). Given a projective module $\mathcal{E}$ over $C^\infty(M)$, the fiber $E_{\tilde{x}}$ of the associated vector bundle over the point $\tilde{x} \in M$ is obtained

$$E_{\tilde{x}} = \mathcal{E}/(\mathcal{E} \otimes \mathcal{I}_{\tilde{x}}) \tag{1.9}$$

1.1.3 Diffeomorphisms $\leftrightarrow$ Automorphisms of $\mathcal{A} = C^\infty(M)$

The group of diffeomorphisms of a manifold M is the set of smooth maps from M in itself, which are invertible and whose inverse is also smooth, and where the group operation is composition of maps

$$\phi \in \mathcal{D}\mathit{iff}(\mathcal{M}),\ \phi : M \to M, \quad \phi(x) \in C^\infty(M) \tag{1.10}$$

Diffeomorphims can be accommodated in the algebraic description of manifold in a straightforward way by considering the action of a diffeomorphism ϕ on a smooth function $f(x) \in \mathcal{A} \equiv C^\infty(M)$,

$$\phi : C^\infty(M) \ni f(x) \to f(\phi^{-1}(x)) \in C^\infty(M) \tag{1.11}$$

Therefore the group $\mathcal{D}\mathit{iff}(\mathcal{M})$ of diffeomorphisms of M is naturally isomorphic to the group $\mathrm{Aut}(C^\infty(M))$ of *automorphisms* of $C^\infty(M)$. The group of $\mathrm{Aut}(\mathcal{A})$ of a given algebra $\mathcal{A}$, is the set of maps from $\mathcal{A}$ into itself which also preserves the multiplication map. The group operation being the composition of maps.

In order to motivate what we stated in the previous chapter about finding solutions of (**??**) for noncommutative spaces, we should study the structure of $\mathrm{Aut}(\mathcal{A})$ for any, commutative or noncommutative, C^*-algebras. If the algebra $\mathcal{A}$ is noncommutative, invertible elements $u \in \mathcal{A}$, define automorphisms of $\mathcal{A}$ by the equality

$$Ad_u(a) = u \circ a \circ u^{-1}, \quad \forall a \in \mathcal{A} \tag{1.12}$$

These automorphisms are not trivial (unless a belongs to the center of $\mathcal{A}$) and form a normal subgroup of $\mathrm{Aut}(\mathcal{A})$. We call the elements of $\mathrm{Aut}(\mathcal{A})$ which can be written as in (**??**) *internal* automorphisms and indicate them as $\mathrm{Int}(\mathcal{A})$. Elements of $\mathrm{Aut}(\mathcal{A})$ which cannot be written as in (**??**) are called *outer* automorphisms and will be indicated as $\mathrm{Out}(\mathcal{A})$. Thus the group of automorphisms of a C^*-algebra has generically the form

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Int}(\mathcal{A}) \ltimes \mathrm{Out}(\mathcal{A}) \quad . \tag{1.13}$$

in the above, $A \ltimes B$ indicates the semi-direct product of A and B, with B being the invariant subgroup.

Specifically if $\mathcal{A} = M_n(C^\infty(M))$, that is $\mathcal{A}$ is the algebra of $n \times n$ matrices with entries in $C^\infty(M)$, and we restrict $\mathrm{Int}(\mathcal{A})$ to the solely unitary elements, (**??**) reduces to

$$\mathrm{Aut}(M_n(C^\infty(M))) = \mathrm{Map}(M, G) \ltimes \mathcal{D}\mathit{iff}(\mathcal{M}) \tag{1.14}$$

where $\mathrm{Map}(M, G)$ is the group of maps from $M \to G$, and G is the group $SU(n)$[1], that is $\mathrm{Map}(M, G)$ is the $SU(n)$ gauge group.

Although we have not fully shown how to find a solution for (**??**), we hope to have provided enough details for the reader to be convinced that if we move to noncommuative C^*-algebras we can construct a theory which naturally provides both gravity and the standard model gauge group. As we stated in the introduction, in the remaining part of the thesis we will not follow Connes et al. approach to noncommutative geometry, therefore we will not discuss this topic any further but refer the reader to the literature for a detailed and explicit resolution of (**??**), see for instance (**?**).

1.1.4 Noncommutative Spacetimes $\leftrightarrow$ Noncommutative C^*-algebras

At this point it should be quite obvious for the reader how we could define a noncommutative space. Following the construction outlined in the beginning of the section, a

[1] To be precise, since (**??**) is trivial if a belongs to the center of the algebra, G is the coset of $SU(n)$ and its finite center.

***noncommutative space* is the space of characters $\widehat{C}$ of a C^*-algebra $\mathcal{C}$ when $\mathcal{C}$ is now noncommutative.**

In particular in the approach to noncommutative geometry followed in this thesis, we will be dealing with a very specific type of non-commutative C^*-algebras: The ones obtained from a deformation of the product of the algebra of functions over $\mathbb{R}^4$. In fact from (**??**) we obtain that on noncommutative spaces

$$x^\mu \star x^\nu \neq x^\nu \star x^\mu \tag{1.15}$$

that is the product among functions is changed and it is now noncommutative. We will encounter different deformations of the algebra of functions on $\mathbb{R}^4$, but we will often use $\mathcal{A}_\star$ to indicate that we are dealing with a noncommutative deformation of $C^\infty(\mathbb{R}^4)$.

A final remark is in order. At this point the reader might wondering whether or not we are really dealing with a spacetime without points since we still constructed the noncommutative spacetime in terms of old tools, that is functions over the spacetime $\mathbb{R}^4$. Although we will be still using $\mathbb{R}^4$ coordinates and we will be able to exactly evaluate a function at a point $\tilde{x}$, we want to remind the reader that the spacetime associated with $\mathcal{A}_\star$ is no longer $\mathbb{R}^4$. As explained previously, such a spacetime should be constructed from space of characters of $\mathcal{A}_\star$. In general the evaluation map ϕ_x in (**??**) is not a character anymore since for a generic $\star$ product, ϕ_x does not preserve the multiplication map on $\mathcal{A}_\star$:

$$\phi_x : f \to f(x) \quad \text{but} \quad \phi_x(f_1 \star f_2) = (f_1 \star f_2)(x) \neq f_1(x) f_2(x) \tag{1.16}$$

Therefore the underlying spacetime associated to $\mathcal{A}_\star$ is not $\mathbb{R}^4$ and it is truly noncommutative.

1.2 Quantum Symmetries and Hopf algebras

The importance of group theory in the formulation of quantum mechanics and quantum field theory can in no way be overstated. The spectrum of particles that we measure (fermions, vector bosons and scalar-0 particles), all comes from theory of representation of the Poincaré symmetry group. What goes usually unnoticed is that we deal with more structure than just a group. A system of two or more particles lives in a tensor product (of single-particle Hilbert spaces) Hilbert space,

$$\mathcal{H}^{(n)} = \underbrace{\mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{N\ factors} \tag{1.17}$$

If a representation ρ of a group $\mathscr{G}$ on $\mathcal{H}$ is given, there is no unique nor canonical way of defining its action on the tensor product space $\mathcal{H}\otimes\mathcal{H}$. Specifying how $\mathscr{G}$ acts on $\mathcal{H}\otimes\mathcal{H}$ enriches the group structure with a new operation, which is called the *co-multiplication* or *co-product* and is indicated with Δ. $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$ is a map which has to satisfy a certain set of properties. Once these properties are satisfied, the group $\mathscr{G}$ acquires the structure of a *Hopf algebra*. Before turning into the formal definition we should convince ourselves of the importance of the co-product Δ.

The theory of angular momentum and the angular momentum addition rules, including *Clebsch-Gordan* coefficient, for instance, is a consequence of a particular choice of Δ on the algebra of rotation group $SO(3)$

$$[J_j, J_k] = i\epsilon_{jkl}J_l, \qquad \Delta_0(J_i) = J_i \otimes \mathbb{1} + \mathbb{1} \otimes J_i \tag{1.18}$$

We call such a choice Δ_0 since it represents the co-product suited for quantum mechanics on commutative spacetime. If $|\pm\rangle$ indicates a state of spin $\pm 1/2$ and use $|a, b\rangle$ for $|a\rangle \otimes |b\rangle$, we then obtain, for instance, the well-known result for $|1, 0\rangle = (|+, -\rangle + |-, +\rangle)/\sqrt{2}$

$$\Delta_0(J_z)|1, 0\rangle = 0 \quad \& \quad \Delta_0(J^2)|1, 0\rangle = \Delta_0(\vec{J}) \cdot \Delta_0(\vec{J})|1, 0\rangle = 2|1, 0\rangle \tag{1.19}$$

where the last property, that is $\Delta_0(a \cdot b) = \Delta_0(a) \cdot \Delta(b)$ is one of the crucial properties that the map Δ should fulfill.

Hopf algebras

An algebra $\mathcal{A}$ endowed with a *co-multiplication map* or *co-product* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, a *co-unit* $\varepsilon : \mathcal{A} \to \mathbb{C}$ and an *antipode* $S : \mathcal{A} \to \mathcal{A}$ is a *Hopf algebra* if the following properties are satisfied

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \quad \text{(coassociativity of } \Delta) \tag{1.20}$$

$$(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a), \quad \forall a \in \mathcal{A} \tag{1.21}$$

$$m(S \otimes id)\Delta(a) = m(id \otimes S)\Delta(a) = \varepsilon(a)\mathbb{1} \quad \forall a \in \mathcal{A} \tag{1.22}$$

$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ represents the multiplication map on $\mathcal{A}$. In addition to the properties above, all three maps should be algebra homomorphisms, that is should preserve the multiplication map m on $\mathcal{A}$

$$\Delta(a_1 a_2) = \Delta(a_1)\Delta(a_2) \tag{1.23}$$

$$\varepsilon(a_1 a_2) = \varepsilon(a_1)\varepsilon(a_2) \tag{1.24}$$

$$S(a_1 a_2) = S(a_2)S(a_1) \tag{1.25}$$

On the RHS's we left the multiplication map implicitly defined by the context, that is in (**??**) the product is given by the multiplication map $m_{\mathcal{A}\otimes\mathcal{A}}$ induced by m on $\mathcal{A}\otimes\mathcal{A}$, in (**??**) is the multiplication on $\mathbb{C}$ whereas in (**??**) is simply m.

An example of a Hopf algebra is the *Group Algebra*, which is roughly speaking a group with extra information on how to act on tensor-product spaces. The situation is slightly more subtle than this, since from the definition above, a Hopf algebra is an algebra in the first place. For a given group $\mathscr{G}$, we can define an algebraic structure as follow. For simplicity we here consider the case $\mathscr{G}$ is discrete so we can have a discrete index labeling its elements. The generalization to continuous groups is straightforward (issues of convergence apart) with the sum replaced by an integral over the group. Consider a formal sum over $\mathbb{C}$ of elements of $\mathscr{G}$

$$\gamma = \sum_i \lambda_i g_i, \quad \lambda_i \in \mathbb{C}, \quad g_i \in \mathscr{G} \tag{1.26}$$

elements of $\mathscr{G}$ are clearly of the form $\lambda_i = 0\ \forall i \neq \bar{i}$ and $\lambda_{\bar{i}} = 1$. We can now define the sum and the multiplication of two elements γ_1 and γ_2 the former inherited by the addition over $\mathbb{C}$ and the latter by the multiplication over $\mathscr{G}$

$$\gamma_1 + \gamma_2 := \textstyle\sum_i (\lambda_i^1 + \lambda_i^2) g_i \tag{1.27}$$

$$\gamma_1 \circ \gamma_2 := \textstyle\sum_{i,j} \lambda_i^1 \lambda_j^2 (g_i \circ g_j) \tag{1.28}$$

The multiplication $\circ$ is clearly distributive over $+$

$$(\gamma_1 + \gamma_2) \circ \gamma_3 = \gamma_1 \circ \gamma_3 + \gamma_2 \circ \gamma_3 \tag{1.29}$$

so the set of all gammas, forms an algebra. We call it the *Group Algebra* $\mathbb{C}\mathscr{G}$. $\mathbb{C}\mathscr{G}$ becomes a Hopf algebra once endowed with the co-product

$$\Delta_0 : \mathbb{C}\mathscr{G} \to \mathbb{C}\mathscr{G} \otimes \mathbb{C}\mathscr{G}, \quad \Delta_0(g_i) := g_i \otimes g_i \tag{1.30}$$

the action of Δ_0 on a generic element $\gamma \in \mathbb{C}\mathscr{G}$ can be obtained by linearity.

The algebra of $\mathbb{C}\mathscr{G}$, which we indicate as $\mathbb{C}\mathfrak{g}$, is also endowed with a Hopf algebra structure once (??) is given[1]. It is possible to show that (??) induces the following map on its algebra $\mathbb{C}\mathfrak{g}$:

$$\Delta_0 : \mathbb{C}\mathfrak{g} \to \mathbb{C}\mathfrak{g} \otimes \mathbb{C}\mathfrak{g}, \quad \Delta_0(\mathbb{X}) = \mathbb{X} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{X}, \quad \forall \mathbb{X} \in \mathfrak{g}. \tag{1.31}$$

endowing $\mathbb{C}\mathfrak{g}$ with a Hopf algebraic structure. The co-product (??) coincides with the usual action of angular momentum operators on multi-particle states, showing indeed that Quantum Mechanics needs a Hopf algebra structure rather than just a group.

1.3 Noncommutative space and deformed Poincaré action

From (??), at first sight it seems that the noncommutativity of spacetime coordinates also violates Poincaré invariance: the L.H.S. of (??) transforms in a non-trivial way under the standard action of the Poincaré group whereas the R.H.S. does not. The issue can be solved by noting that the L.H.S. of (??) is to be interpreted in terms of tensor products and $\star$, the deformed noncommutative product on the algebra $C^\infty(M)$:

$$[x_\mu, x_\nu] = x_\mu \star x_\nu - x_\nu \star x_\mu = m_\theta(x_\mu \otimes x_\nu - x_\nu \otimes x_\mu) \quad . \tag{1.32}$$

Here we introduced a notation which will be used often in the following. The $\star$-product can be seen as a map from $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \to \mathcal{A}_\theta$, which we will indicate by m_θ since in the limit $\theta_{\mu\nu} \to 0$, $\star$ goes to the standard pointwise, commutative product:

$$m_\theta : \mathcal{A}_\theta \otimes \mathcal{A}_\theta \to \mathcal{A}_\theta, \quad m_\theta(f_1 \otimes f_2)(x) := (f_1 \star f_2)(x) \tag{1.33}$$

As we described in the previous section, the way the Poincaré group $\mathscr{P}$ acts on the tensor product space is a further information which is not given by the way elements of the group act on x_μ. For this we need to define a *coproduct* Δ from $\mathscr{P} \to \mathscr{P} \otimes \mathscr{P}$ (More precisely Δ is a homomorphism from the group algebra $\mathbb{C}\mathscr{P}$ to $\mathbb{C}\mathscr{P} \otimes \mathbb{C}\mathscr{P}$.) In physics the standard choice is the diagonal map:

$$\Delta_0 : g \in \mathscr{P} \to \Delta_0(g) = g \otimes g \in \mathscr{P} \otimes \mathscr{P} \quad . \tag{1.34}$$

[1]Although we don't want to make the treatment heavier than it already is, we should mention that $\mathbb{C}\mathfrak{g}$ is very different from $\mathfrak{g}$, the algebra of $\mathscr{G}$. Formally $\mathbb{C}\mathfrak{g}$ is called the *enveloping algebra* of $\mathfrak{g}$. We will not address all the subtleties of such a definition here, referring the reader to the literature for details (? ?).

Endowed with Δ, $\mathscr{P}$ becomes a Hopf algebra, we will indicate it as $H\mathscr{P}$ and call if *Poincaré-Hopf algebra*. We have now enough information to define an action on tensor products. For example, for Δ_0, $\mathscr{P}$ acts on $x_\mu \otimes x_\nu$ according to $x_\mu \otimes x_\nu \to \Delta_0(g) x_\mu \otimes x_\nu := (g x_\mu \otimes g x_\nu)$.

In (**? ? ?**), it has been shown that there exists a choice for Δ, different from (**??**), which allows an action of the Poincaré group algebra (indicated in what follows by $g \triangleright f$) preserving the relations (**??**). The new Poincaré action we get is called the *twisted action*. The coproduct Δ_θ, which defines it, is called the *twisted coproduct*. Finally Δ_θ changes the standard Hopf algebra structure associated with the Poincaré group (the Poincaré-Hopf algebra $H\mathscr{P}$) given by Δ_0 (**??**) to a twisted Poincaré-Hopf algebra $H_\theta\mathscr{P}$. In the next chapter we will explain in details the theory of Hopf algebras twist deformations. For a complete treatment of deformations of Hopf algebras we refer to Appendix **??** or to the literature (**? ? ?**).

We conclude this chapter providing a further motivation why to expect the coproduct for the Poincaré group to be deformed with respect to the standard choice Δ_0, when acting on a noncommutative space. The action of a group on a algebra $\mathcal{A}$ is by definition an automorphism, that is it preserves the multiplication map in the following sense:

$$g \triangleright f_1 \star f_2 : \quad g(f_1 \star f_2) = m_\theta\Big(g \triangleright (f_1 \otimes f_2)\Big) = m_\theta\Big(\Delta(g)(f_1 \otimes f_2)\Big) \tag{1.35}$$

which can be summarized as the product m_θ has to commute with the group action (remember that the action of g on the tensor product $\mathcal{A} \otimes \mathcal{A}$ is always given by the coproduct Δ).

It can be shown that the choices Δ_0 (**??**) and (**??**) are suitable action in the case of the pointwise product among functions. That is for a quantum field theory on a commutative space. On noncommutative spaces, Δ_0 does no longer commutes with m_θ. The coproduct should then be deformed to make the action of the group an automorphism. Interestingly enough, the deformation Δ_θ which commutes with the multiplication map m_θ coincides with the deformed coproduct choice which restores Poincaré invariance.

The explicit construction of such deformation Δ_θ and the theory of quantum fields on a noncommutative spacetime will be the topic of the next chapter.

2

Quantum Fields on noncommutative Spaces: The Twisted Field Approach

So far we have been given the mathematical tools to deal with noncommutative spaces. Our next step is to introduce quantum fields on spaces where spacetime coordinates do not commute as in (**??**).

As we have seen in the last chapter, the action of the symmetry group in general, and the Poincaré group in particular, is modified on noncommutative spacetimes for compatibility with the spacetime coordinates commutation relations. Such a deformation is crucial in the construction of quantum fields on noncommutative spacetimes. We construct the fields à la Weinberg, that is creation and annihilation operators should provide an irreducible representation of the deformed Poincaré group. We also discuss a series of consequences of such a construction like the twisted statistics.

Such a construction might be extremely hard for generic Quantum Groups (deformed Hopf Algebras) so we restrict to a subset of them of which we can provide an explicit representation on a Hilbert space.

2.1 The Drinfel'd twist and deformed coproduct

In the previous chapter we noticed that the relations (**??**) can be implemented by deforming the product of the standard commutative algebra of functions on our n-

dimensional space-time. We indicate such an algebra as $\mathcal{A}_0 \equiv (C^\infty(M), m_0)$, where $M \cong \mathbb{R}^n$, $C^\infty(M)$ are smooth functions on M and m_0 is the point-wise multiplication map:

$$m_0(f \otimes g)(x) = f(x)g(x) = g(x)f(x) = m_0(g \otimes f)(x).$$

There is a general procedure to deform such a product in a controlled way using the so-called *twist deformation* (**?**) which we are now going to introduce.

2.1.1 Twisting the *Product*

Let us denote as $\mathcal{A}_\theta = (C^\infty(M), m_\theta)$ a deformation of $\mathcal{A}_0$ which leads to (**??**). Henceforth we assume that $\mathcal{A}_\theta$ is a *twist deformation*, that is the deformed product can be written as $m_\theta = m_0 \circ \mathcal{F}_\theta$, where m_0 is the point-wise, commutative product, and $\mathcal{F}_\theta$, the map which makes m_0 not commutative anymore, gets the name of *twist map*. The two satisfy:

$$m_\theta(f \otimes g)(x) = m_0 \circ \mathcal{F}_\theta(f \otimes g)(x) := (f \star g)(x), \tag{2.1}$$

$$\mathcal{F}_\theta : \; C^\infty(M) \otimes C^\infty(M) \to C^\infty(M) \otimes C^\infty(M) \qquad \text{and} \qquad \mathcal{F}_\theta \to \mathbb{1} \quad \text{as} \quad \theta \to 0.$$

An explicit form of $\mathcal{F}_\theta$ is, for example,

$$\mathcal{F}_\theta = \exp \frac{i}{2}\theta[\partial_x \otimes \partial_y - \partial_y \otimes \partial_x]. \tag{2.2}$$

In particular the unit is preserved by the deformation. We notice that

1) $\mathcal{F}_\theta$ is one-to-one and invertible;
2) $\mathcal{F}_\theta$ acts on the tensor product in a non-factorizable manner, i.e. the action on $C^\infty(M) \otimes C^\infty(M)$ intertwines the two factors.

The above choice (**??**) is a particular one and it is called the *Moyal* deformation. So $\mathcal{F}_\theta \equiv \mathcal{F}_\theta^{\mathcal{M}}$ leads to the Moyal plane $\mathcal{A}_\theta^{\mathcal{M}}$:

$$m_\theta^{\mathcal{M}}(f \otimes g)(x,y) \equiv f(x,y) \cdot g(x,y) + \frac{i}{2}\theta\big[(\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)\big] + \sum_{n=2} \frac{\big[\frac{i}{2}\theta(\partial_x \otimes \partial_y - \partial_y \otimes \partial_x)\big]^n}{n!}(f \otimes g).$$

But it is not unique. Another choice, leading to the Wick-Voros plane $\mathcal{A}_\theta^V$ is

$$\mathcal{F}_\theta^V = \exp\frac{1}{2}\theta[\partial_x\otimes\partial_x+\partial_y\otimes\partial_y]\mathcal{F}_\theta^{\mathcal{M}} = \mathcal{F}_\theta^{\mathcal{M}}\exp\frac{1}{2}\theta[\partial_x\otimes\partial_x+\partial_y\otimes\partial_y]. \tag{2.3}$$

We will have chance later in the text, to discuss how the choice between the two affects the physics.

2.1.2 Twisting the *Coproduct*

As we already noticed at the end of the last chapter, the noncommutative relations (**??**) bring with them another problem: at first sight it seems that the noncommutativity of spacetime coordinates violates Poincaré invariance: the l.h.s. of (**??**) transforms in a non-trivial way under the standard action of the Poincaré group $\mathscr{P}$ whereas the r.h.s. does not. Yet we can exploit the freedom we have, to choose the action on the tensor product space of the Poincaré group, or rather, of its group algebra $\mathbb{C}\mathscr{P}$, to try to find a way to consistently act on the deformed algebras $\mathcal{A}_\theta$. What we obtain is called twisted action (**? ? ?**). It goes as follows. The l.h.s. of (**??**) is an element of the tensor product space followed by $m_\theta : \mathcal{A}_\theta^{\mathcal{M},V}\otimes\mathcal{A}_\theta^{\mathcal{M},V}\to\mathcal{A}_\theta^{\mathcal{M},V}$. The way in which $\mathbb{C}\mathscr{P}$ acts on the tensor product space requires a coproduct Δ:

$$\Delta:\ \mathbb{C}\mathscr{P}\to\mathbb{C}\mathscr{P}\otimes\mathbb{C}\mathscr{P}.$$

Usually we assume for the coproduct the diagonal map Δ_0:

$$\Delta_0(g) = g\otimes g \qquad \forall\, g\in\mathscr{P}. \tag{2.4}$$

which extends to $\mathbb{C}\mathscr{P}$ by linearity. But (**??**) is not the only possible choice. The idea proposed in (**? ? ?**) is that we can assume a different coproduct on $\mathbb{C}\mathscr{P}$, that is "twisted" or deformed with respect to Δ_0, to modify the action of the Poincaré group on tensor product spaces in such a way that it does preserve relations (**??**). As we already stressed previously, the change of Δ_0 is not a mere mathematical construction, as it affects the way composite systems transform under spacetime symmetries. This observation will have deep consequences in the physical interpretation of the theory as it will be shown later. This modification changes the standard Hopf algebra structure associated with the Poincaré group (the Poincaré–Hopf algebra $H\mathscr{P}$) to a twisted Poincaré–Hopf algebra $H_\theta\mathscr{P}$ ($H_0\mathscr{P}\equiv H\mathscr{P}$).

We already introduced two possible deformations, (**??**) and (**??**), therefore the deformed algebra is not unique. For the Moyal and Wick–Voros cases they are different, although isomorphic[1]. The isomorphism map is

$$\gamma = \mathrm{e}^{-\frac{1}{4}\theta(\partial_x^2+\partial_y^2)}, \qquad \gamma : \mathcal{A}_\theta^{\mathcal{M}} \to \mathcal{A}_\theta^{V} \qquad \text{and} \qquad \mathcal{F}_\theta^{V} = \gamma \otimes \gamma \mathcal{F}_\theta^{\mathcal{M}} \Delta\big(\gamma^{-1}\big).$$

We denote them by $H_\theta^{\mathcal{M},V}\mathscr{P}$ when we want to emphasise that we are working with Moyal and Wick–Voros spacetimes. The question of whether or not they give rise to equivalent QFT's will be addressed in the next chapter.

The explicit form of the deformed coproduct Δ_θ of $H_\theta\mathscr{P}$ is obtained from requiring that the action of $\mathbb{C}\mathscr{P}$ is an *automorphism* of the new algebra of functions $\mathcal{A}_\theta$ on spacetime. That is, the action of $\mathbb{C}\mathscr{P}$ has to be compatible with the new noncommutative multiplication rule (**??**) in the sense of (**??**).

It is easy to see that the standard coproduct choice (**??**) is not compatible (**? ? ?**) with the action of $\mathbb{C}\mathscr{P}$ on the deformed algebra $\mathcal{A}_\theta$. In the cases under consideration, where $\mathcal{A}_\theta$ are twist deformations of $\mathcal{A}_0$, there is a simple rule to get deformations Δ_θ of Δ_0 compatible with m_θ. They are given by the formula:

$$\Delta_\theta = (F_\theta)^{-1}\Delta_0 F_\theta, \tag{2.5}$$

where F_θ is an element in $H_\theta\mathscr{P}\otimes H_\theta\mathscr{P}$ and it is determined by the map $\mathcal{F}_\theta$ introduced before, $\mathcal{F}_\theta$ being the realisation of F_θ on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$.

For $\mathcal{F}_\theta = \mathcal{F}_\theta^{\mathcal{M},V}$, the corresponding $F_\theta^{\mathcal{M},V}$ give us the Hopf algebras $H_\theta^{\mathcal{M},V}\mathscr{P}$.

Without going deeper into the deformation theory of Hopf algebras, which will be discussed in Appendix **??**, we just note that the deformations we are considering here are very specific ones since we keep the multiplication rule unchanged and deform only the co-structure of the underlying Hopf algebra. Thus for $\mathbb{C}\mathscr{P}$, we only change Δ_0 to Δ_θ leaving the group multiplication the same. For a deeper discussion on deformations of algebras and Hopf algebras, we refer again to Appendix **??** and the literature (**? ? ? ?**).

[1] The two deformations are in fact equivalent in Hopf algebra deformation theory. That is they belong to the same equivalence class in the non-Abelian cohomology that classifies Hopf algebra twist-deformations. See Appendix **??** or for even more details (**?**).

2.1.3 Twisting *Statistics*

Lastly we have to introduce the concept of twisted statistics. It is a strict consequence of the twisted action of the Poincaré group on the tensor product space (**??**). In quantum mechanics two kinds of particles, with different statistics, are admitted: fermions, which are described by fully antisymmetrised states, and bosons, which are instead completely symmetric. Let $\mathcal{H}$ be a single particle Hilbert space. Then given a two-particle quantum state, $\alpha \otimes \beta$ with $\alpha, \beta \in \mathcal{H}$, we can get its symmetrised and anti-symmetrised parts as:

$$\alpha \otimes_{S,A} \beta = \frac{\mathbb{1} \pm \tau_0}{2} \alpha \otimes \beta,$$

where the map τ_0 is called the *flip operator* and it simply switches the elements on which it acts,

$$\tau_0(\alpha \otimes \beta) = \beta \otimes \alpha.$$

From the foundations of quantum field theories it can be proved that the statistics of particles have to be superselected, that is Poincaré transformations cannot take bosons (fermions) into fermions (bosons). In other words, a symmetric (antisymmetric) state must still be symmetric (antisymmetric) after the action of any element of the Poincaré group. This requirement implies that the flip operator has to commute with the coproduct of any element of $\mathbb{C}\mathscr{P}$. As can be trivially checked, the action of τ_0 commutes with the coproduct $\Delta_0(g)$ of $g \in \mathscr{P}$, but not with $\Delta_\theta(g)$. If we do not modify the flip operator, we end up with a theory in which, for example, a rotation can transform a fermion into a boson.

If the deformation of the coproduct is of the kind we have been considering so far, that is a *twist deformed* coproduct as in (**??**), again it is easy to find a deformation τ_θ of the flip operator τ_0 which commutes with Δ_θ:

$$\tau_0 \to \tau_\theta = (F_\theta)^{-1} \tau_0 F_\theta \tag{2.6}$$

and, moreover $\tau_\theta^2 = \mathbb{1} \otimes \mathbb{1}$. This equation contains the $\mathcal{R}$-matrix of a quasi-triangular Hopf algebra. By definition, $\mathcal{R}$ is given by

$$\tau_\theta = \mathcal{R} \circ \tau_0. \tag{2.7}$$

Hence:

$$\mathcal{R} := F_\theta^{-1} \circ F_{\theta 21}, \qquad F_{\theta 21} \equiv \tau_0 F_\theta \tau_0^{-1}. \tag{2.8}$$

In quantum physics on a noncommutative spacetime, we then consider symmetrisation (antisymmetrisation) with respect to τ_θ rather then τ_0:

$$\alpha \otimes_{S_\theta, A_\theta} \beta = \frac{\mathbb{1} \pm \tau_\theta}{2} \alpha \otimes \beta. \tag{2.9}$$

2.2 Examples: Moyal and Wick-Voros planes

We now proceed with the study of two explicit examples: Moyal and Wick-Voros planes. We denote by $\mathcal{F}_\theta^{\mathcal{M},V}$ and $m_\theta^{\mathcal{M},V}$ the twists and multiplication maps for the deformed algebras $\mathcal{A}_\theta^{\mathcal{M},V}$ respectively

$$m_\theta^{\mathcal{M},V}(f \otimes g) \equiv m_0 \circ \mathcal{F}_\theta^{\mathcal{M},V}(f \otimes g) \quad . \tag{2.10}$$

In the following, for the sake of simplicity, we will work in two dimensions. The generalization to arbitrary dimensions will be discussed at the end of the section.

In two dimensions, we can always write $\theta_{\mu\nu}$ as

$$\theta_{\mu\nu} \;=\; \theta\epsilon_{\mu\nu} \tag{2.11}$$

$$\epsilon_{01} = \;-\; \epsilon_{10} = 1 \tag{2.12}$$

where θ is a constant. Then the two twists $\mathcal{F}_\theta^{\mathcal{M},V}$ assume the form

$$\mathcal{F}_\theta^{\mathcal{M}} = \exp \tfrac{i}{2}\theta[\partial_x \otimes \partial_y - \partial_y \otimes \partial_x] \;, \tag{2.13}$$

$$\mathcal{F}_\theta^{V} = \exp \tfrac{1}{2}\theta[\partial_x \otimes \partial_x + \partial_y \otimes \partial_y]\mathcal{F}_\theta^{\mathcal{M}} = \mathcal{F}_\theta^{\mathcal{M}} \exp \tfrac{1}{2}\theta[\partial_x \otimes \partial_x + \partial_y \otimes \partial_y] \;. \tag{2.14}$$

Hereafter we will call $\mathcal{F}_\theta^{\mathcal{M}}$ and $\mathcal{F}_\theta^{V}$ the Moyal and Wick-Voros twists, and $\mathcal{A}_\theta^{\mathcal{M}}$ and $\mathcal{A}_\theta^{V}$ the Moyal and Wick-Voros algebras respectively. Both deformations, $\mathcal{A}_\theta^{\mathcal{M}}$ and $\mathcal{A}_\theta^{V}$, realize the commutation relations (**??**). This fact shows how noncommutativity of spacetime does not fix uniquely the deformation of the algebra. There are many more noncommutative algebras of functions on spacetime that realize (**??**) with different twisted products. We will address the study of how this freedom reflects on the quantum field theory side in the next chapter. For these purposes it is enough to work with two of them. Thus hereafter we will only work with $\mathcal{A}_\theta^{\mathcal{M},V}$.

Given the above expressions for the twists, explicit expressions for the noncommutative product of the functions in the two cases follow immediately from (**??**):

$$(f \star_{\mathcal{M}} g)(x) = m_\theta^{\mathcal{M}}(f \otimes g)(x) \equiv f(x) \mathrm{e}^{\frac{i}{2}\theta_{\alpha\beta}\overleftarrow{\partial^\alpha} \otimes \overrightarrow{\partial^\beta}} g(x) \quad , \tag{2.15}$$

$$(f \star_V g)(x) = m_\theta^{V}(f \otimes g)(x) \equiv f(x) \mathrm{e}^{\frac{i}{2}\left(\theta_{\alpha\beta}\overleftarrow{\partial^\alpha} \otimes \overrightarrow{\partial^\beta} - i\theta\delta_{\alpha\beta}\overleftarrow{\partial^\alpha} \otimes \overrightarrow{\partial^\beta}\right)} g(x) \quad . \tag{2.16}$$

If we let the $\star$-product to act on the coordinate functions, we get in both cases the noncommutative relations (**??**).

Once the two twists are given, following (**??**), we can immediately write down the deformations of the two coproducts as well:

$$\Delta_\theta^{\mathcal{M},V}(g) = \left(F_\theta^{\mathcal{M},V}\right)^{-1}\Delta_0(g)F_\theta^{\mathcal{M},V} = \left(F_\theta^{\mathcal{M},V}\right)^{-1}(g\otimes g)F_\theta^{\mathcal{M},V} \tag{2.17}$$

Here $F_\theta^{\mathcal{M},V} \in H\mathscr{P}\otimes H\mathscr{P}$ are:

$$F_\theta^{\mathcal{M}} = \exp\left(-\tfrac{i}{2}\theta[P_x\otimes P_y - P_y\otimes P_x]\right) \ , \tag{2.18}$$

$$F_\theta^{V} = \exp\left(-\tfrac{1}{2}\theta[P_x\otimes P_x + P_y\otimes P_y]\right)F_\theta^{\mathcal{M}} = F_\theta^{\mathcal{M}}\exp\left(-\tfrac{1}{2}\theta[P_x\otimes P_x + P_y\otimes P_y]\right) \tag{2.19}$$

where P_μ are translation generators. Their realisation on $\mathcal{A}_\theta^{\mathcal{M},V}$ is:

$$P_\mu \triangleright f(x) \equiv (P_\mu f)(x) = -i(\partial_\mu f)(x) \quad . \tag{2.20}$$

We end this section with a discussion on how the τ_0 gets twisted in the two cases. As $F_\theta^{\mathcal{M}}$ is skew-symmetric and F_θ^V is the composition of $F_\theta^{\mathcal{M}}$ and a symmetric part, we get:

$$F_{\theta 21}^{\mathcal{M}} = \left(F_\theta^{\mathcal{M}}\right)^{-1} \qquad , \tag{2.21}$$

$$F_{\theta 21}^{V} = \exp\left(-\tfrac{1}{2}\theta[P_x\otimes P_x + P_y\otimes P_y]\right)\left(F_\theta^{\mathcal{M}}\right)^{-1} \qquad . \tag{2.22}$$

In the $\mathcal{R}$-matrix (**??**), any symmetric part of the twist cancels. Thus in both Moyal and Wick-Voros cases, the statistics of particles is twisted in the same way:

$$\mathcal{R}^{\mathcal{M},V} = \left(F_\theta^{\mathcal{M}}\right)^{-2} = \exp\left(-i\theta[P_x\otimes P_y - P_y\otimes P_x]\right) \quad . \tag{2.23}$$

Generalization to N-dimension

We can now briefly outline how to generalize our considerations on the Wick-Voros twist (**??**) to $2N$-dimensions[1]. We can always choose $\hat{x}_\mu$ so that $\theta_{\mu\nu}$, now an $2N\times 2N$ skew-symmetric matrix, becomes a direct sum of N 2×2 ones. These 2×2 matrices are of the form (**??**), but different $2{\times}2$ matrices may have different factors θ. For every such 2×2 block, we have a pair of $\hat{x}$'s which can be treated as in the 2-dimensional case above. (Of course there is no twist in any block with a vanishing θ.)

[1] In $2N+1$-dimensions, we can always choose $\theta_{\mu\nu}$ so that $\theta_{\mu,2N+1}{=}\theta_{2N+1,\mu}{=}0$.

2.3 Quantum Field Theories on Moyal and Wick-Voros planes

It is time now to discuss how to quantize the two theories introduced in the previous section. Our approach to the Moyal plane is discussed in (? ?) and to the Wick-Voros plane can be found in (? ?). For another approach to the latter, see (?).

The quantization procedure consists in finding a set of creation operators, and by adjointness the annihilation counterpart, which create multiparticle states providing a unitary representation of the twisted Poincaré-Hopf symmetry. Such a set of creation and annihilation operators must also implement the appropriate twisted statistics (??-??). Out of them we can construct the twisted quantum fields. It has been in fact proven elsewhere (? ?) that the Hamiltonan constructed out of such twisted fields is Hopf-Poincaré invariant.

Let us first consider the Moyal case. As our previous work (? ?) shows,

$$a_p^{\mathcal{M}} = c_p \exp(-\tfrac{i}{2}p_\mu\theta^{\mu\nu}P_\nu) \tag{2.24}$$

$$a_p^{\mathcal{M}\dagger} = c_p^\dagger \exp(\tfrac{i}{2}p_\mu\theta^{\mu\nu}P_\nu) \tag{2.25}$$

where c_p, $c_p^\dagger$ are the untwisted $\theta^{\mu\nu} = 0$ annihilation and creation operators (we can assume all such operators to refer to in-, out- or free-operators as the occasion demands), provide the operators we were looking for. Let's see that.

Since we are considering only deformations in which the coproduct is changed, the way in which the Poincaré group acts on a single particle state is the usual one. Therefore we expect the creation operator (??) to act on the vacuum like the untwisted operator $c_p^\dagger$. Since $P_\nu|0\rangle = 0$, it is easy to see that this is in fact the case. It is then plausible that the generators of Poincaré transformations on the Hilbert space under consideration have to be the untwisted ones. We will now confirm this: if $(a,\Lambda) \to U(a,\Lambda) \equiv U(\Lambda)U(a)$ is the $\theta = 0$ unitary representation of the Poincaré group, we will show that the multiparticle states created by acting with (??) on the vacuum transform with the Moyal coproduct (??).

If we consider the action of a general group element of the Poincaré group (a,Λ) on a two–particle state $|p,q\rangle_\theta$, we expect

$$\begin{aligned}\Delta_\theta^{\mathcal{M}}\Big((a,\Lambda)\Big) \triangleright |p,q\rangle_\theta &= \Big(F_\theta^{\mathcal{M}}\Big)^{-1}\Big((a,\Lambda)\otimes(a,\Lambda)\Big)F_\theta^{\mathcal{M}} \triangleright |p,q\rangle_\theta \\ &= |\Lambda p,\Lambda q\rangle_\theta \mathrm{e}^{-\frac{i}{2}(p\wedge q-\Lambda p\wedge\Lambda q)}\mathrm{e}^{-i(p+q)\cdot a},\end{aligned} \tag{2.26}$$

where $a \wedge b := a_\mu \theta^{\mu\nu} b_\nu$, and we have used the properties of a momentum eigenstate, $P_\mu \triangleright |p\rangle = p_\mu |p\rangle$ and $(a, \Lambda) \triangleright |p\rangle = |\Lambda p\rangle \mathrm{e}^{-ip\cdot a}$.

For the $\theta = 0$ generators of the unitary representation of the Poincaré group on the Hilbert space under consideration, we have

$$U(a,\Lambda) c_p^\dagger U^{-1}(a,\Lambda) = \mathrm{e}^{ip\cdot a} c_{\Lambda p}^\dagger \tag{2.27}$$

Defining (**?**)[1]

$$|p,q\rangle_\theta^{\mathcal{M}} = a_q^{\mathcal{M}\dagger} a_p^{\mathcal{M}\dagger} |0\rangle \tag{2.28}$$

we can now explicitly compute how $U(a, \Lambda)$ acts on the two–particle state considered in (**??**):

$$\begin{aligned} U(a,\Lambda)|p,q\rangle_\theta &= U(a,\Lambda) c_q^\dagger U^{-1}(a,\Lambda) U(a,\Lambda) \mathrm{e}^{\frac{i}{2} q\wedge P} c_p^\dagger |0\rangle = c_{\Lambda q}^\dagger c_{\Lambda p}^\dagger |0\rangle \mathrm{e}^{-i(p+q)\cdot a} \mathrm{e}^{\frac{i}{2} q\wedge p} \\ &= a_{\Lambda q}^{\mathcal{M}\dagger} \mathrm{e}^{\frac{i}{2} q\wedge P} c_{\Lambda p}^\dagger |0\rangle \mathrm{e}^{\frac{i}{2} q\wedge p} \mathrm{e}^{-i(p+q)\cdot a} = |\Lambda p, \Lambda q\rangle \mathrm{e}^{-\frac{i}{2}(p\wedge q - \Lambda p \wedge \Lambda q)} \mathrm{e}^{-i(p+q)\cdot a} \end{aligned} \tag{2.29}$$

In the above computation we have used $P_\nu|0\rangle = 0$ and a relation which will be used repeatedly in what follows: $\mathrm{e}^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c_q^\dagger \mathrm{e}^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} = \mathrm{e}^{\frac{i}{2} p_\mu \theta^{\mu\nu} [P_\nu, \,\cdot\,]} c_q^\dagger = \mathrm{e}^{\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} c_q^\dagger$. Thus (**??**) coincides with (**??**). It is a remarkable fact that the appropriate deformation of the coproduct naturally appears as the $\theta = 0$ Poincaré group generator acts on the two particle states obtained by the creation operator (**??**). This result generalises to n-particles states.

The two–particle state in (**??**) also fulfills the twisted statistics. From (**??**-**??**) we expect that (throughout this paper we will only consider the bosonic case):

$$|p,q\rangle_\theta^{\mathcal{M}} = \frac{\mathbb{1} + \tau_\theta^{\mathcal{M}}}{2} |p,q\rangle = \frac{1}{2}\Big(|p,q\rangle + \mathrm{e}^{-i\vec{q}\wedge\vec{p}} |q,p\rangle\Big) \tag{2.30}$$

and

$$|q,p\rangle_\theta^{\mathcal{M}} = \frac{\mathbb{1} + \tau_\theta^{\mathcal{M}}}{2} |q,p\rangle = \frac{1}{2}\Big(|q,p\rangle + \mathrm{e}^{-i\vec{p}\wedge\vec{q}} |p,q\rangle\Big) = \mathrm{e}^{-i\vec{q}\wedge\vec{p}} |p,q\rangle_\theta^{\mathcal{M}} \tag{2.31}$$

On the other hand using the definition (**??**) of two–particle states in terms of the creation operators $a_p^{\dagger\mathcal{M}}$'s:

$$|q,p\rangle_\theta^{\mathcal{M}} = a_p^{\mathcal{M}\dagger} a_q^{\mathcal{M}\dagger} |0\rangle = c_p^\dagger \exp(\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu) c_q^\dagger |0\rangle = \mathrm{e}^{-\frac{i}{2}\vec{q}\wedge\vec{p}} a_q^\dagger \exp(-\frac{i}{2} q_\mu \theta^{\mu\nu} P_\nu) c_p^\dagger |0\rangle = \mathrm{e}^{-i\vec{q}\wedge\vec{p}} |p,q\rangle_\theta \tag{2.32}$$

[1] In the case under study, because of the twisted statistics, the creation operators, and likewise their adjoints, do not commute. The order in which they act on a state becomes then an issue. The choice made here is motivated by asking for consistency (**?**). The scalar product we consider for the definition of the adjoint is the one associated with the untwisted creation and annihilation operators.

where we have used the relation introduced above and the fact that $c_p^\dagger$ and $c_q^\dagger$ commute. So these creation operators do implement the statistics we want.

The operators defined in (**??**) and (**??**) are referred to as *dressed operators* in the literature (**? ? ?**). They are obtained from the $\theta = 0$ ones by *dressing* them using the exponential term.

We can now introduce the quantum field on the Moyal plane $\varphi_\theta^{\mathcal{M}}$:

$$\varphi_\theta^{\mathcal{M}}(x) = \int \mathrm{d}\mu(p)\big[a_p^{\mathcal{M}}\mathrm{e}_{-p}(x) + a_p^{\mathcal{M}\dagger}\mathrm{e}_p(x)\big] \tag{2.33}$$

where $\mathrm{e}_p(x)$ denotes $\mathrm{e}^{ip\cdot x}$ as usual, through a similar dressing of the standard $\theta = 0$ scalar field:

$$\varphi_\theta^{\mathcal{M}} = \varphi_0 \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \quad . \tag{2.34}$$

This formula is first deduced for in-, out- or free-fields. For example,

$$\varphi_\theta^{\mathcal{M},\mathrm{in}} = \varphi_0^{\mathrm{in}} \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \quad . \tag{2.35}$$

But since the Heisenberg field becomes the 'in' field as $x_0 \to -\infty$,

$$\varphi_0(x) \to \varphi_0^{\mathrm{in}} \quad \text{as} \quad x_0 \to -\infty \quad , \tag{2.36}$$

and P_μ is time-independent, we (at least heuristically) infer (**??**) for the fully interacting Heisenberg field.

Products of the field (**??**) have a further remarkable property which we have called *self-reproducing property*:

$$(\varphi_\theta^{\mathcal{M}} \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M}})(x) = \Big[(\varphi_0 \cdot \varphi_0)(x)\Big] \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \quad . \tag{2.37}$$

where the $\cdot$ represents the standard point-wise product. This property generalises to products of N fields

$$\underbrace{\varphi_\theta^{\mathcal{M}} \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M}} \star_{\mathcal{M}} \dots \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M}}}_{N-factors} = \varphi_0^N \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \tag{2.38}$$

where again φ_0^N indicates the N-th power with respect the commutative product m_0. This self-reproducing property plays a significant role in general theory. It is the basis for the proof of the absence of UV-IR mixing in Moyal field theories (with no gauge fields) (**? ?**).

Now consider the Wick-Voros case. The twisted creation operators, which correctly create states from the vacuum transforming under the twisted coproduct, are (?)

$$a_p^{V\dagger} = c_p^\dagger \mathrm{e}^{\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu - i\theta p_\nu P_\nu)} \tag{2.39}$$

where $p_\nu P_\nu$ uses the Euclidean scalar product. Its adjoint is

$$a_p^V = \mathrm{e}^{-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu)} c_p \quad . \tag{2.40}$$

We prove in Chapter **??** that (**??**) and (**??**) are also dictated by the covariance of quantum fields.

It can be shown that, like in Moyal case, the states obtained by the action of $a_\theta^{V\dagger}$ reproduce the appropriate twisted statistics too.

Although we have obtained, like in Moyal, (**??**) and (**??**) by dressing the $\theta = 0$ operators, the quantum field φ_θ^V on Wick-Voros plane:

$$\varphi_\theta^V(x) = \int \mathrm{d}\mu(p) \left[a_p^V \mathrm{e}_{-p}(x) + a_p^{V\dagger} \mathrm{e}_p(x) \right] \tag{2.41}$$

cannot be obtained from an overall dressing like in (**??**). This property fails due to the fact that is not possible to factorise the same overall exponential out of both (**??**) and (**??**), since it is not possible in (**??**) and (**??**) to move the exponential from right (left) to left (right), that is:

$$a_p^{V\dagger} \neq \exp\left(-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu)\right) c_p^\dagger \quad , \tag{2.42}$$

$$a_p^V \neq c_p \exp\left(-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu)\right) \quad . \tag{2.43}$$

This is because $c_p^\dagger$ and c_p do not commute with the exponentials in (**??**) and (**??**), in fact moving $c_p^\dagger$ (c_p) to the right (left) will bring a factor $\mathrm{e}^{-\frac{\theta}{2} p_\nu p_\nu}$:

$$a_p^{V\dagger} = \exp\left(\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu - i\theta p_\nu P_\nu) - \frac{\theta}{2} p_\nu p_\nu\right) c_p^\dagger \tag{2.44}$$

$$a_p^V = c_p \exp\left(-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu) - \frac{\theta}{2} p_\nu p_\nu\right) \tag{2.45}$$

A consequence is that we have to twist the creation-annihilation parts $\varphi_0^{(\pm)I}$ ($I=$ in-, out- or free-) fields separately:

$$\varphi_\theta^{(+)V,\mathrm{I}} = \int \mathrm{d}\mu(p) a^{V,\mathrm{I}\dagger} \mathrm{e}_p = \varphi_0^{(+)\mathrm{I}} e^{\frac{1}{2}(\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu - i\theta \overleftarrow{\partial}_\mu P_\mu)} \quad , \tag{2.46}$$

$$\varphi_\theta^{(-)V,\mathrm{I}} = \int \mathrm{d}\mu(p) a^{V,\mathrm{I}} \mathrm{e}_{-p} = e^{\frac{1}{2}(\overrightarrow{\partial}_\mu \theta^{\mu\nu} P_\nu + i\theta \overrightarrow{\partial}_\mu P_\mu)} \varphi_0^{(-)\mathrm{I}} \quad , \tag{2.47}$$

$$\mathrm{e}_p(x) = \mathrm{e}^{ip\cdot x}, \quad \mathrm{d}\mu(p) := \frac{\mathrm{d}^3 p}{2\sqrt{\vec{p}^2 + m^2}} \quad m = \text{mass of the field } \varphi_0^{\mathrm{I}} \quad , \tag{2.48}$$

where now we have added the superscript I to φ_0^{I}, $a_p^{V,\mathrm{I}\dagger}$, and $a_p^{V,\mathrm{I}}$.

Therefore to obtain the field (**??**) we have to twist creation and annihilation parts separately

$$\varphi_\theta^{V,\mathrm{I}} = \varphi_\theta^{(+)V,\mathrm{I}} + \varphi_\theta^{(-)V,\mathrm{I}} \tag{2.49}$$

A further point relates to the self-reproducing property of these Wick-Voros fields. The quantum fields $\varphi_\theta^{(\pm)\mathrm{I}V}$ enjoy the self-reproducing property, but in different ways. Thus

$$\underbrace{\varphi_\theta^{(+)V,\mathrm{I}} \star_V \varphi_\theta^{(+)V,\mathrm{I}} \star_V \ldots \star_V \varphi_\theta^{(+)V,\mathrm{I}}}_{M-factors} = \left(\varphi_0^{(+)\mathrm{I}}\right)^M \mathrm{e}^{\frac{1}{2}(\overleftarrow{\partial}_\mu)\theta^{\mu\nu}P_\nu - i\theta\overleftarrow{\partial}_\mu P_\mu} \tag{2.50}$$

$$\underbrace{\varphi_\theta^{(-)V,\mathrm{I}} \star_V \varphi_\theta^{(-)V,\mathrm{I}} \star_V \ldots \star_V \varphi_\theta^{(-)V,\mathrm{I}}}_{M'-factors} = \mathrm{e}^{\frac{1}{2}(\overrightarrow{\partial}_\mu)\theta^{\mu\nu}P_\nu + i\theta\overrightarrow{\partial}_\mu P_\mu} \left(\varphi_0^{(-)\mathrm{I}}\right)^{M'} \tag{2.51}$$

where, as in (**??**), $\left(\varphi_0^{(\pm)\mathrm{I}}\right)^N$ is the N-th power of $\varphi_0^{(\pm)I}$ with respect to the commutative product m_0. Given (**??**), it follows that the full field, φ_θ^V, does not have the self-reproducing property.

3

Weak and Strong equivalence: Moyal versus Wick-Voros

In the previous chapter we introduced the Moyal and Wick-Voros quantum field theories. Given that we presented at least two way of realizing relations (**??**), it is a legitimate question to ask how these different ways of implementing quantum fields on a noncommutative spacetime relate to each other. More specifically whether or not give rise to equivalent quantum theory.

We try to systematically address this question in the present chapter in the context of dressing transformation, that is the specific quantization procedure outlined in the previous chapter. Most of the material in the present chapter is based on two papers, (**?**) and (**?**).

3.1 Weak Equivalence

We already addressed the question of equivalence of two quantum field theories on noncommutative spaces in (**?**) and (**?**). We want to recall briefly here what we called "classical equivalence" in the former manuscript and "weak equivalence" in the latter.

Mathematically, in the theories we are dealing with, there are two deformations involved. The first one is at the product (algebraic) level because of the $\star$-product which makes the algebra of functions on spacetime noncommutative. The second is the Hopf algebraic deformation of the symmetry group acting on the deformed algebra of functions. We have shown in (**?**) how the two are strongly tied, but still mathematically different.

Let us denote by $\mathcal{A}_\theta, H_\theta$ and $\mathcal{A}'_\theta, H'_\theta$ two different pairs of deformations of spacetime and of the Hopf algebras of the kinematical group acting on them. We will say that the two theories constructed from them are "weakly equivalent" if both pair of algebras are equivalent $\mathcal{A}_\theta \cong \mathcal{A}'_\theta$ and $H_\theta \cong H'_\theta$, where the notion of equivalence of deformations of algebras and Hopf algebras can be found respectively in (**?**) and (**?**). (In (**?**), this equivalence was called "classical equivalence", but "weak equivalence" seems more appropriate).

In (**?**), we have shown that if the pair of deformations are equivalent both at the algebraic and Hopf algebraic level, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}_\theta \otimes \mathcal{A}_\theta & \xrightarrow{\mathrm{T}\otimes\mathrm{T}} & \mathcal{A}'_\theta \otimes \mathcal{A}'_\theta \\ \big\downarrow{\scriptstyle \Delta_\theta(g)} & & \big\downarrow{\scriptstyle \Delta'_\theta(\mathrm{T}g\mathrm{T}^{-1})} \\ \mathcal{A}_\theta \otimes \mathcal{A}_\theta & \xrightarrow{\mathrm{T}\otimes\mathrm{T}} & \mathcal{A}'_\theta \otimes \mathcal{A}'_\theta \end{array} \tag{3.1}$$

for all $g \in H_\theta$.

Here the map T is the one which maps $\mathcal{A}_\theta$ to $\mathcal{A}'_\theta$ (**?**). In Appendix **??** we will prove that if $\mathcal{A}_\theta \cong \mathcal{A}'_\theta$, then the two Hopf algebra deformations which are compatible with the product in each deformed algebra are also equivalent *provided* $\mathrm{T} \in H_\theta$. This result reduces the "weak equivalence" of two field theories on noncommutative spacetimes to the requirement that the two algebras of functions are equivalent under the action of H_θ.

The meaning of diagram (**??**) is simple. It is just the requirement that the map T which implements the isomporphism $\mathcal{A}_\theta \to \mathcal{A}'_\theta$ also correctly implements the isomorphism $H_\theta \to H'_\theta$.

We call (**??**) "weak equivalence" because (**??**) is a necessary condition but not sufficient to establish the equivalence of quantum field theories on H_θ and H'_θ. We call the obstruction blocking the implementation of this weak equivalence in quantum field theories a "quantum field anomaly". It is discussed in what follows. It does not appear in quantum mechanics as already shown in (**?**).

3.2 Quantum Field Theory on a noncommutative spacetime

We want now to proceed to a comparison between the two quantum field theories, namely Wick-Voros and Moyal. We already anticipated that in the former construction new features arise with respect to the standard quantum field theory on the Moyal plane (**? ?**). In this section, we are going to recall the main points of a quantum field theory on a noncommutative spacetime. Then we will show that quantum field theory on the Wick-Voros spacetime is not consistent if constructed using dressing transformation the way outlined in the previous chapter.

The twisted quantum fields should carry a unitary representation of the Poincaré group which implements the twisted coproduct. These fields should also implement the twisted statistics. We shown the details in the previous chapter.

Let us first consider the Moyal case. From (**??**) and (**??**),

$$a_p^{\mathcal{M}} = c_p \exp(-\tfrac{i}{2} p_\mu \theta^{\mu\nu} P_\nu) \tag{3.2}$$

$$a_p^{\mathcal{M}\dagger} = c_p^\dagger \exp(\tfrac{i}{2} p_\mu \theta^{\mu\nu} P_\nu) \tag{3.3}$$

where c_p, $c_p^\dagger$ are the untwisted $\theta^{\mu\nu} = 0$ annihilation and creation operators. (We can assume all such operators to refer to in, out or free operators as the occasion demands), p_μ is the four momentum of the particle whereas P_μ is the momentum operator (of the fully interacting theory). We have also explicitly shown in (**??**) that if $(a, \Lambda) \to U(a, \Lambda)$ is the $\theta = 0$ unitary representation of the Poincaré group, then these operators acting on the vacuum create states which transform with the Moyal coproduct under conjugation by $U(a, \Lambda)$.

We remark that the Fock space we use here is "standard" and can be created by applying $c_p^\dagger$'s on the vacuum. The unitarity of $U(a, \Lambda)$ is with regard to the scalar product on this Fock space.

Transformations of the form (**??**) and (**??**) from c_p, $c_p^\dagger$ to $a_p^{\mathcal{M}}$, $a_p^{\mathcal{M}\dagger}$ appeared in the context of integrable models in 1+1 dimensions (**? ? ?**) where they are called "dressing transformations". A discretised version of these formulas has in fact appeared there. For this reason, here too, we will call them dressing transformations.

In these equations, the dressing transformation could have been changed to

$$a_p^{\mathcal{M}} = \exp\left(-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu\right) c_p \tag{3.4}$$

$$a_p^{\mathcal{M}\dagger} = \exp\left(\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu\right) c_p^\dagger \tag{3.5}$$

But in fact (**??**) equals (**??**) and (**??**) equals (**??**) because $\theta^{\mu\nu} = -\theta^{\nu\mu}$ (**? ?**). This observation is important. It ensures that $a_p^{\mathcal{M}}$ is the adjoint of $(a_p^{\mathcal{M}})^\dagger$ for the standard scalar product on Fock space.

One can deduce from (**??**) and (**??**) that the twisted Moyal quantum field is

$$\varphi_\theta^{\mathcal{M}} = \varphi_0 \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \quad . \tag{3.6}$$

Although this formula is first deduced for in, out or free fields, we previously showed how it can be inferred for fully interacting Heisenberg as well, see (**??**-**??**). The explicit expression for a Moyal field is then

$$\varphi_\theta^{\mathcal{M},\mathrm{in}} = \int \frac{\mathrm{d}^3 p}{2|p_0|} \left[a_p^{\mathcal{M}} e_{-p} + a_p^{\mathcal{M}\dagger} e_p\right] \quad , \tag{3.7}$$

$$\mathrm{e}_p(x) = \mathrm{e}^{ip\cdot x}, \quad \mathrm{d}\mu(p) := \frac{\mathrm{d}^3 p}{2\sqrt{\vec{p}^2 + m^2}} \quad m = \text{mass of the particle} \quad , \tag{3.8}$$

An important feature of (**??**) was shown in (**??**), that is what we called self-reproducing property:

$$\underbrace{\varphi_\theta^{\mathcal{M}} \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M}} \star_{\mathcal{M}} \dots \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M}}}_{N-factors} = \varphi_0^N \mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \tag{3.9}$$

This property plays a significant role in general theory. It is the basis for the proof of the absence of UV-IR mixing in Moyal field theories (with no gauge fields) (**? ?**).

Now consider the Wick-Voros case. The twisted creation operators which correctly create states from the vacuum transforming by the twisted coproduct are (**??**)-(**??**)

$$a_p^{V\dagger} = c_p^\dagger \mathrm{e}^{\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu - i\theta p_\nu P_\nu)} \tag{3.10}$$

$$a_p^V = \mathrm{e}^{-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu)} c_p \quad . \tag{3.11}$$

where $p_\nu P_\nu$ uses the Euclidean scalar product. We will show in Chapter **??** that (**??**) and (**??**) are also dictated by the covariance of quantum fields.

The Moyal twist of $\varphi_0^{\mathcal{M}}$ is compatible with the adjointness operation since from (**??**) and (**??**) we have for the adjoint $(a_p^{\mathcal{M}\dagger})^\dagger$ of $a_p^{\mathcal{M}\dagger}$,

$$(a_p^{\mathcal{M}\dagger})^\dagger = \exp\left(-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu\right) c_p = a_p^{\mathcal{M}} \quad . \tag{3.12}$$

Thus we can put the dressing transformation on the right or on the left, and such flexibility is needed to preserve the †-operation: the dressed operator $a_p^{\mathcal{M}}$ is equal to the adjoint of the dressed operator $a_p^{\mathcal{M}\dagger}$. This is the significance of the remark following (**??**-**??**).

As it was noticed previously, the above property fails for the Wick-Voros case. Thus

$$a_p^V = c_p \exp\left(-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu) - \frac{\theta}{2} p_\nu p_\nu\right) \neq c_p \exp\left(-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu + i\theta p_\nu P_\nu)\right) \quad . \tag{3.13}$$

A consequence is that we have to twist the creation-annihilation parts $\varphi_0^{(\pm)I}$ (I=in, out or free) fields separately:

$$\varphi_\theta^{(+)V,\mathrm{I}} = \int \mathrm{d}\mu(p) a^{V,\mathrm{I}\dagger} \mathrm{e}_p = \varphi_0^{(+)\mathrm{I}} e^{\frac{1}{2}(\overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu - i\theta \overleftarrow{\partial}_\mu P_\mu)} \quad , \tag{3.14}$$

$$\varphi_\theta^{(-)V,\mathrm{I}} = \int \mathrm{d}\mu(p) a^{V,\mathrm{I}} \mathrm{e}_{-p} = e^{\frac{1}{2}(\overrightarrow{\partial}_\mu \theta^{\mu\nu} P_\nu + i\theta \overrightarrow{\partial}_\mu P_\mu)} \varphi_0^{(-)\mathrm{I}} \quad , \tag{3.15}$$

$$\mathrm{e}_p(x) = \mathrm{e}^{ip\cdot x}, \quad \mathrm{d}\mu(p) := \frac{\mathrm{d}^3 p}{2\sqrt{\vec{p}^2 + m^2}} \quad m = \text{mass of the field } \varphi_0^{\mathrm{I}} \quad , \tag{3.16}$$

where now we have added the superscript I to φ_0^{I}, $a_p^{V,\mathrm{I}\dagger}$, and $a_p^{V,\mathrm{I}}$.

Therefore the field

$$\varphi_\theta^{V,\mathrm{I}} = \varphi_\theta^{(+)V,\mathrm{I}} + \varphi_\theta^{(-)V,\mathrm{I}} \tag{3.17}$$

cannot be obtained by an overall twist acting on φ_0^{I}. As we have to twist the creation and annihilation parts separately, we have to separately twist its positive and negative frequency parts $\varphi_0^{(\pm)\mathrm{I}}$. But we cannot decompose the Heisenberg field φ_0 for $\theta^{\mu\nu} = 0$ into $\varphi^{(\pm)}$ such that $\varphi_0^{(\pm)} \to \varphi_0^{(\pm)\mathrm{I}}$ as $x_0 \to \mp\infty$. That means that we do not know how to write the twisted Heisenberg field or develop the LSZ formalism for the Wick-Voros case. (The LSZ formalism for the Moyal case was developed from (**??**) in (**?**).)

But that is not all. The states created by the Wick-Voros quantum fields $\varphi_\theta^{(\pm)V}$ are not normalised in the same way as in the Moyal case. For instance

$$\langle 0| a_{k_1}^{V,\mathrm{I}} a_{k_2}^{V,\mathrm{I}} a_{p_2}^{V,\mathrm{I}\dagger} a_{p_1}^{V,\mathrm{I}\dagger} |0\rangle = \tag{3.18}$$

$$= \mathrm{e}^{\theta k_1 \cdot k_2} 4\sqrt{(\vec{k}_1^2 + m^2)(\vec{k}_2^2 + m^2)} \left[\delta^3(k_1 - p_1)\delta^3(k_2 - p_2) + \mathrm{e}^{\frac{i}{2} k_{1\mu}\theta^{\mu\nu} k_{2\nu}} \delta^3(k_1 - p_2)\delta^3(k_2 - p_1)\right],$$

$$\mathrm{I} = \text{in, out}, \quad |0\rangle_{\text{in}} = |0\rangle_{\text{out}}; \quad m = \text{mass of the field } \varphi_0^I \quad . \tag{3.19}$$

For scattering theory, normalisation is important. If we normalise the states as in the Moyal case, since the normalisation constant in (**??**) is momentum dependent, the normalised states no longer transform with the Wick-Voros coproduct.

The normalisation (**??**) has been computed using the standard scalar product in the Fock space. We can try changing it (**?**) so that the states become correctly normalised. But then the representation $(a, \Lambda) \to U(a, \Lambda)$ ceases to be unitary.

A further point relates to the self-reproducing property of these Wick-Voros fields. $\varphi_\theta^{(\pm)\mathrm{I}V}$ enjoy the self-reproducing property, but in different ways. Thus

$$\underbrace{\varphi_\theta^{(+)V,\mathrm{I}} \star_V \varphi_\theta^{(+)V,\mathrm{I}} \star_V ... \star_V \varphi_\theta^{(+)V,\mathrm{I}}}_{M-factors} = \left(\varphi_0^{(+)\mathrm{I}}\right)^M \mathrm{e}^{\frac{1}{2}(\overleftarrow{\partial}_\mu)\theta^{\mu\nu}P_\nu - i\theta\overleftarrow{\partial}_\mu P_\mu} \tag{3.20}$$

$$\underbrace{\varphi_\theta^{(-)V,\mathrm{I}} \star_V \varphi_\theta^{(-)V,\mathrm{I}} \star_V ... \star_V \varphi_\theta^{(-)V,\mathrm{I}}}_{M'-factors} = \mathrm{e}^{\frac{1}{2}(\overrightarrow{\partial}_\mu)\theta^{\mu\nu}P_\nu + i\theta\overrightarrow{\partial}_\mu P_\mu}\left(\varphi_0^{(-)\mathrm{I}}\right)^{M'} \tag{3.21}$$

So $\varphi_\theta^{V,\mathrm{I}}$ does not have sellf-reproducing property as in (**??**).

3.3 On a Similarity transformation

There is no similarity transformation transforming $a_p^{\mathcal{M},\mathrm{I}}$, $a_p^{\mathcal{M},\mathrm{I}\dagger}$, $a_p^{V,\mathrm{I}}$, $a_p^{V,\mathrm{I}\dagger}$. One way to quickly see this is to examine the operators without the Moyal part of the twist. So we consider c_p^{I}, $c_p^{\mathrm{I}\dagger}$ and

$$a_p^{V,\mathrm{I}'} = \mathrm{e}^{\frac{1}{2}\theta p_\nu P_\nu} c_p^{\mathrm{I}} \quad , \tag{3.22}$$

$$a_p^{V,\mathrm{I}'\dagger} = c_p^{\mathrm{I}\dagger}\mathrm{e}^{\frac{1}{2}\theta p_\nu P_\nu} \quad . \tag{3.23}$$

Now

$$[c_p^{\mathrm{I}}, c_k^{\mathrm{I}\dagger}] = 2|p_0|\delta^3(p-k)\mathbb{1} \quad , \tag{3.24}$$

$$p_0 = \sqrt{\vec{p}^2 + m^2}, \quad m = \text{mass of the field } \varphi_0^{\mathrm{I}} \quad . \tag{3.25}$$

If there existed a W such that

$$W c_p^{\mathrm{I}} W^{-1} = a_p^{V,\mathrm{I}'}, \quad W c_p^{\mathrm{I}\dagger} W^{-1} = a_p^{V,\mathrm{I}'\dagger} \quad , \tag{3.26}$$

then we would have

$$[a_p^{V,\mathrm{I}'}, a_k^{V,\mathrm{I}'\dagger}] = 2|p_0|\delta^3(p-k)\mathbb{1} \quad . \tag{3.27}$$

But a direct calculation of the L.H.S. using (**??**) and (**??**) shows that is is not equal to the R.H.S..

But there exists an S which transforms $a_p^{\mathcal{M},\mathrm{I}\dagger}$ to $a_p^{V,\mathrm{I}\dagger}$:

$$S = \mathrm{e}^{\frac{\theta}{4}(P_\mu P_\mu + K)}, \quad K = -\int \mathrm{d}\mu(k) k_\mu k_\mu c_k^{\mathrm{I}\dagger} c_k^{\mathrm{I}} \tag{3.28}$$

$$S a_p^{\mathcal{M},\mathrm{I}\dagger} S^{-1} = a_p^{V,\mathrm{I}\dagger} \quad . \tag{3.29}$$

where, as usual, I on $c_k^{\mathrm{I}\dagger}$, c_k^{I} denotes in, out or free while in $P_\mu P_\mu$ and $k_\mu k_\mu$ we use the Euclidean scalar product.

But

$$S a_p^{\mathcal{M},\mathrm{I}} S^{-1} = \mathrm{e}^{-\frac{i}{2}(p_\mu \theta^{\mu\nu} P_\nu - i\theta p_\nu p_\nu)} c_p^{\mathrm{I}} = \tilde{a}_p^{V,\mathrm{I}} \neq a_p^{V,\mathrm{I}} \quad . \tag{3.30}$$

Let us pursue the properties of this operator further.

The operator S leaves the vacuum invariant and shows that certain correlators in the Moyal and Wick-Voros cases are equal. From the explicit expression (**??**) follows also that the map induced by the operator S is isospectral, but not unitary in the standard Fock space scalar product. It is possible to define a new scalar product which makes S unitary (**?**). But again $U(a,\Lambda)$ is not unitary in this scalar product.

Now consider the twisted fields

$$\varphi_\theta^{\mathcal{M},\mathrm{I}} = \int \mathrm{d}\mu(p) \big[a_p^{\mathcal{M},\mathrm{I}} \mathrm{e}_{-p} + a_p^{\mathcal{M},\mathrm{I}\dagger} \mathrm{e}_p\big] \quad , \tag{3.31}$$

$$\tilde{\varphi}_\theta^{V,\mathrm{I}} = \int \mathrm{d}\mu(p) \big[\tilde{a}_p^{V,\mathrm{I}} \mathrm{e}_{-p} + a_p^{V,\mathrm{I}\dagger} \mathrm{e}_p\big] \quad , \tag{3.32}$$

where $\mathrm{e}_p(x)$ denotes as usual $\mathrm{e}^{ip\cdot x}$. Then of course,

$$S : \varphi_\theta^{\mathcal{M},\mathrm{I}} \to S \triangleright \varphi_\theta^{\mathcal{M},\mathrm{I}} := \int \mathrm{d}\mu(p) S \big[a_p^{\mathcal{M},\mathrm{I}} \mathrm{e}_{-p} + a_p^{\mathcal{M},\mathrm{I}\dagger} \mathrm{e}_p\big] S^{-1} = \tilde{\varphi}_\theta^{V,\mathrm{I}} \quad . \tag{3.33}$$

Also

$$S|0\rangle = S^{-1}|0\rangle = 0 \tag{3.34}$$

From (**??**) and (**??**) we obtain trivially the equality of the n-points correlation functions:

$$\langle \varphi_\theta^{\mathcal{M},\mathrm{I}}(x_1) \varphi_\theta^{\mathcal{M},\mathrm{I}}(x_2) ... \varphi_\theta^{\mathcal{M},\mathrm{I}}(x_N) \rangle_0 = \langle \tilde{\varphi}_\theta^{V,\mathrm{I}}(x_1) \tilde{\varphi}_\theta^{V,\mathrm{I}}(x_2) ... \tilde{\varphi}_\theta^{V,\mathrm{I}}(x_N) \rangle_0 \quad . \tag{3.35}$$

Consider simple interaction densities such as

$$\mathscr{H}_{\mathrm{I}}^{\mathcal{M}} = \underbrace{\varphi_\theta^{\mathcal{M},\mathrm{I}} \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M},\mathrm{I}} \star_{\mathcal{M}} ... \star_{\mathcal{M}} \varphi_\theta^{\mathcal{M},\mathrm{I}}}_{N-factors} \quad \text{and} \quad \mathscr{H}_{\mathrm{I}}^{V} = \underbrace{\tilde{\varphi}_\theta^{V,\mathrm{I}} \star_V \tilde{\varphi}_\theta^{V,\mathrm{I}} \star_V ... \star_V \tilde{\varphi}_\theta^{V,\mathrm{I}}}_{N-factors} \quad . \tag{3.36}$$

in either fields.

Since S only acts on the operator parts of the fields, the similarity transformation in (**??**) will not map $\mathscr{H}_{\rm I}^{\mathcal{M}}$ to $\mathscr{H}_{\rm I}^{V}$:

$$S \triangleright \mathscr{H}_{\rm I}^{\mathcal{M}} \neq \mathscr{H}_{\rm I}^{V} \quad . \tag{3.37}$$

Hence

$$\langle \varphi_\theta^{\mathcal{M},{\rm I}}(x_1)...\varphi_\theta^{\mathcal{M},{\rm I}}(x_j)\mathscr{H}_{\rm I}^{\mathcal{M}}(x_{j+1})\varphi_\theta^{\mathcal{M},{\rm I}}(x_{j+2})...\varphi_\theta^{\mathcal{M},{\rm I}}(x_N)\rangle_0 \tag{3.38}$$

$$\neq \langle \tilde{\varphi}_\theta^{V,{\rm I}}(x_1)...\tilde{\varphi}_\theta^{V,{\rm I}}(x_j)\mathscr{H}_{\rm I}^{V}(x_{j+1})...\tilde{\varphi}_\theta^{V,{\rm I}}(x_{j+2})...\tilde{\varphi}_\theta^{V,{\rm I}}(x_N)\rangle_0 \tag{3.39}$$

So we can immediately conclude that also in this case the two theories are different.

There is no such S for mapping $\varphi_\theta^{\mathcal{M},{\rm I}}$ to $\varphi_\theta^{V,{\rm I}}$, so that the correlators are not equal even at the free level.

3.4 A criterion for the strong equivalence of Twisted QFT's

It seems reasonable to assert that two twisted quantum field theories obtained by twisting the same quantum field φ_0 are *strongly* equivalent if they give the same answer for the same scattering cross sections. This criterion is logically distinct from the criterion requiring the equality of Wightman functions, but is perhaps physically more compelling. The reason that the equality of Wightman functions and that of scattering cross sections need not mutually imply each other is the following. Below, in (**??**) and (**??**), we have given the scattering amplitudes in the Moyal and Wick-Voros cases. Even if they were equal due to equality of Wightman functions, it does not mean that the corresponding cross sections are equal, as the states in the two cases are not normalised in the same way.

Let us first recall the expression for a general scattering amplitude of spinless particles of mass m_i in the Moyal case using the LSZ formalism.

In (**?**) the LSZ formalism for Moyal field $\varphi_\theta^{\mathcal{M}}$ (**??**), constructed via dressing transformations, was presented. For scattering amplitudes it leads to:

$$\begin{aligned} S_\theta^{\mathcal{M}}(k_1,...,k_N) &= \langle -k_M, -k_{M-1},...,-k_1; {\rm out} \,|\, k_N, k_{N-1},...,k_{N-M}; {\rm in}\rangle_{\mathcal{M}} \\ &= \int \mathcal{I}\; G_N^{\mathcal{M}}(x_1,x_2,...,x_N) \end{aligned} \tag{3.40}$$

where

$$G_N^{\mathcal{M}}(x_1, ..., x_N) = T\mathrm{e}^{\frac{i}{2}\sum_{I<J}\partial_I\wedge\partial_J}W_N^0(x_1, ..., x_N) = T\ W_N^{\mathcal{M}}(x_1, ..., x_N) \tag{3.41}$$

$$\mathcal{I} = \prod_{i=1}^N \mathrm{d}x_i \mathrm{e}^{-iq_i\cdot x_i} i(\partial_i^2 + m^2) \tag{3.42}$$

The momenta k_i are taken to be in-going so that $\sum k_i = 0$. Also since

$$a_k^{\mathcal{M}\dagger}|0\rangle = c_k^\dagger|0\rangle \quad , \tag{3.43}$$

the single particle states are normalised canonically:

$$\langle 0|a_{k'}^{\mathcal{M}} a_k^{\mathcal{M}\dagger}|0\rangle = 2|k_0|\delta^3(k-k') \tag{3.44}$$

while the normalisation of the multiparticle states

$$a_{k_1}^{\mathcal{M}\dagger}\cdots a_{k_N}^{\mathcal{M}\dagger}|0\rangle \tag{3.45}$$

is consistent with what is required by twisted statistics.

For the Wick-Voros case, we can tentatively construct an in, out or free Wick-Voros field $\varphi_\theta^{V,\mathrm{I}''}$ following the construction (**??**) of $\varphi_\theta^{(+)V,\mathrm{I}}$:

$$\varphi_\theta^{V,\mathrm{I}''} = \varphi_0^{\mathrm{I}}(x)\mathrm{e}^{\frac{1}{2}\left(\overleftarrow{\partial}_\mu\theta^{\mu\nu}P_\nu - i\theta\overleftarrow{\partial}_\mu P_\mu\right)} \quad . \tag{3.46}$$

The annihilation part of the $\varphi_\theta^{V,\mathrm{I}''}$ differs from $\varphi_\theta^{(-)V,\mathrm{I}}$ so that $\varphi_\theta^{V,\mathrm{I}''}$ does not have correct adjointness properties. But the formula (**??**) does generalise to Heisenberg fields. Using (**??**), we can obtain a formula like (**??**) for scattering amplitudes. It is

$$\begin{aligned} S_\theta^{V''}(k_1, ..., k_N) &= \langle -k_M, -k_{M-1}, ..., -k_1; \mathrm{out}\,|\, k_N, k_{N-1}, ..., k_{N-M}; \mathrm{in}\rangle_V \\ &= \int \mathcal{I} G_N^{V''}(x_1, ..., x_N), \end{aligned} \tag{3.47}$$

$$\begin{aligned} G_N^{V''}(x_1, ..., x_N) &= T\mathrm{e}^{\frac{i}{2}\sum_{I<J}\partial_I\wedge\partial_J}\mathrm{e}^{\frac{\theta}{2}\sum_{I<J}\partial_I\cdot\partial_J}W_N^0(x_1, ..., x_N) \\ &= T\ W_N^V(x_1, ..., x_N) \end{aligned} \tag{3.48}$$

where $\partial_I \cdot \partial_J$ uses the Euclidean scalar product.

There is no reason to expect that $S_\theta^{V''}(k_1, ..., k_N) = S_\theta^{\mathcal{M}}(k_1, ..., k_N)$. In particular there is a problem with the normalisation of the states associated with $a^{V,\mathrm{I}\dagger}$ as was pointed out already in (**??**).

We note however that the field (**??**) does have the self-reproducing property.

4

Twisted-fields coming from Covariance

The construction of QFT's on noncommutative spacetimes presented so far, relies on finding a unitary representation of the deformed Hopf-Poincaré group $H_\theta^{\mathcal{M},V}\mathscr{P}$. Such a choice is motivated by generalizing, to the $\theta \neq 0$ case, the standard construction of quantum fields on commutative Minkowski spacetime (see e.g. § 1 or 5 of (**?**)).

In this chapter we study further the constraints that the symmetry group imposes on quantum fields. In particular we recall in what sense quantum fields are Poincaré covariant in the $\theta = 0$ case and use such a notion to present another possible way of constructing quantum fields on noncommutative spacetimes. We will also show that the results obtained here coincides with what we obtained in the previous chapters. The work presented here is mostly based on (**?**).

4.1 Poincaré covariance on commutative spacetimes

The Poincaré group $\mathscr{P}$ acts on Minkowski space $\mathcal{M}$ by transforming its coordinates (or coordinate functions), $x = (x_\mu)$ to $\Lambda x + a$

$$(a, \Lambda) \in \mathscr{P} : \quad (a, \Lambda)x = \Lambda x + a \quad . \tag{4.1}$$

If the spacetime algebra of functions on $\mathcal{M}$ is the commutative $C^\infty(M)$, and φ is a quantum relativistic scalar field on $\mathcal{M}$, we require that there exists a unitary representation

$$U : (a, \Lambda) \to U(a, \Lambda) \tag{4.2}$$

on the Hilbert space $\mathcal{H}$ of states vectors such that

$$U(a,\Lambda)\varphi(x)U(a,\Lambda)^{-1} = \varphi\big((a,\Lambda)x\big) \quad . \tag{4.3}$$

There are similar requirements on relativistic fields of all spins. They express the requirement that the spacetime transformations (**??**) can be unitarily implemented in quantum theory. It is analogous to the requirement in nonrelativistic quantum mechanics that infinitesimal spatial rotations are to be implemented by the (self-adjoint) angular momentum operators.

A field φ fulfilling (**??**) is said to be a "covariant field" and the condition in (**??**) is the covariance condition. We call it "primitive" as we later extend it to products of fields.

We can write (**??**) in the equivalent form

$$U(a,\Lambda)\varphi\big((a,\Lambda)^{-1}x\big)U(a,\Lambda) = \varphi(x) \tag{4.4}$$

Now in this form, covariance can be readily understood in terms of the coproduct on the Poincaré group. Thus

$$\varphi \in L(\mathcal{H}) \otimes S(\mathcal{M}) \tag{4.5}$$

where $L(\mathcal{H})$ are linear operators on $\mathcal{H}$ and $S(\mathcal{M})$ are distributions on the spacetime $\mathcal{M}$. There is an action of $\mathscr{P}$ on both, that on $L(\mathcal{H})$ being the adjoint action $\mathrm{Ad}U(a,\Lambda)$ of $U(a,\Lambda)$,

$$\mathrm{Ad}U(a,\Lambda)\varphi = U(a,\Lambda)\varphi U(a,\Lambda)^{-1} \tag{4.6}$$

and that on $S(\mathcal{M})$ being

$$\alpha \to (a,\Lambda) \triangleright \alpha, \quad \big[(a,\Lambda)\alpha\big](x) = \alpha\big((a,\Lambda)^{-1}x\big), \quad \alpha \in S(\mathcal{M}) \quad . \tag{4.7}$$

We call the latter action as V. So

$$V\big((a,\Lambda)\big)\alpha(x) = \alpha\big((\alpha,\Lambda)^{-1}x\big) \tag{4.8}$$

Now the coproduct on $\mathscr{P}_+^\uparrow$ for commutative spacetimes is Δ_0, where

$$\Delta_0\big((a,\Lambda)\big) = (a,\Lambda) \otimes (a,\Lambda) \quad . \tag{4.9}$$

Then by (**??**)

$$(\mathrm{Ad}U \otimes V)\Delta_0\big((a,\Lambda)\big)\varphi = \varphi \quad . \tag{4.10}$$

We will have occasion to use both the versions (**??**) and (**??**,**??**) of covariance.

4.2 Covariance for Tensor Products: Commutative Spacetimes

We saw in the previous section that for a single field, covariance ties together spacetime transformations and its implementation on the quantum Hilbert space. Products of fields bring in new features which although present for commutative spacetimes, assume prominence on quantum spacetimes. We now briefly examine these features in the former case

4.2.1 Tensor Products

Consider

$$\varphi(x_1)\varphi(x_2)...\varphi(x_N) \quad . \tag{4.11}$$

This can be understood as the element $\varphi \otimes \varphi ... \otimes \varphi$ belonging to $L(\mathcal{H}) \otimes \big(S(\mathcal{M}) \otimes S(\mathcal{M}) \otimes ... \otimes S(\mathcal{M})\big)$ evaluated at $x_1, x_2, ..., x_N$

$$\varphi \otimes \varphi \otimes ... \otimes \varphi \in L(\mathcal{H}) \otimes \big(S(\mathcal{M})\big)^{\otimes N}, \quad (\varphi \otimes \varphi \otimes ... \otimes \varphi)(x_1, x_2, ..., x_N) = \varphi(x_1)\varphi(x_2)...\varphi(x_N) \quad . \tag{4.12}$$

Note that tensoring refers only to $S(\mathcal{M})$, there is no tensoring involving $L(\mathcal{H})$. There is only one Hilbert space $\mathcal{H}$ which for free particles is the Fock space and $U(a,\Lambda)$ acts by conjugation on the L.H.S. for all N.

But that is not the case for $S(\mathcal{M})^{\otimes N}$. The Poincaré group acts on it by the coproduct

$$(\underbrace{\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \otimes \Delta_0}_{N-1})(\underbrace{\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \otimes \Delta_0}_{N-2})...\Delta_0 \tag{4.13}$$

of (a,Λ). Thus

$$(\mathbf{??}) \text{ on } (a,\Lambda) = (a,\Lambda) \otimes (a,\Lambda) \otimes ... \otimes (a,\Lambda) \tag{4.14}$$

and

$$\Big((\mathbf{??}) \text{ on } (a,\Lambda) \triangleright \varphi^{\otimes N}\Big)(x_1, x_2, ..., x_N) = \varphi^{\otimes N}\big((a,\Lambda)^{-1}x_1, (a,\Lambda)^{-1}x_2, ..., (a,\Lambda)^{-1}x_N\big) \quad . \tag{4.15}$$

Covariance is now the demand

$$U(a,\Lambda)\Big(\varphi^{\otimes N}\big((a,\Lambda)^{-1}x_1, (a,\Lambda)^{-1}x_2, ..., (a,\Lambda)^{-1}x_N\big)\Big)U(a,\Lambda)^{-1} = \varphi^{\otimes N}(x_1, x_2, ..., x_N) \quad . \tag{4.16}$$

It is evidently fulfilled for the coproduct (**??**) if the primitive covariance (**??**,**??**) is fulfilled.

For free fields (or in and out-fields), covariance can be verified in a different manner. Thus for a free real scalar field φ of mass m, we have

$$\begin{aligned} \varphi = \int \mathrm{d}\mu(p)\Big(c_p^\dagger \mathrm{e}_p + c_p \mathrm{e}_{-p}\Big) = \varphi^{(-)} + \varphi^{(+)} \\ \mathrm{e}_p(x) = \mathrm{e}^{-ip\cdot x}, \quad |p_0| = (\vec{p}^2 + m^2)^{\frac{1}{2}}, \quad \mathrm{d}\mu(p) = \tfrac{\mathrm{d}^d p}{2|p_0|} \end{aligned} \tag{4.17}$$

where c_p, $c_p^\dagger$ are the standard annihilation and creation operators, and $\varphi^{(\mp)}$ refer to the annihilation and creation parts of φ.

Now $\varphi^{(\mp)}$ must separately fulfill the covariance requirement. Let us consider $\varphi^{(-)}$. We have that

$$\varphi^{(-)}(x_1)\varphi^{(-)}(x_2)...\varphi^{(-)}(x_N)|0\rangle = \int \prod_i \mathrm{d}\mu(p_i) c_{p_1}^\dagger c_{p_2}^\dagger ... c_{p_N}^\dagger |0\rangle \mathrm{e}_{p_1}(x_1)\mathrm{e}_{p_2}(x_2)...\mathrm{e}_{p_N}(x_N) \tag{4.18}$$

Let us first check translations. Let P_μ be the translation generators on the Hilbert space,

$$[P_\mu, c_p^\dagger] = p_\mu c_p^\dagger, \quad P_\mu|0\rangle = 0 \tag{4.19}$$

and let $\mathcal{P}_\mu = -i\partial_\mu$ be the translation generator on $S(\mathcal{M})$:

$$\mathcal{P}_\mu \mathrm{e}_p = -p_\mu \mathrm{e}_p \tag{4.20}$$

The coproduct Δ_0 gives for the Lie algebra element $\mathcal{P}_\mu$[1],

$$\Delta_0(\mathcal{P}_\mu) = \mathbb{1} \otimes \mathcal{P}_\mu + \mathcal{P}_\mu \otimes \mathbb{1} \tag{4.21}$$

It follows that

$$(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \otimes \Delta_0)...\Delta_0(\mathcal{P}_\mu)\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes ... \otimes \mathrm{e}_{p_N} = -\sum_i p_{i\mu} \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes ... \otimes \mathrm{e}_{p_N} \tag{4.22}$$

Covariance for translations is the requirement

$$P_\mu c_{p_1}^\dagger c_{p_2}^\dagger ... c_{p_N}^\dagger |0\rangle \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes ... \otimes \mathrm{e}_{p_N} + c_{p_1}^\dagger c_{p_2}^\dagger ... c_{p_N}^\dagger |0\rangle \big(-\sum_i p_{i\mu}\big) \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes ... \otimes \mathrm{e}_{p_N} = 0 \tag{4.23}$$

[1] If $\underline{v}$ is the representation of the Lie algebra of $\mathscr{P}_+^\uparrow$ on functions, and $\hat{P}_\mu$ is the Lie algebra generator in the abstract group $\mathscr{P}_+^\uparrow$ so that $\underline{v}(P_\mu) = \mathcal{P}_\mu$, the L.H.S. here should strictly read $\underline{v}\big(\Delta_0(\hat{P}_\mu)\big)$. So we have simplified the notation in (**??**).

which is clearly fulfilled.

Next consider Lorentz transformations. A Lorentz transformation Λ acts on e_p according to

$$(\Lambda \mathrm{e}_p)(x) = \mathrm{e}_p(\Lambda^{-1}x) = \mathrm{e}_{\Lambda p}(x) \tag{4.24}$$

or $\Lambda \mathrm{e}_p = \mathrm{e}_{\Lambda p}$.

For Lorentz transformations Λ, covariance is thus the identity

$$\int \prod_i \mathrm{d}\mu(p_i) c^\dagger_{\Lambda p_1} c^\dagger_{\Lambda p_2} ... c^\dagger_{\Lambda p_N} |0\rangle \mathrm{e}_{\Lambda p_1} \otimes \mathrm{e}_{\Lambda p_2} ... \otimes \mathrm{e}_{\Lambda p_N} = \int \prod_i \mathrm{d}\mu(p_i) c^\dagger_{p_1} c^\dagger_{p_2} ... c^\dagger_{p_N} |0\rangle \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} ... \otimes \mathrm{e}_{p_N} \tag{4.25}$$

which is true because of the Lorentz invariance of the measure:

$$\mathrm{d}\mu(\Lambda^{-1}p_i) = \mathrm{d}\mu(p_i) \quad . \tag{4.26}$$

4.3 Covariance for products: Multiplication Map and Self-Reproducing Property

The multiplication map involves products of fields at the *same* point and hence the algebra of the underlying manifold. It is not the same as the tensor product which involves products of fields at *different* points.

There is a further property of φ, involving now the multiplication map, which is easily understood on commutative spacetimes. It has much importance for both commutative and noncommutative spacetimes. It is the self-reproducing property. Let us first understand this property for $C^\infty(\mathcal{M})$, the set of smooth functions on a manifold $\mathcal{M}$. If $\alpha : p \to \alpha p,\ p \in \mathcal{M}$, is a diffeomorphism of $\mathcal{M}$, it acts on $f \in C^\infty(\mathcal{M})$ by pull-back:

$$(\alpha^* f)(p) = f(\alpha p) \quad . \tag{4.27}$$

But $C^\infty(\mathcal{M})$ has a further property, routinely used in differential geometry: $C^\infty(\mathcal{M})$ is closed under point-wise multiplication:

If $f_1, f_2 \in C^\infty(\mathcal{M})$, then

$$f_1 f_2 \in C^\infty(\mathcal{M}) \tag{4.28}$$

where

$$\big(f_1 f_2\big)(p) = f_1(p) f_2(p) \quad . \tag{4.29}$$

This property is very important for noncommutative geometry: the completion of this algebra under the supremum norm gives the commutative algebra of $C^0(\mathcal{M})$, a commutative C^*-algebra. As we showed in the first Chapter, by the Gel'fand-Naimark theorem (**?** **?**) it encodes the topology of $\mathcal{M}$.

Now by (**??**) and (**??**), we see that multiplication of functions preserves transformation under diffeos. This simple property gets generalised to covariant quantum field thus:

The pointwise product of covariant quantum fields is covariant.

That means in particular that

$$U(a,\Lambda)\varphi^2\big((a,\Lambda)^{-1}x\big)U(a,\Lambda)^{-1} = \varphi^2(x) \quad . \tag{4.30}$$

This result is obviously true modulo renormalization problems. It is at the basis of writing invariant interactions in quantum field theories on $\mathcal{A}_0(\mathcal{M})$.

Note that generally we require covariance of the product of any two covariant fields, distinct or the same.

4.3.1 The $*$-covariance

In quantum field theories on $\mathcal{A}_0(\mathcal{M})$, another routine requirement is that covariance and the $*$- or the adjoint operation be compatible. Thus if ψ is a covariant complex field,

$$U(a,\Lambda)\psi\big((a,\Lambda)^{-1}x\big)U(a,\Lambda)^{-1} = \psi(x) \quad , \tag{4.31}$$

we require that $\psi^\dagger$ is also a covariant complex field. That is fulfilled if $U(a,\Lambda)$ is unitary.

Thus $*$-covariance is linked to unitarity of time-evolution and the S-matrix and many more physical requirements.

The covariance requirements on quantum fields for commutative spacetimes (ignoring the possibility of parastatistics of order 2 or more) can be then summarized as: ***A quantum field should be $*$- covariant with commutation or anti-commutation relations (symmetrisation postulates) compatible with $*$-covariance.***

4.4 Covariance on the Moyal Plane

We discussed lengthly the Moyal plane $\mathcal{A}_\theta^{\mathcal{M}}$. The Poincaré group $\mathscr{P}$ acts on smooth functions $\alpha \in \mathcal{A}_\theta^{\mathcal{M}}$ by pull-back as before:

$$\mathscr{P} \ni (a, \Lambda) : \alpha \to (a, \Lambda)\alpha, \quad \big((a, \Lambda)\alpha\big)(x) = \alpha\big((a, \Lambda)^{-1}x\big) \tag{4.32}$$

We recall now the explicit form of the Moyal deformation on $C^\infty(\mathcal{M})$, (cf. (**??**)):

$$(f \star_{\mathcal{M}} g)(x) = m_\theta^{\mathcal{M}}(f \otimes g)(x) \equiv f(x) \mathrm{e}^{\frac{i}{2}\theta_{\alpha\beta}\overleftarrow{\partial^\alpha}\otimes\overrightarrow{\partial^\beta}} g(x) \tag{4.33}$$

and the deformed co-product turning the Poincaré group algerbra $\mathbb{C}\mathscr{P}$ into the deformed Poincaré-Hopf algebra $H_\theta^{\mathcal{M}}\mathscr{P}$ (cf. (**??**)):

$$\Delta_\theta^{\mathcal{M}}(g) = (F_\theta^{\mathcal{M}})^{-1}(g \otimes g)(F_\theta^{\mathcal{M}}), \quad F_\theta^{\mathcal{M}} = \mathrm{e}^{-\frac{i}{2}\hat{P}_\mu \otimes \theta_{\mu\nu}\hat{P}_\nu} = \text{Drinfel'd twist} \tag{4.34}$$

Here $\hat{P}_\mu$ is as before the translation generator in $\mathscr{P}$ with representatives $\mathcal{P}_\mu = -i\partial_\mu$ and P_μ on functions and $L(\mathcal{H})$ respectively.

Equation (**??**) is the starting point for further considerations.

Let $\varphi_\theta^{\mathcal{M}}$ be the twisted analogue of the field φ of section 2. Also let U_θ be the unitary operator implementing $\mathscr{P}$ in $L(\mathcal{H})$. Covariance then is the requirement

$$U_\theta(a, \Lambda)\varphi_\theta^{\mathcal{M}}\big((a, \Lambda)^{-1}x\big)U_\theta(a, \Lambda)^{-1} = \varphi_\theta^{\mathcal{M}}(x) \tag{4.35}$$

and its multifield generalisation, while compatibility with * or unitarity requires that $\varphi_\theta^{\mathcal{M}\dagger}$ is also covariant. There is also one further requirement, namely compatibility with symmetrisation postulate.

The analysis of these requirements becomes transparent on working with the mode expansion of $\varphi_\theta^{\mathcal{M}}$ which is assumed to exist:

$$\varphi_\theta^{\mathcal{M}} = \int \mathrm{d}\mu(p)\big[a_p^{\mathcal{M}\dagger}\mathrm{e}_p + a_p^{\mathcal{M}}\mathrm{e}_{-p}\big] = \varphi_\theta^{(-)\mathcal{M}} + \varphi_\theta^{(+)\mathcal{M}}, \quad \mathrm{d}\mu(p) = \frac{\mathrm{d}^d p}{2|p_0|} \quad . \tag{4.36}$$

The expansion can refer to in- , out- or free fields.

We also assume the existence of vacuum $|0\rangle$:

$$a_p^{\mathcal{M}}|0\rangle = 0, \ \forall p \quad . \tag{4.37}$$

4.4.1 The Primitive Covariance of a Single Field

We are here referring to (**??**). It requires that

$$U_\theta(a,\Lambda)a_p^{\mathcal{M}\dagger}U_\theta(a,\Lambda)^{-1} = a_{\Lambda p}^{\mathcal{M}\dagger}, \quad U_\theta(a,\Lambda)a_p^{\mathcal{M}}U_\theta(a,\Lambda)^{-1} = a_{\Lambda p}^{\mathcal{M}} \tag{4.38}$$

A particular consequence of (**??**,**??**) is that single particle states transform for all θ in the same manner or assuming that $U_\theta(a,\Lambda)|0\rangle = |0\rangle$:

$$U_\theta(a,\Lambda)a_p^{\mathcal{M}\dagger}|0\rangle = a_{\Lambda p}^{\mathcal{M}\dagger}|0\rangle \tag{4.39}$$

New physics can be expected only in multi-particle sectors.

4.4.2 Covariance in Multi-Particle Sectors

On the Moyal plane, multi-particle wave functions $\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N}$ transform under $\mathscr{P}$ with the twisted coproduct. This affects the properties of $a_p^{\mathcal{M}}$, $a_p^{\mathcal{M}\dagger}$ in a $\theta_{\mu\nu}$-dependent manner.

Let us focus on the two-particle sector:

$$\int \prod_i \mathrm{d}\mu(p_i) a_{p_1}^{\mathcal{M}\dagger} a_{p_2}^{\mathcal{M}\dagger}|0\rangle \mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} \tag{4.40}$$

Since translations act in the usual way on $\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}$,

$$\Delta_\theta(\mathcal{P}_\mu)\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} = (\mathbb{1}\otimes\mathcal{P}_\mu + \mathcal{P}_\mu\otimes\mathbb{1})\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} = -(\sum_i p_{i\mu})\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} \tag{4.41}$$

translational covariance requires the standard transformation of $a_{p_i}^\dagger$:

$$[P_\mu^\theta, a_p^{\mathcal{M}\dagger}] = p_\mu a_p^{\mathcal{M}\dagger} \quad , \tag{4.42}$$

P_μ^θ is the possibly θ dependent translation generator.

Lorentz transformations are more interesting. We have that

$$\Delta_\theta^{\mathcal{M}}(\Lambda)\triangleright\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} = (\mathscr{F}_\theta^{\mathcal{M}})^{-1}(\Lambda\otimes\Lambda)(\mathscr{F}_\theta^{\mathcal{M}})\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} = \mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2}\mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2} \quad . \tag{4.43}$$

(We do not consider the anti-unitary time-reversal in what follows.) Covariance thus requires that

$$\int \prod_i \mathrm{d}\mu(p_i) U_\theta(\Lambda) a_{p_1}^{\mathcal{M}\dagger} a_{p_2}^{\mathcal{M}\dagger}|0\rangle \mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2}\mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2} = \int \prod_i \mathrm{d}\mu(p_i) a_{p_1}^{\mathcal{M}\dagger} a_{p_2}^{\mathcal{M}\dagger}|0\rangle \mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2} \tag{4.44}$$

4.4.3 The Dressing Transformation

We can solve this requirement, as well as (**??**), by writing $a_p^{\mathcal{M}\dagger}$ in terms of the $c_p^\dagger$ and P_μ:

$$a_p^{\mathcal{M}\dagger} = c_p^\dagger \mathrm{e}^{\frac{i}{2} p\wedge P} \tag{4.45}$$

and setting

$$U_\theta(a,\Lambda) = U_0(a,\Lambda) = U(a,\Lambda) \quad . \tag{4.46}$$

The adjoint of (**??**) is

$$a_p^{\mathcal{M}} = \mathrm{e}^{-\frac{i}{2} p\wedge P} c_p = c_p \mathrm{e}^{-\frac{i}{2} p\wedge P} \quad , \tag{4.47}$$

where the equality in the last step uses the anti-symmetry of $\theta_{\mu\nu}$.

As we can twist c_p on left *or* on right, we can write $\varphi_\theta^{\mathcal{M}}$ as a twist applied to $\varphi_0 \equiv \varphi$:

$$\varphi_\theta^{\mathcal{M}} = \varphi_0 \mathrm{e}^{-\frac{1}{2} \overleftarrow{\partial} \wedge P} \tag{4.48}$$

which is the same result we obtained in (**??**).

It is important to note that (**??**) *is well-defined for a fully interacting Heisenberg field* φ_0 *if* P_μ *stands for the total four momentum of the interacting theory.* In that case $\varphi_\theta^{\mathcal{M}}$ is the twisted Heisenberg field.

We can now check that

$$U(a,\Lambda)\varphi_\theta^{\mathcal{M}}(x_1)...\varphi_\theta^{\mathcal{M}}(x_N)U(a,\Lambda)^{-1}|0\rangle = \varphi_\theta^{\mathcal{M}}\big((a,\Lambda)x_1\big)...\varphi_\theta^{\mathcal{M}}\big((a,\Lambda)x_N\big)|0\rangle \tag{4.49}$$

with a similar equation for the vacuum $\langle 0|$ put on the left. Since vacuum is a cyclic vector, we can then be convinced that (**??**) fully solves the problem of constructing a covariant quantum field on the Moyal plane at the multi-field level as well.

A particular implication of (**??**) is that

$$U_\theta(a,\Lambda) = U(a,\Lambda) = U_0(a,\Lambda) \quad . \tag{4.50}$$

Its expression in terms of in-, out- or free fields looks the same as in the commutative case. It has no $\theta_{\mu\nu}$- dependence.

4.4.4 Symmetrization and Covariance

We will now show that the dressing transformations (**??**,**??**-**??**) are exactly what we need to be compatible with appropriate symmetrisation postulates.

At the level of the particle dynamics (functions on $\mathcal{M}$ and their tensor products), we already showed that for the coproduct Δ_θ, symmetrisation and anti-symmetrisation should be based on the twisted flip operator (cf. (**??**))

$$\tau_\theta = \mathscr{F}_\theta^{-1}\tau_0\mathscr{F}_\theta \tag{4.51}$$

$$\tau_0\alpha\otimes\beta := \beta\otimes\alpha \tag{4.52}$$

where α, β are single particle wave functions.

As defined, τ_0 and τ_θ act on two-particle wave functions and generate S_2 since

$$\tau_0^2 = \mathbb{1} \quad \Rightarrow \quad \tau_\theta^2 = \mathbb{1} \quad . \tag{4.53}$$

But soon we will generalise them to N-particles to get S_N.

Thus twisted bosons (fermions) have the two-particle plane wave states

$$\mathrm{e}_{p_1}\otimes_{S_\theta}\mathrm{e}_{p_2} = \frac{\mathbb{1}\pm\tau_\theta}{2}\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} \quad . \tag{4.54}$$

Let us focus on $S_\theta^{\mathcal{M}}$, that is the twisted flip operator on the Moyal plane:

$$\begin{aligned}
\mathrm{e}_{p_1}\otimes_{S_\theta^{\mathcal{M}}}\mathrm{e}_{p_2} &= \frac{1}{2}\left[\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} + (\mathscr{F}_\theta^{\mathcal{M}})^{-2}\mathrm{e}_{p_2}\otimes\mathrm{e}_{p_1}\right] && (4.55)\\
&= \frac{1}{2}[\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} + \mathrm{e}^{ip_2\wedge p_1}\mathrm{e}_{p_2}\otimes\mathrm{e}_{p_1}] && (4.56)\\
&= \mathrm{e}^{ip_2\wedge p_1}\mathrm{e}_{p_2}\otimes_{S_\theta^{\mathcal{M}}}\mathrm{e}_{p_1} && (4.57)
\end{aligned}$$

This gives

$$\begin{aligned}
&\int\prod_{i=1}^{2}\mathrm{d}\mu(p_i)a_{p_1}^{\mathcal{M}\dagger}a_{p_2}^{\mathcal{M}\dagger}|0\rangle\mathrm{e}_{p_1}\otimes_{S_\theta^{\mathcal{M}}}\mathrm{e}_{p_2} && (4.58)\\
&= \int\prod_{i=1}^{2}\mathrm{d}\mu(p_i)a_{p_1}^{\mathcal{M}\dagger}a_{p_2}^{\mathcal{M}\dagger}|0\rangle\mathrm{e}^{ip_2\wedge p_1}\mathrm{e}_{p_2}\otimes_{S_\theta^{\mathcal{M}}}\mathrm{e}_{p_1} && (4.59)\\
&= \int\prod_{i=1}^{2}\mathrm{d}\mu(p_i)\big(\mathrm{e}^{ip_1\wedge p_2}a_{p_2}^{\mathcal{M}\dagger}a_{p_1}^{\mathcal{M}\dagger}\big)|0\rangle\mathrm{e}_{p_1}\otimes_{S_\theta^{\mathcal{M}}}\mathrm{e}_{p_2} && (4.60)
\end{aligned}$$

Thus we require that

$$a_{p_1}^{\mathcal{M}\dagger}a_{p_2}^{\mathcal{M}\dagger} = \mathrm{e}^{ip_1\wedge p_2}a_{p_2}^{\mathcal{M}\dagger}a_{p_1}^{\mathcal{M}\dagger} \tag{4.61}$$

which is fulfilled by (**??**).

We can extend this demonstration regarding the consistency of the twist to multinomials in $a^{\dagger}$'s and a's. The necessary tools are in (**?**). We just note one point. In the N-particle sector, call $\mathscr{F}_{\theta}^{ij}$ the Drinfel'd twist (**??**) where in $\partial_{\mu} \otimes \partial_{\nu}$, ∂_{μ} acts on the i^{th} and ∂_{ν} on the j^{th} factor in the tensor product.

Define a generalization of the *twisted flip operator* (**??**)

$$\tau_{\theta}^{ij} = \mathscr{F}_{\theta}^{-1} \tau_{0}^{ij} \mathscr{F}_{\theta} = \mathscr{F}_{\theta}^{-2} \tau_{0}^{ij} \tag{4.62}$$

where τ_{0}^{ij} flips the entries of an N-fold tensor product by flipping the i^{th} and j^{th} entries as in (**??**). Then

$$\left(\tau_{0}^{ij}\right)^{2} = \mathbb{1} \tag{4.63}$$

which is obvious and

$$\tau_{\theta}^{i,i+1} \tau_{\theta}^{i+1,i+2} \tau_{\theta}^{i,i+1} = \tau_{\theta}^{i+1,i+2} \tau_{\theta}^{i,i+1} \tau_{\theta}^{i+1,i+2} \tag{4.64}$$

which is not obvious. S_N has the presentation

$$S_N = \langle \tau_{i,i+1} : i \in [1, 2, ..., N-1], \tau_{i,i+1}^{2} = \mathbb{1}, \tau_{i,i+1} \tau_{i+1,i+2} \tau_{i,i+1} = \tau_{i+1,i+2} \tau_{i,i+1} \tau_{i+1,i+2} \rangle \tag{4.65}$$

It follows then that $\tau_{\theta}^{i,i+1}$'s generate S_N in this sector.

One can check that the Poincaré group action with the twisted coproduct commutes with this action of S_N.

4.4.5 $*$-Covariance

Covariance requirements on the Moyal plane has led us to the dressed field (**??**). We now require it to be compatible with the $*$-operation. That is if $\varphi_0^* = \varphi_0$, we want that $(\varphi^{\mathcal{M}})_{\theta}^{*} = \varphi_{\theta}^{\mathcal{M}}$. Now

$$(\varphi_{\theta}^{\mathcal{M}})^{*} = \mathrm{e}^{-\frac{1}{2}\partial \wedge P} \varphi_0 \tag{4.66}$$

where ∂_{μ} acts just on φ_0, P_{ν} acts on φ_0 and all that may follow. But since P_{ν} acting on φ_0 is $-i\partial_{\nu}\varphi_0$ and $\partial \wedge \partial = 0$, we see that

$$(\varphi^{\mathcal{M}})_{\theta}^{*} = \varphi_0^{*} \mathrm{e}^{-\frac{1}{2}\overleftarrow{\partial} \wedge P} \quad . \tag{4.67}$$

So the dressing transformations preserves $*$-covariance. The antisymmetry of θ plays a role in this process.

We can also understand these statements from (**??**). That gives

$$a_p^{\mathcal{M}} = \mathrm{e}^{-\frac{i}{2}p\wedge P}c_p = c_p\mathrm{e}^{-\frac{i}{2}p\wedge P} \tag{4.68}$$

since $p \wedge p = 0$. So *we can twist both creation and annihilation operators on the same side because θ is antisymmetric*. It is only because of this that we can get the twisted quantum Heisenberg field (**??**). The importance of its existence has been emphasised before.

4.5 Covariance on the Wick-Voros plane

We have already introduced the Wick-Voros plane $\mathcal{A}_\theta^V$ previously. The deformed product on $\mathcal{A}_\theta^V$ acts on plane waves as (cf. (**??**))

$$\mathrm{e}_p \star_V \mathrm{e}_q = \mathrm{e}^{-\frac{1}{2}\hat{\theta}p\cdot q}\mathrm{e}^{-\frac{i}{2}p\wedge q}\mathrm{e}_{p+q} \tag{4.69}$$

whereas the deformed coproduct on $H_\theta^V\mathscr{P}$ looks (cf. (**??**)):

$$\Delta_{\theta,V}(g) = (\mathscr{F}_\theta^V)^{-1}(g\otimes g)(\mathscr{F}_\theta^V), \quad \text{where} \quad \mathscr{F}_\theta^V = \mathrm{e}^{\frac{i}{2}\partial_\mu\otimes\theta_{\mu\nu}\partial_\nu+\hat{\theta}\partial_\mu\cdot\partial_\mu} \tag{4.70}$$

Let us first assume that the Wick-Voros $\star$ also admits twisted creation-annihilatin operators and associated (in-, out-, or free-) field φ_θ^V as in (**??**):

$$\varphi_\theta^V = \int \mathrm{d}\mu(p)\left[a_p^{V\dagger}\mathrm{e}_p + a_p^V\mathrm{e}_{-p}\right] := \varphi_\theta^{(-)V} + \varphi_\theta^{(+)V} \tag{4.71}$$

Primitive covariance gives as before

$$U(a,\Lambda)a_p^V U(a,\Lambda)^\dagger = a_{\Lambda p}^V \tag{4.72}$$

and

$$U(a,\Lambda)a_{p,V}^{V\dagger}U(a,\Lambda)^\dagger = a_{\Lambda p}^{V\dagger} \tag{4.73}$$

where we did not attach a θ to U.

In the two-particle sector, the coproduct is given in (**??**). As $\mathscr{F}_\theta^V$ is translationally invariant, the coproduct for P_μ is not affected by the twist. So we focus on Lorentz transformations.

For Lorentz transformations, (**??**) is modified to

$$\left(\int\prod \mathrm{d}\mu(p_i)U(\Lambda)a_{p_1}^{V\dagger}a_{p_2}^{V\dagger}|0\rangle\right)\mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)-\frac{\hat{\theta}}{2}(\Lambda p_1)\cdot(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2-\frac{\hat{\theta}}{2}p_1\cdot p_2}\mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2} \tag{4.74}$$

giving the dressing equation

$$a_p^{V\dagger} = c_p^\dagger \mathrm{e}^{\frac{i}{2}p\wedge P - \frac{\hat{\theta}}{2}p\cdot P} \quad , \tag{4.75}$$

scalar products being Euclidean.

The adjoint of (**??**) is

$$a_p^{V} = \mathrm{e}^{-\frac{i}{2}p\wedge P - \frac{\hat{\theta}}{2}p\cdot P} c_p = \mathrm{e}^{\frac{\hat{\theta}}{2}p\cdot p} c_p \mathrm{e}^{-\frac{i}{2}p\wedge P - \frac{\hat{\theta}}{2}p\cdot P} \tag{4.76}$$

which is not what we get by dressing c_p on the right.

We again came to the conclusion that φ_θ^V is not the outcome of dressing φ_0 by a single twist. Its parts $\varphi_\theta^{(\mp)V}$ get separate twists. Therefore it appears that there is no way to dress a fully interacting Heisenberg field Φ_0 since Φ_0 cannot decomposed into positive and negative frequency parts.

5

Twisting over a finite group and Geons

So far we have only considered the case of trivial spacetime topology, that is in the algebra of functions $C^\infty(M)$ we begin with, $M \cong \mathbb{R}^{d+1}$, where $\mathbb{R}^d$ represents the spatial topology. The spatial slice $\mathbb{R}^d$ is not the only admissible spatial slice for asymptotically flat spacetimes. Friedman and Sorkin (**?**) have studied generic asymptotically flat spatial slices and have come up with their remarkable interpretation in terms of gravitational topological excitations called "topological" or "Friedman-Sorkin" "geons". The diffeomorphisms (diffeos) of geon spacetimes are much richer than those from the topologically trivial ones. In particular, they contain discrete subgroups encoding the basic physics of geons. It was a striking discovery of Friedman and Sorkin that the geon spin even in pure gravity can be 1/2 or its odd multiples (**? ? ? ?**). The statistics groups of identical geons are also novel. Their precise identification requires further considerations as we shall see.

In this chapter, we develop a machinery to construct Drinfel'd twists for generic and in particular discrete diffeos. The notion of covariant quantum fields for generic spacetimes (**?**), discussed in the previous chapter, helps us construct covariant twisted fields for geons using the above twists. The requirement of covariance puts conditions on acceptable twists for quantum fields and eliminates many (**? ?**).

Spacetimes emergent from these twists are noncommutative as is appropriate at geon scales according to DFR (**?**). There is a diffeo-invariant way to define the size of a geon (**?**) and it is expected to be of Planck-scale. Spacetime noncommutativity in this case is localised at geons and is of this scale just as we wish for.

As we indicate, several novel spacetimes including non-associative spacetimes and new sorts of statistics algebras arise naturally (**?**). These matters are discussed only in a preliminary manner. But already, new phenomena like non-Pauli transitions are suggested as we will see. Most of the work presented here is based on (**?**).

To the seek of keeping the amount of mathematical details as limited as possible, we devote a separate Appendix **??** to review what Geon spacetimes are, and the definition of diffeos on a non-trivial topological spacetime.

5.1 Quantum Fields

In standard quantum physics, there is a relation between spacetime symmetries like the Poincaré group $\mathscr{P}_+^\uparrow$ and the statistics group that implements the identity of particles. It can be described as follows. An element α of the Poincaré group acts on a member ψ of the single particle Hilbert space $\mathcal{H}$ by pullback:

$$\big(\alpha\psi\big)(x) = \psi(\alpha^{-1}x) \quad . \tag{5.1}$$

This action extends to the N-particle Hilbert space $\mathcal{H}^{\otimes N}$ via the coproduct Δ_0:

$$\Delta_0(\alpha) = \alpha \otimes \alpha \tag{5.2}$$

Thus on $\mathcal{H}^{\otimes N}$, it acts by

$$\underbrace{(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \Delta_0)}_{(N-1)\ factors}\underbrace{(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \Delta_0)}_{(N-2)\ factors}...\Delta_0(\alpha) = \underbrace{\alpha \otimes \alpha \otimes ... \otimes \alpha}_{N\ factors} \tag{5.3}$$

The statistics group expressing the identity of particles must commute with the action of the symmetry group. This requirement just says that symmetry transformations, such as Lorentz transformations, should not spoil particle identity. It is fulfilled by the permutation group S_N which permutes the factors in the tensor product

$$\psi_1 \otimes \psi_2 \otimes ... \otimes \psi_N \in \mathcal{H}^{\otimes N} \tag{5.4}$$

Quantum fields compatible with the symmetry group such as $\mathscr{P}_+^\uparrow$ and implementing statistics exist. For these fields, the permutation group S_N and say the Poincaré transformation commute when acting on N-particle in- or out- states.

In the case of geon spacetimes the situation is more involved. As we discussed in Appendix **??**, in the geon case the first homotopy group of the configuration space, $\pi_1(\mathcal{Q})$, whose representation label inequivalent quantization $\mathcal{H}^{(l)}$, is not trivial (cf. (**??**)):

$$\pi_1\Big(\mathrm{Riem}(\mathcal{M}_\infty)/D^\infty\Big) = D^\infty/D_0^\infty \tag{5.5}$$

The equation above is true for a single geon spacetime. The generalization to the N-*geons* case is discussed in details in section **??** where we also set up the notation. From (**??**), $\pi_1(\mathcal{Q})$ for the $N - geons$ case is:

$$D^{(N)\infty}/D_0^{(N)\infty} \equiv \Big(\mathscr{S} \rtimes \Big[\times^N D^{(1)\infty}/D_0^{(1)\infty}\Big]\Big) \rtimes S_N \tag{5.6}$$

Where $\mathscr{S}$ are called *slides*, $D^{(1)\infty}/D_0^{(1)\infty}$ *internal symmetries* and S_N is the standard N-*dimensional* permutation group. While for the standard coproduct like that in (**??**), the internal symmetry $D^{(1)\infty}/D_0^{(1)\infty}$ acts by its diagonal map into $D^{(N)\infty}/D_0^{(N)\infty}$ and that action commutes with S_N, the slides present a more complex story. They do not commute with S_N (nor with $\alpha \otimes ... \otimes \alpha$ for $\alpha \in D^{(1)\infty}/D_0^{(1)\infty}$) and can change representations of S_N: they can convert bosons into fermions! For such reasons, Sorkin and Surya have suggested that elements of $\mathscr{S}$ represent interactions of geons. But for now we let $\mathscr{S}$ act by the identity representation on quantum states. That means that we will work with $\Big[\times^N D^{(1)\infty}/D_0^{(1)\infty}\Big] \rtimes S_N$ and their group algebra.

Below we will work with the group algebra $\mathbb{C}\big(D^{(1)\infty}/D_0^{(1)\infty}\big)$ with a twisted coproduct. In that case too, the algebra defining statistics is in the commutant of the coproduct. It is still S_N, but acts differently on $\mathcal{H}^{\otimes N}$.

From (**??**), slides form an invariant subgroup in $D^{(N)\infty}/D_0^{(N)\infty}$. For this reason, slides can be represented by identity on quantum states. Sorkin and Surya (**? ?**) have suggested that we do so motivated by the considerations above. We follow their suggestion.

5.2 Twists of Geon Spacetimes: Motivation

As we have repeatedly seen, the effect of noncommutativity can be encompassed via the *twist* F_θ

$$F_\theta = \mathrm{e}^{-\frac{i}{2}P_\mu\otimes\theta^{\mu\nu}P_\nu}\,. \tag{5.7}$$

The twist is the map which can be used to twist both the product and the coproduct in a coherent and consistent manner. Its realization on the algebra of functions is just the term appearing in between the two functions f_1 and f_2 in the deformed product. We will indicate it by a script $\mathscr{F}_\theta$:

$$\mathscr{F}_\theta = \mathrm{e}^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta_{\mu\nu} \overrightarrow{\partial}_\nu} \quad . \tag{5.8}$$

whereas F_θ deforms the action of the symmetry group by twisting Δ_0 into the deformed coproduct

$$\Delta_\theta(g) = F_\theta^{-1}(g \otimes g)F_\theta \quad . \tag{5.9}$$

In this section we want to provide some motivation to generalize such a construction to the case of geonic diffeos and in particular to $D^{(1)\infty}/D_0^{(1)\infty}$. It goes as follows.

If a sphere S^{d-1} encloses the prime in $\mathbb{R}^d \# \mathcal{P}_\alpha$ in the sense that the complement of this sphere in $\mathbb{R}^d \# \mathcal{P}_\alpha$ is homeomorphic to $\mathbb{R}^d / \mathscr{B}^d$ where $\mathscr{B}^d$ is the d–dimensional ball, then by suitably adjoining elements of D_0^∞, we can ensure that D^∞ / D_0^∞ acts as the identity outside S^{d-1}. So these diffeos can be taken to be localised on the geon. If the geon size is of the order of the Planck volume, the action of $D^{(1)\infty}/D_0^{(1)\infty}$ is also confined to such Planck volumes (It is possible to define geon sizes in a diffeo-invariant way (**?**)). As explained in the introduction, at these scales we expect the spacetime to be noncommutative and the action of the symmetry group to be consequently twisted.

We will generalise F_θ to $D^{(1)\infty}/D_0^{(1)\infty}$ and after that twist using elements of $D^{(1)\infty}/D_0^{(1)\infty}$. Then, as we shall see, spacetimes become noncommutative on the above Planck-scale volumes. This is in accordance with the arguments of DFR (**?**).

Thus the choice of twists using $D^{(1)\infty}/D_0^{(1)\infty}$ appears to be one good way to implement the DFR ideas.

It is also one way to incorporate aspects of the topology of geons in these basic quantum field theories as we shall see.

5.3 Twists of Geon Spacetimes: Coassociative Coproducts

The generalisation of F_θ to $D^{(1)\infty}/D_0^{(1)\infty}$ is not immediate since $D^{(1)\infty}/D_0^{(1)\infty}$ is discrete. It can be finite or infinite, but it is certainly discrete. So we must know how to adapt F_θ to discrete groups. The difficulty comes from the fact that for Lie groups, we write F_θ in terms of the exponential of the tensor product of Lie algebra elements, as in (**??**). There is no analogue of the Lie algebra for discrete groups. As we will shortly see, writing the twist F_θ in momentum space sheds light on the path to follow for the generalization.

The plane waves e_p, $\mathrm{e}_p(x) = \mathrm{e}^{ip\cdot x}$, carry the irreducible representations of the translation subgroup of $\mathbb{C}\mathscr{P}_+^\uparrow$. Since

$$\mathcal{P}_\mu \mathrm{e}_p = p_\mu \mathrm{e}_p \quad , \tag{5.10}$$

the restriction of $\mathscr{F}_\theta$ (**??**) to $\mathrm{e}_p \otimes \mathrm{e}_q$ is given by

$$\mathscr{F}_\theta \mathrm{e}_p \otimes \mathrm{e}_q = \mathrm{e}^{-\frac{i}{2}p_\mu \theta^{\mu\nu} q_\nu} \mathrm{e}_p \otimes \mathrm{e}_q \ . \tag{5.11}$$

Let $\mathfrak{P}_p$ be the projection operator which acting on functions of $\mathbb{R}^d$ projects to the subspace spanned by e_p. It is thus the projector to the irreducible representation of the translation subgroup identified by the real vector "p". For a particle of mass m, for which $p_0 = \sqrt{\vec{p}^2 + m^2}$, we can define $\mathfrak{P}_p$ by requiring that

$$\mathfrak{P}_p \mathrm{e}_q = 2|p_0| \delta^{(3)}(\vec{p} - \vec{q}) \mathrm{e}_p \quad . \tag{5.12}$$

Then we can see that

$$\mathscr{F}_\theta = \int \mathrm{d}\mu(p) \mathrm{d}\mu(q) \mathrm{e}^{-\frac{i}{2}p \wedge q} \mathfrak{P}_p \otimes \mathfrak{P}_q, \qquad \mathrm{d}\mu(p) := \frac{\mathrm{d}^3 p}{2\sqrt{\vec{p}^2 + m^2}} \tag{5.13}$$

where $p \wedge q := p_\mu \theta_{\mu\nu} q_\nu$, and that

$$F_\theta = \int \mathrm{d}\mu(p) \mathrm{d}\mu(q) \mathrm{e}^{-\frac{i}{2}p \wedge q} \mathfrak{P}_p \otimes \mathfrak{P}_q \quad . \tag{5.14}$$

If e_p is off–shell, so that p_0 is not constrained to be $\sqrt{\vec{p}^2 + m^2}$, we can still write $\mathscr{F}_\theta$ in terms of projections by slightly modifying (**??**).

5.3.1 A Simple Generalisation to Discrete Abelian Groups

It is possible to find a simple generalisation of (**??**-**??**) to discrete abelian groups. We first discuss this generalisation.

Consider first the group

$$\mathbb{Z}_n = \{\xi^k \equiv \mathrm{e}^{i\frac{2\pi}{n}k} : k = 0, 1, ..., (n-1)\} \quad . \tag{5.15}$$

Its IRR's ϱ_m are all one-dimensional and given by its characters χ_m:

$$\chi_m(\xi) = \xi^m, \quad m \in \{0, 1, ..., (n-1)\} \quad . \tag{5.16}$$

Then if $\hat{\xi}$ is the operator representing ξ on the space on which it acts, the projector $\mathfrak{P}_m$ to the IRR ϱ_m is

$$\mathfrak{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k)\hat{\xi}^k \tag{5.17}$$

This follows from

$$\hat{\xi}^l \mathfrak{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k)\hat{\xi}^{k+l} = \frac{1}{n} \sum_{k=l}^{n+l-1} \bar{\chi}_m(\xi^{k-l})\hat{\xi}^k = \chi_m(\xi^l)\mathfrak{P}_m \tag{5.18}$$

where we used the fact that

$$\bar{\chi}(\xi^l)\chi(\xi^l) = 1, \qquad \bar{\chi}(\xi^l) = \chi(\xi^{-l}), \tag{5.19}$$

and the orthogonality relations,

$$\frac{1}{n} \sum_{\xi} \bar{\chi}_m(\xi)\chi_n(\xi) = \delta_{m,n}\,, \tag{5.20}$$

that imply,

$$\mathfrak{P}_m \mathfrak{P}_n = \delta_{m,n}\mathfrak{P}_n\,. \tag{5.21}$$

Note that $\mathfrak{P}_m$ is the image of

$$\mathbb{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k)\xi^k \tag{5.22}$$

in the group algebra $\mathbb{C}\mathbb{Z}_n$ and that

$$\mathbb{P}_m\mathbb{P}_n = \delta_{m,n}\mathbb{P}_n, \qquad \sum_{m=0}^{n-1} \mathbb{P}_m = \mathbb{1} \quad . \tag{5.23}$$

5.3.2 The case of $D^{(1)\infty}/D_0^{(1)\infty}$

From $D^{(1)\infty}/D_0^{(1)\infty}$, we pick its maximal abelian subgroup A and assume for the moment that A is finite. Then A is the direct product of cyclic groups:

$$A = \mathbb{Z}_n \times \mathbb{Z}_{n_2} \times ... \times \mathbb{Z}_{n_k} \quad . \tag{5.24}$$

Its IRR's are given by:

$$\varrho_{m_1} \otimes \varrho_{m_2} \otimes ... \otimes \varrho_{m_k}, \quad m_j \in \{0, 1, .., n_j - 1\}, \tag{5.25}$$

with characters

$$\chi_{\vec{m}} = \prod_i \chi_{m_i} \tag{5.26}$$

and projectors $\mathfrak{P}_{\vec{m}} = \otimes_i \mathfrak{P}_{m_i}$ on the representation space or projectors

$$\mathbb{P}_{\vec{m}} = \otimes_i \mathbb{P}_{m_i}, \quad \mathbb{P}_{\vec{m}} \mathbb{P}_{\vec{m}'} = \delta_{\vec{m},\vec{m}'} \mathbb{P}_{\vec{m}}, \quad \sum_{\vec{m}} \mathbb{P}_{\vec{m}} = \text{identity of } A \tag{5.27}$$

in the group algebra $\mathbb{C}A$. (The summation of m_j in (**??**) is from 0 to $n_j - 1$).

Let $\theta = [\theta_{ij} = -\theta_{ji} \in \mathbb{R}]$ be an antisymmetric matrix with constant entries. Following (**??**), we can write a Drinfel'd twist using elements of $\mathbb{C}A$:

$$F_\theta = \sum_{\vec{m},\vec{m}'} \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}} \otimes \mathbb{P}_{\vec{m}'} \quad . \tag{5.28}$$

But there are quantisation conditions on θ_{ij}. That is because ϱ_m and ϱ_{m+n} give the same IRR for $\mathbb{Z}_n$ as (**??**) shows. That means that $\vec{m}$ and $\vec{m} + (0, ..., 0, n_i, 0, ...0)$ give the same IRR $\varrho_{\vec{m}}$, n_i being the i^{th} entry. Since F_θ must be invariant under these shifts, we find that θ_{ij} is restricted to the values

$$\theta_{ij} = \frac{4\pi}{n_{ij}} \tag{5.29}$$

where

$$\frac{n_i}{n_{ij}}, \ \frac{n_j}{n_{ij}} \in \mathbb{Z} \quad . \tag{5.30}$$

The twist (**??**) of the canonical coproduct of $\mathscr{P}_+^\uparrow$ using F_θ leads to a coassociative coproduct. Similarly the twist of the coproduct of $D^{(1)\infty}/D_0^{(1)\infty}$ or any of its subgroups leads to a coassociative coproduct. That is because the twist involves the abelian algebra $\mathbb{C}A$. As we will further discuss later on, the spacetime algebra is associative, but not commutative if a $\theta_{ij} = -\theta_{ji} \neq 0$ (**?**).

Remarks

a) The condition (**??**) has a solution $n_{ij} \neq \pm 1$ only if n_i and n_j have a common factor ($\neq \pm 1$). Thus if say $n_i = 2$, $n_j = 3$ for some i, j then $n_{ij} = \pm 1$. For either of these solutions,

$$\mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} = 1 \quad \text{or} \quad \theta_{ij} \text{ is effectively equivalent to } 0 \quad . \tag{5.31}$$

b) There are many instances where A contains factors of $\mathbb{Z}$. The IRR's ϱ_φ of $\mathbb{Z}$ are given by points of $S^1 = \{\mathrm{e}^{i2\pi\varphi} : 0 \leq \varphi \leq 1\}$:

$$\varrho_\varphi : n \in \mathbb{Z} \to \mathrm{e}^{i2\pi n\varphi}. \tag{5.32}$$

Note that

$$\varrho_\varphi = \varrho_{\varphi+1} \quad . \tag{5.33}$$

Suppose now that $A = \times_{i=1}^{k} \mathbb{Z}_{n_i} \times \mathbb{Z}$. Now its IRR's are labelled by the vector $(\vec{m}, \varphi) = (m_1, ..., m_k, \varphi)$. The twist F_θ is written as

$$F_\theta = \sum_{\vec{m},\vec{m}'} \int_0^1 \mathrm{d}\varphi \int_0^1 \mathrm{d}\varphi' \; \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \times$$
$$\times \mathrm{e}^{-\frac{i}{2}[m_i(\theta_{m_i,k+1})\varphi' - \varphi(\theta_{m_i,k+1}) m'_i]} \mathbb{P}_{(\vec{m},\varphi)} \otimes \mathbb{P}_{(\vec{m}',\varphi')} \tag{5.34}$$

But the periodicity in φ, φ' is 1 and hence $\theta_{m_i,k+1} = \pm 4\pi$ and the second exponential in (**??**) is $\mathbb{1} \otimes \mathbb{1}$. In short, F_θ has no twist factor involving $\mathbb{Z}$ and F_θ reduces back to the earlier expression (**??**). If there are say *two factors* of $\mathbb{Z}$ so that $A = \times_{i=1}^{k-1} \mathbb{Z}_{n_i} \otimes \mathbb{Z} \otimes \mathbb{Z}$ the second exponential in (**??**) is replaced by

$$\mathrm{e}^{\varphi \theta_{k,k+1} \varphi'} \tag{5.35}$$

and we require its periodicity in φ and φ'. Hence $\theta_{k,k+1} \simeq 0$. In this way, we see that F_θ depends nontrivially only on compact abelian discrete groups.

c) Later in section **??**, we will argue that the twists found above seem general so long as we insist on the coassociativity of the coproduct (or equivalently the associativity of the spacetime algebra).

5.4 On Twisted Symmetrisation and Antisymmetrisation

Let $\mathcal{H}$ be a one-geon Hilbert space. It carries a representation of $D^{(1)}$ or more generally of D. The "momentum constraint" is implemented by requiring that $D_0^\infty \to \mathbb{1}$ in this representation which we assume is satisfied.

Let τ_0 be the flip operator on $\mathcal{H} \otimes \mathcal{H}$:

$$\tau_0 \; \alpha \otimes \beta = \beta \otimes \alpha, \quad \alpha, \beta \in \mathcal{H} \quad . \tag{5.36}$$

When the coproduct is Δ_0, $\Delta_0(d) = d \otimes d$ for $d \in D$, τ_0 commutes with Δ_0 (by d here we mean the representation of d on $\mathcal{H}$.). So the subspaces $\frac{\mathbb{1}\pm\tau_0}{2}\mathcal{H} \otimes \mathcal{H}$ are invariant under diffeos and carry the identity representation of D_0^∞. We can then use them to define bosonic and fermionic geons.

But if we deform Δ_0 into (**??**), τ_0 does not commute with $\Delta_\theta(d)$ for all d anymore if $F_\theta \neq \mathbb{1} \otimes \mathbb{1}$. So the subspaces $\frac{\mathbb{1}\pm\tau_0}{2}\mathcal{H} \otimes \mathcal{H}$ are not diffeomorphism invariant, nor need they fulfill the constraint $\Delta_\theta(d)\Big[\tau_0(\alpha \otimes \beta)\Big] = \tau_0\Big[\Delta_\theta(d)(\alpha \otimes \beta)\Big]$ for $d \in D^{(1)}$. That means that bosons and fermions cannot be associated with the subspaces $\frac{\mathbb{1}\pm\tau_0}{2}\mathcal{H} \otimes \mathcal{H}$.

Instead, as discussed in previous chapters, one should use the twisted flip operator

$$\tau_\theta = F_\theta^{-1} \tau_0 F_\theta, \quad \tau_\theta^2 = \mathbb{1} \otimes \mathbb{1} \tag{5.37}$$

which commutes with the twisted coproduct $\Delta_\theta(d)$. Bosonic and fermionic geons are thus associated with the subspaces $\frac{\mathbb{1}\pm\tau_\theta}{2}\mathcal{H} \otimes \mathcal{H}$.

The twist depends on D^∞ / D_0^∞. So these twisted subspaces incorporate at least aspects of the internal diffeos of geons unlike τ_0. Such a twist of flip is a consequence of deforming the coproduct to Δ_θ. As we will discuss, this deformation introduces spacetime noncommutativity localised at the geon. Further there are outlines available for an approach to build an orderly quantum field theory (compatibly with the DFR suggestion) incorporating this noncommutativity and deformed statistics, and transforming by the twisted coproduct. These are all attractive aspects of introducing the twist F_θ.

5.5 Covariant Quantum Fields: Geons on Commutative Spacetimes

In the previous chapter, we have carefully discussed the notion of covariant fields in general including in particular the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^d)$. This concept in the limit $\theta \to 0$ reduces to the corresponding well-known concept for $\theta = 0$. We now want to generalize such a construction to geons' spacetimes.

We assume that a covariant quantum field φ_0 can be associated with a geon when the underlying spacetime is commutative. Diffeomorphism invariance implies that $D_0^{(1)\infty}$ acts trivially on φ_0. So the group $D^{(1)}/D_0^{(1)\infty}$ acts nontrivially on φ_0 by the pull-back of the action of $D^{(1)}$ on spacetime: if $g \in D/D_0^{\infty}$[1] and $\hat{g} = g g_0^\infty$, $g_0^\infty \in D_0^\infty$ is any member of the equivalence class gD_0^∞, then

$$g : \varphi_0 \to g\varphi_0, \quad (g\varphi_0)(p) = \varphi_0(\hat{g}^{-1}p) \tag{5.38}$$

This action does not depend on the choice of g_0^∞ since $g_0^\infty \varphi_0 = \varphi_0$ for all $g_0^\infty \in D_0^\infty$, and hence is consistent.

Equation (**??**) has been written for scalar geon fields for simplicity. It is easily generalised to spinorial and tensorial fields.

Also for simplicity, we will henceforth write

$$(g\varphi_0)(p) = \varphi_0(g^{-1}p) \tag{5.39}$$

even though on the r.h.s., we should write $\hat{g}^{-1}p$.

Covariance implies that there exists a representation U of D/D_0^∞ so that

$$U(g)\varphi_0(g^{-1}p)U(g)^{-1} = \varphi_0(p) \tag{5.40}$$

or

$$U(g)\varphi_0(p)U(g)^{-1} = \varphi_0(gp) \quad . \tag{5.41}$$

The twist we now consider is based on abelian discrete compact groups A: as we saw, its dependence on representations of $\mathbb{Z}$ is trivial. Let $f_{\vec{m}}^{(\pm)}$ furnish the orthonormal basis

[1] From now on we will only refer to the single geon diffeo group. Therefore we will use D, D^∞ and D_0^∞ instead of $D^{(1)}$, $D^{(1)\infty}$ and $D_0^{(1)\infty}$ to simplify the notation.

on the geon spacetime which carry the UIRR $\vec{m} = (m_1, m_2, \ldots, m_k)$ of $A = \times_{i=1}^{k}\mathbb{Z}_{n_i}$ and which have positive and negative frequencies $\pm|E_{\vec{m}}|^2$:

$$f_{\vec{m}}^{(\pm)}(h^{-1}p) = f_{\vec{m}}^{(\pm)}(p)\chi_{\vec{m}}(h), \quad h \in A \quad , \tag{5.42}$$

$$i\partial_0 f_{\vec{m}}^{(\pm)} = \pm|E_{\vec{m}}|f_{\vec{m}}^{(\pm)} \tag{5.43}$$

Here $\chi_{\vec{m}}$ is the character function of A. Since $\bar{\chi}_{\vec{m}} = \chi_{-\vec{m}}$, we can assume that

$$\bar{f}_{\vec{m}}^{(\pm)} = f_{-\vec{m}}^{(\mp)} \tag{5.44}$$

If $g \in D/D_0^\infty$, we can then write

$$f_{\vec{m}}^{(\pm)}(g^{-1}p) = \sum_{\vec{m}'} f_{\vec{m}'}^{(\pm)}(p)\mathscr{D}_{\vec{m}'\vec{m}}(g) \tag{5.45}$$

where $\mathscr{D}$ is a unitary representation of D/D_0^∞.

The untwisted quantum field φ_0, assumed real for simplicity, and also assumed to be in, out or free field, can be written as

$$\varphi_0 = \sum_{\vec{m}} \left[c_{\vec{m}} f_{\vec{m}}^{(+)} + c_{\vec{m}}^\dagger f_{-\vec{m}}^{(-)}\right] \tag{5.46}$$

Here $c_{\vec{m}}$, $c_{\vec{m}}^\dagger$ are annihilation and creation operators:

$$[c_{\vec{m}}, c_{\vec{n}}^\dagger] = \delta_{\vec{m},\vec{n}} \quad , \tag{5.47}$$

$$[c_{\vec{m}}, c_{\vec{n}}] = [c_{\vec{m}}^\dagger, c_{\vec{n}}^\dagger] = 0 \quad . \tag{5.48}$$

Covariance is the requirement that there is a unitary representation of D/D_0^∞ on the Hilbert space of vector states such that

$$U(g)\varphi_0(g^{-1}p)U(g)^{-1} = \varphi_0(p) \quad . \tag{5.49}$$

Hence since $\bar{\mathscr{D}}_{\vec{m}'\vec{m}}(g)\mathscr{D}_{\vec{n}'\vec{m}}(g) = \delta_{\vec{m}',\vec{n}'}$ (with sum over $\vec{m}$ being implicit),

$$U(g)c_{\vec{m}}U(g)^{-1} = c_{\vec{m}'}\bar{\mathscr{D}}_{\vec{m}'\vec{m}}(g) \quad , \tag{5.50}$$

$$U(g)c_{\vec{m}}^\dagger U(g)^{-1} = c_{\vec{m}'}^\dagger \mathscr{D}_{\vec{m}'\vec{m}}(g) \quad . \tag{5.51}$$

For untwisted fields, the symmetrisation postulates on $f_{\vec{m}}^{(\pm)}$ are based on Bose statistics for tensorial fields. They are incorporated in (**??**,**??**) and are compatible with covariance.

[2] We assume their existence as is normally the case.

5.5.1 Covariance for Abelian Twists

The twisted quantum field φ_θ associated with φ_0 is written as

$$\varphi_\theta = \sum_{\vec{m}} \left[a_{\vec{m}} f^{(+)}_{\vec{m}} + a^\dagger_{\vec{m}} f^{(-)}_{-\vec{m}} \right] \tag{5.52}$$

We will as before deduce the relation of $a_{\vec{m}}$, $a^\dagger_{\vec{m}}$ to $c_{\vec{m}}$, $c^\dagger_{\vec{m}}$ using covariance.

First consider

$$\varphi^{(-)}_\theta |0\rangle = \sum_{\vec{m}} a^\dagger_{\vec{m}} |0\rangle f^{(-)}_{-\vec{m}} = \sum_{\vec{m}} a^\dagger_{\vec{m}} |0\rangle \bar{f}^{(+)}_{\vec{m}} \quad . \tag{5.53}$$

Covariance implies the requirement

$$\sum_{\vec{m}} U(g) a^\dagger_{\vec{m}} U(g)^{-1} |0\rangle \bar{f}^{(+)}_{\vec{m}'} \bar{\mathscr{D}}_{\vec{m}'\vec{m}}(g) = \varphi^{(-)}_\theta |0\rangle \tag{5.54}$$

where $U(g)$ represents g on the vector states and we use $U(g)|0\rangle = |0\rangle$. Hence

$$U(g) a^\dagger_{\vec{m}} U(g)^{-1} |0\rangle = \sum_{\vec{m}''} a^\dagger_{\vec{m}''} |0\rangle \mathscr{D}_{\vec{m}''\vec{m}}(g) \tag{5.55}$$

Next consider the two-particle case:

$$\varphi^{(-)}_\theta \otimes \varphi^{(-)}_\theta |0\rangle = \sum_{\vec{m},\vec{n}} a^\dagger_{\vec{m}} a^\dagger_{\vec{n}} |0\rangle \bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}} \tag{5.56}$$

The action of $g \in D/D^\infty_0$ on $\bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}}$ is via the twisted coproduct:

$$\begin{aligned} g \triangleright \bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}} &= F^{-1}_\theta (g \otimes g) F_\theta \bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}} \\ &= F^{-1}_\theta (g \otimes g) \bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}} \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} n_j} \\ &= F^{-1}_\theta \sum_{\vec{m}',\vec{n}'} \bar{f}^{(+)}_{\vec{m}'} \otimes \bar{f}^{(+)}_{\vec{n}'} \bar{\mathscr{D}}_{\vec{m}'\vec{m}}(g) \bar{\mathscr{D}}_{\vec{n}'\vec{n}}(g) \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} n_j} \\ &= \sum_{\vec{m}',\vec{n}'} \bar{f}^{(+)}_{\vec{m}'} \otimes \bar{f}^{(+)}_{\vec{n}'} \mathrm{e}^{\frac{i}{2} m'_i \theta_{ij} n'_j} \bar{\mathscr{D}}_{\vec{m}'\vec{m}}(g) \bar{\mathscr{D}}_{\vec{n}'\vec{n}}(g) \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} n_j} \quad . \end{aligned} \tag{5.57}$$

The covariance requirement

$$\sum_{\vec{m},\vec{n}} U(g) a^\dagger_{\vec{m}} a^\dagger_{\vec{n}} |0\rangle \left(g \triangleright \bar{f}^{(+)}_{\vec{m}} \otimes \bar{f}^{(+)}_{\vec{n}} \right) = \varphi^{(-)}_\theta \otimes \varphi^{(-)}_\theta |0\rangle \tag{5.58}$$

can thus be fulfilled by setting

$$a^\dagger_{\vec{m}} = \sum_{\vec{m}'} c^\dagger_{\vec{m}} \mathrm{e}^{\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \tag{5.59}$$

and identifying $U(g)$ as the untwisted operator with the action (**??**,**??**) on $c_{\vec{m}}$, $c^\dagger_{\vec{m}}$.

The adjoint of (**??**) gives

$$a_{\vec{m}} = \sum_{\vec{m}'} \left(\mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \right) c_{\vec{m}} \equiv V_{-\vec{m}} \; c_{\vec{m}} \tag{5.60}$$

Now $V_{-\vec{m}}$ is unitary with inverse

$$V^{-1}_{-\vec{m}} = V_{\vec{m}} = \sum_{\vec{m}'} \mathrm{e}^{\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \tag{5.61}$$

It is the unitary operator on the quantum HIlbert space representing the element

$$\times_j \mathrm{e}^{\frac{i}{2} m_i \theta_{ij}} \in A \tag{5.62}$$

(The quantisation condition on θ_{ij} is also manifest from here.) Hence

$$V_{\vec{m}} a_{\vec{m}} V^{-1}_{\vec{m}} = \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m_j} a_{\vec{m}} = a_{\vec{m}} \tag{5.63}$$

and

$$a_{\vec{m}} = \sum_{\vec{m}'} c_{\vec{m}} \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{m'_j} \tag{5.64}$$

so that we can freely twist on left or right.

The twisted symmetrisation properties (statistics) of the multigeon states

$$a^\dagger_{\vec{m_1}} a^\dagger_{\vec{m_2}} ... a^\dagger_{\vec{m_N}} |0\rangle \tag{5.65}$$

follows from (**??**).

Self-reproduction under the $\star$-product can also be easily verified:

$$\left(a_{\vec{m}} f^{(+)}_{\vec{m}} \right) \star \left(a_{\vec{n}} f^{(+)}_{\vec{n}} \right) = \left(c_{\vec{m}} c_{\vec{n}} \right) \mathrm{e}^{\frac{i}{2} (m_i + n_i) \theta_{ij} m'_j} f^{(+)}_{\vec{m}+\vec{n}} \mathfrak{P}_{m'_j} \tag{5.66}$$

There are similar equation involving creation operators. Here again $\mathfrak{P}_{\vec{m}} = \prod_j \mathfrak{P}_{m_j}$ be the projection operator which acting on functions, projects out the IRR $\vec{m}$ of A. With this notation, we can incorporate the dressing transformation directly in φ_θ:

$$\varphi_\theta = \sum_{\vec{m}, \vec{m}'} \left(\mathfrak{P}_{\vec{m}} \varphi \right) \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \quad . \tag{5.67}$$

This equation is the analogue of the dressing transformation for the Moyal field.

5.6 How may we generalise?

Physical considerations outlined below suggest that the twist discussed above (and its generalisations such as that in the Moyal case) is unique upto unitary equivalence if we require the spacetime algebra to be associative. We do have nonassociative examples (**?**), they are associated with quasi-Hopf algebras as symmetries. We will now briefly consider them as well.

5.6.1 Abelian Twists $\Rightarrow$ Associative Spacetimes

For the abelian algebra, we retain $A = \times_{i=1}^{k}\mathbb{Z}_{n_i}$. If $f_{\vec{m}}^{(\eta)}$, $(\eta = \pm)$, denote the same functions as before, then for the $\star$-product, we assume the general form

$$f_{\vec{m}}^{(\eta)} \star f_{\vec{m}'}^{(\varrho)} = \sigma(\vec{m},\vec{m}') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}, \quad \eta,\varrho = \pm, \quad \sigma(\vec{m},\vec{m}') \in \mathbb{C} \quad , \tag{5.68}$$

where on the right, $f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}$ denotes point-wise product.

Now $f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}$ transforms by the representation $\vec{m}+\vec{m}'$ (modulo n_i in each entry). Taking this into account we require associativity:

$$f_{\vec{m}}^{(\eta)} \star \left(f_{\vec{m}'}^{(\varrho)} \star f_{\vec{m}''}^{(\zeta)}\right) = \left(f_{\vec{m}}^{(\eta)} \star f_{\vec{m}'}^{(\varrho)}\right) \star f_{\vec{m}''}^{(\zeta)} \quad . \tag{5.69}$$

The l.h.s. and r.h.s. of this equation are

$$\text{l.h.s.} = \sigma(\vec{m},\vec{m}'+\vec{m}'')\sigma(\vec{m}',\vec{m}'') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)} f_{\vec{m}''}^{(\zeta)} \tag{5.70}$$

$$\text{r.h.s.} = \sigma(\vec{m},\vec{m}')\sigma(\vec{m}+\vec{m}',\vec{m}'') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)} f_{\vec{m}''}^{(\zeta)} \tag{5.71}$$

Therefore

$$\sigma(\vec{m},\vec{m}'+\vec{m}'')\sigma(\vec{m}',\vec{m}'') = \sigma(\vec{m},\vec{m}')\sigma(\vec{m}+\vec{m}',\vec{m}'') \quad . \tag{5.72}$$

It has the solution

$$\sigma(\vec{m},\vec{m}') = \mathrm{e}^{-\frac{i}{2} m_i \hat{\theta}_{ij} m'_j} \tag{5.73}$$

where $\hat{\theta}_{ij}$ is quantised as before:

$$\hat{\theta}_{ij} = \frac{4\pi}{n_{ij}}, \quad \frac{n_i}{n_{ij}}, \frac{n_j}{n_{ij}} \in \mathbb{Z} \quad . \tag{5.74}$$

Note that the quantisation requirement forces $\hat{\theta}_{ij}$ to be real, but not necessarily antisymmetric. Hence we can in general write

$$\hat{\theta}_{ij} = \theta_{ij} + s_{ij}, \quad \theta_{ij} = -\theta_{ij} = \frac{4\pi}{n_{ij}}, \quad s_{ij} = s_{ji} = \frac{4\pi}{m_{ij}} \tag{5.75}$$

where both n_{ij} and m_{ij} divide n_i and n_j, that is fulfill the analogue of (**??**).

Thus associativity and quantisation conditions reduce σ to the form

$$\sigma(\vec{m},\vec{m}') = \mathrm{e}^{-\frac{i}{2}m_i\theta_{ij}m'_j}\ \mathrm{e}^{\frac{i}{2}m_i s_{ij} m'_j} \tag{5.76}$$

with the constraints on θ_{ij} and s_{ij} stated above.

The corresponding Drinfel'd twist is

$$F_\sigma = \sum_{\vec{m},\vec{m}'} \sigma(\vec{m},\vec{m}')\mathbb{P}_{\vec{m}} \otimes \mathbb{P}_{\vec{m}'} \tag{5.77}$$

Note that

$$|\sigma(\vec{m},\vec{m}')| = 1, \quad F_\sigma^{-1} = F_{\bar{\sigma}} \quad . \tag{5.78}$$

If ϵ is the counit, then there is the normalisation condition (**?**)

$$(\epsilon \otimes \mathbb{1})F_\sigma = (\mathbb{1} \otimes \epsilon)F_\sigma = \mathbb{1} \quad . \tag{5.79}$$

Here ϵ is the map to the "trivial" representation, so $\vec{m}$ and $\vec{m}'$ become $\vec{0}$ (mod $\vec{n} = (n_1,..,n_k)$) under ϵ and $\epsilon(\mathbb{P}_{\vec{m}}) = \delta_{\vec{m},\vec{0}}$, $\epsilon(\mathbb{P}_{\vec{m}'}) = \delta_{\vec{m}',\vec{0}}$. Since $\sum_{\vec{m}} \mathbb{P}_{\vec{m}} = \sum_{\vec{m}'} \mathbb{P}_{\vec{m}'} = \mathbb{1}$, the above requirement is fulfilled by (**??**).

Next we show that the symmetric factor with s_{ij} can be eliminated by requiring that the twist preserves the adjoint operation.

For the twist F_σ above, the dressed annihilation and creation operators are

$$a_{\vec{m}} = \sum_{\vec{m}'} c_{\vec{m}} \mathrm{e}^{-\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j}\mathbb{P}_{\vec{m}'} \tag{5.80}$$

$$a^*_{\vec{m}} = \sum_{\vec{m}'} c^\dagger_{\vec{m}} \mathrm{e}^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j}\mathbb{P}_{\vec{m}'} \tag{5.81}$$

where $*$ denotes that it is not necessarily the adjoint † of $a_{\vec{m}}$, and we have used the fact that $a^*_{\vec{m}}$ transforms by the representation $-\vec{m}$.

Now

$$a^\dagger_{\vec{m}} = \Big(\sum_{\vec{m}'} \mathrm{e}^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j}\mathbb{P}_{\vec{m}'}\Big)c^\dagger_{\vec{m}} \tag{5.82}$$

The prefactor is the unitary operator $U_{\vec{m}}$ representing the element

$$\times_j\ \mathrm{e}^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})} \tag{5.83}$$

in A. Hence

$$U_{\vec{m}} c^\dagger_{\vec{m}} U^{-1}_{\vec{m}} = \mathrm{e}^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m_j} c^\dagger_{\vec{m}} = \mathrm{e}^{\frac{i}{2}m_i s_{ij} m_j} c^\dagger_{\vec{m}} \tag{5.84}$$

since $\theta_{ij} = -\theta_{ji}$. Thus

$$a^{\dagger}_{\vec{m}} = \mathrm{e}^{\frac{i}{2} m_i s_{ij} m_j} a^{*}_{\vec{m}} \quad . \tag{5.85}$$

The requirement

$$a^{*}_{\vec{m}} = a^{\dagger}_{\vec{m}} \,. \tag{5.86}$$

imposes the constraint

$$\mathrm{e}^{\frac{i}{2} m_i s_{ij} m_j} = 1 \tag{5.87}$$

From this we can infer that $s_{ij} = 0 \bmod 4\pi/m_{ij}$ where n_i/m_{ij}, $n_j/m_{ij} \in \mathbb{Z}$. For example the successive choices $\vec{m} = (1, \mathbf{0}), (0, 1, \vec{0}), (1, 1, \vec{0})$, shows that $s_{ij} = 0$ if $i, j \leq 2$. Thus (**??**) reduces σ to

$$\sigma(\vec{m}, \vec{m}') = \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \tag{5.88}$$

It thus appears that our previous considerations are general for associative spacetime algebras.

5.6.2 Nonabelian Generalisations of Drinfel'd Twists

We now discuss nonabelian generalisations of the above considerations. They generally lead to quasi-Hopf algebras based on D^{∞}/D^{∞}_0 as the symmetry algebras and non-associative spacetimes.

Here is an approach to such a generalisation. Let us consider the following nested groups:

$$D^{\infty}/D^{\infty}_0 \equiv G_0 \supset G_1 \supset \ldots \supset G_N = A \tag{5.89}$$

Here $A = \times_{i=1}^{k} \mathbb{Z}_{n_i}$ is the maximal abelian subgroup of G_0 (quotiented by factors of $\mathbb{Z}$) while the rest, G_k for $k < N$, can be nonabelian. The chain is supposed to be such that there exists an orthonormal basis $\{b^{(\vec{\varrho})}\}$ $\Big(\vec{\varrho} = (\varrho_0, \varrho_1, \ldots, \varrho_N)\Big)$ for the vector space $V^{(\varrho_0)}$ for the IRR ϱ_0 of G_0 where $b^{(\vec{\varrho})}$ is a vector in the representation space for the IRR ϱ_j of G_j. In this notation, $\varrho_N =$ our previous $\vec{m}$. Thus the chain (**??**) leads to a complete system of labels for the basis vectors.

Let $\mathbb{P}_{\vec{\varrho}}$ be the projector to the space $\mathbb{C} b^{(\vec{\varrho})}$:

$$\mathbb{P}_{\vec{\varrho}}\, b^{(\vec{\varrho})} = b^{(\vec{\varrho})} \quad . \tag{5.90}$$

Then

$$\mathbb{P}_{\vec{\varrho}} \mathbb{P}_{\vec{\varrho}'} = \delta_{\vec{\varrho}, \vec{\varrho}'} \mathbb{P}_{\vec{\varrho}}, \quad \sum_{\vec{\varrho}} \mathbb{P}_{\vec{\varrho}} = \mathbb{1} \quad . \tag{5.91}$$

Let $\vec{\varrho}_\epsilon$ label the IRR associated with the counit ϵ. Then

$$\epsilon(\mathbb{P}_{\vec{\varrho}}) = \delta_{\vec{\varrho},\vec{\varrho}_\epsilon} \tag{5.92}$$

Now consider

$$F_\sigma = \sum_{\vec{\varrho},\vec{\varrho}'} \sigma(\vec{\varrho},\vec{\varrho}')\mathbb{P}_{\vec{\varrho}} \otimes \mathbb{P}_{\vec{\varrho}'}, \quad \sigma(\vec{\varrho},\vec{\varrho}') \in \mathbb{C} \quad . \tag{5.93}$$

We plan to use F_σ as the Drinfel'd twist. Its realization used to deform the $\star$-product of functions will be indicated as usual as $\mathscr{F}_\sigma$. It involves the realization of $\mathbb{P}_{\vec{\varrho}}$'s on functions which again we will call $\mathfrak{P}_{\vec{\varrho}}$'s. The Drinfel'd twist of the coproduct as in (**??**), requires F_σ to be invertible so that

$$\sigma(\vec{\varrho},\vec{\varrho}') \neq 0 \quad \text{for any } \tilde{\varrho}, \tilde{\varrho}' \quad . \tag{5.94}$$

For then,

$$F_\sigma^{-1} = \sum_{\vec{\varrho},\vec{\varrho}'} \frac{1}{\sigma(\vec{\varrho},\vec{\varrho}')}\mathbb{P}_{\vec{\varrho}} \otimes \mathbb{P}_{\vec{\varrho}'} \quad . \tag{5.95}$$

The next requirement on F_σ is the normalisation condition

$$(\epsilon \otimes \mathbb{1})F_\sigma = (\mathbb{1} \otimes \epsilon)F_\sigma = \mathbb{1} \quad . \tag{5.96}$$

In view of (**??**), this requires that

$$\sigma(\vec{\varrho}_\epsilon, \vec{\varrho}) = \sigma(\vec{\varrho}, \vec{\varrho}_\epsilon) = 1 \quad . \tag{5.97}$$

According to Majid (**?**), there is no further requirement on F_σ if quasi-Hopf algebras are acceptable. The spacetime algebra with its star product governed by F_σ as in previous sections is then its module algebra which is generally nonassociative (with an associator) (**? ? ?**). It is associative only if its symmetry algebra is Hopf.

The spacetime orthonormal basis is now denoted by $b^{(\pm)}_{\vec{\varrho}}$ instead of by $f^{(\pm)}_{\vec{m}}$ while the twisted quantum field is written as

$$\varphi_\theta = \textstyle\sum_{\vec{\varrho},\vec{\varrho}'} \left[a_{\vec{\varrho}}\, b^{(+)}_{\vec{\varrho}} + a^*_{\vec{\varrho}}\, b^{(-)}_{\vec{\varrho}}\right] \tag{5.98}$$

$$a_{\vec{\varrho}} = \textstyle\sum_{\vec{\varrho}'} c_{\vec{\varrho}}\, \sigma(\vec{\varrho},\vec{\varrho}')\mathbb{P}_{\vec{\varrho}'} \tag{5.99}$$

$$a^*_{\vec{\varrho}} = \textstyle\sum_{\vec{\varrho}'} c^\dagger_{\vec{\varrho}}\, \bar{\sigma}(\vec{\varrho},\vec{\varrho}')\mathbb{P}_{\vec{\varrho}'} \tag{5.100}$$

where $c_{\vec{\varrho}}$, $c^\dagger_{\vec{\varrho}}$ are the untwisted annihilation and creation operators.

Unitarity requires that

$$a^*_{\vec{\varrho}} = a^\dagger_{\vec{\varrho}} = \sum_{\vec{\varrho}'} \bar{\sigma}(\vec{\varrho}, \vec{\varrho}')\mathbb{P}_{\vec{\varrho}'} c^\dagger_{\vec{\varrho}} = \sum_{\vec{\varrho}'} c^\dagger_{\vec{\varrho}}\, \bar{\sigma}(\vec{\varrho}, \vec{\varrho})\ \bar{\sigma}(\vec{\varrho}, \vec{\varrho}')\mathbb{P}_{\vec{\varrho}'} \quad . \tag{5.101}$$

Hence we have also

$$\sigma(\vec{\varrho}, \vec{\varrho}) = 1 \quad . \tag{5.102}$$

Thus it appears that we have an approach to a quantum field theory if the normalisation condition (**??**) and the unitary condition (**??**) are fulfilled.

If $\mathfrak{P}_{\vec{\varrho}}$ is the projector on the space of functions to the IRR $\vec{\varrho}$, the twisted field can be written without a mode expansion:

$$\varphi_\theta = \sum_{\vec{\varrho}, \vec{\varrho}'} \sigma(\vec{\varrho}, \vec{\varrho}') \Big(\mathfrak{P}_{\vec{\varrho}} \varphi_0 \Big) \mathbb{P}_{\vec{\varrho}'} \quad . \tag{5.103}$$

It is then easily verified that the dressed field (**??**) coincides with (**??**) and it has the self-reproducing property:

$$\varphi_\theta \star \varphi_\theta = \sum_{\vec{\varrho}, \vec{\varrho}'} \sigma(\vec{\varrho}, \vec{\varrho}') \Big(\mathfrak{P}_{\vec{\varrho}} \varphi_0^2 \Big) \mathbb{P}_{\vec{\varrho}} \quad . \tag{5.104}$$

But there is in general no associativity:

$$(\varphi_\theta \star \varphi_\theta) \star \varphi_\theta \neq \varphi_\theta \star (\varphi_\theta \star \varphi_\theta) \quad . \tag{5.105}$$

Such quantum fields merit study. They seem to lead to Pauli principle violations with testable experimental consequences.

5.7 On Rings and Their Statistics (Motion) Groups

A theoretical approach to the investigation of statistics of a system of identical constituents is based on the properties of the fundamental group of its configuration space. For N spinless identical particles in a Euclidean space of three or more dimensions, for example, this group is known to be the permutation group S_N (**?**). There is furthermore an orderly method for the construction of a distinct quantum theory for each of its unitary irreducible representations (UIRs). As these theories describe bosons, fermions and paraparticles according to the choice of the representation, the study of the fundamental group leads to a comprehensive account of the possible statistics of structureless particles in three or more dimensions.

It has been appreciated for some time that the statistical possibilities of a particle species confined to the plane $\mathbb{R}^2$ can be quite different from those in three or more dimensions. This is because the fundamental group for N identical spinless particles in a plane is not S_N. It is instead an infinite group B_N, known as the braid group. Since S_N is a factor group of B_N, and hence representations of S_N are also those of B_N, it is of course possible to associate Bose, Fermi or parastatistics with a particle species in a plane. But since B_N has many more UIRs which are not UIRs of S_N, there are also several possibilities for exotic planar statistics. One such possibility of particular interest, for instance, is that of fractional statistics, which is of importance in the context of fractional quantum Hall effect.

As we discussed earlier, it was pointed out some time ago that configuration space with unusual fundamental groups, and hence exotic statistical possibilities, occur not merely for point particles on a plane, but also for topological geons. It was also emphasized elsewhere (**?**) that there are many remarkable properties associated with the quantum version of geons, such as the failure of the spin-statistics connection and the occurrence of states in three spatial dimensions which are not bosons, fermions or paraparticles.

In (**? ?**) the investigation of exotic statistics was continued by examining another system of extended objects, namely a system of identical closed strings assumed to be unknots and imbedded in three spatial dimensions. Using known mathematical results on motion groups (**?**), it was shown that the fundamental group of the configuration space of two or more such strings is not the permutation group either. It is instead an infinite non-Abelian group which bears a certain resemblance to the gravitational fundamental groups mentioned a moment ago. It was further shown that quantum strings as well may not be characterized by permutation group representations. Thus they may not obey Bose, Fermi or parastatistics. They may also fail to obey the familiar spin-statistics connection.

Thus identical geons and identical knots share certain topological properties. For this reason, in this section we briefly examine the statistics of identical unknots. We here consider only the configuration spaces of one and two unknots and their fundamental groups.

We denote the configuration space of N unknots in $\mathbb{R}^3$ as $\mathcal{Q}^{(N)}$ and consider $N = 1$ and 2. These unknots can be unoriented or oriented. These cases will be discussed

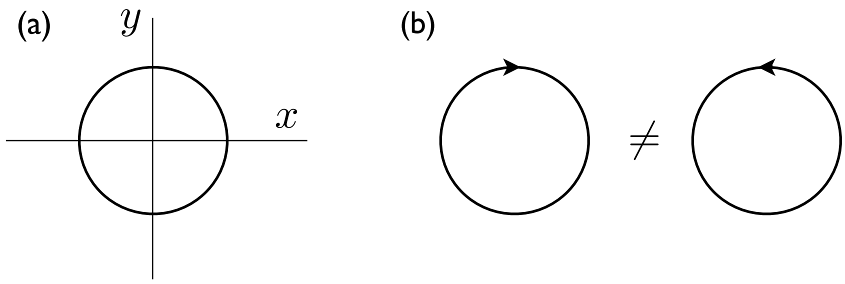

Figure 5.1: (a) Circle of unit radius centered at the origin. (b) Two oriented inequivalent knots.

separately.

5.7.1 *The case of one unoriented unknot*

An unoriented unknot is a map of a circle S^1 into $\mathbb{R}^3$ where the image is the unoriented unknot. That means the following:

a) It can be deformed to the standard map where the image is say the circle $\{(x, y, 0) : \sum x^2 + y^2 = 1\}$ in the 1-2 plane. (Here we chose the flat metric δ_{ij}).

b) Two maps which differ by an orientation reversal of S^1 are identified.

Intuitively, an unknot is a closed loop deformable to the above standard loop.

The configuration space $\mathcal{Q}^{(1)}$ of the unknot consists of all such maps.

We now consider the fundamental group $\pi_1(\mathcal{Q}^{(1)})$.

The construction of $\pi_1(\mathcal{Q}^{(1)})$ involves the choice of a fixed ("base") point $\bar{q}$ in $\mathcal{Q}^{(1)}$. As $\mathcal{Q}^{(1)}$ is the space of maps from S^1 to $\mathbb{R}^3$, $\bar{q}$ in this case is one particular choice of such maps. If $\mathcal{Q}^{(1)}$ is connected, as is the case for us, it can be any point $\bar{q}$ of $\mathcal{Q}^{(1)}$. The resultant group $\pi_1(\mathcal{Q}^{(1)}, \bar{q})$, where we have put in the base point $\bar{q}$ in the notation for the fundamental group, does not depend on $\bar{q}$. So we can talk of $\pi_1(\mathcal{Q}^{(1)})$ and omit $\bar{q}$.

But there is no canonical isomorphism between $\pi_1(\mathcal{Q}^{(1)}; \bar{q})$ and $\pi_1(\mathcal{Q}^{(1)}; \bar{q}')$ with $\bar{q} \neq \bar{q}'$. Any isomorphism depends on the choice of the path from $\bar{q}$ to $\bar{q}'$ (**?**).

For $\bar{q}$, we can for convenience choose the flat metric δ_{ij} in $\mathbb{R}^3$ as we did above, and choose $\bar{q}$ to be a circle of unit radius centered at the origin in the $x - y$ plane as in Fig. **??** (a).

Consider rotating this figure by π around the x-axis. It maps $\bar{q}$ to $\bar{q}$ and creates a loop T in $\mathcal{Q}^{(1)}$ as the rotation evolves from 0 to π. The loop cannot be deformed to a point, the point loop based at $\bar{q}$. So [T], the homotopy class of this loop is a non-trivial element of $\pi_1(\mathcal{Q}^{(1)})$.

Rotating $\bar{q}$ around another axis $\hat{n}$ ($\hat{n} \cdot \hat{n} = 1$) through the origin generates a loop which however is homotopic to T: just consider the sequence of loops got by rotating $\hat{n}$ to the x-axis $\hat{i}$ to this result.

By repeating T k-times, we get a $k\pi$ rotation loop call it T^k, of $\bar{q}$. If J_1 is the angular momentum of $SO(3)$, then

$$\{\mathrm{e}^{i\theta J_1} \ : \ 0 \leq \theta \leq 2\pi\} \tag{5.106}$$

is a 2π-rotation loop in $SO(3)$ and T^2 is just $\{\mathrm{e}^{i\theta J_1}\bar{q} \ : \ 0 \leq \theta \leq 2\pi\}$.

But this loop can be deformed to a point. For consider the sequence of loops

$$\{\mathrm{e}^{i\theta \hat{n}\vec{J}}\bar{q} : \ 0 \leq \theta \leq 2\pi, \ \hat{n} \cdot \hat{n} = 1\} \tag{5.107}$$

as $\hat{n}$ varies from (1,0,0) to (0,0,1). The starting loop is T, the final loop is a point. Thus $[\mathrm{T}^2]=e$.

We thus see that

$$\pi_1(\mathcal{Q}^{(1)}) = \mathbb{Z}_2 = \langle [T], [\mathrm{T}^2] = e\rangle \tag{5.108}$$

5.7.2 *The case of the oriented unknot*

In this case we drop the identification *b*) above so that there is an arrow attached to the unknot like in Fig. **??** (b). Otherwise, its configuration space $\mathcal{Q}^{(1)}$ is defined as above.

As for $\pi_1(\mathcal{Q}^{(1)})$, the base point $\bar{q}$ is as above, but there is now an arrow on the circle in Fig. **??** (b). Hence the curve

$$T = \langle \mathrm{e}^{i\theta J_1}\bar{q} \ : \ 0 \leq \theta \leq \pi\rangle \tag{5.109}$$

does not close (is not a loop). We conclude that

$$\pi_1(\mathcal{Q}^{(1)}) = \{e\} \tag{5.110}$$

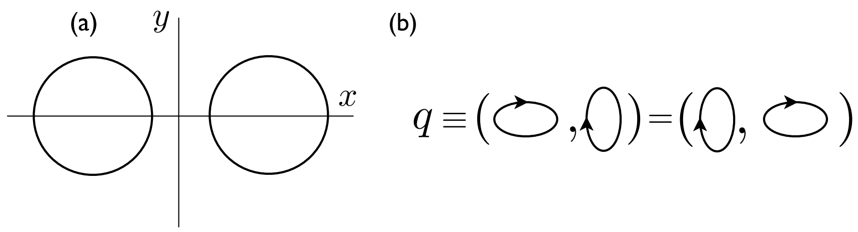

Figure 5.2: (a) The configuration space of two identical unoriented unknots. (b) Pair of unordered identical knots.

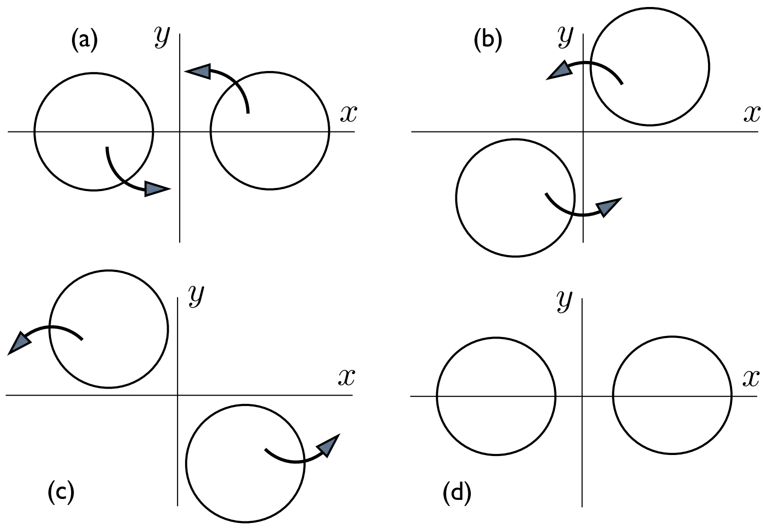

Figure 5.3: The loop E defining "exchange".

5.7.3 *The case of two identical unoriented unknots*

Its configuration space $\mathcal{Q}^{(2)}$ can be informally described (see Fig. **??**) (a) as follows: A point $q \in \mathcal{Q}^{(2)}$ consists now of 2 unlinked unknots in $\mathbb{R}^3$. The pair is unordered as the knots are "identical", see Fig. **??** (b). This requirement is as for identical particles (**?**).

For $\bar{q}$, using our flat metric, we choose two circles of unit radius on the $x - y$ plane centered in ± 2.

The discussion of identical unknots here is to be compared with the corresponding discussion of identical geons (**?**).

We can now recognize the following elements of $\pi_1(\mathcal{Q}^{(2)})$:

- *Exchange* [E]: The loop E defining "exchange" rotates $\bar{q}$ from 0 to π around third axis:

$$\mathrm{E} = \langle \mathrm{e}^{i\theta J_3}\bar{q} \ : \ 0 \leq \theta \leq \pi \rangle \quad . \tag{5.111}$$

Its evolving pictures are depicted in Fig. ??.

There are standard proofs that

$$\mathrm{E}^2 \equiv \langle \mathrm{e}^{i\theta J_3}\bar{q} \ : \ 0 \leq \theta \leq 2\pi \rangle \tag{5.112}$$

is deformable to e and that the loop with $\theta \to -\theta$ in (??) is homotopic to E.

The homotopy class [E] of E in $\pi_1(\mathcal{Q}^{(2)})$ is the exchange. The corresponding group is S_2.

- *The π-rotations* $[\mathrm{T}^{(1)}], [\mathrm{T}^{(2)}]$.

The loop $\mathrm{T}^{(1)}$ rotates the ring 1 (on left) by π around x-axis, $\mathrm{T}^{(2)}$ does so for the ring 2 on right. They are inherited from $\mathcal{Q}^{(1)}$ and generate the elements $[\mathrm{T}^{(i)}]$ in $\pi_1(\mathcal{Q}^{(2)})$. Clearly

$$[\mathrm{E}][\mathrm{T}^{(1)}][\mathrm{E}^{-1}] = [\mathrm{T}^{(2)}], \quad [\mathrm{E}][\mathrm{T}^{(2)}][\mathrm{E}^{-1}] = [\mathrm{T}^{(1)}] \tag{5.113}$$

where the products in $\pi_1(\mathcal{Q}^{(2)})$ are as usual defined by concatenation of loops in $\mathcal{Q}^{(2)}$.

$[\mathrm{T}^{(1)}]$ and $[\mathrm{T}^{(2)}]$ commute.

- *The Slide:* Let us first consider the loop $\mathscr{S}_{12}$ or the slide $[\mathscr{S}_{12}]$ of 2 through 1. Figure ?? above explains the loop:

The homotopic class $[\mathscr{S}_{12}]$ of $\mathscr{S}_{12}$ is the slide of 2 through 1.

The slide $[\mathscr{S}_{21}]$ of 1 through 2 is similarly defined. We can show that

$$[\mathrm{E}][\mathscr{S}_{12}][\mathrm{E}]^{-1} = [\mathscr{S}_{21}] \quad , \tag{5.114}$$

$$[\mathrm{E}][\mathscr{S}_{21}][\mathrm{E}]^{-1} = [\mathscr{S}_{12}] \quad . \tag{5.115}$$

The full group $\pi_1(\mathcal{Q}^{(2)})$ is thus generated by $[\mathrm{T}^{(1)}]$, $[\mathrm{T}^{(2)}]$, [E], $[\mathscr{S}_{12}]$, $[\mathscr{S}_{21}]$ with the relations

$$[\mathrm{T}^{(1)}]^2 = [\mathrm{E}]^2 = [\mathrm{T}^{(2)}]^2 = e, [\mathrm{E}][\mathrm{T}^{(1)}][\mathrm{E}]^{-1} = [\mathrm{T}^2], [\mathrm{T}^{(1)}][\mathrm{T}^{(2)}] = [\mathrm{T}^{(2)}][\mathrm{T}^{(1)}]\,, \tag{5.116}$$

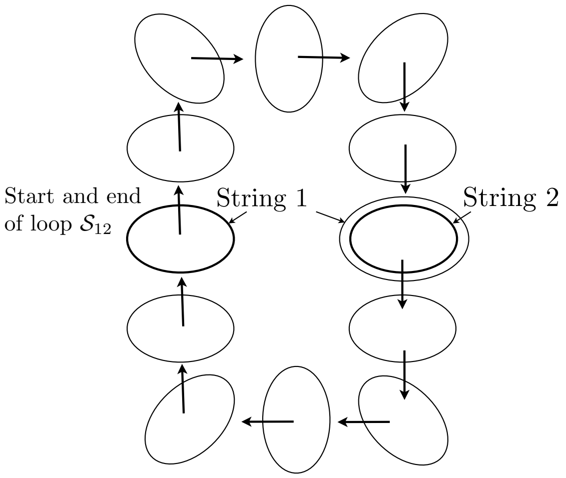

Figure 5.4: The "slide" loop $\mathcal{S}_{12}$.

and (**??**,**??**). There are no further relations.

If $\mathscr{S}$ is the group that $[\mathscr{S}_{ij}]$ generate, we have the semi-direct product structure

$$\pi_1(\mathcal{Q}^{(2)}) = \left\{ \mathscr{S} \rtimes \left(\pi_1(\mathcal{Q}^{(1)}) \times \pi_1(\mathcal{Q}^{(1)}) \right) \right\} \rtimes S_2 \tag{5.117}$$

Here $G_1 \rtimes G_2$ is the semi-direct product of G_1 and G_2 with G_1 being the invariant subgroup. Also $\pi_1(\mathcal{Q}^{(1)})$ acts trivially on $\mathscr{S}$.

Eq. (**??**) is to be compared with the corresponding equation for the mapping class group $D^{(2)\infty}/D_0^{(2)\infty}$ of two identical geons (**? ?**) if $\mathscr{S}$ is its group of slides,

$$D^{(2)\infty}/D_0^{(2)\infty} = \left\{ \mathscr{S} \rtimes \left(D^{(1)\infty}/D_0^{(1)\infty} \times D^{(1)\infty}/D_0^{(1)\infty} \right) \right\} \rtimes S_2 \tag{5.118}$$

5.7.4 *The case of two identical oriented unknots*

Orienting the knots reduces $\pi_1(\mathcal{Q}^{(1)})$ to $\{e\}$. With that in mind, we can repeat the above discussion (with $\bar{q}$ chosen analogously to above) to find

$$\pi_1(Q^{(2)}) = \mathscr{S} \rtimes S_2 \quad . \tag{5.119}$$

Some discussion about the quantum theory of these unknots and their unusual statistical features can be found in (**? ?**).

Conclusion

Here it comes the end of our fairly long journey through some of the subtleties of noncommutative quantum field theory. We hope to have provided a clear enough presentation of some of the aspects of quantum field theory on spacetimes whose coordinate operators do not commute, yet we are aware that many more issues are left unaddressed by the present manuscript.

After providing some mathematical background, we presented a throughout explanation of the twisted field formalism. Such an approach can be summarized in a few key steps. At first we noticed that the commutation relations (**??**) can be implemented by a deformation of the pointwise product of the commutative algebra of functions. The new noncommutative product has throughout been indicated with a $\star$. The *twist* (**??**) makes its first appearance when writing down an explicit form for a $\star$-deformation which is suited for (**??**). Consistency with the new deformed product requires the action of the Poincarè group to be modified as well (cf. Section **??**). Specifically the $\star$-product carries a very peculiar modification of the group: is the way in which the group acts on the tensor product space and not the group multiplication itself which gets deformed. This is done in order to keep the Poincarè group an automorphism of the, now noncommutative, algebra of functions. For a generic $\star$-product, it is not an easy task to find the appropriate modification of the group co-structure. Twist deformation are different in this respect, and we can immediately write the deformed *coproduct* (**??**), again using the *twist* previously introduced to write down the deformed product. Before addressing the quantization of the classical fields, we need to ensure that the new action on tensor product spaces, preserves the notion of statistics, that is being fermionic or bosonic remains an intrinsic property of a state on noncommutative spacetimes as well, and cannot be changed by the new action of the Poincarè group. Surprisingly enough, the standard definition of either bosonic or fermionic states, is not left invariant by the new action of the group. Again for twist deformations, it is straightforward to account for this last modification which brings the notion of *twisted statistics* (cf. Section **??**).

The quantization of the classical fields, proceeds then in the standard way. The creation and annihilation operators ought now to provide a representation of the deformed Poincarè group. The new creation and annihilation operators can be written in terms of the ones which provide a representation of the Poincarè group in the commutative case. The difference lying in the former ones being the "dressed" version of the latter (**??**-**??**). Creation and annihilation which provide the appropriate representation, also satisfy the twisted commutation relations, they are then consistent with the twisted statistics (**??**-**??**) as well.

The quantization of the fields can be also carried out by asking for covariance. Chapter **??** is devoted to give a detailed account of the notion of covariant fields in the commutative case and its generalization to noncommutative spacetiemes. Both quantization procedures give rise to equivalent results, providing some ground for confidence in the twisted field formalism.

Since the introduction of the *twist*, we pointed out that the commutation relations (**??**) do not constrain the choice uniquely. Different twist choices are allowed (**??**-**??**). We study in details two of the most popular twists: Moyal and Wick-Voros. Chapter **??** addresses lengthly the comparison between the quantum theories obtained from quantizing fields on the Moyal and Wick-Voros planes. The two are shown to be inequivalent, the reason being the impossibility to carry through the equivalence of the two Twist deformations established at the level of deformation of algebras and Hopf algebras (cf. Appendix **??** and section **??**). Also the Moyal twist appears to be more suitable for quantum theory. As shown in (**??**), the normalization of states in the Wick-Voros case is momentum dependent. This fact causes problems for a coherent definition of scattering theory, suggesting that the Wick-Voros quantum theory might be inconsistent. Throughout, in order to derive our conclusions, we made heavily use of the specific "dressing transformation" quantization technique.

Lastly we presented an interesting scenario in which non-trivial topological excitations of the spacetime, called Geons by Friedman and Sorkin, could be the source of noncommutativity. We argued that the non trivial structure of the group of diffeomorphisms of a multi-geon spacetime, which cannot be decomposed as a tensor product of diffeos of a single geon spacetime, could be an evidence for the twisted coproduct structure. So the existence of geons might motivate a twisted coproduct and therefore a noncommutative spacetime. The geon case, though, differs in a few, interesting, aspects from the more standard noncommutative quantum field theory. They are worth pointing out. We first noticed that the spacetime noncommutativity which naturally arises in the geon case, is localised at the geons' scale. The reason being that we can completely encircle all the topological "defects" with d-dimensional balls $\mathscr{B}^d$ outside of which the spacetime is isomorphic to $\mathbb{R}^d/\mathscr{B}^d$. The second aspect is the nature of the group $\mathscr{G}$ we are twisting. In standard twisted quantum field theory, $\mathscr{G}$ is always a Lie group. The definition of the *twist* strongly relies on such a property: $\mathscr{F}_\theta$ is a map which is defined as a function of the Lie algebra elements. In the geons case the group which we need to twist is instead generically discrete. In order to carry through the twisting Hopf algebra theory, we developed a non-trivial generalization of the *twist map* in the discrete case. This technique in itself might find different application in future studies.

Many aspects of noncommutativity have not been addressed here. One of the deepest question pertains to whether or not noncommutativity is a fundamental or effective description of reality. Although questions of this kind are in general very hard to tackle, the different approaches to noncommutativity briefly listed in the introduction seem to suggest that different ideas have been developed on the matter. There are approaches to noncommutativity which do not treat it as a low-energy effective description of a possibly quite differently looking theory of Quantum Gravity, but as fundamental property of nature. To be more explicit, in Connes' et al., the matter content of the theory can be derived from first principle, choosing the appropriate spectral triple. Although seeing noncommutativity as a guiding principle for Quantum Gravity is an appealing one, the $\star$-product noncommutative quantum field theory presents itself more coherently as a low-energy effective theory. The commutation relations (**??**) can in fact be derived as a four-dimensions low energy limit of certain String theories, the fundamental theory being the fullfledged 10 or 11 dimensional superstring theory. Even the more quantum field theoretical derivation of (**??**), presented in section **??**, uses semiclassical arguments involving low-energy properties of both Quantum Mechanics and General Relativity.

The argument, for as abstract as it could appear, is not condemned to remain a purely speculative one. For instance SUSY searches at the LHC could say a word in such a dispute, pushing away or towards an effective view point. In case of any SUSY discovery, for example, it seems unlikely that the wonderful and coherent description of the Standard Model presented by Connes et al., could include, without drastic modifications, the doubling of the matter fields which SUSY would compel.

Quantum Gravity does remain the ultimate challenge for theoretical physicists, though the path to get there might be not as straight as we hope. There are experiments which specifically addressed the seek of evidences for the "Theory of Everything", see for example (**? ? ? ? ?**), yet we believe Quantum Gravity effects to become dominant at energy scales of the order of $M_P \sim 10^{19}$ GeV. Such an extremely high energy scale, more than 15 orders of magnitude higher than the highest energy scale human beings have ever been able to probe via an Earth located laboratory, relegates, currently, direct probing of the ultimate theory almost in the fantascientific realm. Yet a better understanding of much lower energy physics, can tell us a lot about what could or could not happen at the Plank scale.

Probing the physics at the TeV scale is definitely the task of this decade. The list of direct and indirect experiments aiming at it is extremely rich and exciting. The LHC has definitely taken a big part of attention, but not less care should be devoted to the, at the moment contradictory, results coming from the different Dark Matter searches, like Dama, Crest, Xenon, etc.. Not last the extremely recent result of superluminal neutrinos, obtained by the OPERA collaboration (**?**). Which, although hardly likely to be confirmed by future measurements, has the potential of completely turn upside down the setting in which to develop our speculations about Quantum Gravity.

Finally interesting insights could come from cosmological observations which during the last ten years have improved dramatically our understanding of the universe. Just to mention few experiments, the WMAP 7 year (**?**) which provides the best WMAP observation to date, the 2dF and SDSS galaxy surveys (**? ?**) which look at the number and brightness of galaxies to infer the baryonic content of the universe, and the Planck satellite whose results are not out yet, but they will improve on the WMAP.

The inflationary expansion of the very early universe, stretched a minuscule region of spacetime of the Plank size into cosmological scales. Arguably for the former, quantum spacetime fluctuations were considerably important. Inflation could then act as a incredibly powerful magnifying glass to probe quantum effects whose length scale would otherwise be of the order of the Plank Legnth, $L_P \sim 10^{-33}$ cm, orders of magnitudes away from our current reach. Many have therefore studied what kind of constraints can cosmology provide in the context of noncommutative geometry (**? ? ? ? ? ? ? ? ? ? ?**). For instance in (**? ?**) from the minuscule temperature fluctuations in the CMB data, a pretty stringent bound on the noncommutative parameter θ was derived:

$$\sqrt{\theta} < 1.36 \times 10^{-19} \text{ m} \sim 10 \text{ TeV}$$

This work is meant to be a small contribution in the incredibly exciting task of understanding and develop the theory of Quantum Gravity. Whether or not the commutation relations (**??**) would turn out to be an accurate description of nature at any energy scale, it is hard to tell at the current stage. Yet it is very likely to expect the most fundamental structure of the spacetime to show unexpected behaviours as we approach the Plank scale M_P. In this respect noncommutative geometry, and the study presented in this manuscript of quantum field theories on noncommutative spacetimes, represent a very useful arena in which to understand how quantum fluctuations of spacetime texture could generically appear at the field theoretic level.

Appendix A

Algebraic Preliminaries

The material presented in this appendix is quite standard, and our presentation follows quite closely the standard literature (? ? ? ? ? ?). Here and in the following an algebra will be indicated as $\mathcal{A}$ and it will be over the field of complex number $\mathbb{C}$. That is $\mathcal{A}$ is a vector space over $\mathbb{C}$,

$$\text{if } a, b \in \mathcal{A}, \quad \alpha a + \beta b \in \mathcal{A} \quad \forall \alpha, \beta \in \mathbb{C}. \tag{A.1}$$

Moreover, there is a further internal operation, which we call *the product*

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad \mathcal{A} \times \mathcal{A} \ni (a, b) \to ab \in \mathcal{A} \tag{A.2}$$

The product above is distributive over the addition (**??**) but generically noncommutative, $ab \neq ba$. Furthermore the algebra $\mathcal{A}$ is *unital* if it possesses a *unit*, that is an element $\mathbb{1}$ such that

$$a\mathbb{1} = \mathbb{1}a = a, \quad \forall a \in \mathcal{A} \quad . \tag{A.3}$$

A *topological algebra* is an algebra with a Hausdorff topology for which both the operations, addition and multiplication, are continuous. Where a Hausdorff topological space M, or simply *separated space*, is a space where any two distinct points $x, y \in M$ can be separated by neighbourhoods. That is there exists a neighbourhood U of x and a neighbourhood V of y, such that U and V are disjoint, $U \cap V = \emptyset$. A *normed algebra* $\mathcal{A}$ is an algebra with a *norm* $\|\cdot\| : \mathcal{A} \to \mathbb{R}$. $\|\cdot\|$ should satisfy the following properties:

$$\|a + b\| \leq \|a\| + \|b\|, \quad \|\alpha a\| = |\alpha|\|a\|, \tag{A.4}$$

$$\|ab\| \leq \|a\|\|b\|, \tag{A.5}$$

$$\|a\| \geq 0, \quad \|a\| = 0 \quad \Longleftrightarrow \quad a = 0 \tag{A.6}$$

for any $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. A normed algebra is more restricted than a topological one. That is a normed algebra can be made into a topological algebra with the *norm topology*, the topology defined by its norm $\|\cdot\|$. It is a *metric topology* with metric given by $d(a,b) = \|a-b\|, \quad \forall a, b \in \mathcal{A}$. The corresponding $\varepsilon-$neighbourhoods of a are given by $U(a,\varepsilon) = \{b \in \mathcal{A} \mid \|a-b\| < \varepsilon\}$. The norm function is continuous since $|\ \|a\| + \|b\|\ | \leq \|a-b\|$.

A (proper, norm closed) subspace $\mathcal{I}$ of the algebra $\mathcal{A}$ is a *left ideal* (*right ideal*) if $a \in \mathcal{A}$ and $b \in \mathcal{I}$ imply that $ab \in \mathcal{I}$ ($ba \in \mathcal{I}$). The sets $\{0\}$ and $\mathcal{A}$ are trivial ideals. A *two-sided ideal* is a subspace which is both a left and a right ideal. The ideal $\mathcal{I}$ (left, right or two-sided) is called *maximal* if there exists no other ideal which contains $\mathcal{I}$. Each ideal is automatically an algebra.

A normed algebra which is complete in the norm topology is called a *Banach algebra*. Finally a C^*-algebra is a Banach algebra, endowed with an *involution* *, $(a^*)^* = a$, and whose norm satisfies the additional identities

$$\|a^*\| = \|a\|, \quad \|a^*a\| = \|a\|^2 \tag{A.7}$$

in most cases, as $\mathcal{A}$ is over $\mathbb{C}$, the * operation is simply complex conjugation. A C^*-algebra $\mathcal{A}$ is called *simple* if it has no nontrivial two-sided ideals.

Example #1

An example of a commutative C^*-algebra is provided by the algebra of complex valued continous functions $C_0(M)$ on a Hausdorff topological space M, which vanish at infinity. The product being the pointwise multiplication and the * involution the complex conjugation. The norm is the supremum norm

$$\|f\|_\infty = \sup_{x\in M}|f(x)| \tag{A.8}$$

This algebra is not *unital*. If M is compact then $C_0(M)$ has a unit, $\mathbb{1}$ = the constant function $f = 1$, which in fact does not vanish at infinity. One can prove that $C_0(M)$ is complete in the suprimum norm. Indeed the algebra $C_0(M)$ is the closure in the norm (**??**) of the algebra of functions with compact support.

Example #2

An example of a noncommutative C^*-algebra is provided by the algebra of bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The involution * is the adjoint and the norm is given by the operator norm

$$\|B\| = \sup\{\|Bh\|_{\mathcal{H}} | h \in \mathcal{H}, \|h\| \leq 1\} \tag{A.9}$$

where $\|\cdot\|_{\mathcal{H}}$ is the Hilbert space norm.

In the case that $\mathcal{H}$ has finite dimension n, $\mathcal{B}(\mathcal{H})$ is the noncommutative algebra of $n \times n$ matrices with complex entries $\mathbb{M}_n(\mathbb{C})$. Now T^* is the Hermitian conjugate of $T \in \mathbb{M}_n(\mathbb{C})$. And the norm reduces to

$$\|T\| = \text{positive square root of the largest eigenvalue of } \mathrm{T^*T}. \tag{A.10}$$

We conclude this appendix with a discussion of algebra morphisms. A *-morphism between two C^*-algebras $\mathcal{A}$ and $\mathcal{A}'$, is any $\mathbb{C}$-linear map $\pi : \mathcal{A} \to \mathcal{A}'$ which is a *homomorphism* of algebras, that is it preserves the multiplication on $\mathcal{A}$, $\pi(ab) = \pi(a)\pi(b) \quad \forall a, b \in \mathcal{A}$, and it is *-preserving, $\pi(a^*) = \pi(a)^*$. It can be shown that π is automatically continuous in the norm topology and the image $\pi(\mathcal{A}) \subset \mathcal{A}'$ is a C^*-subalgebra of $\mathcal{A}'$. A morphism from $\mathcal{A}$ in itself, is an *automorphism.* The sunbset of $\mathrm{Aut}(\mathcal{A})$, the set of automorphisms, which contains automorphisms which can be written as conjugation by elements of $\mathcal{A}$

$$\pi : \quad \exists\, b \in \mathcal{A} : \pi(a) = bab^{-1} \quad \forall a \in \mathcal{A} \tag{A.11}$$

is indicated by $\mathrm{Int}(\mathcal{A})$ and called the set of *inner* or *internal* automorphisms. The complement of $\mathrm{Int}(\mathcal{A})$ in $\mathrm{Aut}(\mathcal{A})$ is indicated as $\mathrm{Out}(\mathcal{A})$ and called the set of *outer automorphisms.* A *-morphism which is also bijective as a map is a *-*isomorphism.* A *representation* of a C^*-algebra is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π a *-morphism into the algebra of bounded operator $\mathcal{B}(\mathcal{H})$:

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \tag{A.12}$$

The representation $(\mathcal{H}, \pi)$ is called *faithful* if $\ker(\pi) = \{0\}$, so that π is a *-isomorphism. It can be shown that a representation is faithful if and only if $\|\pi(a)\| =$

$\|a\|$ for any $a \in \mathcal{A}$ or $\pi(a) > 0$ for all $a > 0$. The representation $(\mathcal{H}, \pi)$ is called *irreducible* if the only closed subspaces which are left invariant under the action of $\pi(\mathcal{A})$ are the trivial subspace $\{0\}$ and the whole $\mathcal{H}$. Alternatively a representation is irreducible if and only if the commutant of $\pi(\mathcal{A})$, i.e. the set of all elements in $\mathcal{B}(\mathcal{H})$ which commute with each element in $\pi(\mathcal{A})$, consists of multiples of the identity operator. Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said to be *unitary equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$, such that $\pi_1(a) = U^* \pi_2(a) U$, for any $a \in \mathcal{A}$.

Appendix B

Deformation theory & Cohomology

This appendix is by no mean aimed to be a self-cointened treatment of neither cohomology nor deformation theory. We just want to provide the essential information for the interested reader to understand how cohomology enters in relations with deforming the product of the algebra of functions or the coproduct of the symmetry group. For a more detailed and complete treatment of the subject we refer to (**?** **?**) as for cohomology in general, (**?**) for algebra deformation theory and Hochschild cohomology and (**?** **?**) for Hopf algebra deformation and cohomology. In the following $\mathcal{A}$ and H represent respectively an algebra and a Hopf algebra over the field of complex number $\mathbb{C}$.

B.1 Hochshild cohomology and Algebras' deformation

The Hochshild cohomology is naturally suited in order to study algebras' deformation, as we shall show in a few moments. In the case in exam, a *p-chain* $C \in C^p(\mathcal{A}, \mathcal{A})$ is a *p-linear* map from $\mathcal{A}^{\otimes p}$ into $\mathcal{A}$, that is

$$C : (a_1, ..., a_p) \in \mathcal{A}^{\otimes p} \to C(a_1, ..., a_p) \in \mathcal{A} \tag{B.1}$$

$$C(a_1, ...\alpha_i a_i + \beta_i b_i ...a_p) = \alpha_i C(a_1, ...a_i...a_p) + \beta_i C(a_1, ...b_i...a_p),\ \alpha_i, \beta_i \in \mathbb{C} \tag{B.2}$$

and its coboundary $bC \in C^{p+1}(\mathcal{A},\mathcal{A})$ is a $p+1$-chain. It acts on $\mathcal{A}^{\otimes^{p+1}}$ and its action is defined as

$$bC(a_0, a_1 ..., a_p) = a_0 C(a_1, ..., a_p) - C(a_0 a_1, ..., a_p) + ... \\ ... + (-1)^p C(a_0, ..., a_{p-1} a_p) + (-1)^{p+1} C(a_1, ..., a_{p-1}) a_p \tag{B.3}$$

It can be shown that the *boundary map* above is nihilpotent, i.e. $b^2 = 0$. We now proceed as usual. A *p-chain* C is a *p-cocycle* if $bC = 0$. We denote by $Z^p(\mathcal{A},\mathcal{A})$ the space of *p-cocycles* and by $B^p(\mathcal{A},\mathcal{A})$ the subspace of $C^p(\mathcal{A},\mathcal{A})$ of *p-chains* which are coboundaries of *(p-1)-cochain*[1]. We are now ready to define the p-th Hochshild cohomology space $H^p(\mathcal{A},\mathcal{A})$

$$H^p(\mathcal{A},\mathcal{A}) = Z^p(\mathcal{A},\mathcal{A})/B^p(\mathcal{A},\mathcal{A}) \tag{B.4}$$

We can now turn into the theory of deformations of algebras. We will indicate as ab the undeformed product and as $a \star b$ its deformation. Formally a deformation $a \star b$ of a multiplication map ab is a power series in a "small parameter" θ that reduces to ab in the $\theta \to 0$, or

$$a \star b = ab + \sum_{i=1}^{\infty} \theta^i C_i(a,b), \quad C_i(a,b) \in C^2(\mathcal{A},\mathcal{A}) \tag{B.5}$$

therefore deformations can be written as a series of *2-cochain*. In order for the new product to be associative, we need

$$(a \star b) \star c = a \star (b \star c), \quad \forall a, b, c \in \mathcal{A} \tag{B.6}$$

This property sets strong restrictions on the possible *2-cochain* which could give rise to associative deformed products. In fact expanding both sides of (**??**), and equating on both sides same powers of θ, we find the constraint:

$$D_r(a,b,c) = bC_r(a,b,c), \quad \text{where} \quad D_r(a,b,c) := \sum_{j+k=r} \big[C_j(C_k(a,b),c) - C_j(C_k(a,b),c)\big] \tag{B.7}$$

After some algebra it is possible to show that $bD_r(a,b,c,d) = 0$, that is the LHS is a *3-cocycle* for any r and which can be written as a function of just the precedent $r-1$

[1] As usual it's immediate to show that $B^p(\mathcal{A},\mathcal{A}) \subset Z^p(\mathcal{A},\mathcal{A})$, that is any *p-cochain* which can be written as a *co-boundary* of a $(p-1)$-*cochain* is a *p-cocycle*. This follows from $b^2 = 0$.

2-cochains. Thus in order to construct a deformation we need to satisfy (**??**) for any r. Starting from $r = 1$, we immediately obtain that $bC_1(a,b,c) = 0$ or, C_1 should be a 2-cocycle. In order to proceed to higher order in θ, we should make sure that the 3-cocycle $D_2(a,b,c)$, which is uniquely defined once our 2-cocycle C_1 has been chosen, can be written as a coboundary of a 2-cochain C_2 which will then be our choice for the θ^2 term in (**??**). At every order we should then be able to write the 3-cochain D_r as a coboundary of a 2-cochain C_r which is going to be the θ^r contribution for the deformed product. In conclusion, the obstructions to build a deformed associative product, lie in the third Hochshild cohomology space $H^3(\mathcal{A},\mathcal{A})$. If $H^3(\mathcal{A},\mathcal{A})$ is trivial, than any 3-cocycle is a 3-coboundary and any deformation comes up to be associative.

Hochshild cohomology helps also in studying the equivalence of deformations of the kind (**??**). First of all, two deformation $\star$ and $\star'$ are said equivalent if there exists an isomorphism $T = \mathbb{1} + \sum_{i=1}^{\infty} \theta^i T_i$, where $T_i \in C^1(\mathcal{A},\mathcal{A})$, i.e. T_i are linear maps of $\mathcal{A}$ in itself, such that

$$T(a \star' b) = (Ta \star Tb) \tag{B.8}$$

Expanding both sides in power θ we can carry out a similar analysis as before. After some algebra it is possible to show that at $\mathcal{O}(\theta)$ the deformation is trivial if the 2-cocycle C_1 is also a 2-coboundary. More generally, exactly as above, we can show (**? ?**) that if two deformations are equivalent up to some order t, the condition to extend the equivalence one step further is that a 2-cocycle (uniquely defined using the $T_k,\ k \le t$) is the coboundary of the required T_{t+1} and therefore the obstructions to equivalence lie in the 2-cohomology $H^2(\mathcal{A},\mathcal{A})$. In particular, if that space is null, all deformations are trivial.

B.2 Nonabelian cohomology and Hopf Algebras' deformations

In a similar manner, we can introduce a cohomological theory which can help us studying Hopf algebras' deformation. We tackle this task in this section. Generic Hopf algebras deformation involve deformation of possibly both the product and the coproduct. Here we will only deal with a much more restricted set of deformations, namely the ones we have considered in the main text where the product is left unchanged and the coproduct is deformed by conjugation:

$$\chi \in H \otimes H, \quad \Delta_\theta = \chi \Delta_0 \chi^{-1} \tag{B.9}$$

so χ can be associated with what we call the *twist* in (**??**). The material presented here can be found in a more extensive form in (**?**). We will call the cohomology suited for the study of deformations (**??**) *Hopf Algebra cohomology.* For the study of generic Hopf algebras' deformation we refer the reader to (**?**) instead.

Let χ be an element of $H^{\otimes^p}$, χ is a *p-chain* if in addition is invertible. We indicate its coboundary by $\partial\chi$ which is defined as

$$\partial\chi = (\partial_+\chi)(\partial_-\chi^{-1}) = \Big(\prod_{i=0}^{i\ even} \Delta_i\chi\Big)\Big(\prod_{i=1}^{i\ odd} \Delta_i\chi^{-1}\Big) \tag{B.10}$$

where

$$\Delta_i : H^{\otimes^n} \to H^{\otimes^{n+1}}, \quad \Delta_i = \mathbb{1} \otimes ... \otimes \Delta \otimes ... \otimes \mathbb{1} \tag{B.11}$$

and the Δ is in the ith position. χ is a *p-cocycle* if $\partial\chi = \mathbb{1}$.

Definition (**??**) at first is quite obscure. In order to gain a better understanding of it, we here write explicitly what 1 and 2-cocycles look like. A 1-cocycle is an invertible element of H which satisfies

$$\chi \otimes \chi = \Delta\chi \tag{B.12}$$

that is χ is *group-like.* A 2-cocycle is instead an invertible element of $H \otimes H$ such that

$$(\mathbb{1} \otimes \chi)(\mathbb{1} \otimes \Delta)\chi = (\chi \otimes \mathbb{1})(\Delta \otimes \mathbb{1})\chi \tag{B.13}$$

this property, although not as self-explanatory as the 1-cocylce condition, will be the main constraint for χ to give rise to a proper Hopf algebra deformation.

The definition of the cohomology groups $\mathcal{H}^p(H,H)$ is more involved in this case. The coboundary map (**??**) involves possibly noncommutative operations so ∂^2 is not necessarily $\mathbb{1}$. It can be shown that $\partial^2 = \mathbb{1}$ for 1 and 2-cochains. The 1st and 2nd Hopf algebra cohomology group are then defined as before as $\mathcal{Z}^p(H,H)/\mathcal{B}^p(H,H)$, where $\mathcal{Z}^p(H,H)$ is the set of *p-cocycles* and $\mathcal{B}^p(H,H)$ the set of *p-coboundaries.* Because of the noncommutativity of (**??**), the coset operation is not trivial. Specifically two p-cocycles χ and χ' are cohomologous if there exists a (p-1)-cochain γ:

$$\chi' = (\partial_+\gamma)\chi(\partial_-\gamma^{-1}) \tag{B.14}$$

where $\partial_+\gamma$ and $\partial_-\gamma^{-1}$ are defined via (**??**).

We are ready to state the main result of the section (cf. Th. 2.3.4. (**?**)). Let H be a co-associative Hopf algebra and let χ be a 2-cocycle (**??**). Then there is a new co-associative Hopf algebra H_χ, defined by the same algebra and counit and a deformed coproduct

$$\Delta_\chi h = \chi(\Delta h)\chi^{-1}, \quad \forall h \in H \tag{B.15}$$

The new antipode, S_χ, can similarly defined in terms of χ but such definition is not very transparent and not needed in the present discussion.

Also let χ and ψ be two 2-cocyles and H_χ and H_ψ the Hopf algebras obtained by the deformations (**??**) using respectively χ and ψ. H_χ and H_ψ are isomorphic via an inner automorphism if χ and ψ are cohomologous as in (**??**). The map γ in (**??**) provides the inner automorphism. It follows that if χ is a coboundary, then H_χ is isomorphic to the initial, underformed, Hopf algebra. Therefore if $\mathcal{H}^2(H,H)$ is trivial, all the 2-cocycles are 2-coboundaries and all the deformation which we can construct by conjugation are isomorphic to the trivial one.

B.3 A Technical Result

Here we want to show a technical result which connects the cohomologies describing deformations of Hopf algebras and their module algebras which have been introduced in the previous two sections. Here we restrict our studies mainly to the deformations of the algebra of smooth function on $\mathbb{R}^4$ and deformations of the Hopf Poincaré group algebra, denoted as $H\mathscr{P}_\theta$.

Given two deformations of the spacetime algebra of functions $\mathcal{A}_\theta$ and $\mathcal{A}'_\theta$ and the compatible deformations of the action of Poincaré group algebra $H\mathscr{P}_\theta$ and $H\mathscr{P}'_\theta$, the condition of equivalence of two algebraic deformations is (**??**):

$$\forall\ f_1, f_2 \in \mathcal{A}_\theta, \quad \mathrm{T}\Big[(f_1 \star f_2)\Big](x) = \Big[\mathrm{T}(f_1) \star' \mathrm{T}(f_2)\Big](x) \quad . \tag{B.16}$$

where $\star$ and $\star'$ are the deformed products in, respectively, $\mathcal{A}_\theta$ and $\mathcal{A}'_\theta$ and $\mathrm{T} : \mathcal{A}_\theta \to \mathcal{A}'_\theta$ is the invertible map which implements the equivalence.

The condition of equivalence of the two Hopf algebras characterized by the twists F_θ and F'_θ, call them $H\mathscr{P}_\theta$ and $H'\mathscr{P}_\theta$ (**?**), is (**??**)

$$F'_\theta = \Delta_0(\mathrm{T}')F_\theta {\mathrm{T}'}^{-1} \otimes {\mathrm{T}'}^{-1} \quad . \tag{B.17}$$

Here T' maps H to H' according to

$$H \ni h \to \mathrm{T}' h \mathrm{T}'^{-1} = h' \in H' \tag{B.18}$$

where T' is an element of the Hopf algebra. Notice that in general $\mathrm{T} \neq \mathrm{T}'$. This discussion is very relevant for what we called "weak equivalence" in the main text (cf. section **??**) since the condition of "weak equivalence" only involves a further requirement: the map T and T' must be the same.

This condition of weak equivalence can be formulated for any two Hopf algebras H and H' acting on two algebras $\mathcal{A}$ and $\mathcal{A}'$ respectively if the following conditions are fulfilled:

1) $\mathcal{A}$ and $\mathcal{A}'$ as vector (topological) spaces are the same and differ only in their multiplications maps m and m'.

2) The Hopf algebras H and H' as algebras are the same and act on elements of $\mathcal{A}$ and $\mathcal{A}'$ in the same way. They differ only in their coproducts.

3) The products m and m' in $\mathcal{A}$ and $\mathcal{A}'$ are given by twists $\mathcal{F} \in H \otimes H$ and $\mathcal{F}' \in H' \otimes H'$ and a common multiplication map m_0 as follows:

$$m(f \otimes g) = m_0 \mathcal{F}(f \otimes g) \tag{B.19}$$

$$m'(f \otimes g) = m_0 \mathcal{F}'(f \otimes g) \tag{B.20}$$

4) The algebra $\mathcal{A}_0$ with the multiplication map m_0 is also a module for a Hopf algebra H_0. H_0 differs from H and H' only in its coproduct. It acts on elements of $\mathcal{A}_0$ just as H and H' act on elements of $\mathcal{A}$ and $\mathcal{A}'$

We consider only such algebras below. They cover the case of Moyal and Wick-Voros algebras and their corresponding Poincaré-Hopf algebras.

We are now going to show that if the two algebras $\mathcal{A}$ and $\mathcal{A}'$ are equivalent, that is (**??**) is satisfied, then the equivalence automatically lifts to the equivalence of the corresponding Hopf algebras *provided that* $\mathrm{T} \in H$. In that case $\mathrm{T}' = \mathrm{T}$ We can hence say that what has been called "weak equivalence" is nothing but the equivalence of the two algebras $\mathcal{A}$ and $\mathcal{A}'$ under a map $\mathrm{T} \in H$.

Let Δ_0, Δ and Δ' be the coproducts for H_0, H and H'. The proof is easily obtained by writing (**??**) using (**??**):

$$\mathrm{T}\Big[m_0 \circ \mathcal{F}(f_1 \otimes f_2)\Big](x) = m_0 \circ \mathcal{F}'\Big[\mathrm{T} \otimes \mathrm{T}(f_1 \otimes f_2)\Big](x) \quad . \tag{B.21}$$

Since the co-product compatible with the point-wise product is Δ_0, we get:

$$\Big[m_0 \circ \Delta_0(\mathrm{T})\mathcal{F}_\theta(f_1 \otimes f_2)\Big](x) = m_0 \circ \mathcal{F}'_\theta\Big[\mathrm{T} \otimes \mathrm{T}(f_1 \otimes f_2)\Big](x) \tag{B.22}$$

which translates exactly into the equivalence condition (**??**) on the twists.

Appendix C

Geons & Mapping Class Group

C.1 What are geons

This is a short review section on the topology of low–dimensional manifolds leading up to those which support geons ("geon manifolds"). The original literature is best consulted for detailed information (**? ? ? ?**).

Given two closed (compact and boundary–less) manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of dimension d, their connected sum $\mathcal{M}_1\#\mathcal{M}_2$ is defined as follows. Remove two balls $\mathscr{B}_1$ and $\mathscr{B}_2$ from $\mathcal{M}_1$ and $\mathcal{M}_2$, leaving two manifolds $\mathcal{M}_i\backslash\mathscr{B}_i$ with spheres S_i^{d-1} ($S_i^{d-1} \sim S^{d-1}$) as boundaries $\partial(\mathcal{M}_i\backslash\mathscr{B}_i)$. Then $\mathcal{M}_1\#\mathcal{M}_2$ is obtained by identifying these spheres. If $\mathcal{M}_i$ are oriented, this identification must be done with orientation-reversal so that $\mathcal{M}_1\#\mathcal{M}_2$ is oriented.

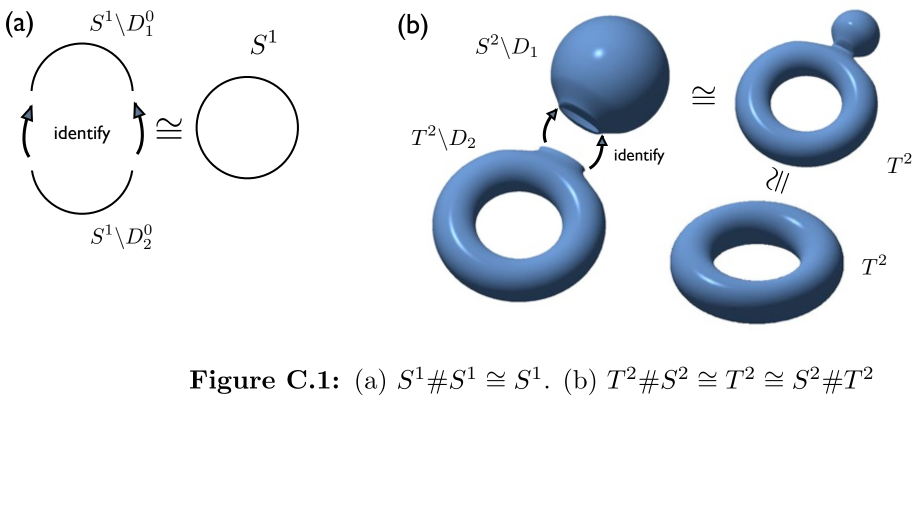

Figure C.1: (a) $S^1\#S^1 \cong S^1$. (b) $T^2\#S^2 \cong T^2 \cong S^2\#T^2$

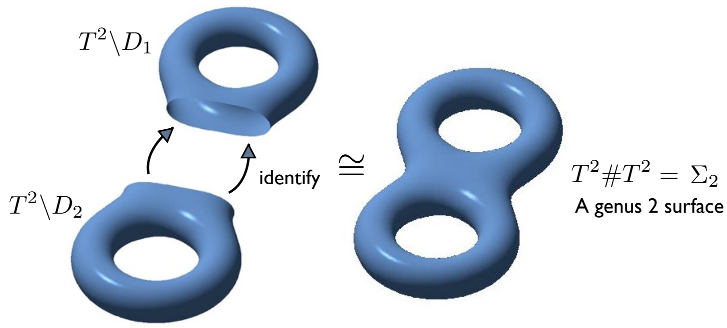

Figure C.2: $T^2 \# T^2 \cong \Sigma_2$. Genus 2 surface.

Connected summing, #, is associative and commutative:

a) $\mathcal{M}_1 \#(\mathcal{M}_2 \# \mathcal{M}_3) \cong (\mathcal{M}_1 \# \mathcal{M}_2) \# \mathcal{M}_3$ so that we can write $\mathcal{M}_1 \# \mathcal{M}_2 \# \mathcal{M}_3$;

b) $\mathcal{M}_1 \# \mathcal{M}_2 \cong \mathcal{M}_2 \# \mathcal{M}_1$.

Here are some simple examples:

- $d = 1$. $S^1 \# S^1 \cong S^1$. (See Fig. **??** (a)).
- $d = 2$. $S^2 \# S^2 \cong S^2$.
- $d = 2$. $T^2 \# S^2 = T^2 \cong S^2 \# T^2$. (See Fig. **??** (b)).
- $d = 2$. $T^2 \# T^2 \cong \Sigma_2$. Genus two manifold. (See Fig. **??**).

As the examples here suggest, for any dimension d, $\mathcal{M} \# S^d \cong S^d \# \mathcal{M} \cong \mathcal{M}$.

These considerations can be extended to asymptotically flat manifolds. If $\mathcal{M}_1$ is asymptotically flat and $\mathcal{M}_2$ is closed and both are oriented (and of the same dimension), then $\mathcal{M}_1 \# \mathcal{M}_2$ is obtained by removing balls $\mathscr{B}_i$ from $\mathcal{M}_i$ and identifying the boundaries $\partial(\mathcal{M}_i \backslash \mathscr{B}_i)$ compatibly with orientation as pointed out above. The connected sum $\mathcal{M}_1 \# \mathcal{M}_2$ is asymptotically flat and oriented.

We will now state certain basic results in low–dimensional topology considering only closed or asymptotically flat, and oriented manifolds $\mathcal{M}$. In the asymptotically flat case, we will insist that there is only one asymptotic region. That is, the asymptotic region of $\mathcal{M}$ is homeomorphic to the complement of a ball $\mathscr{B}^d$ in $\mathbb{R}^d$. In other words $\mathcal{M}$ has one asymptotic region if all its topological complexities can be encompassed within a sphere $S^{d-1} \subset \mathcal{M}$.

The case $d = 1$ is trivial, there being only two such manifolds S^1 and $\mathbb{R}^1$. ($\mathbb{R}^1$ has "one" asymptotic region in the above sense even though it is not connected.)

The basic results of interest for $d = 2$ and 3 are as follows.

C.1.1 *Closed Manifolds*

In $d = 2$ and 3, there is a class of special closed manifolds called *prime* manifolds. Any closed manifold $\mathcal{M} \neq S^d$ for $d = 2$ or 3 is a *unique* connected sum of prime manifolds $\mathcal{P}_\alpha$ (with the understanding that spheres are not inserted in the connected sum):

$$\mathcal{M} = \#_\alpha \mathcal{P}_\alpha \,. \tag{C.1}$$

(All manifolds have the same dimension. If $\mathcal{M} = S^d$, then (**??**) is substituted by the triviality $S^d = S^d$, hence, a better way to write (**??**) is $\mathcal{M} = \#_\alpha \mathcal{P}_\alpha$ mod S^d.)

The uniqueness of (**??**) implies that a prime $\mathcal{P}_\alpha$ cannot be decomposed as the connected sum of two or more primes. (It is indecomposable just like a stable elementary particle.)

For $d = 2$, there is just one prime, namely the torus. In that case, $T^2 \# T^2 \# ... \# T^2$ with k terms is just a genus k surface (see Fig. **??** for $k = 2$).

For $d = 3$, there are an infinity of prime manifolds. They are not fully known. Representative examples are the following:

a) *Spherical Space Forms.* Notice that $S^3 \cong SU(2)$ by writing

$$SU(2) \ni g = \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}, \ |\xi_1|^2 + |\xi_2|^2 = 1 \,. \tag{C.2}$$

Then $SO(4) = \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$ acts on S^3 by

$$g \to hgh'^{-1}, \quad h, h' \in SU(2) \,. \tag{C.3}$$

There are several discrete subgroups of $SO(4)$ which act freely on S^3. Such free actions are given for example by the choices $h \in \mathbb{Z}_p$, $h' \in \mathbb{Z}_q$ where p and q are relatively prime. The quotients of S^3 by the free actions of discrete subgroups of $SO(4)$ are called spherical space forms. For the above example with cyclic groups

$\mathbb{Z}_{p,q} := \mathbb{Z}_p \times \mathbb{Z}_q$ the quotients are Lens spaces $L_{p,q}$ (**?**). Of these $L_{1,2}$ and $L_{2,1}$ are $\mathbb{R}P^3$.

Spherical space forms are prime and admit metrics with constant positive curvature. They have been studied exhaustively from the point of view of quantum gravity by Witt (**?**).

b) *Hyperbolic spaces.* Consider the hyperboloid

$$\mathcal{H}^+ : \{x = (x_0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4 : (x_0)^2 - (\vec{x})^2 = 1, x_0 > 0\} \tag{C.4}$$

in $\mathbb{R}^4$. The connected Lorentz group $\mathscr{L}_+^\uparrow$ acts transitively on $\mathcal{H}^+$. Let $D \subset \mathscr{L}_+^\uparrow$ be a discrete subgroup acting freely on $\mathcal{H}^+$. Then $\mathcal{H}^+/D$ is a hyperbolic space.

Hyperbolic spaces are prime and admit metrics with constant negative curvature.

There are other primes as well such as $S^2 \times S^1$ which do not fall into either of these classes.

C.1.2 *Manifolds with one asymptotic region*

These manifolds $\mathcal{M}_\infty$ also have a unique decomposition of the form

$$\mathcal{M}_\infty = \mathbb{R}^d \#_\alpha \mathcal{P}_\alpha \tag{C.5}$$

where $\mathcal{P}_\alpha$ are the prime manifolds we discussed previously. Manifolds with one asymptotic region can be obtained from closed manifolds $\mathcal{M}$ by removing a point (“point at ∞”).

C.2 On Diffeos

Spatial manifolds of interest for geons are $\mathcal{M}_\infty$. They serve as Cauchy surfaces in globally hyperbolic spacetimes. Spacetime topology is taken to be $\mathcal{M}_\infty \times \mathbb{R}$ where $\mathbb{R}$ accounts for time.

In the standard Drinfel’d twist approach, the twist F_θ belongs to $\mathbb{C}\mathscr{G} \otimes \mathbb{C}\mathscr{G}$, where $\mathscr{G}$ represents the symmetry group which in relativistic quantum field theory is taken to be $\mathscr{P}$ or its identity component $\mathscr{P}_+^\uparrow$. In order to let the twist act on a geon spacetime we should identify the substitute for $\mathscr{P}$ or $\mathscr{P}_+^\uparrow$. To achieve that we need to recall a few properties of quantization of diffeomorphism-invariant theories. We will present a summary of the main ideas here. For a self-contained treatment of the topic we refer the reader to (**?**).

It is a result of quantization on multiply connected configuration spaces (**?**) that there is an action on the Hilbert space $\mathcal{H}$ of $\pi_1(\mathcal{Q})$, where $\mathcal{Q}$ is the configuration space of the classical system we want to quantize. This action can also be shown to commute with the action of any observable on $\mathcal{H}$. Now $\mathcal{H}$ can be decomposed into the direct sum $\mathcal{H} \cong \bigoplus \mathcal{H}^{(l)}$ of carrier spaces of irreducible representation of $\pi_1(\mathcal{Q})$. (More precisely this is so only if $\pi_1(\mathcal{Q})$ is abelian. Otherwise $\mathcal{H}$ carries only the action of the center of the group algebra $\mathbb{C}\pi_1(\mathcal{Q})$, see for instance (**?**).) Since all the observables commute with the action of $\pi_1(\mathcal{Q})$, they take each $\mathcal{H}^{(l)}$ into itself. These quantizations for different l are generally inequivalent. In other words each $\mathcal{H}^{(l)}$ provides an inequivalent quantization of the classical system (**?**). These results have been widely used from molecular physics to quantum field theory. The θ angle of QCD it is in fact understood in such a topological way.

In a theory of quantum gravity we consider $\pi_1(\mathcal{Q})$ as the group to twist. We turn now into the study of what this group looks like.

In general relativity the configuration space is very different from the usual $\mathbb{R}^{3n}$, as it is in the n-particle case. Specifically it is constructed from the set of all possible Riemannian metrics on a given space-like Riemannian manifold $\mathcal{M}$, which we will indicate as $\mathrm{Riem}(\mathcal{M})$. We also require, in order to make sense of concepts constantly used in physics like energy, that $\mathcal{M}$ is asymptotically flat. So we restrict $\mathcal{M}$ to what has been called $\mathcal{M}_\infty$ above. We indicate by $\mathrm{Riem}(\mathcal{M}_\infty)$ the space of metrics on it.

Not all possible metrics on $\mathcal{M}_\infty$ represent physically inequivalent "degrees of freedom" though. Because of diffeomorphism invariance we should consider only $\mathrm{Riem}(\mathcal{M}_\infty)$ upto the action of D^∞, the diffeos which act trivially at infinity. We thus find for the configuration space $\mathcal{Q}$ of general relativity: $\mathcal{Q} \equiv \mathrm{Riem}(\mathcal{M}_\infty)/D^\infty$.

The next step is to compute the fundamental group of $\mathcal{Q}$. We first quote the result:

$$\pi_1\Big(\mathrm{Riem}(\mathcal{M}_\infty)/D^\infty\Big) = D^\infty/D_0^\infty := \mathrm{MCG}(\mathcal{M}_\infty) \tag{C.6}$$

where D_0^∞ is the (normal) subgroup of D^∞ which is connected to the identity and MCG denotes the Mapping Class Group. This group is an important invariant of topological spaces.

Here is the proof of (**??**). It can be shown that the action of D^∞ on Riem$(\mathcal{M}_\infty)$ is free. Thus $\mathcal{Q}$ is the base manifold of a principal bundle Riem$(\mathcal{M}_\infty)$ with structure group D^∞. By a well-known theorem of homotopy theory (**?**), the following sequence of homotopy groups is then exact:

$$... \to \pi_1\Big(\text{Riem}(\mathcal{M}_\infty)\Big) \to \pi_1(\mathcal{Q}) \to \pi_0(D^\infty) \to \pi_0\Big(\text{Riem}(\mathcal{M}_\infty)\Big) \to ... \tag{C.7}$$

As the space of Riemmanian asymptotically flat metrics is topologically "trivial", that is $\pi_n\Big(\text{Riem}(\mathcal{M}_\infty)\Big) \equiv \mathbb{1},\ \forall n$, (**??**) becomes:

$$\mathbb{1} \to \pi_1(\mathcal{Q}) \to \pi_0(D^\infty) \to \mathbb{1} \tag{C.8}$$

from which (**??**) follows.

The nontrivial structure of the MCG$(\mathcal{M}_\infty)$ leads to striking results like the possibility of spinorial states from pure gravity. Let us briefly discuss this interesting result. The group D^∞ contains a diffeo called the 2π-rotation diffeo $R_{2\pi}$. It becomes a 2π rotation on quantum states. It may or may not be an element of D_0^∞. Now "the momentum constraints" of general relativity imply that D_0^∞ acts as identity on all quantum states. Thus it is only the group D^∞/D_0^∞ (or more generally D/D_0^∞ where D may contain elements which asymptotically act for example as rotations and translations) which can act nontrivially on quantum states. The conclusion in the following relies on this fact.

If $R_{2\pi} \in D_0^\infty$ then it maps to the identity in D^∞/D_0^∞ and on quantum states.

If $R_{2\pi} \notin D_0^\infty$, then it does not map to identity in D^∞/D_0^∞ and can act nontrivially on quantum states.

For $d \geq 3$, $R_{2\pi}^2$ is always in D_0^∞ and hence always acts trivially on quantum states.

Thus if $d \geq 3$ and $R_{2\pi} \notin D_0^\infty$, there can exist quantum geons with 2π rotation$=-\mathbb{1}$ on their Hilbert space. In fact suppose that $\psi \in \mathcal{H}$ is a physical state on which $R_{2\pi}$ does not act trivially, $\hat{R}_{2\pi}\psi \neq \psi$. But $R_{4\pi} = R_{2\pi}^2$ acts trivially on $\mathcal{H}$. Then the state $\psi' := (\frac{\mathbb{1}-\hat{R}_{2\pi}}{2})\psi$ is spinorial:

$$\hat{R}_{2\pi}\psi' = \frac{1}{2}(\hat{R}_{2\pi} - \hat{R}_{4\pi})\psi = -\psi' \quad . \tag{C.9}$$

It was a remarkable observation of Friedman and Sorkin (**? ? ?**) that there exist primes $\mathcal{P}_\alpha$ such that $R_{2\pi} \notin D_0^\infty$ for $\mathcal{M} = \mathbb{R}^d \# \mathcal{P}_\alpha$. These are the "spinorial" primes. The quantisation of the metric of such $\mathbb{R}^3 \# \mathcal{P}_\alpha$ can lead to vector states with spin $\frac{1}{2}+n$ $(n \in \mathbb{Z}^+)$. Thus we can have "spin $\frac{1}{2}$ from gravity".

For $d = 2$, the situation is similar, but $R^2_{2\pi}$ or any nontrivial power of $R_{2\pi}$, need not be in D^∞_0. That is indeed the case for $\mathbb{R}^2 \# T^2$ (?). That means that the quantum states for such geon manifolds can have fractional spin, can be anyons.

C.2.1 *Notation.*

Here we introduce some notation. We will call the diffeo groups of $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha$ which are asymptotically Poincaré, asymptotically identity and the component connected to the identity of the latter as $D^{(1)}$, $D^{(1)\infty}$ and $D^{(1)\infty}_0$ respectively. We will also refer to $D^{(1)\infty}/D^{(1)\infty}_0$ as the internal diffeos of the prime $\mathcal{P}_\alpha$. Similarly $D^{(N)}$,$D^{(N)\infty}$ and $D^{(N)\infty}_0$ will refer to the corresponding groups in the case of N-geon manifolds $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha \# ... \# \mathcal{P}_\alpha$, where the primes are all the same. They are appropriate for constructing vector states of several identical geons.

The MCG of an N-geon manifold can be decomposed into semi-direct products involving three groups:

$$D^{(N)\infty}/D^{(N)\infty}_0 \equiv \left(\mathscr{S} \rtimes \left[\times^N D^{(1)\infty}/D^{(1)\infty}_0 \right] \right) \rtimes S_N \quad . \tag{C.10}$$

Here $A \rtimes B$ indicates the semi-direct product of A with B where A is the normal subgroup.

In the above we could remove the brackets as it has been shown in (? ?) that the above semi-direct product is associative.

The last two factors in (**??**) are easily understood, the second term being the N-th direct product of the MCG of the single geon manifold $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha$ and S_N being the usual permutation group of N elements that consists of elements which permute the geons.

The first term, namely $\mathscr{S}$, is called the group of "slides" and consists of diffeos which take one prime through another along non-contractible loops. The existence of such a term is strictly linked with the fact that the primes are not simply connected. In fact elements of $\mathscr{S}$ can be described using elements of fundamental groups of the single primes $\mathcal{P}_\alpha$. Since we are not interested in the full details of the MCG, we refer the reader to the literature for further details (? ?) while we now move on to the analysis of the $N = 2$ case where we can also get a better understanding of what slides represent.

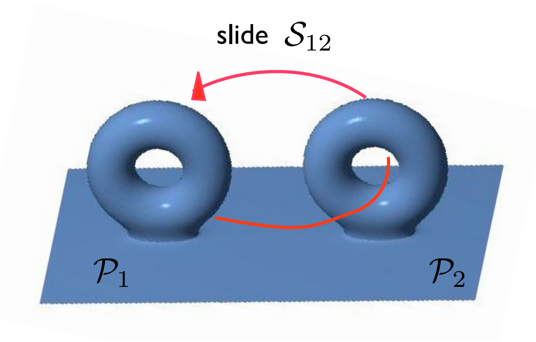

Figure C.3: $\mathbb{R}^d\#\mathcal{P}_1\#\mathcal{P}_2$

As we said the group $D^{(2)\infty}/D_0^{(2)\infty}$ of the manifold $\mathbb{R}^d\#\mathcal{P}_\alpha\#\mathcal{P}_\alpha$ appropriate for two identical geons contains diffeos corresponding to the exchange $E^{(2)}$ of geons and a new type of diffeos called slides besides the diffeos $D^{(1)\infty}/D_0^{(1)\infty}$ of $\mathbb{R}^d\#\mathcal{P}_\alpha$.

If $\mathbb{R}^d\#\mathcal{P}_\alpha\#\mathcal{P}_\alpha$ is represented as in Fig. **??** with bumps representing $\mathcal{P}_\alpha$, the exchange diffeo $E^{(2)}$ can be regarded as moving the geons so that they exchange places. This diffeo (mod $D_0^{(2)\infty}$) is the generator of S_2 in (**??**). For $d = 3$, $E^{(2)2} \in D_0^{(2)\infty}$, but for $d = 2$ that is not so. Thus for $d = 2$, we can have geons with fractional statistics (**?**).

Slides $\mathscr{S}^{(2)}$ arise because for $\mathcal{P}_\alpha \neq S^d$, $\pi_1(\mathcal{P}_\alpha) \neq \{e\}$ for $d = 2$ (where $\mathcal{P}_\alpha = T^2$), and $d = 3$ (in view of the now-proved Poincaré conjecture). Thus let L be a non-contractable loop threading $\mathbb{R}^d\#\mathcal{P}_\alpha^{(1)}$, where $\mathcal{P}_\alpha^{(j)}$ are primes and let $\mathscr{B}_p$ be a ball containing a point P on L in its interior. Then $\mathscr{S}_{21}^{(2)}$, the slide of $\mathcal{P}_\alpha^{(2)}$ along L through $\mathcal{P}_\alpha^{(1)}$, is obtained by attaching $\mathcal{P}_\alpha^{(2)}$ to $\partial\mathscr{B}_p$ and dragging it along L by moving p in a loop around L. Note that the slide $\mathscr{S}_{12}^{(2)}$ of $\mathcal{P}_\alpha^{(1)}$ through $\mathcal{P}_\alpha^{(2)}$ is not equal to $\mathscr{S}_{21}^{(2)}$.

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