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Titolo della Tesi:

# *Chaotic Dynamics in Solow-type Growth Models.*


**Candidato**: *dott. Cesare Palmisani*
**Supervisor**: *Prof. Vincenzo Aversa*
**Coordinatore**: *Prof.ssa Emilia Di Lorenzo*


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To my sons Gianluca and Alessandro

# Introduction

The thesis, entitled "*Chaotic dynamics in Solow-type growth models*", explores some discrete time models of economic growth and research in them elements of dynamic complexity. It consists of three chapters. In the *first chapter* we recall the definitions and the results of the theory of *Discrete Systems Dynamics* and *Chaos Theory* preliminary to the study of certain models of economic growth in discrete time. In particular, we give a brief overview of the definitions of one-dimensional chaos more widely used in the literature; then we present some significant one-dimensional discrete time models. Concerning two-dimensional dynamical systems we describe a technique for the linearization of a two-dimensional map and we study a Kaldorian model of business cycle that presents aspects of complex dynamic such as the "*Arnold's tongues*". In the *second chapter* we review significant models of growth. The discussion begins with the Solow model (1956) and continues with the study of the contribution by R.H. Day (1982) in which chaotic dynamics emerges in a discrete time versions of the Solow model through appropriate changes in the production function - or in the propensity to save, which is no longer regarded as a constant parameter but as a endogenous variable. Later we study a model of growth, due to V. Böhm and L. Kass (2000), which extends the Solow model by introducing differentiated propensities to save as in Kaldor (1955, 1956) and Pasinetti (1962). The model of V. Böhm and L. Kass (2000), although has in common with the Solow model the important characteristic of being a one-sector (only one good is produced in the economy) and one-dimensional model (the law of accumulation is represented by a single discrete time equation), it differs from it because it assumes two different types of economic agents (the "classes" of "workers" and "capitalists") instead of a representative agent. A second aspect of differentiation that is evident in the model of V. Böhm and L. Kass (2000) consists of a production function, an approximation to the Leontief production function, that unlike the Cobb-Douglas used in the Solow model, relaxes the Inada condition by introducing weaker assumptions concerning its properties. Also while in the Solow model representative agent save according to a unique constant propensity to save - which corresponds to the aggregate average propensity to save of the economy - in the model of V. Böhm and L. Kass (2000), following Kaldor (1955; 1956) and Pasinetti's (1962) approach, two saving propensities are introduced, both constant and relating to the two classes considered. Finally, the authors show that their model meets the conditions of the theorem of Li-Yorke (1975) and, therefore, that it can generate chaotic dynamics. The discussion of the models of economic growth is followed by the presentation of a recent model developed by P. Commendatore (2005), which represents a discrete time version of the Solow model proposed by Samuelson and Modigliani (1966). It is possible to note that even if it is a model with two classes, differs significantly from the work of Böhm and Kass (2000). Distinctive features involve in the first place the use of a CES production function, which does not meet the 'weak' conditions of Inada and that only asymptotically behaves like a Cobb-Douglas. A second element of differentiation is the time map describing the accumulation of capital that is two-dimensional. The properties of this map are studied by using some sophisticated techniques such as the linear approximation theorem of Hartman and Grobman. Moreover, the author, following Chiang (1973), introduces three saving propensities and studies the different types of existing equilibria (the equilibrium of Pasinetti, the dual equilibrium (or anti-Pasinetti equilibrium) and the trivial equilibrium) and the asymptotic local and global stability properties of the system. The analysis of this model proposed in this thesis it is not a mere exposition but it refines some demonstrations and provides explanations for certain observations only mentioned by the author and not fully investigated.

In the *third chapter*, which represents the most innovative part of the thesis, a discrete time model of economic growth is presented. The model also represents a discrete time version of the Solow and Samuelson and Modigliani model. However, unlike other models described in the second chapter, it assumes that workers and capitalists save on the basis of a rational choice. Following Thomas R. Michl (2005), we assume that workers' saving choices, which are egoistic, follow a pattern based on an overlapping generation structure, whereas capitalists behave like a dynasty a la Barro. The solutions of the model generates a two-dimensional time map for the accumulation of capital. About this map, we study in depth the local asymptotic stability properties.

# Chapter 1

## 1.1 Contents

## 1.2 Introduction

Few branches of mathematics can boast exact origins as *Chaos Theory*. As a matter of fact its history begins in 1889[1], when, in order to commemorate the 60th birthday of King Oscar II of Sweden and Norway (Jan. 21, 1889), a competition was held to produce the best research in celestial mechanics concerning the stability of the solar system, a particularly relevant $n$-body problem. The contest was organized by Magnus Gösta Mittag-Leffler, editor of the journal of *Acta Mathematica* and supported as judges by two mathematicians, Charles Hermite and Karl Weierstrass. The prize was a gold medal, a sum of 2500 crowns and the publication of the work on the journal of *Acta Mathematica*. The winner was declared to be Jules Henri Poincaré, a young [2] professor at the University of Paris. Poincaré submitted a memory about the *three-body problem*, making the following assumptions: the three bodies move in a plane, two of the bodies are massive and the third has negligible mass in comparison, so small not affecting the motion of the others two (*planar restricted three-body problem*). Moreover the two bodies move in circles, at a constant speed, circling around their combined mass centre of mass. Among the many seminal ideas in the entry of the winner there were the crucial notions of "*stable and unstable manifolds*". However a colleague of Mittag-Leffler, Lars Edvard Phragmen, "*after Poincaré was declared the winner but before his memory was published*"[3], detected a serious mistake in a proof of Poincaré's entry. Poincaré did not entirely understand the nature of the stable and the unstable manifolds: these manifolds may cross each other in a so-called *homoclinic point*. The Poincaré's article, *Sur les équations de la dinamique et le probleme des trois corps*, revised, was published in 1890. Recently Poincaré' s ideas were applied by NASA "*to send a spacecraft with a minimal fuel through the tail of a comet. In this application the three bodies were Earth, Moon, and a spacecraft*" (Kennedy, Kocak, Yorke, 2001)[4]. We encounter the notions of stable and unstable manifolds into several economic models (Brock, Hommess (1997); Grandmont, Pintus, de Vilder (1998); Yokoo (2000); Onozaki, Sieg, Yokoo (2003); Puu (2003), Agliari, Dieci, Gardini (2005)).

[1] For this reconstruction we follow Alligood et al. (1996) and June Barrow-Green (1997)

[2] Poincaré (1854-1912) became professor at the age of 27.

[3] Alligood et alt.(1996), or "... (the) *long process of editing, typesetting, printing took place from July to November 1889"*. Ivars Petersons, *The Prophet of Chaos*, MathTrek.

[4] See also Koon, Lo, Marsden and Ross (1999).

## 1.3 Preliminaries

Let $(X, d)$ a compact metric space without isolated points and $f : X \to X$ a continuous map.

**Definition 3.1**[5] Let $A = \{t_j\}$ be an increasing sequence of positive integers, let $m > 0$ be an integer and $\epsilon > 0$. A set $E \subset X$ is an $(A, m, f, \epsilon)-$span, if for any $x \in X$ there is some $y \in X$ such that

$$d(f^{t_j}(x), f^{t_j}(y)) < \epsilon, \text{ for } 1 \leq j \leq m.$$

Let $S(A, m, f, \epsilon)$ be an $(A, m, f, \epsilon)-$span with a minimal possible number of points.

**Definition 3.2** The topological sequence entropy of $f$ with respect to $A$ is

$$h_A(f) = lim_{\epsilon \to 0} \limsup_{m \to \infty} \frac{1}{\mathrm{m}} log\#S(A, m, f, \epsilon)$$

where $\#\{\cdot\}$ means 'the number of elements in the set'. In the case $A = N = \{0, 1, \ldots\}$ we obtain the topological entropy $h(f)$ of $f$.

**Definition 3.3** $F_{xy}^{(n)(t)} = \frac{1}{\mathrm{n}}\#\{m : 0 \leq m \leq n-1, \delta_{xy}(m) < t\}$

The definition of the lower distribution is

$$F(t) = \liminf_{n \to \infty} F_{xy}^{(n)}(t)$$

and of the upper distribution is

$$F_{xy}^{*}(t) = \limsup_{n \to \infty} F_{xy}^{(n)}(t)$$

[5] Forti (2005) defines the topological entropy in the sense of Bowen (1971) and Dinaburg (1970). The original definition was given by Adler, Koneheim and McAndrew (1965).

**Definition 3.4** *A set* $S \subset X$ *(which has at least two point) such that for any* $x, y \in S$, $x \neq y$,

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0,$$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

*is called a scrambled set.*

Let $\omega_f(x)$ be the set of limit points of the sequence $f^n(x)$.

**Definition 3.5** *A set* $S \subset X$ *(which has at least two point) such that for any* $x, y \in S$, $x \neq y$,

*i)* $\omega_f(x) \setminus \omega_f(y)$ *is uncountable,*

*ii)* $\omega_f(x) \cap \omega_f(y)$ *is non-empty,*

*iii)* $\omega_f(x)$ *is not contained in the set of periodic points*

*is called an* $\omega-$*scrambled set.*

**Definition 3.6** *The orbit of a point* $x \in X$ *is said to be unstable if there exists* $r > 0$ *such that for every* $\epsilon > 0$ *there are* $y \in X$ *and* $n \leq 1$ *satisfying the inequalities* $d(x, y) < \epsilon$ *and* $d(f^n(x), f^n(y)) > r$.

**Definition 3.7** *Let* $\epsilon > 0$. *The map* $f$ *is called Lyapunov* $\epsilon$*-unstable at a point* $x \in X$ *if for every neighbourhood* $U$ *of* $x$, *there is* $y \in U$ *and* $n \geq 0$ *with* $d(f^n(x), f^n(y)) > \epsilon$. *The map is called unstable at a point* $x$ *(or the point* $x$ *itself is called unstable) if there is* $\epsilon > 0$ *such that* $f$ *is Lyapunov* $\epsilon$*-unstable at* $x$.

**Definition 3.8** *The map* $f$ *is topologically transitive if for every pair of non-empty open sets* $U$ *and* $V$ *in* $X$ *there is a positive integer* $k$ *such that* $f^n(U) \cap V \neq \emptyset$.

**Definition 3.9** *The map* $f$ *is called sensitive on initial conditions in* $X$ *if exists* $r > 0$ *such that for every* $x_0 \in X$ *and every* $\epsilon > 0$, *we can find a* $y_0 \in X$ *such that* $d(x_0, y_0) < \epsilon$ *and, for some integer* $m$, $d(f^n(x_0), f^n(y_0)) > r$.

**Definition 3.10** *A map $f$ is called accessible if for every pair of non-empty open sets $U$ and $V$ of $X$, there exist points $x \in U$, $y \in V$ and a positive integer $n$ such that $d(f^n(x), f^n(y)) < \epsilon$.*

## 1.4 Various notions of chaos for dynamical systems

Following Martelli, Dang and Seph (1998), Forti (2005), Alligood (1996) and other authors, we will present and compare various definitions of chaos and we will observe that someone of them are equivalent under particular conditions. We begin from Martelli, Dang and Seph (1998) and from now on we will shortly refer to it as MDS (1998). The author tell us that, in the last thirty years, different definitions of chaos have been proposed by scientists belonging to distinct fields of research: Chemistry, Physics, Biology, Medicine, Engineering and Economics, leading to the "*non desiderable situation*" in which "*there are as many definitions of chaos as experts in this new area of knowledge*". Every definition of chaos tries to capture the peculiar characteristic of the discipline that originated it. However, MDS (1998) notices that there is a trade-off between the needs of the experimentalist and the theoretician: the former requires that the chaos' definition may be tested in laboratory instead the latter cares for "*characterizing chaotic behaviour uniquely*". According to MDS (1998), the more common[6]definitions of chaos in literature are: the Li-Yorke chaos, the experimentalist' definition of chaos, the Devaney's chaos, the Wigging's chaos and the Martelli's chaos. To compare the previous definitions of chaos, MDS (1998) use the following table:

[6]and "easily accessible to undergraduates", says MDS (1998), p. 112.

| Definition | map | domain | requirements | advantages | weak points |
|---|---|---|---|---|---|
| Li-Yorke | continuous | bounded interval | periodic orbit of period 3 | easy to check | can be used only in $\Re$ |
| Experimentalist' | continuous | $X \subset \Re^q$, bounded, closed, invariant | sensitivity on initial conditions | easy to check | defines as chaotic systems which are not |
| Devaney | continuous | $X \subset \Re^q$, bounded, closed, invariant | sensitivity, transitivity, dense periodic orbits | goes to the roots of chaotic behaviour | redundancy |
| Wiggins | continuous | $X \subset \Re^q$, bounded, closed, invariant | sensitivity, transitivity | goes to the roots of chaotic behaviour | admits degenerate chaos |
| Martelli | continuous | $X \subset \Re^q$, bounded, closed, invariant | dense orbit in $X$ which is unstable | "equivalence" with Wiggins, easy to check numerically | none of above |

About the weak points of the definition of Li-Yorke, Martelli (1998) shows that:

- the Li-Yorke's Theorem does not hold in dimensions higher than one. For example, a map on $\Re^2$, which, thought admits a three-period cycle, has none of the properties which the Li-Yorke's Theorem (1975) refers to[7];
- there are discontinuous maps, defined on $[0, 1]$, with period three;
- the chaos characterized by the Li-Yorke' Theorem (1975) is not an *observable or ergodic chaos*[8].

The survey of Forti (2005) does not consider the *experimentalists' chaos* and the *Wiggin's chaos*. However it includes the definitions of chaos by Li-Yorke, Devaney and Martelli, and introduces other four definitions: *topological chaos*, *distributional chaos*, *$\omega$-chaos* and *Block-Coppel chaos*.

The seven definitions of chaos presented in the Forti's paper are:

[7]Let $F$ be a rotation in $\Re^2$ of $120^\circ$ around the origin. Then $F$ has period three.
[8]See also the Lasota-Yorke's Theorem (1973) or Boldrin and Woodford (1990).

**Topological chaos**

**Definition 4.1** *A map $f$ is topologically chaotic if its topological entropy $h(f)$ is positive.*

**Distributional chaos**

**Definition 4.2** *The map $f$ is distributionally chaotic (d−chaotic) in the sense of B. Schweizer and J. Smítal if there exist a pair $x, y \in X$ such that*

$F_{xy}(t) < F^*_{xy}(t)$,

*for $t$ in some non-degenerate interval.*

**Li-Yorke chaos**

**Definition 4.3** *The map $f$ is chaotic in the sense of Li and Yorke if it has a scrambled set $S$.*

**$\omega$-chaos**

**Definition 4.4** *The map $f$ is $\omega$-chaotic if there exists an uncountable $\omega$−scrambled set $S$.*

**Martelli's chaos**

**Definition 4.5** *The map $f$ is chaotic in the sense of Martelli if there exist a point $x_0 \in X$ such that*

*i) the orbit of $x_0$ is dense in $X$;*

*ii) the orbit of $x_0$ is unstable.*

**Devaney's chaos**

**Definition 4.6** *The map is chaotic in the sense of Devaney if it* *i) is topologically transitive;*

*ii) periodic points are dense in X;*

*iii) is sensitive on initial conditions.*

Bank et al.(1992) prove that *i) and ii) imply iii)* for any metric space $X$. Furthemore Block and Coppel (1992) show that, *if $X$ is a real interval, i) implies ii)* . Thus, *the Devaney's chaos has reduced to the topological transitivity.*

**Block-Coppel's chaos**

**Definition 4.7** *The map $f$ is chaotic in the sense of Block and Coppel if there exist disjoint non-empty compact subsets $J$, $K$ of $X$ and a positive integer $n$ such that $J \cup K \subseteq f^n(J) \cap f^n(K)$.*

## 1.5 Additional notions of chaos

In literature we encounter other definitions of chaos. The aim of a recent line of research followed by Rongbao Gu (2005); Roman Flores (2003); Alessandro Fedeli (2005), is to extend a chaotic map $f$ from a metric space fixed $X$, with a metric $d$, to a continuous map defined on the metric space given by the family of all non-empty compact subsets of $X$ and equipped by the Hausdorff metric [9]. Particularly, the notions of chaos involved (and not included in the Forti's survey) are the *Kato's chaos*, *Robinson's chaos*, *Ruelle-Takens' chaos*, *Knudsen's chaos*, *Touhey's chaos*. Let $(X, d)$ a metric space fixed, the previous definitions of chaos are:

**Kato's chaos**

**Definition 5.1** *The map $f$ is chaotic in the sense of Kato if it*

*i) sensitive on initial conditions,*

[9] The topology induced by the Hausdorff metric is called *Vietoris topology*

*ii) accessible.*

**Robinson's chaos (or Wiggin's chaos)**

**Definition 5.2** *The map $f$ is chaotic in the sense of Robinson if it is*

*i) topologically transitive,*

*ii) sensitive on initial conditions.*

**Ruelle-Taken's chaos (or Auslander-Yorke's chaos)**

**Definition 5.3** *The map $f$ is chaotic in the sense of Ruelle and Takens if*

*i) it is surjective;*

*ii) every point is unstable (in the sense of Lyapunov);*

*iii) $X$ contains a dense orbit.*

**Knudsen's chaos**

**Definition 5.4** *The map $f$ is chaotic in the sense of Knudsen if*

*i) there is a dense orbit in $X$;*

*ii) it is sensitive on initial conditions.*

**Touhey's chaos**

**Definition 5.5** *The map $f$ is chaotic in the sense of Touhey if for every non-empty pair $U$ and $V$ of open subsets of $X$,*

*i) exists a periodic point $p \in U$;*

*ii) exists a non-negative number $k$ such that $f^k(p) \in V$.*

## 1.6 Comparison among the different notions of chaos

We can observe that *six among the seven chaotic maps presented into the Forti's paper are equivalent if the domain of them is an interval* (Theorem 6.1.1). More in detail, the Li-Yorke's chaos is the only definition not equivalent to the other definitions but it is implicated by them(*the particular case of the real interval*). In dimension higher than one (*the general case*) Forti (2005) tells us that from topological chaos, or $\omega$-chaos, or Devaney's we deduce the Li-Yorke's chaos and that Devaney's chaos implies Martelli's chaos.

The Robinson's chaoticity implies the Kato's chaoticity on the complete metric space, but the converse is not true in general (H. Román-Flores and Y. Chalco-Cano, 2005).

According to MDS (1998), Wiggin's (or Robinson's) chaos is equivalent to Martelli's chaos on any metric space $X$. Thus when $X = I$, with $I$ a real interval, the Robinson's (or Wiggins's) chaos is equivalent to topological chaos.

The definition of the Knudsen's chaos is equivalent to the definition of the Kato's chaos on a compact metric space (Rongbao Gu, 2005).

If $X$ is a metric space and $f$ is a map continuous from $X$ in itself, *Devaney's chaos is equivalent to Touhey's chaos* (Touhey, 1997). By Banks et al.(1992) it is sufficient to prove that:

**Theorem 6.2.1** *$f$ is chaotic in the sense of Touhey if and only if*

*i) the set of periodic points of $f$ is dense in $X$;*

*ii) $f$ is topologically transitive.*

**Proof** Assume that $f$ is chaotic in the sense of Touhey. Then every pair of non-empty open set $A$ and $B$ of $X$ shares a periodic orbit. In particular, if $B = A$, every non-empty open set $A$ must contain a periodic point. Thus the periodic points of $f$ are dense in $X$. The transitivity of $f$ follows immediately from definition of chaos in the sense of Touhey. Now we suppose that the conditions i) and ii) hold. We set a pair of non-empty open subsets $U$ and $V$ in $X$. From i)-condition, exists a point $u \in U$ and a non-negative integer $k$ such that $f^k(u) \in V$. We set $W = f^{-k}(U) \cap V$. We note that $W$ is a non-empty open set in $X$ [10]. Moreover $W \subset U$ and $f^k(W) \subset V$. From i)-condition a periodic point $p$ belongs to $W$. We summarize the previous results saying that exists a periodic point $p \in W \subset U$ such that $f^k(W) \subset V$, i.e. $f$ is chaotic in the sense of Touhey.

**Remark** $W$ is non-empty since $u \in f^{-k}(U) \cap V$.

Furthemore, in particular, let $I$ be a real interval and set $X = I$, *Tohey's chaos is equivalent to the topological entropy and for $f$ chaotic in the sense of Touhey the Theorem 6.1.1 holds .*

## 1.7 Discrete Dynamical Systems: One-Dimensional, Autonomous, First-Order Systems

### 1.7.1 Linear Systems

We consider the following linear dynamical system (Galor, 2006)

$y_{t+1} = ay_t + b$, $t = 0, 1, 2, 3, \ldots$ (1)

where $a, b \in \Re$ are constant parameters, and $y_t$ is a state-variable such that $y_t \in \Re$ for all $t \in \Re$ (one-dimensional).

**Definition** We say that a point $\overline{y} \in \Re$ is a steady-state of (1) if

$\overline{y} = a\overline{y} + b$ (2)

By (2) immediately we derive the equation

$(1 - a)\overline{y} = b$ (3).

We note that

[10] See Block-Coppel (1992), Lemma 37.

**Proposition 1** If $\overline{y}$ denotes a steady-state of (1) occurs that

- Case 1 Let $a \neq 1$. Then $\overline{y}$ exists and it is unique.
- Case 2 Let $a = 1$ and $b = 0$. Then there is a continuum of steady-states $\overline{y}$.
- Case 3 Let $a = 1$ and $b \neq 0$. Then $\overline{y}$ does not exists.

*Proof*

- (Case 1) Dividing both hand-sides of (3) by $1 - a \neq 0$, we have $\overline{y} = \frac{b}{1-a}$ ;
- (Case 2) Equation 3) is equivalent to equation $0 \cdot \overline{y} = 0$, therefore every $\overline{y} \in \Re$ satisfies (3);
- (Case 3) Equation (3) reduces to impossible equation $0 = b$, where $b \neq 0$.

**Definition** Given $y_0$ (initial condition), any sequence $y_0, y_1, y_2, \ldots$ that satisfies (1) is called trajectory.

From (1) we get:

$y_1 = ay_0 + b,$

$y_2 = ay_1 + b = a(ay_0 + b) + b = a^2 y_0 ab + b,$

$y_3 = ay_2 + b = a(ay_1 + b) + b = a^3 y_0 a^2 b + ab + b,$

$\ldots$

$y_t = a^t y_0 + a^{t-1} b + a^{t-2} b + +ab + b,$

$= a^t y_0 + b(a^{t-1} + a^{t-2} + \ldots + a + 1).$

By recurrence it is possible to prove that

$$1 + a + \ldots + a^{t-1} = \begin{cases} \frac{1-a^t}{1-a} & \text{if } a \neq 1, \\ t & \text{if } a = 1, \end{cases}$$

from which

$$y_t = \begin{cases} a^t y_0 + b\frac{1-a^t}{1-a} = a^t(y_0 - \frac{b}{1-a}) & \text{if } a \neq 1, \\ y_0 + bt & \text{if } a = 1, \end{cases}$$

and [11]

$$\overline{y} = \begin{cases} \frac{b}{1-a} & \text{if } a \neq 1, \\ y_0 & \text{if } a = 1 \text{ and } b = 0, \end{cases}$$

Thus we can rewrite $y_t$ as

$$y_t = \begin{cases} (y_0 - \overline{y})a^t + \overline{y} & \text{if } a \neq 1, \\ y_0 + bt & \text{if } a = 1 \text{ and } b \neq 0, \\ y_0 & \text{if } a = 1 \text{ and } b = 0. \end{cases}$$

**Remark** If $a = 1$ and $b = 0$ the system (1) becomes $y_{t+1} = y_t$ for all $t$, from which $y_t = y_{t-1} = \ldots = y_1 = y_0$, that is $y_t = y_0$. Thus *the system does not deviate from the initial condition and it is in the steady-state* $\overline{y} = y_0$. Instead if $a = 1$ and $b \neq 1$, the system take a form $y_{t+1} = y_t + b = y_0 + bt$ and *it increases indefinitely if* $b > 0$ *and decreases indefinitely if* $b < 0$.

**Definition 2** A steady-state is *globally* (asymptotically) stable if the system converges to this steady-state regardless the level of the initial condition, whereas a steady-state is *locally* (asymptotically) stable if there exists at least an $\epsilon$-neighborhood of the steady-state such that from every initial condition within this neighborhood, the system converges to this steady-state.

[11] We observe that $a^t \to 0$ for $t \to -\infty$ if $a > 1$ or $t \to +\infty$ if $0 < a < 1$.

We observe that the concepts of global and local stability of a steady-state require respectively global uniqueness and local uniqueness of the steady-state. Therefore

**Corollary** A steady-state of (1) is globally stable only if the steady-state is unique.

From the behaviour of the absolute value of the system (1) as time approaches to infinity we can derive the global or local stability of a steady-state. As a matter of fact, since

$$\lim_{t\to\infty} y_t = \begin{cases} (y_0 - \overline{y}) \lim_{t\to\infty} a^t + \overline{y} & \text{if } a \neq 1, \\ y_0 + b \lim_{t\to\infty} t & \text{if } a = 1. \end{cases}$$

we get that $\lim |y_t|$ is equal to

- $|\overline{y}|$ if $|a| < 1$;
- $|y_0|$ if $a = 1$ and $b = 0$;
- $|y_0|$ for $t = 0, 2, 4, \ldots$ if $a = -1$;
- $|b - y_0|$ for $t = 1, 3, 5, \ldots$ if $a = -1$;
- $\infty$ otherwise.

From the previous results we note that the parameter $a$ plays a central role in determining if a steady-state is globally stable. Precisely we can say that

- if $|a| < 1$, *the system is globally stable.* Moreover if $0 < a < 1$ the trajectory *converges monotonically* from the initial level $y_0$ to the steady-state level $\overline{y}$: in particular, if $y_0 < \overline{y}$ the sequence $y_t$ is *monotonically increasing*, otherwise it is *monotonically decreasing* (See **Figure 1.1** and **Figure 1.2**). Instead, if $-1 < a < 0$, the convergence of the sequence $y_t$ is *oscillatory* (See **Figure 1.3** and **Figure 1.4**).
- If $a = 1$ and $b = 0$ there is a continuum set of steady-states but the system is neither globally nor locally stable (See **Figure 1.5**).
- If $a = 1$ and $b \neq 1$ then there aren't steady-states (See **Figure 1.6**).
- If $a = -1$ the system has a continuum of two-period cycles. Each cycle is unstable and also $\overline{y} = b/2$ is unstable. The trajectory is $y_0, b - y_0$ (See **Figure 1.7** and **Figure 1.8**).

- If $|a| > 1$ the system has a diverging path. If $a > 1$ we may distinguish another two sub-cases: if $y_0 > \overline{y}$ the divergence is positive (See **Figure 1.9**), otherwise is negative. Moreover $y_t$ diverges with oscillations (See **Figure 1.10**).

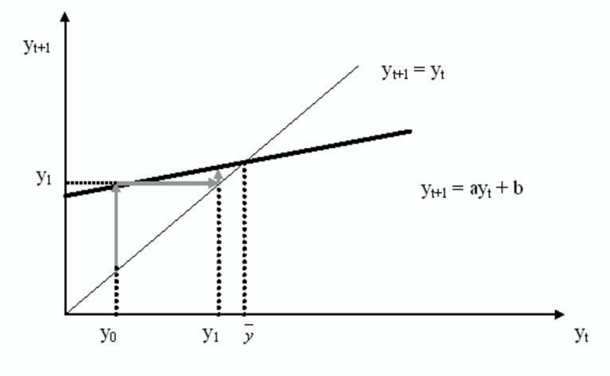


Figure 1.1: *Monotonic Convergence*

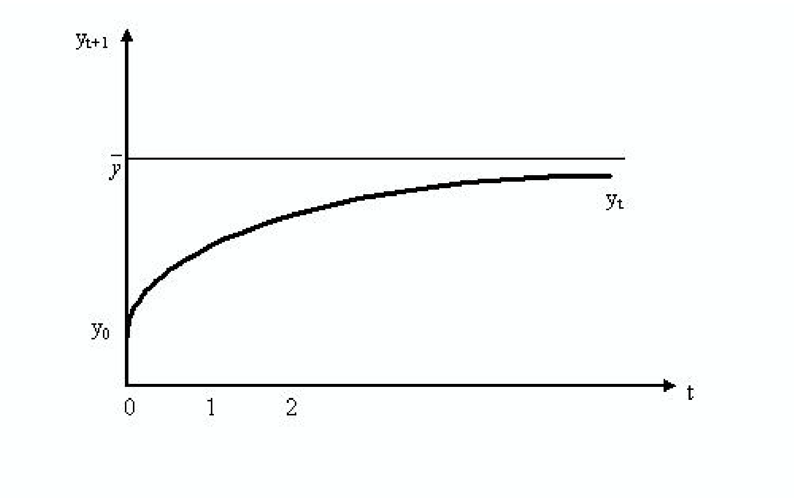


Figure 1.2: *Evolution of the State Variable in following Monotonic Convergence*

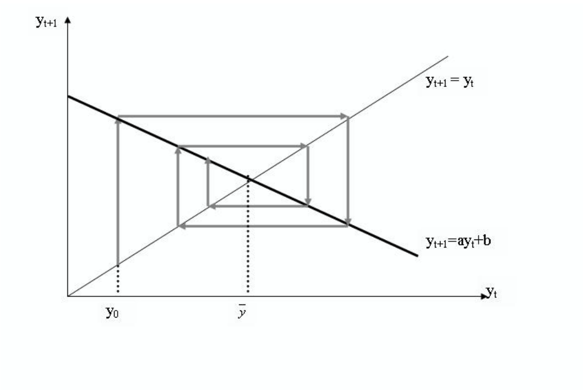


Figure 1.3: *Oscillatory Convergence*

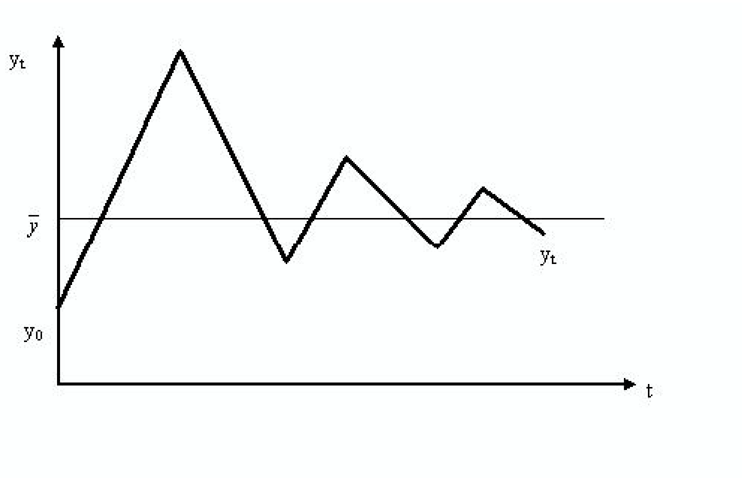


Figure 1.4: *Evolution of the State Variable in following the Oscillatory Convergenge*

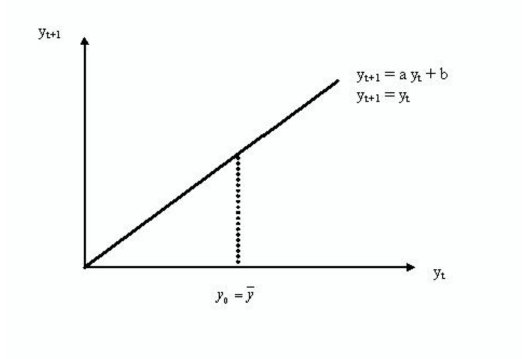


Figure 1.5: *Continuum of Unstable Steady-State Equilibria*

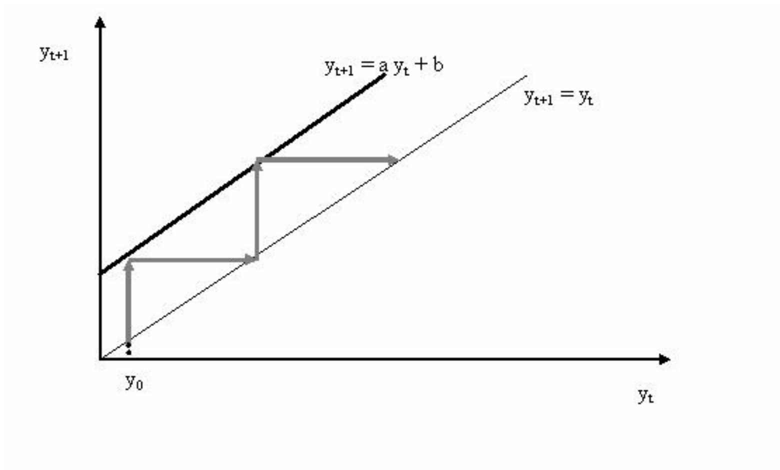


Figure 1.6: *Non-Existence of a Steady-State Equilibrium*

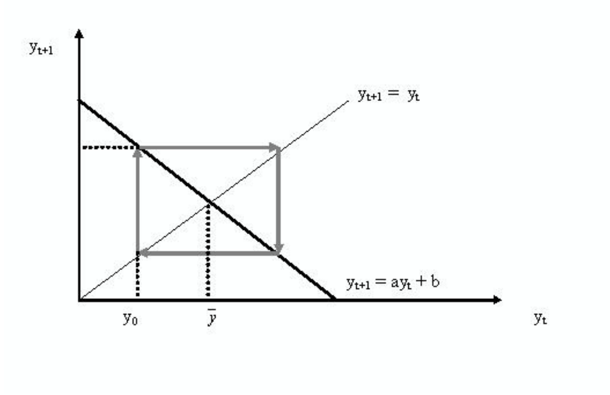


Figure 1.7: *Unstable Two-Period Cycle*

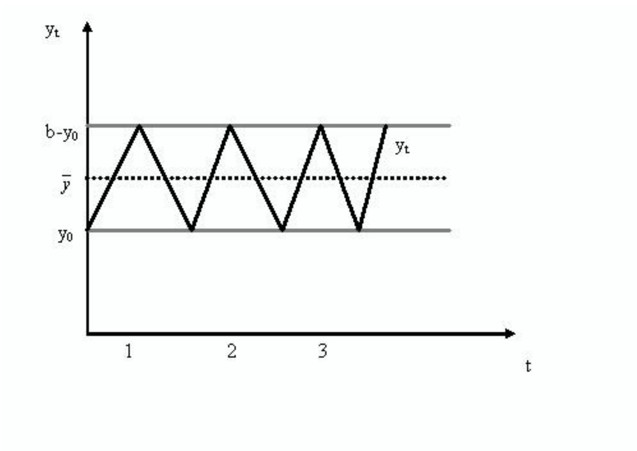


Figure 1.8: *The Evolution of the State-Variable in the Two-Period Cycle*

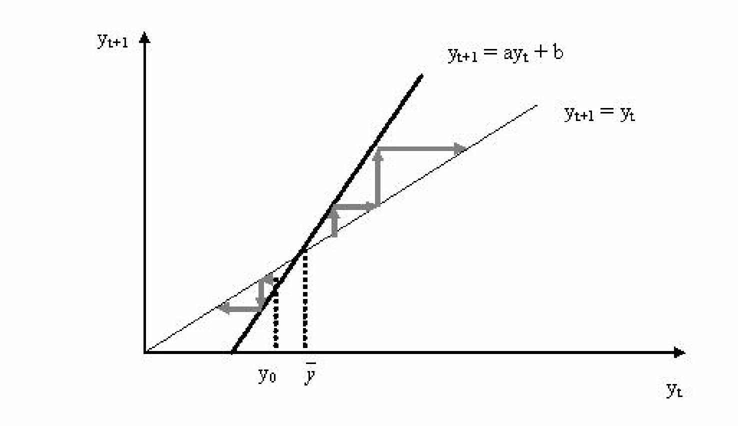


Figure 1.9: *Monotonic Divergence*

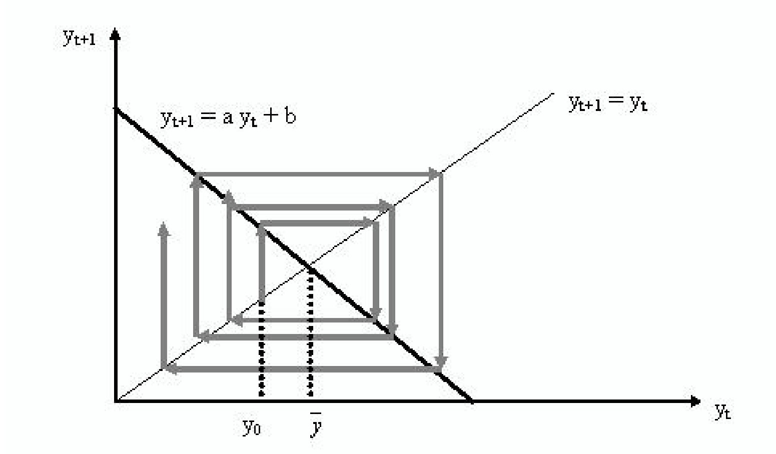


Figure 1.10: *Oscillatory Divergence*

**Remark** To describe the oscillatory behaviour of a discrete time and one-dimensional system A.Medio and M.Lines (2001) observe that the form of a trajectory of a variable is *kinky* and that the oscillations "*do not describe the smoother ups and downs of real variables*". Therefore they use the term *improper oscillations* to differentiate is them from those that occurs in continuous time.

### 1.7.2 Nonlinear Systems

Let $f : \Re \rightarrow \Re$ a map continuously differentiable. We indicate with $y_t$ the state variable. We suppose that the evolution of $y_t$ follows the law

$y_{t+1} = f(y_t)$; $t = 0, 1, 2, \ldots$ (2)

Given $y_0$, the trajectory of the state variable $y_t$ is:

$y_1 = f(y_0)$, $y_2 = f(y_1) = f((y_0))) = f^2(y_0))$, $\ldots$ , $y_t = f^t(y_0))$, $\ldots$ .

We define *steady-state* of the nonlinear system (2) a value $\overline{y}$ such that $\overline{y} = f(\overline{y})$.

In order to study the behaviour of the system (2), we approximate linearly (2) in the proximity of a steady-state $\overline{y}$ (*linearization of a nonlinear dynamical system*) with a Taylor expansion:

$y_{t+1} = f(y_t) = f(\overline{y}) + f^{'}(\overline{y})(y_t - \overline{y})$

$= f^{'}(\overline{y})y_t + f(\overline{y}) - f^{'}(\overline{y})\overline{y}$

$= ay_t + b,$

where $a = f^{'}(\overline{y})$ and $b = f(\overline{y}) - f^{'}(\overline{y})\overline{y}$ are constants.

Thus, like the linear system (1), the non linear system (2) is locally stable around the steady-state $\overline{y}$ if and only if $|a| < 1$, that is $|f^{'}(\overline{y})| < 1$.

In order to analyze the global stability of the nonlinear system (2) we will use the concept of *contraction map* from a given metric space into itself and the theorem of existence and uniqueness of a fixed-point for contraction mappings on a complete metric space.

**Definition** Let $(S, \rho)$ be a metric space, we say that a mapping $T : S \rightarrow S$ is a contraction mapping if exists a constant $0 < \beta < 1$ such that

$\rho(Tx, Ty) \leq \beta\rho(x, y)$.

**Example** Let $f : [a, b] \to [a, b]$ be a continuous function with positive slope smaller than one. Since $\frac{f(x)-f(y)}{y-x} \leq \beta < 1$, then $f$ is a contraction and its graph must cut the 45° line.

**Theorem** We suppose that $(S, \rho)$ is a complete metric space and $T : S \to S$ a contraction mapping for S. Then (1) exists a unique $v \in S$ such that $Tv = v$ (fixed point for $T$); (2) for all $v_0 \in S$ and $0 < \beta < 1$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$ for all $n = 1, 2, ....$

**Corollary** A steady-state of nonlinear system $y_{t+1} = f(y_t)$ exists and is unique and globally (asymptotically) stable if $f : \Re \to \Re$ is a contraction mapping, i.e., if

$\frac{f(y_{t+1} f(y_t))}{y_{t+1} - y_t} < 1$, for all $t = 1, 2, \ldots$

or if $f \in C^1$ and $|f^{'}(y_t)| < 1$, for all $y_t \in \Re$.

## 1.8 Continuous dynamical systems in the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $x, y \in \Re$, $a_{i,j}$ real constant.

We consider the set $E$ of points $(x, y)$ such that $\dot{x} = 0$ and $\dot{y} = 0$ and we note that if $det(A) \neq 0$ we have $E = \{(0, 0)\}$. The point $(0, 0)$ is called *equilibrium.*

The characteristic equation is

$$0 = det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$

$= \lambda^2 - tr(A)\lambda + det(A),$

and the eigenvalues are

$\lambda_{1,2} = \frac{1}{2}(tr(A) \pm \sqrt(\Delta))$

where $\Delta = ((tr(A))^2 - 4det(A))$ is called the discriminant. The different types of dynamic behaviour depends upon the sign of the discriminant. We distinguish three cases.

Case 1 $\Delta > 0$ We have that *eigenvalues and eigenvectors are real.*

To Case 1 corresponds three subcases.

(i) $tr(A) < 0$, $det(A) > 0$ Eigenvalues and eigenvectors are real and both eigenvectors are negative. The two-dimensional state space coincides with the stable eigenspace. The equilibrium is called a **stable node**.

(ii) $tr(A) > 0$, $det(A) > 0$ Eigenvalues and eigenvectors are real and both eigenvectors are positive. The two-dimensional state space coincides with the unstable eigenspace. The equilibrium is called a **unstable node**.

(iii) $det(A) < 0$ One eigenvalue is positive and the other is negative. Thus there is a one-dimensional stable and one-dimensional unstable eigenspace. The equilibrium is called **saddle node**.

Case 2 $\Delta < 0$ We have that *eigenvalues and eigenvectors are complex coniugate pairs.* There are three subcases:

(i) $tr(A) < 0$, $Re(\lambda) = \alpha < 0$. We have that the oscillations are *dampened.* Moreover the dynamical system converges to equilibrium known as **focus** or **vortex**.

(ii) $tr(A) > 0$, $Re(\lambda) = \alpha > 0$. The amplitude of the oscillations gets larger with time and the system diverges from an unstable equilibrium called **unstable focus** or **vortex**.

(iii) $tr(A) = 0$, $Re(\lambda) = \alpha = 0$, $det(A) > 0$.

$Case3$ $\Delta = 0$ We have that *the eigenvalues are real and equal.*

## 1.9 Discrete dynamical systems in the plane

### 1.9.1 Homogeneous systems

We consider the following homogeneous system

$x_{t+1} = a_{11}x_t + a_{12}y_t,$

$y_{t+1} = a_{21}x + a_{22}y_t.$

We can rewrite the previous system such that

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

or $z_{t+1} = Az_t$,

where $z_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ and $A$ is the coefficient matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Unlike the discrete and one-dimensional systems, we must solve simultaneously all the equation in the system. However if $A$ is diagonal, the system becomes

$x_{t+1} = a_{11}x_t,$

$y_{t+1} = a_{22}y_t,$

that is it is reduced to two independent equations which we can solve separately as one-dimensional systems. We call the previous system *uncoupled system* and its general solution is given by

$x_t = c_1 a_{11}^t;$

$y_t = c_2 a_2^t.$

The previous case suggests that if the coefficient matrix $A$ can be transformed into a diagonal matrix $D$ then the system becomes an uncoupled system and its solution can be used to solve the original system.

The square matrix $A$ is said *diagonalizable* if exists an invertible matrix $P$ such that $P^{-1}AP = D$, where $D$ is a diagonal matrix. Moreover *if all eigenvalues of $A$ are distinct then it is possible to prove that $A$ is diagonalizable and the corresponding eigenvectors are linear indipendent.* In order to diagonalize $A$ we use the matrix $E$ formed by eigenvectors of $A$. In the $2 \times 2$, we set $E = (e_1, e_2)$, where $e_1 = \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$ and $e_2 = \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}$, and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $\lambda_1$ and $\lambda_2$ are the distinct eigenvalues of $A$. We obtain

$E^{-1}AE = \Lambda$ iff $AE = A$.

We have

$E^{-1}z_{t+1} = E^{-1}Az_t = E^{-1}AIz_t$

$= E^{-1}A(EE^{-1})z_t = (E^{-1}AE)(E^{-1}z_t)$

$= \Lambda E^{-1}z_t.$

If we set $\hat{z}_t = E^{-1}z_t$, the system becomes $\hat{z}_{t+1} = \Lambda \hat{z}_t$, that is

$\begin{pmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \hat{x}_t \\ \hat{y}_t \end{pmatrix}$ and it takes the form of an uncoupled system:

$\hat{x}_{t+1} = \lambda_1 \hat{x}_t,$

$\hat{y}_{t+1} = \lambda_2 \hat{y}_t.$

As above, we find immediately the general solution:

$\hat{x}_t = c_1 \lambda_1^t$, $\hat{y}_t = c_2 \lambda_2^t$.

In order to obtain the solution of the original system we must invert the original transformation:

$\hat{z}_t = E^{-1}z_t \Rightarrow z_t = E\hat{z}_t$, from which we have

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} \begin{pmatrix} c_1\lambda_1^t \\ c_2\lambda_2^t \end{pmatrix}.$$

Hence the general solution of the original system is

$$x_t = c_1 e_{11}\lambda_1^t + c_2 e_{21}\lambda_2^t,$$

$$y_t = c_1 e_{12}\lambda_1^t + c_2 e_{22}\lambda_2^t.$$

If the eigenvalues are complex conjugates we can write them in *algebraic form*, that is

$$\lambda_1 = \gamma + i\mu = r(\cos\theta - i\sin\theta) = re^{i\theta},$$

$$\lambda_1 = \gamma - i\mu = r(\cos\theta + i\sin\theta) = re^{-i\theta},$$

and the corresponding eigenvectors are

$e_1 = d + if$, $e_2 = d - if$, where $d$ and $f$ are also vectors.

$$z_t^1 = e_1\lambda_1^t$$

$$= (d + if)(re^{i\theta})^t$$

$$= (d + if)r^t[cos(\theta t) + isin(\theta t)]$$

$$= r^t[dcos(\theta t) + idsin(\theta t) + ifcos(\theta t) + i^2 fsin(\theta t)]$$

$$= r^t[dcos(\theta t) - fsin(\theta t)] + ir^t[dsin(\theta t) + fcos(\theta t)].$$

If we proceed as above we have

$$z_t^2 = e_1\lambda_2^t$$

$$= r^t[dcos(\theta t) - fsin(\theta t)] - ir^t[dsin(\theta t) + fcos(\theta t)].$$

Setting

$$u_t = r^t[dcos(\theta t) - fsin(\theta t)],$$

$$v_t = r^t[dsin(\theta t) + fcos(\theta t)],$$

we derive

$$z_t^1 = u_t + iv_t,$$

$z_t^2 = u_t - iv_t$.

Consider the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and the discrete dynamical system

$$x_{n+1} = b_{11}x_n + b_{12}y_n$$

$$y_{n+1} = b_{21}x_n + b_{22}y_n. \ (.1)$$

We can write the system in such a way that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = B \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \ (.2) \text{ or}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \ (.3)$$

We assume that the matrix $(I - B)$ is nonsingular. Then exists the unique equilibrium point $(0, 0)$ for (.1). We recall that $trB = b_{11} + b_{22}$ and $detB = b_{11}b_{12} - b_{21}b_{12}$.

We call *characteristic equation* the following equation

$$p(\lambda) = |B - \lambda I| = \det \begin{vmatrix} b_{11} - \lambda & b_{12} \\ b_{21} & b_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - (b_{11} + b_{22})\lambda + (b_{11}b_{12} - b_{21}b_{12}) =$$

$$= \lambda^2 - (TrB)\lambda + (detB) = 0. \ (.4)$$

We determine the roots $\lambda_{1,2}$ of (.4)

$$\lambda_{1,2} = \frac{TrB \pm \sqrt{(TrB)^2 - 4detB}}{2},$$

and we call $\lambda_{1,2}$ the *eigenvalues* of $p(\lambda) = 0$.

We consider three cases.

**Case 1** $\Delta > 0$ *The eigenvalues are real* and take the form

$$x(n) = c_1 \lambda_1^n v_1^{(1)} + c_2 \lambda_2^n v_2^{(1)}$$

$$y(n) = c_1 \lambda_1^n v_1^{(2)} + c_2 \lambda_2^n v_2^{(2)}$$

- If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the fixed point is a stable node (See **Figure 1.11** and **Figure 1.12**).
- If $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then the fixed point is a unstable node (See **Figure 1.11** and **Figure 1.12** and consider the arrows point in the opposite direction).
- If $|\lambda_1| > 1$ and $|\lambda_2| < 1$ then the fixed point is a saddle node (See **Figure 1.13** and **Figure 1.14**).

**Case 2** $\Delta < 0$ Then $detB > 0$ and *the eigenvalues are a complex conjugate pair*

$$(\lambda_1, \lambda_2) = (\lambda, \overline{\lambda}) = \sigma \pm i\theta$$

The solutions are sequence of points situated on spirals whose amplitude increase and decrease in time according to the factor $r^n$ $(n = 0, 1, \ldots)$, where $r = |\sigma \pm i\theta| = \sqrt{\sigma^2 + \theta^2} = \sqrt{detB}$, is the modulus of the complex conjugate pair.

The solutions are

$x(n) = Cr^n cos(\omega n + \phi)$

$y(n) = Cr^n sin(\omega n + \phi)$

- If $r < 1$ , the solutions converge to equilibrium and the equilibrium point is a *stable focus* (See **Figure 1.15**).
- If $r > 1$ , the solutions diverge and the equilibrium is an *unstable focus* (See **Figure 1.15** and consider the arrows point in the opposite direction).
- If $r = 1$ the eigenvalues lie on a *unit circle*. We set $\omega = arccos[tr(B)/2]$. If $\omega/2\pi$ is rational then the orbit is a *periodic* sequence (See **Figure 1.16**), otherwise the sequence is *quasiperiodic* (See **Figure 1.17**).

**Case 3** $\Delta = 0$ There is a repeated real eigenvalue $\lambda = trB/2$.

$x(n) = (c_1 v^{(1)} + c_2 u^{(1)})\lambda^n + nc_2 v^{(1)}\lambda^n$

$y(n) = (c_1 v^{(2)} + c_2 u^{(2)})\lambda^n + nc_2 v^{(2)}\lambda^n$

- If $|\lambda| < 1$, $\lim_{n\to} n\lambda^n = 0$.
- If the repeated eigenvalue is equal to one in absolute value, the equilibrium is unstable. However, divergence is linear not exponential.

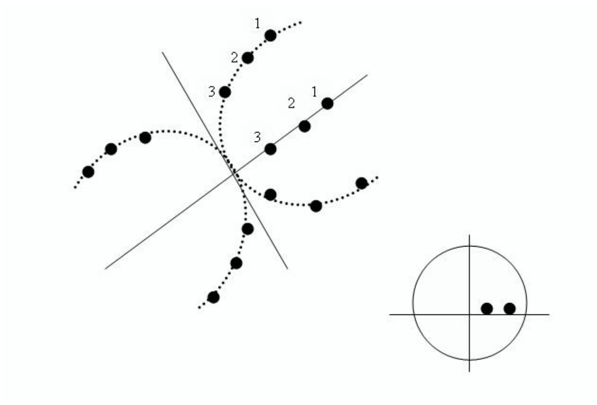


Figure 1.11: *Stable Node*

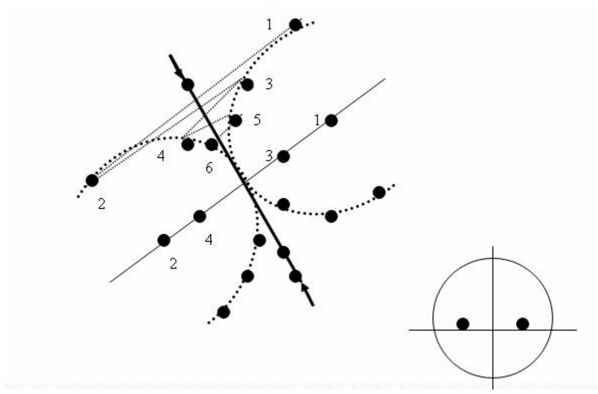


Figure 1.12: *Stable Node*

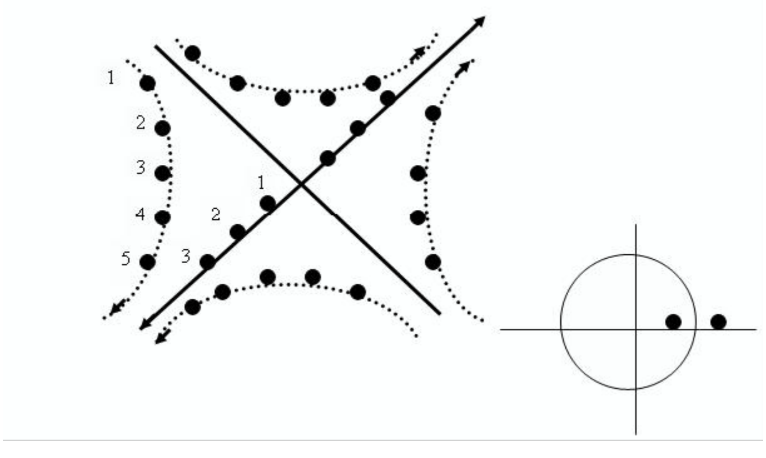


Figure 1.13: *Saddle-Points*

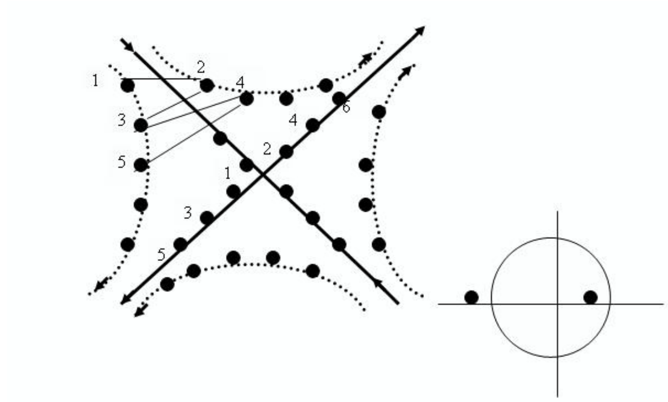


Figure 1.14: *Saddle-Points*

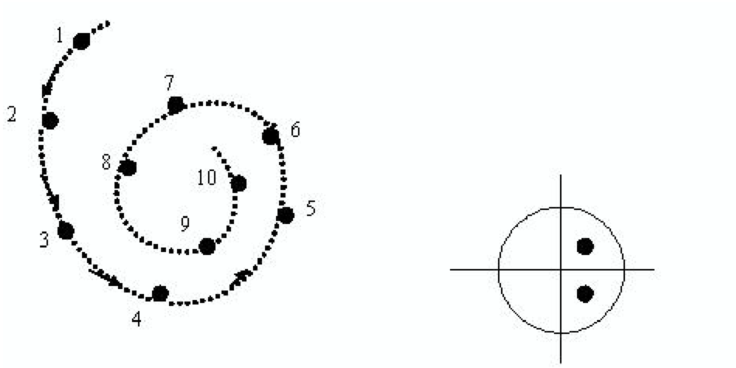


Figure 1.15: *Stable Focus*

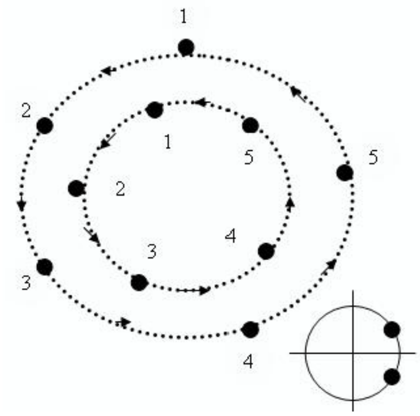


Figure 1.16: *Periodic Cycles*

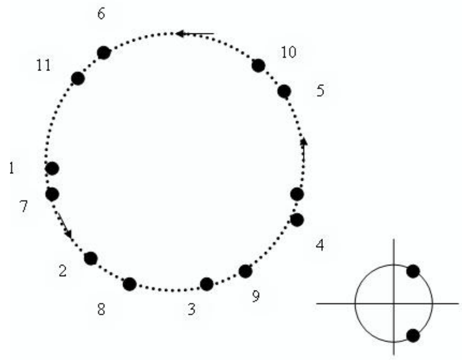


Figure 1.17: *Quasiperiodic Orbit*

## 1.10 Stability of planar discrete systems

**Definition 1** A point $\overline{x} \in X$ is a steady state of the system $x_{t+1} = f(x_t)$, that is $\overline{x} = f(\overline{x})$.

**Definition 2** (Stability) The steady state $\overline{x}$ is a stable fixed point of the map $f$ if for any $\epsilon > 0$ there exist some $\delta \in (0, \epsilon)$ such that

$||x_t - \overline{x}|| < \delta \Rightarrow ||x_t - \overline{x}|| < \epsilon$

for all integers $t \geq s$ (See **Figure 1.18**).

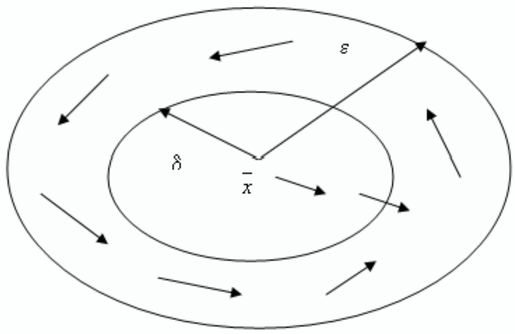


Figure 1.18: *Stability*

**Definition 3** (Asymptotically stability) The steady state $\overline{x}$ is asymptotically stable if it is stable and a constant $\delta$ can be chosen so that, if $||x_s - \overline{x}|| < \delta$ for any $s$, then $||x_t - \overline{x}|| \to 0$ as $t \to \infty$ (See **Figure 1.19**).

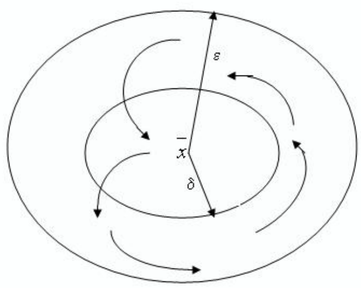


Figure 1.19: *Asymptotical Stability*

**Definition 4** (Topological or flow equivalence) Let $f$ and $g$ be continuously differentiable maps from $X \subseteq \Re^n$ into $\Re^n$. Then we say that the discrete dynamical systems $x_{t+1} = f(x_t)$ and $x_{t+1} = g(x_t)$ are topologically equivalent if there exists a homeomorphism $h : \Re^n \to \Re^n$ that maps $f$ orbits into g orbits while preserving the sense of direction in time.

**Definition 5** (Hyperbolic equilibrium) Let $\overline{x}$ be a steady state of the system $x_{t+1} = f(x_t)$. We say that $\overline{x}$ is a hyperbolic equilibrium if none of the eigenvalues of the Jacobian matrix of the partial derivatives $Df(\overline{x})$, evaluated at $\overline{x}$, falls on the unit circle in the complex plane, that is, if no eigenvalue has modulus exactly equal to 1.

**Linearization**

Let $f : \Re^n \supseteq X \to \Re^n$ and $f \in C^1$. We consider the non linear system

$x_{t+1} = f(x_t)$ (.1)

We applying the Taylor's formula to equation (.1). We obtain

$f(x) = f(\overline{x}) + Df(\overline{x})(x - \overline{x}) + O(||x - \overline{x}||)$.

We suppose $\overline{x} \in X$ and $f(\overline{x}) = \overline{x}$. We expect that the linear system

$x_{t+1} = \overline{x} + Df(\overline{x})(x_t - \overline{x})$ (.2)

approximates well the system (.1) near the steady state $\overline{x}$.

**Theorem** (Hartman-Grobman) Let $\overline{x}$ be the hyperbolic equilibrium of equation (.1 ). If the Jacobian matrix $Df(\overline{x})$ is invertible, there is a neighbourhood $U$ of $\overline{x}$ in which the nonlinear system (.1) is topologically equivalent to the linear system (.2).

**Theorem** (Nonlinear stability) Let $\overline{x}$ be a steady state of (.1).

- If the modulus of each eigenvalue of $Df(\overline{x})$ is less than 1, $\overline{x}$ is asymptotically stable (a sink).
- If at least one eigenvalue has the modulus greater than 1 then $\overline{x}$ is unstable. If this holds for all eigenvalues, $\overline{x}$ is a source, otherwise is a saddle.

- If no eigenvalue of the Jacobian matrix is outside the unit circle but at least one is on the boundary (has modulus 1), then $\overline{x}$ may be stable, asymptotically stable, or unstable.

We consider a non linear system of the form

$$x_{t+1} = f(x_t, y_t),$$

$$y_{t+1} = g(x_t, y_t),$$

where $f : \Re^2 \rightarrow \Re$ and $g : \Re^2 \rightarrow \Re$ are continuously differentiable. We suppose that $s = (x, y)$ is a steady state of the system and that $f_x, f_y, g_x, g_y$ are the partial derivatives of $f$ and $g$ at steady state $s$, and we write the *Jacobian matrix* $J$ at $s$:

$$J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}.$$

The *characteristic polynomial* $p(\lambda)$ is

$$p(\lambda) = |J - \lambda I| = \det \begin{vmatrix} f_x - \lambda & f_y \\ g_x & g_y - \lambda \end{vmatrix}$$

$$= (f_x - \lambda)(g_y - \lambda) - f_y g_x$$

$$= \lambda^2 - (f_x + g_y)\lambda + f_x g_y - f_y g_x$$

$$= \lambda^2 - (trJ)\lambda + detJ = 0.$$

The steady state $s$ is said

- a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;
- a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$;

- a *saddle* if ($|\lambda_1| > 1$ and $|\lambda_2| < 1$) or ($|\lambda_1| < 1$ and $|\lambda_2| > 1$),

where $|\lambda_i|$ $(i = 1, 2)$ denotes a *modulus* of $\lambda_i$.

In order to interpret the eigenvalues from a geometric viewpoint, we introduce the $TD$-plane, where $T = trJ$ is the horizontal axis and $D = detJ$ is the vertical axis. We indicate the discriminant of $p(\lambda) = 0$ with $\Delta = T^2 - 4D$.

We observe that the eigenvalues are real if $\Delta \geq 0$ and are complex conjugate if $\Delta < 0$ . In the $TD$-plane the curve $\Gamma : \ \Delta = T^2 - 4D = 0$, that is $D = \frac{1}{4}T^2$, represents a parabola. Then the real and distinct eigenvalues, the real and repeated eigenvalues, and the complex conjugate eigenvalues are respectively below, on and above $\Gamma$ (See **Figure 1.20**).

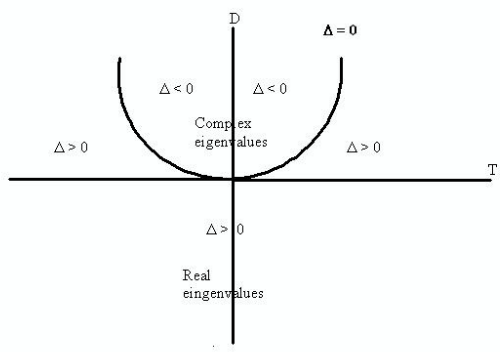


Figure 1.20: *The parabola* $\Gamma : T^2 - 4D = 0$ *in the TD-plane*

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of Jacobian matrix $J$. Then we can consider $p(\lambda)$ as a product of two linear factors, that is $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$, and we evaluate $p(\lambda)$ at a constant $c$.

We may observe that $p(c) = (c-\lambda_1)(c-\lambda_2) > 0$ if and only if the factors $(c-\lambda_i)$ $(i = 1, 2)$ have the same sign: thus *the eigenvalues fall on the same side of* $c$. In particular if $p(1) > 0$ (resp. $p(-1) > 0$) implies that the eigenvalues are at the same side of 1 (resp. $-1$) on the real line.

We will draw the hyperplanes $p(1) = 0$ and $p(-1) = 0$ into $TD$-plane (See **Figure 1.21** and **Figure 1.22**). Recalling that $p(\lambda) = \lambda^2 - (Trace)\lambda + detJ$, we obtain that $p(1) = 0$ and only if $1 - T + D = 0$ and $p(-1) = 0$ if and only if $1 + T + D = 0$. The line $p(1)$ goes through the points $(0, -1)$ and $(1, 0)$, instead the line $p(-1) = 0$ goes through the points $(-1, 0)$ and $(0, -1)$ and $p(1)$ is perpendicular to $p(-1)$.

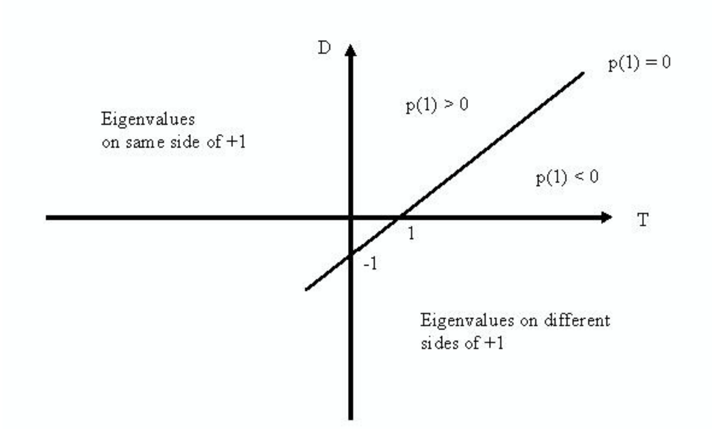


Figure 1.21: *The Hyperplane* $p(1)$

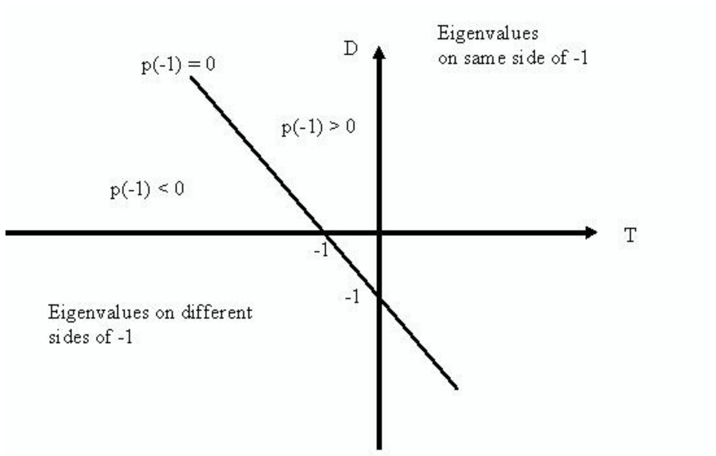


Figure 1.22: *The Hyperplane* $p(-1)$

The four lines $p(1) = 0$, $p(-1) = 0$, $D = \frac{1}{4}T^2$ and the horizontal segment given by pairs $(T, D)$ such that $-2 \leq T \leq 2$ and $det = 1$, divide the $TD$-plane into eight regions.

We can now identify the stability type of the steady-state. The regions that correspond to real eigenvalues are $1, 2, 3, 4, 7, 8$; instead the regions that correspond to the complex eingenvalues are $5, 6$. We obtain that (See **Figure 1.23**):

- **Region 1** Since $p(1) < 0$ and $p(-1) > 0$, the eigenvalues fall on the same side of $-1$ and on different sides of 1. Hence $-1 < \lambda_1 < 1$ and $\lambda_2 > 2$ and *the steady-state is a saddle-point.*

- **Region 2** Since $p(1) < 0$ and $p(-1) < 0$, the eigenvalues lie on different sides of $-1$ and $+1$. Hence $\lambda_1 < -1$ and $\lambda_2 > 1$ and *the steady-state is a source.*

- **Region 3** Since $p(1) > 0$ and $p(-1) < 0$, the eigenvalues fall on the same side of $+1$ and on the different sides of $-1$. Hence $\lambda_1 < -1$ and $-1 < \lambda_2 < 1$ and *the steady-state is a saddle-point.*

- **Region 4** Since $p(1) > 0$ and $p(-1) > 0$, then the eigenvalues lie on the same side of $+1$ and $-1$. The pairs $(T, D)$ that belong to Region 4 are such that $det > 1$ and $T < -2$, therefore both eigenvalues are negative and are smaller than $-1$. Thus *the steady state is a source.*

- **Region 5** The eigenvalues that fall in the regions 5 and 6 are complex conjugate, i.e. $\lambda_1 = \alpha + i\mu$ and $\lambda_2 = \alpha - i\mu$, from which $Trace = 2\alpha$ and $detJ = \alpha^2 + \mu^2 = |\lambda_1|^2 = |\lambda_2|^2$. For all pairs $(T, D)$ belongs to Region 5 we have $D > 1$, therefore $|\lambda_1| > 1$ and $|\lambda_2| > 1$ and *the steady-state is a source.*

- **Region 6** Instead, since in the Region 6, $0 < D < 1$, we obtain that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and *the steady-state is a sink.*

- **Region 7** Since $p(1) > 0$ and $p(-1) > 0$ we deduce that the eigenvalues are on the same side of $+1$ and $-1$. But in the Region 7 for all pair $(T, D)$ we have $-2 < T < 2$ and $-1 < det < 1$. Thus $-1 < \lambda_i < 1$ $(i = 1, 2)$ and *the steady-state is a source.*

- **Region 8** From conditions $p(1) > 0$ and $p(-1) > 0$ we derive that the eigenvalues are on the same side of $+1$ and $-1$. Since $det > 1$ and $trace > 2$ then $\lambda_i$ are both positive and greater than 1. Thus *the steady-state is a source.*

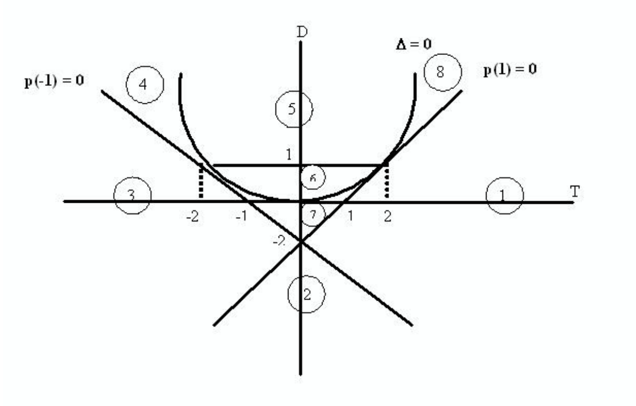


Figure 1.23: *The Triangle of Stability*

## 1.11 Planar Systems: Stability Triangle and Bifurcations

### 1.11.1 The Implicit Function Theorem and Bifurcations

Following Azariadis (1993) and de la Fuente (2000), let $F \in C^1$, and $F : \Re^2 \to \Re$. We consider the system constituted by a single equation in one unknown $x$ and one parameter $\alpha$:

$F(x; \alpha) = 0$ (1)

We observe that the graph of $F$ is a three-dimensional surface in the space $(x; \alpha; z)$ and the solutions $(x; \alpha)$ of (1) correspond to the intersection of the surface with the horizontal plane $x\alpha$. The set of pairs that satisfies (1) is called *zero level set of* $F$ and it describes a planar curve if $F$ has certain regularity properties.

In general, given a value of parameter $\alpha$, we do not interpret (1) as a graph of function $x(\alpha)$.

Now we study the zero-level set from a different point of view, fixing the value of $\alpha$ at $\alpha^0$ and plotting $F(x; \alpha^0)$ as function only of $x$. We may image that $F(x; \alpha^0)$ shows two types of behavior: it crosses the axis *tranversally* and the equilibrium is locally unique (*regular equilibria*) or it is only *tangent* to it (*critical equilibria*).

In the former case $\frac{\partial F(x;\alpha^0)}{\partial x}$ is positive or negative and *the equilibria are preserved under small perturbations*; in the latter case $\frac{\partial F(x;\alpha^0)}{\partial x} = 0$ and *the equilibria will be fragile, that is they tend to disappear or infold into two different equilibria.*

**Definition** When small perturbations lead to qualitative changes to dynamic behavior of the system we say that a *bifurcation* has occurred.

The Implicit Function Theorem guarantees the existence of a isolated equilibrium that "smoothly changes" if we little perturb the parameter $\alpha$ of a hyperbolic dynamical system, and it, in the simplest case, is usually stated in the following way (See **Figure 1.24**):

**Theorem** (Implicit Function Theorem) Let $F : \Re^2 \to \Re$ be and suppose that $F$ is a $C^1$ map on an open neighborhood $A$ of a point $(x^0, \alpha^0)$ such that $F(x^0, \alpha^0) = 0$ and $F_x(x^0, \alpha^0) \neq 0$. Then exist open intervals $I_x$ and $I_\alpha$ centered at $x^0$ and $\alpha^0$, respectively, such that the following hold:

(a) For all $\alpha \in I_\alpha$, there exists a unique $x_\alpha \in I_x$ such that $F(x_\alpha, \alpha)$. That is, the restriction of the zero-level curve of $F$ to the rectangle $I_x \times I_\alpha$ defines a function $x^\star : I_x \to I_\alpha$ with $x^\star(\alpha) = x_\alpha$; (b)$x^\star$ is differentiable in $I_\alpha$, and its derivative is a continuous function given by

$x^{\star\prime} = -\frac{F_\alpha(x,\alpha)}{F_x(x,\alpha)}$.

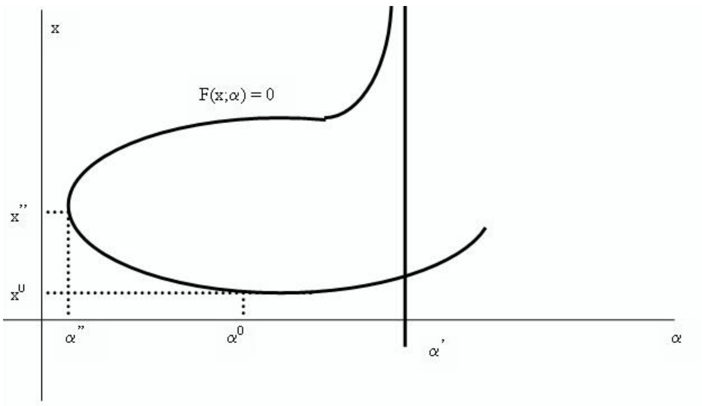


Figure 1.24: *Implicit Function Theorem*

Usually (Medio-Lines, 2001) we denote the domain of state variable $x$ and the set values of parameter $\alpha$ with $X$ and $\Omega$ respectively. Moreover $X$ and $\Omega$ are referred as *state space* and *parameter space*. $X$ is also known as *phase space* or, sometimes, *configuration space*, and we appeal the system $F$ as *parametrized*. The Implicit Function Theorem still holds if set $X = \Re^n$ and $\Omega = \Re^m$ and we replace the condition $F_x(x^0, \alpha^0) \neq 0$ with $|DF_x(x^0, \alpha^0)| \neq 0$, where $DF_x(x^0, \alpha^0)$ is the Jacobian matrix of $F$ at $(x^0, \alpha^0)$.

If we restate the *Implicit Function Theorem* from the bifurcation point of view we can say that:

**Theorem** (Implicit Function Theorem and Bifurcations) Given the system $F(x; \alpha) = 0$, we consider the pair $(x^c, \alpha^c)$ such that $F(x^c, \alpha^c) = 0$. A necessary condition for $(x^c, \alpha^c)$ to be a bifurcation point for $F(x; \alpha) = 0$, at which at least one steady state appears or disappears, is that $x^c$ is a critical point of the function $f(x) = F(x; \alpha^c)$ (See **Figure 1.25**).

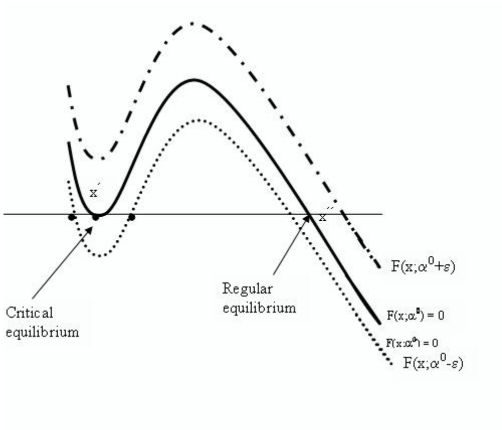


Figure 1.25: *Implicit Theorem and Small Perturbations*

In order to find the bifurcations we can proceed as follows. Before we consider the set of equilibria $M = \{(x, \alpha) \in X \times \Omega | F(x, \alpha) = 0\}$, and we define the singularity set of the system as $S = \{(x, \alpha) \in M || D_x F(x, \alpha)| = 0\}$. After, we eliminate the state variables from the equations $F(x, \alpha) = 0$ and $|D_x F(x, \alpha)| = 0$, that is, geometrically, we project $S$ onto the parameter space $\Omega$, and we obtain the bifurcation set $B = \{\alpha \in \Omega | (x, \alpha) \in S, x \in X\}$.

Let a discrete family of dynamical system

$x_{t+1} = F(x_t, \alpha)$ $(F : X \times \Omega \rightarrow X,\ F \in C^1)$ (2)

We observe that the equilibria of (2) are solutions of the system

$G(x_t;\alpha) = F(x_t;\alpha) - Ix_{t+1} = 0$, where $I$ is an identity matrix. Suppose that $(x^0;\alpha^0)$ is a solution of (2).

If we indicate with $\lambda_f$ and $\lambda_g$ respectively an eigenvalues of Jacobian $D_xF(x^0;\alpha^0)$ and the Jacobian $D_xG(x^0;\alpha^0)$, we observe that they are related by $\lambda_g = \lambda_f - 1$.

Moreover $|D_xG(x^0;\alpha^0)| = \Pi_i\lambda_g^i = \Pi_i(\lambda_f^i - 1)$.

Thus if $\lambda_f^i$ are real, we can say that $DG_x$ *vanishes and the implicit-function theorem fails only if at least one of the eigenvalues of* $F$ *is one.*

We suppose that some eigenvalues are complex, for example let $\lambda_f^1 = a + jb$ be and let$\lambda_f^2 = a - jb$ be, where $j = \sqrt{-1}$, and consider $\lambda_f^i \in \Re$ for $i = 3, 4, \ldots$. We have

$|D_xG(x^0;\alpha^0)| = [(a-1)+jb][(a-1)-jb]\Pi_{i=3,\ldots,n}(\lambda_f^i - 1)$.

We note that $D_xG$ vanishes for ($a = 1$ and $b = 0$) or $|\lambda_f^i| = 1$ ($j = 3, \ldots, n$).

Following Hirsch, Smale, Devaney (2004), in continuous time we prove:

**Theorem** (Saddle-Node Bifurcations) Suppose $x^{'} = f_a(x)$ is a first-order differential equation for which

1. $f_{a_0}(x_0) = 0$;
2. $f_{a_0}^{'}(x_0) = 0$;
3. $f_{a_0}^{''}(x_0) \neq 0$;
4. $\frac{\partial f_{a_0}}{\partial a} \neq 0$.

Then the differential equation undergoes a saddle-node bifurcation at $a = a_0$.

**Proof** Let $G(x, a) = f_a(x)$. We have

$$G(x_0, a_0) = 0,\ \frac{\partial G}{\partial a}(x_0, a_0) = \frac{\partial f_{a_0}}{\partial a}(x_0) \neq 0.$$

Thus, applying the implicit-function theorem, we can say that there is a function $a = a(x)$ such that $G(x, a(x)) = 0$. In particular if $x^\star$ falls into domain of $a(x)$, then $f_{a(x^\star)}(x^\star) = 0$, from which $x^\star$ is an equilibrium for $x^{'} = f_{a(x^\star)}(x)$. If we differentiate $G(x, a)$ with respect to $x$, we have $a^{'}(x) = -\frac{\partial G/\partial x}{\partial G/\partial a}$. From the assumptions 2. and 4. we deduce that $a^{'}(x) = 0$. Since

$$a^{''}(x) = \frac{-\frac{\partial^2 G}{\partial x^2}\frac{\partial G}{\partial a} + \frac{\partial G}{\partial x}\frac{\partial^2 G}{\partial a^2}}{(\frac{\partial G}{\partial a})^2},$$

using the assumptions 2. and 3. we derive

$$a^{''}(x_0) = \frac{\frac{\partial^2 G}{\partial x^2}(x_0, a_0)}{\frac{\partial G}{\partial a}(x_0, a_0)} \neq 0.$$

We conclude that

- the graph of $a = a(x)$ is either concave up or concave down;
- there are two equilibria near $x_0$ for $a$-values on one side of $a_0$ and there aren't equilibria for $a$-values on the other side.

### 1.11.2 Local bifurcations for discrete and nonlinear maps

To describe the local bifurcations we will follow A.Medio and M.Lines (2001). We recall that a fixed point loses the hyperbolicity if it happens that the Jacobian matrix calculated at the fixed point

(i) has one real eigenvalue equal to one;

(ii) or the eigenvalue is equal to minus one;

(iii) or the pair of complex conjugate eigenvalues have modulus equal to one.

We observe that the *centre manifold theorem* allows to reduce the dimensionality to a one-dimensional map in cases (i) and (ii) and to a two-dimensional map in case (iii).

**Case (i)** We distinguish three types of local bifurcation: fold, transcritical and pitchfork (supercritical or subcritical).

Let $x_{n+1} = G(x_n; \mu)$ be a general one-dimensional family of map, where $x_n \in \Re$ and $\mu \in \Re$, and, if $\mu_c$ indicates a value of controlling parameter, let $\overline{x}(\mu_c)$ be a corresponding equilibrium value.

To detect the local bifurcations we use the following conditions:

- $\frac{\partial G(\overline{x};\mu_c)}{\partial x_n} = 1$ simultaneously for fold, transcritical and pitchfork;
- $\frac{\partial^2 G(\overline{x};\mu_c)}{\partial x_n^2} \neq 0$ simultaneously for fold and transcritical;
- $\frac{\partial^3 G(\overline{x};\mu_c)}{\partial x_n^3} = 0$ and $\frac{\partial^3 G(\overline{x};\mu_c)}{\partial x_n^3} \neq 0$ for pitchfork;
- $\frac{\partial G(\overline{x};\mu_c)}{\partial \mu} \neq 0$ for fold;
- $\frac{\partial G(\overline{x};\mu_c)}{\partial \mu} = 0$ and $\frac{\partial^2 G(\overline{x};\mu_c)}{\partial \mu \partial x_n} \neq 0$ simultaneously for transcritical and pitchfork.

We consider now some *prototypes* of bifurcations:

(A) $x_{n+1} = G(x_n; \mu) = \mu - x_n^2$: fold;

(B) $x_{n+1} = G(x_n; \mu) = \mu x_n - x_n^2$: transcritical;

(C) $x_{n+1} = G(x_n; \mu) = \mu x_n - x_n^3$: pitchfork.

**Prototype (A)** To find the equilibria we impose $\overline{x}_{n+1} = \overline{x}_n = \overline{x}$. We have $\overline{x} = \mu - \overline{x}^2$, that is $\overline{x}^2 + \overline{x} - \mu = 0$, and, if $\mu > -1/4$, we derive two real solutions

$\overline{x}_{1,2} = \frac{1}{2}(-1 \pm \sqrt{1 + 4\mu})$. We note that

- if $\mu > 0$ the solutions are nonzero and opposite sign;
- if $-1/4 < \mu < 0$ the solutions are both negative;
- if $\mu = 0$ we obtain that $\overline{x}_1 = -1$ and $\overline{x}_2 = 0$;
- if $\mu < -1/4$ there are not real solutions.

Moreover if $\mu = -1/4$ occurs that the two equilibria coalesce and become equal to $-1/2$. Instead, when $\mu$ decreases further, the equilibria disappear.

We consider the fixed point $(\overline{x};\mu_c) = (-1/2;-1/4)$ be. *We can say that it is a fold.* As a matter of fact $\partial G/\partial x_n = 1 > 0$ (equilibrium nonhyperbolic), $\partial^2 G/\partial x_n^2 = -2 \neq 0$, $\partial G/\partial \mu = 1 \neq 0$.

**Prototype (B)** From equation $\overline{x}^2 + \mu\overline{x} = 0$ we deduce the existence of two equilibria: $\overline{x}_0 = 0$ and $\overline{x}_1 = \mu - 1$. Since $|\partial G/\partial x_n| = |\mu - x_n|$ we deduce that

- if $-1 < \mu < 1$ then $\overline{x}_0 = 0$ is stable and $\overline{x}_1 < 0$ is unstable;
- if $1 < \mu < 3$ then $\overline{x}_0$ is unstable and $\overline{x}_1 > 0$ is stable.

We observe that if $\mu = 1$ then the equilibria coalesce and $\overline{x}_0 = \overline{x}_1 = 0$. Let $(\overline{x};\mu_c) = (0;1)$. We have $\partial G/\partial x_n = 1$ (nonhyperbolic equilibrium); $\partial^2/\partial x_n^2 = -2 \neq 0$; $\partial G/\partial \mu = x_n = 0$ and $\partial^2 G/\partial \mu \partial x_n = 1$. Thus $(0;1)$ *is a transcritical bifurcation.*

**Prototype (C)** We solve the equation $\overline{x} = \mu\overline{x} - \overline{x}^3$. It is equivalent to $\overline{x}^3 - (\mu - 1)\overline{x} = \overline{x}[\overline{x}^2 - (\mu - 1)] = 0$. We observe that $\overline{x}_1 = 0$ is an equilibrium for all real $\mu$ and if the condition $\mu > 1$ holds there are further two equilibria at $\overline{x}_{2,3} = \pm\sqrt{\mu - 1}$. We have $|\partial G/\partial x_n| = |\mu - 3x_n^2|$. Thus $\overline{x}_1$ is stable if $(-1 < \mu < 1)$ and unstable otherwise. Instead the branches of $\overline{x}_{2,3}$ are stable if $(1 < \mu < 2)$. We note that at $\mu = 1$ the three equilibria coalesce.

Let $(\overline{x};\mu_c) = (0,1)$. Then $(\overline{x};\mu_c)x_n = 1$, $\partial G/\partial \mu = 0$ and $\partial^2 G/\partial \mu \partial x_n = 1$. Moreover $\partial G^2/\partial x_n^2 = -6\overline{x} = 0$; $\partial G^3/\partial x_n^3 = -6$. Thus *the fixed point is a (supercritical) pitchfork bifurcation.*

**Case (ii)** A prototype of flip bifurcation is given by *the family of logistic maps* $x_{n+1} = G(x_n) = \mu x_n(1 - x_n)$, $x \in \Re$, $\mu \in \Re$. From equation $\mu\overline{x}^2 - (\mu - 1)\overline{x} = 0$ we derive two equilibria: $\overline{x}_1 = 0$ and $\overline{x}_2 = 1 - (1/\mu)$ $(\mu \neq 0)$. We find that $|\partial G/\partial x_n| = |\mu(1 - 2x_n)|$. Because $|\partial G(\overline{x}_1;\mu)/\partial x_n| = |\mu(1 - 2\overline{x}_1)| = |\mu|$ and $|\partial G(\overline{x}_2;\mu)/\partial x_n| = |\mu(1-2\overline{x}_2)| = |\mu(1-2(1-1/\mu))| = |2-\mu| = |\mu-2|$, then $\overline{x}_1$ is stable if $1 < \mu < 1$ and $\overline{x}_2$ is stable if $1 < \mu < 3$. Let $(\overline{x};\mu_c) = (0;1)$. Because at $(0;1)$ we obtain that $\partial^2 G/\partial x_n^2 = -1 \neq 0$, $\partial G/\partial \mu = 0$ and $\partial G^2/\partial \mu \partial x_n = 1 \neq 0$, we can say that $(0;1)$ *is a transcritical bifurcation.*

Moreover from $\mu = 3$ we have $\overline{x}_2 = 1 - (1/3) = 2/3$ and we observe that at $(\overline{x}_2;3)$ the eigenvalue $\partial G(2/3;3)/\partial x_n = 3(1 - 2(2/3)) = -1$. Thus $(2/3;3)$ is nonhyperbolic.

Even if the Hartman-Grobman theorem is not true because $|\partial G(2/3;3)/\partial x_n| = 1$, however now we approach linearly $G$ around $\overline{x}_2$. We obtain $G(x_n) = G(\overline{x}_2)+$

$G^{'}(\overline{x}_2)(x_n - \overline{x}_2)$. Since $x_{n+1} = G(x_n)$, $G(\overline{x}_2) = \overline{x}_2$ and $G^{'}(\overline{x}_2) = -1$ we derive $x_{n+1} - \overline{x}_2 = -(x_n - \overline{x}_2)$. If we set $\xi_n = x_n - \overline{x}_2$ for all $n$, we can rewrite the previous relation such that : $\xi_{n+1} = -\xi_n$. Given the initial value $\xi_0$, the one-dynamical system $\{\xi_0, -\xi_0, \xi_0, -\xi_0, \ldots\}$ is equal to the period-2 cycle $\{\xi_0, -\xi_0\}$.

Alternatively we solve the equation $\overline{x} = G(G(\overline{x}))$, that is $\overline{x} = \mu G(\overline{x})(1 - G(\mu\overline{x}(1-\overline{x}))) = \mu(\mu\overline{x}(1-\overline{x}))(1-(\mu\overline{x}(1-\overline{x})))$, or

$$\mu^3\overline{x}^4 - 2\mu^3\overline{x}^3 + \mu^2(1+\mu)\overline{x}^2 + (1-\mu^2)\overline{x} = 0.$$

Solving the last equation we find $\overline{x}_1 = 0$,$\overline{x}_2 = 1-(1/\mu)$, and $\overline{x}_{3,4} = \frac{(1+\mu)\pm\sqrt{\mu^2-2\mu-3}}{2\mu}$ for ($\mu \leq -1$ and) $\mu \geq 3$.

**Remark** We have that

$$\overline{x}_3 + \overline{x}_4 = \frac{2(1+\mu)}{2\mu} = \frac{1+\mu}{\mu};$$

$$\overline{x}_3\overline{x}_4 = \frac{(1+\mu)^2-(\mu^2-2\mu-3)}{4\mu^2} = \frac{1+2\mu+\mu^2-\mu^2+2\mu+3}{4\mu^2} = \frac{4(1+\mu)}{4\mu^2} = \frac{1+\mu}{\mu^2}.$$

We note that $G(\overline{x}_3;\mu) = \overline{x}_4$ and $G(\overline{x}_4;\mu) = \overline{x}_3$. We will verify only the first.

We have $G(\overline{x}_3;\mu) = \mu\overline{x}_3(1-\overline{x}_3)$

$$= \mu\frac{(1+\mu)+\sqrt{\mu^2-2\mu-3}}{2\mu}(1 - \frac{(1+\mu)+\sqrt{\mu^2-2\mu-3}}{2\mu})$$

$$= \frac{(1+\mu)+\sqrt{\mu^2-2\mu-3}}{2}\ \frac{(\mu-1)-\sqrt{\mu^2-2\mu-3}}{2\mu} = \frac{\mu^2-1-(\mu^2-2\mu-3)-2\sqrt{\mu^2-2\mu-3}}{4\mu}$$

$$= \frac{(2\mu+2)-2\sqrt{\mu^2-2\mu-3}}{4\mu} = \overline{x}_4.$$

Thus the set $\{\overline{x}_3, \overline{x}_4\}$ is a period-2 cycle for $G$.

We recall that the *Chain Rule* states that $(g\circ f)^{'}(p) = g^{'}(f(p))f^{'}(p)$ for all $f$ and $g$ differentiable in $g(p)$ and $p$ respectively, where $p$ belongs to an interval $X$ and $g(p) \in X$.

Then for $f = g$, we deduce that $(g^2)^{'}(p) = (g\circ g)^{'}(p) = g^{'}(g(p))g^{'}(p)$. But if $p = p_1$, $g(p_1) = p_2$ and $g(p_2) = p_1$ then $(g\circ g)^{'}(p_1) = g^{'}(g(p_1))g^{'}(p_1) = g^{'}(p_2)g^{'}(p_1)$. Obviously $(g\circ g)^{'}(p_2) = (g\circ g)^{'}(p_1)$.

Thus

$$\frac{\partial G^2(\overline{x}_3;\mu)}{\partial x_n} = \frac{\partial G^2(\overline{x}_4;\mu)}{\partial x_n} = \frac{\partial G(\overline{x}_4;\mu)}{\partial x_n}\frac{\partial G(\overline{x}_3;\mu)}{\partial x_n} = \mu(1-2\overline{x}_4)\mu(1-2\overline{x}_3)$$

$= \mu^2[1 + 4\overline{x}_3\overline{x}_4 - 2(\overline{x}_3 + \overline{x}_4)]$

$= \mu^2(1 + 4\frac{1+\mu}{\mu^2} - 2\frac{1+\mu}{\mu})$

$= -\mu^2 + 2\mu + 4.$

We observe that *at* $\mu = 3$ *and for* $\mu$ *slightly larger of three the equilibria* $\overline{x}_{3,4}$ *are stable for* $G^2$.

As a matter of fact, evaluating $(-\mu^2 + 2\mu + 4)$ at $\mu = 1$ the eigenvalues $\frac{\partial G^2(\overline{x}_3;\mu)}{\partial x_n} = \frac{\partial G^2(\overline{x}_4;\mu)}{\partial x_n}$ are equal to one and they are minus one for $\mu$ slightly larger of three.

**Definition** We call flip bifurcation a fixed point for $G$ such that

- its eigenvalue goes through minus one;
- the nonzero equilibrium loses the stability;
- a stable period 2-cycle appears.

The conditions for a flip bifurcation to occur are:

$(F1)$ $\frac{\partial G(\overline{x};\mu_c)}{\partial x_n} = -1;$

$(F2)$ $\frac{\partial^2 G^2(\overline{x};\mu_c)}{\partial x_n^2} = 0$ and $\frac{\partial^3 G^2(\overline{x};\mu_c)}{\partial x_n^3} \neq 0;$

$(F3)$ $\frac{\partial^2 G^2(\overline{x};\mu_c)}{\partial \mu} = 0$ and $\frac{\partial^2 G^2(\overline{x};\mu_c)}{\partial \mu \partial x_n} \neq 0.$

We observe that *the flip bifurcation for the map* $G$ *corresponds to a pitchfork bifurcation for the map* $G^2$.

At $\mu = 1 + \sqrt{6}$ the period-2 cycle for $G$ loses the stability and a new flip bifurcation appears. Moreover initially will have a new stable period-2 cycle for $G^2$ that corresponds to a new stable period-4 cycle for $G$. By increasing $\mu$ this *period-doubling* scenario continues.

**Case (iii)** Neimark (1959) and Sacker (1965) stated relevant results about the case in which a pair of complex eigenvalues of the Jacobian matrix at the fixed point of a discrete map has modulus one.

**Definition** In a planar and discrete system, we say that a saddle-node, a flip-bifurcation, a Neimark-Saker bifurcation occur respectively if one of eigenvalue is unity and the other is less than unity in absolute value, if one of eigenvalue is equal to $-1$ and the other is less than unity in absolute value and if the eigenvalues are complex conjugates and both are equal to unity in absolute value.

We enunciate the following

**Theorem** (Neimark-Sacker) Let $G_p : \Re^2 \to \Re^2$ be a family of maps of a class $C^k$, $k \geq 5$, depending on a real parameter $\mu$, so that for $\mu$ near 0, $x = 0$ is a fixed point of $G_p$ and the following conditions are satisfied

(i) for $\mu$ near zero, the Jacobian matrix has two complex, conjugate eigenvalues $k(\mu)$ and $\overline{k}(\mu)$ with $|k(0)| = 1$ ;

(ii) $\frac{|k(\mu)|}{d\mu} \neq 0$;

(iii) $[k(\mu)]^i \neq 1$, for $i = 1, 2, 3, 4$.

Then, after a trivial change of the $\mu$ coordinate and a smooth, $\mu$-dependent coordinate change on $\Re^2$,

(i) the map $G_p$ in polar coordinates takes the form:
$$\begin{pmatrix} r_{n+1} \\ \phi_{n+1} \end{pmatrix} = \begin{pmatrix} (1+r)r_n - \alpha(\mu)r_n^3 \\ \phi_n + \beta(\mu) + \gamma(\mu)r_n^2 \end{pmatrix} + O\left(\det \begin{vmatrix} r_n \\ \phi \end{vmatrix}^6\right)$$
where $\alpha, \beta, \gamma$ are smooth functions of $\mu$ and $\alpha(0) \neq 0$;

(ii) for $\alpha > 0$ (respectively, for $\alpha < 0$) and in a sufficiently small right (left) neighborhood of $\mu = 0$, for the map $G_p$ there exists an invariant attractive (repelling) circle $\Gamma_\mu$ bifurcating from the fixed point at $\overline{x} = 0$ and enclosing it.

### 1.11.3 Stability triangle and bifurcations

**Proposition** Into $TD$-plane we consider the stability triangle $ABC$, where $A(0,1), B(1,1), C(-1,0)$ (See **Figure 1.26**). We have that

1. the saddle-node bifurcation occurs on line segment $BC$;
2. the flip-bifurcation occurs on line segment $AC$;
3. the Neimark-Saker bifurcation occurs on line segment $AB$.

*Proof*

1. Let $\lambda_1 = 1$ be and let and $|\lambda_2| < 1$ be. If we evaluate the characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ at $\lambda = 1$ we derive

   $p(1) = (1 - \lambda_1)(1 - \lambda_2) = (1 - 1)(1 - \lambda_2) = 0.$

   Moreover $-1 < \lambda_2 < 1 \Rightarrow \lambda_1 - 1 < \lambda_1 + \lambda_2 < \lambda_1 + 1 \Rightarrow 1 - 1 < \lambda_1 + \lambda_2 < 1 + 1$ $\Rightarrow 0 < Trace < 2$, and $|det| = |\lambda_1||\lambda_2| = |1||\lambda_2| < 1 \Rightarrow -1 < det < 1$.

2. Let $\lambda_1 = -1$ be and let and $|\lambda_2| < 1$ be. As above, $p(-1) = (-1 - \lambda_1)(-1 - \lambda_2) = (-1 + 1)(-1 - \lambda_2) = 0$. Further $-1 < \lambda_2 < 1 \Rightarrow \lambda_1 - 1 < \lambda_1 + \lambda_2 < \lambda_1 + 1 \Rightarrow -1 - 1 < Trace < -1 + 1 \Rightarrow -2 < Trace < 0$, and $|det| = |\lambda_1||\lambda_2| = |-1||\lambda_2| < 1 \Rightarrow -1 < det < 1$.

3. Let $\lambda_1 = a + ib$ be, $\lambda_2 = a - ib$ be, with $|\lambda_1| = |\lambda_2| = 1$. We observe that $det = \lambda_1 \lambda_2 = a^2 + b^2 = |\lambda|^2 = 1$. Then $|a| < 1$. As a matter of fact, if $a > 1$ then $a^2 > 1$, from which $a^2 + b^2 > 1 + b^2 > 1$ and if $a < -1$ we have also $a^2 < 1$, therefore $a^2 + b^2 > 1$. Thus, since $Trace = \lambda_1 + \lambda_1 = 2a$ we obtain $-2 < Trace < 2$.

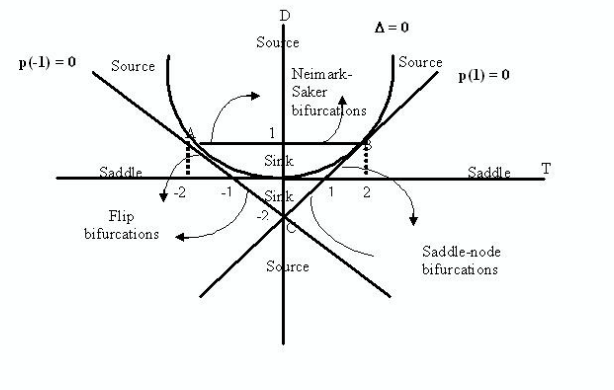


Figure 1.26: *The Triangle of Stability and Bifurcations*

## 1.12 The Lyapunov Characteristic Exponents

### 1.12.1 The Sensitive Dependence On Initial Conditions

Following Martelli, Dang and Seph (1998), we notice that many scientists non-mathematician consider *chaotic* a dynamical system when it shows *a sensitive dependence on initial conditions*. We recall that a discrete dynamical system $x_{n+1} = F(x_n)$ has a sensitive dependence on initial conditions (SDIC) if there exists $r_0 > 0$ such that for every $\delta > 0$ we can find $y_0 \in X$ and $n \geq 1$ satisfying the property that $d(x_0, y_0) < \delta$ and $d(x_n, y_n) > r_0$, where $(X, d)$ is a metric space, $X \subseteq \Re^q$, $F : X \to X$ is a continuous map on $X$. For example (See **Figure 1.27**), if we pose $X = [0, 1]$, $F(x) = 4x(1-x)$, $x_0 = 0.3$, $y_0 = 0.300001$, and we plot the points $(n; |x_n - y_n|)$ $(n = 0, 1, ..., 100)$, we observe that the two sequences $x_n$ and $y_n$ of iterates

- are very close for $n = 0, 1, \ldots, 15$;
- they separate for all almost $n > 15$;
- sometimes they become very close, for example for $n = 45$ and $n = 60$.

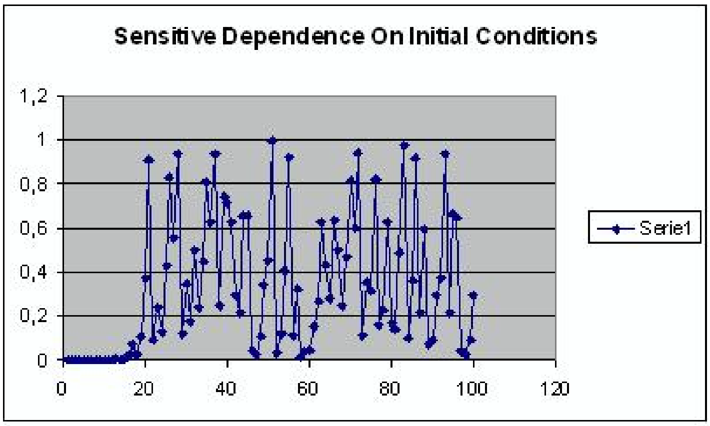


Figure 1.27: *Sensitive Dependence on Initial Conditions*

Thus, for the experimentalists, *the divergence* in different directions of the orbits $O(x_0)$ and $O(y_0)$ is the hallmark of the sensitivity of the dynamical system $x_{n+1} = F(x_n)$ to small changes and *the impossibility* to know exactly the initial states $x_0$ and $y_0$ in the experimental sciences because they are affected by measurement errors, lead to conclude that the evolution of dynamical system is *unpredictable*.

### 1.12.2 The Lyapunov Characteristic Exponents In One Dimension

Following Medio and Lines (2001), to make more precise the notion of sensitive dependence on initial conditions we will use the concept of Lyapunov characteristic exponents (LCE) requiring that the divergence of nearby orbits occurs at an exponential rate.

Let $G : U \rightarrow E$ be a continuously differentiable map, where $U$ is an open subset of $\Re$, and we consider the dynamical system $x_{n+1} = G(x_n)$.

We define recursively the iterates of $G$ by $G^0(x) = x$, $G^1 = G$, $G^k = G \circ G^{k-1}$ for all $k > 1$.

We pose $x_1 = G(x_0)$, $x_2 = G(x_1)$, ..., $x_{k-1} = G(x_k)$, from which $G^{k-1}(x_0) = x_{k-1}$, $G^{k-2}(x_0) = x_{k-2}$, ... for all $k > 1$.

Let $G^{'}(x_i) \neq 0$ be for all $i = 0, 1, \ldots, k-1$.

By *the chain rule* $(g \circ f)^{'}(x^{\star}) = g^{'}(f(x^{\star}))f^{'}(x^{\star})$, we state that for all $k > 1$

$DG^k(x_0) = DG(G^{k-1}(x_0)) = G^{'}(G^{k-1}(x_0))DG^{k-1}(x_0)$

$= G^{'}(x_{k-1})DG^{k-1}(x_0)$

$= G^{'}(x_{k-1})DG(G^{k-2})(x_0)$

$= G^{'}(x_{k-1})G^{'}(G^{k-2})(x_0)DG^{k-2}(x_0)$

$= G^{'}(x_{k-1})G^{'}(x_{k-2})DG^{k-2}(x_0)$

...

$= G^{'}(x_{k-1})G^{'}(x_{k-2})\ldots G^{'}(x_0)$

$= G^{'}(x_0)G^{'}(x_1)\ldots G^{'}(x_{k-1})$.

We consider now in $U$ two nearby points $x_0$ and $\overline{x_0}$, where $x_0$ is a fixed point and $\overline{x_0}$ is a variable point, and we expand the nth iterate $G^n(\overline{x_0})$ in a Taylor series around $x_0$. We have

$G^n(\overline{x_0}) = G^n(x_0) + \frac{dG^n}{dx}|_{\overline{x_0}=x_0}(\overline{x_0} - x_0) + \ldots$

Stopping the Taylor's expansion at the first order-term and using the previous result for k = n we obtain that

$|\overline{x_n} - x_n| = |G^n(\overline{x_0}) - G^n(x_0)| \approx |\frac{dG^n}{dx}|_{\overline{x_0}=x_0}(\overline{x_0} - x_0)|$

$= |G^{'}(x_0)G^{'}(x_1)\ldots G^{'}(x_{n-1})||\overline{x_0} - x_0|$

$= \exp^{\ln|G^{'}(x_0)G^{'}(x_1)\ldots G^{'}(x_{n-1})|}|\overline{x_0} - x_0|$

$$= \exp^{\ln\left[|G'(x_0)G'(x_1)...G'(x_{n-1})|^{1/n}\right]^n} |(\overline{x_0} - x_0)|$$

$$= \exp^{n \ln |G'(x_0)G'(x_1)...G'(x_{n-1})|^{1/n}} |\overline{x_0} - x_0|$$

$$= \exp^{n\left[\frac{\ln |G'(x_0)|+\ln |G'(x_1)|+...+\ln |G'(x_{n-1})|}{n}\right]} |\overline{x_0} - x_0|.$$

We put $\lambda(x_0) = \lim_{n\to\infty} \frac{\ln |G'(x_0)|+\ln |G'(x_1)|+...+\ln |G'(x_{n-1})|}{n}$

and if the limit $\lambda(x_0)$ exists we call $\lambda(x_0)$ the *Lyapunov characteristic exponent (LCE).*

Then $\lim_{n\to\infty} |\overline{x_n} - x_n| \approx \exp^{n\lambda(x_0)} |\overline{x_0} - x_0|$.

Because LCE's are obtained around $x_0$ they represent a *local* average. Moreover taking $n \to \infty$ LCE's are an *asymptotic* rate of separation of orbits. Finally we denote LCE with the term *exponential rate* because the rate $|\frac{\overline{x_n}-x_n}{\overline{x_0}-x_0}|$ tends to $\exp^{n\lambda(x_0)}$.

The main features of LCE are:

- $\lambda(x_0) < 0$ if $|G'(x_0)G'(x_1)\ldots G'(x_{n-1})| < 1$, i.e., if the orbit of $x_0$ is *stable*;
- $\lambda(x_0) > 0$ if $|G'(x_0)G'(x_1)\ldots G'(x_{n-1})| > 1$, i.e., if the orbit of $x_0$ is *unstable*;
- $\lambda(x_0) = 0$ if $x_0$ *converges to a quasiperiodic orbit* or $x_0$ *converges to a periodic orbit which is nonhyperbolic.*

### 1.12.3 The Lyapunov Exponents In Two Dimensions

#### The Strain Ellipse

In order to extend to higher dimensions the concept of Lyapunov Exponent we need to introduce some preliminary notions of geometry and linear algebra. Following Lang (1966), we start from a very straightforward case.

In the plane we consider the unitary circle $S^1$ centered at the origin, i.e., the set of points $P = (x, y)$ such that $x^2 + y^2 = 1$ and the linear map $F : \Re^2 \to \Re^2$ defined by $F(x, y) = (ax, by)$, where $a$ and $b$ are fixed positive real constant. We put $u = ax$ and $v = by$ and we deduce that $F(S^1)$ is the set of points

$P' = F(P) = (u, v)$ such that $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$, i.e., the ellipse centered at origin and length of semi-axes equal respectively to $a$ and $b$ (See **Figure 1.28**).

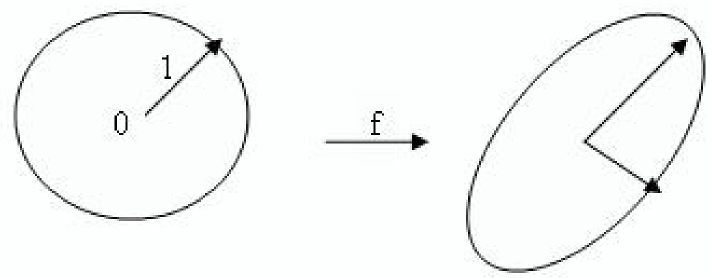


Figure 1.28: *The disk is mapped into an ellipse*

Obviously if $a = b$ then $F(S^1)$ is also a circle, i.e., an ellipse with axes of equal length. We note that

- if $a > 1$ and $b < 1$, *the ellipse grows along the x-axis and shrinks along the y-axis*;
- if $a < 1$ and $b > 1$, *the ellipse grows along the y-axis and shrinks along the x-axis*;
- if $a > 1$ and $b > 1$ (respectively, $a < 1$ and $b < 1$), *the ellipse grows (respectively, shrinks) along x-axis and y-axis.*

In *Geology* the ellipse obtained under the action of the $F$-deformation is sometimes called *strain.*

Now, following Alligood, Sauer and Yorke (1996), we translate the previous case into the language of linear algebra extending it to *nth* iteration map.

We recall that we say linear a map $T$ from $\Re^m$ to $\Re^m$ such that $T(av + by) = aT(v) + bT(w)$ for each scalar $a$, $b \in \Re$ and for each vector $v$, $w \in \Re^m$. Moreover we may view *every matrix* $A$ *on* $\Re^2$ *(or on* $\Re^m$*) as a linear map* ; by definition $v \rightarrow Av$, i.e.

$$A(v) = Av = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix},$$

where $v = \begin{pmatrix} x \\ y \end{pmatrix}$ and $T = A$. We say that a scalar $\lambda$ is an *eigenvalue* of the matrix $A$ if there is a non-zero vector $v$ such that $Av = \lambda v$. We denote $v$ an *eigenvector*. We distinguish three cases.

- **Case I**: $A$ has distinct real eigenvalues. Let

  $$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

  Then $a$ and $b$ are the eigenvalues of $A$ and the correspondent eigenvectors are $(1, 0)$ and $(0, 1)$. Moreover if we consider $A^n$, i.e., the $n$-iterate of $A$, we have

  $$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}.$$

  Geometrically the product $A^n N$ maps a disk $N$ with radius one and centered at the origin into a strain ellipse with semi-major axes of length $|a|^n$ and $|b|^n$. If we deform the disk $N_\epsilon(0, 0)$ with radius $\epsilon > 0$ and center $(0, 0)$ we obtain a strain ellipse with semi-major axes of length $\epsilon|a|^n$ and $\epsilon|b|^n$. Then, as in the introductive example, when $n \to \infty$, the ellipse

  - *shrinks* toward the origin $(0, 0)$ if $|a| < 1$ and $|b| < 1$ and the origin is a *sink*;
  - *grows* along the axes if $|a| > 1$ and $|b| > 1$ and the origin is a *source*;
  - *grows* along the x-axis and shrinks along the y-axis if $|a| > 1$ and $|b| < 1$ and the origin is a *saddle*;
  - *shrinks* along the x-axis and grows along the y-axis if $|a| < 1$ and $|b| > 1$ and the origin is a *saddle*.

- **Case II**: $A$ has repeated real eigenvalues. Let

  $$A = \begin{pmatrix} a & n \\ 0 & a \end{pmatrix}.$$

  Then, by recurrence,

  $$A^n = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix}.$$

The ellipse (with axes of equal length) $AN$ shrinks toward the origin if $|a| < 1$ and $|b| < 1$, instead grows along the x-axis and the y-axis if $|a| > 1$ and $|b| > 1$. In the former case the origin is a sink, in the latter it is a source.

- **Case III**: $A$ has complex and conjugate eigenvalues. Let

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We pose $r = \sqrt{a^2 + b^2}$ and we observe that

$$A = r \begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

As a matter of fact the entries $a/r$ and $b/r$ satisfy the identity $c^2 + s^2 = 1$ because $\frac{a^2}{r^2} + \frac{b^2}{r^2} = \frac{a^2+b^2}{r^2} = 1$ and, from the relation $b/a = \tan\theta$, we deduce that $\theta = \arctan(b/a)$. Moreover $c = \cos\theta$ and $s = \sin\theta$. Thus $A$ is a dilatation followed by a rotation: the factor $\sqrt{a^2 + b^2}$ stretches or shrinks a vector and the other factor rotates a vector around the origin by an angle $\theta$ given by $\arctan(b/a)$.

From the characteristic equation $|A - \lambda I| = 0$, i.e., $(a - \lambda)^2 + b^2 = 0$, from which $a - \lambda = \pm ib$, where $i = \sqrt{-1}$. We have the eigenvalues $\lambda_1 = a - bi$ and $\lambda_2 = a + bi$.

Now, we will present a complete view on the main features of the ellipse $AN$ recalling basic results of Linear Algebra.

We note that being $(A^TA)^T = A^T(A^T)^T = A^TA$ for each matrix $A$ $m \times m$, where $A^T$ is the transpose of $A$, we can conclude that the product $A^TA$ is a symmetric matrix.

Moreover

**Lemma** Let $A$ be an $m \times m$ matrix. The eigenvalues of $A^TA$ are nonnegative.

**Proof** Let $v$ be a unit eigenvector of $A^TA$. Thus there is a scalar $\lambda$ (eigenvalue) such that $A^TAv = \lambda v$ with $|v| = 1$. We have

$0 \le |Av|^2 = v^TA^TAv = v^T\lambda v = \lambda$,

from which $\lambda \ge 0$.

The next result shows explicitly the link between the ellipse $AN$ and the matrix $A^TA$.

**Theorem 1** Consider a unit disk $N$ in $\Re^m$ and an $m\times m$ matrix $A$. Suppose that $s_1^2, s_2^2, \ldots, s_m^2$ and $u_1, u_2, \ldots, u_m$ are the eigenvalues and the unit eigenvectors, respectively, of the $m \times m$ matrix $A^T A$. Then

1. $u_1, u_2, \ldots, u_m$ are mutually orthogonal unit vectors;

2. the axes of ellipse $AN$ are $s_1 u_1, s_2 u_2, \ldots, s_m u_m$.

We apply the previous theorem to **Case I**. We recall that $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then, since $A^T = A$, we have $A^T A = A^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$. The eigenvalues of $A^2$ are $s_1^2 = a^2$ and $s_2^2 = b^2$ and the eigenvector of $A^2$ are the unit vectors $u_1 = (1,0)$ and $u_2 = (0,1)$. Thus the axes of the ellipse $AN$ are $(a,0)$ and $(0,b)$ and the length of axes are $a$ and $b$. We observe that the length of axes of the ellipse $A^n N$ are given by the square root of the eigenvalue of matrix $(A^n)^T A^n = A^{2n}$. The eigenvalues of $(A^n)^T A^n$ are $a^{2n}$ and $b^{2n}$, from which we obtain that the length of the axes are $a^n$ and $b^n$.

Another application of the **Theorem 1** refers to the **Case III**.

Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then $A^T A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$.

Thus the strain ellipse $AN$ is a circle centered at the origin with radius equal to $\sqrt{a^2+b^2}$: the original disk $N$ rotates by $\arctan(b/a)$ and stretches (or shrinks) by a factor $\sqrt{a^2+b^2}$.

We enunciate the following

**Theorem 2** Consider an $m \times m$ matrix $A$. Then there exist two orthonormal bases of $\Re^m$, $\{v_1, v_2, \ldots, v_m\}$ and $\{u_1, u_2, \ldots, u_m\}$, and real numbers $s_1 \geq s_2 \geq \ldots \geq s_m \geq 0$ such that $Av_i = s_i u_i$ for $i = 1, \ldots, m$.

The **Theorem 2** implies that the matrix $A$ can be written as $USV^T$, where $U$ is the matrix whose columns are $u_1, u_2, \ldots, u_m$, the matrix $S$ is a diagonal matrix whose entries are $s_1, s_2, \ldots, s_m$ and $V$ is the matrix whose columns are $v_1, \ldots, v_m$. The previous factorization of $A$ is called *singular value decomposition*. The matrices $U$ and $V$ are *orthogonal*, i.e., $U^T U = I$ and $V^T V = I$, where $I$ is the identity matrix.

**The Definition of Lyapunov Exponents in Two Dimensions**

We suppose that $f$ is a smooth map on $\Re^m$ and $v_0$ is the initial point of the orbit. We consider the unit sphere $N$ and we indicate with $J_n$ the first derivative $Df^n(v_0)$ (i.e. the *the Jacobian matrix of* $f^n$ *at* $v_0$) of the nth iterate of $f$. Then $J_nN$ is an ellipsoid with $m$ orthogonal axes. The length of axes are given by the square roots of the $m$ eigenvalues of the matrix $J_n^T J_n$. Let $r_k^n$ be the length of the $k$th orthogonal axis of the ellipsoid $J_nN$. Thus $r_k^n$ is the measure of the contraction of expansion near the orbit of initial point $v_0$ when the map $f$ is iterated. We define the $k$th *Lyapunov number of* $v_0$ as $L_k = \lim_{n\to\infty}(r_k^n)^{1/n}$ and the $k$th *Lyapunov exponent of* $v_0$ as $h_k = lnL_k$.

We now will give an example of the application of definition of Lyapunov exponent for two-dimensional maps. We consider a *skinny baker map*, i.e., the map on unit square $S$ of $\Re^2$ defined by

$$B(x,y) = \begin{cases} (\frac{1}{2}x, 2y) & \text{if } 0 \le y \le \frac{1}{2}; \\ (\frac{1}{2}x + \frac{2}{3}, 2y-1) & \text{if } \frac{1}{2} < y \le 1. \end{cases}$$

We observe that (See **Figure 1.28**)

- $B(S)$ lies in the left one-third and right one-third of $S$;
- $B^2(S)$ is the union of four stripes.

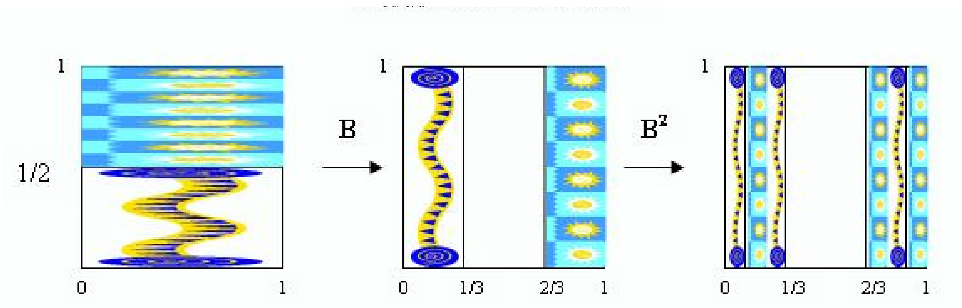


Figure 1.29: *The Skinny-Baker Map*

Since for all $v \in S$

$$DB(v) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix},$$

we can say that *the Jacobian matrix is constant in each point of* $S$ (except along the discontinuity line $y = 1/2$). We deduce that $J_n N$, where $N$ is a circle of radius $r$ centered at a point of $S$, is an ellipse with length of axes equal to $(\frac{1}{3})^n r$ in the horizontal direction and equal to $2^n r$ in the vertical direction. We take $r$ very small to avoid that the ellipses across the $y = 1/2$-line. We obtain that the Lyapunov exponent of $B$ are $-\ln 3$ and $\ln 2$.

## 1.13 The Dynamic Complexity: the Arnold tongues introduced with a discrete nonlinear business cycle model

Samuelson (1939) formalized the well-known model of Alvin H.Hansen "which ingeniously combines the multiplier analysis with that of acceleration principle". For this model the following assumption holds: a period $t$, the national income $Y_t$ consists of three components: the government deficit spending $g_t$, the private consumption expenditure $C_t$ and the private investment $I_t$. Moreover $C_t$ is in given proportion of the income of the previous period $Y_{t-1}$, that is $C_t = \alpha Y_{t-1}$, where $\alpha$ is the marginal propensity to consume and $0 < \alpha < 1$, and that $I_t$ is a given proportion $\beta$ ($\beta > 0$) of the change of consumption $\Delta C_t = C_t - C_{t-1}$, that is $I_t = \beta(C_t - C_{t-1})$. The constant $\beta$ is called *accelerator*. We consider $g_t$ as exogenous. Therefore we can think $g_t$ as a positive constant and denote it with the symbol $G_0$. In the original model Samuelsons set $g_t$ equal to 1.

Thus the model is described by the equations:

$Y_t = G_0 + C_t + I_t$,

$C_t = \alpha Y_{t-1}$, $(0 < \alpha < 1)$,

$I_t = \beta(C_t - C_{t-1})$, $(\beta > 0)$.

Following Chiang (1974), we observe that

$I_t = \beta(\alpha Y_{t-1} - \alpha Y_{t-2}) = \alpha\beta(Y_{t-1} - Y_{t-2})$, from which the previous relations are equivalent to the following linear nonhomogeneous difference equation of second-order

$Y_t - \alpha(1+\beta)Y_{t-1} + \alpha\beta Y_{t-2} = G_0$. (9.1)

We can rewrite the previous equation as

$Y_{t+2} - \alpha(1+\beta)Y_{t+1} + \alpha\beta Y_t = G_0$.

If we leave out the component $I_t$ from the national income, the last equation becomes a difference equation of first order, that is

$Y_t = \alpha Y_{t-1} + G_0$. (9.2)

We obtain the particular integral replacing $Y_t$ with $Y_P$. As a matter of fact

$Y_P = \alpha Y_P + G_0$, from which $Y_P = \frac{G_0}{1-\alpha}$.

If $G_0 = 1$, the intertemporal equilibrium $Y_P$ becomes $\frac{1}{1-\alpha}$ and usually it is called *multiplier of Keynes-Kahn-Clark.*

Given $Y_0 > 0$, the general solution of (9.2) is $Y_t = Y_0\alpha^t + \frac{G_0}{1-\alpha}$. Being $0 < \alpha < 1$, for $t \to \infty$, $Y_t \to Y_P$, that is, $Y_P$ is a stable equilibrium.

Proceeing as above we find that $Y_P = \frac{G_0}{1-\alpha(1+\beta)+\alpha\beta} = \frac{G_0}{1-\alpha}$ is the intertemporal equilibrium for (9.1). We consider the homogeneous equation

$b^2 - \alpha(1+\beta)b + \alpha\beta = 0$. (9.3)

If $\alpha^2(1+\beta)^2 > 4\alpha\beta$, or $\alpha(1+\beta)^2 > 4\beta$, or $\alpha > \frac{4\beta}{(1+\beta)^2}$, the equation (9.3) has two distinct and real solutions (Case I). Instead if $\alpha = \frac{4\beta}{(1+\beta)^2}$ the (9.3) admits two coincident and real roots (Case II), otherwise the roots are complex conjugate (Case III). We can interpret geometrically the previous results telling that in the plane $(\beta, \alpha)$ all solutions lie in the strip $S = ]0, +\infty[\times]0, 1[$. In particularly, denoted with $\Gamma$ the curve $\alpha = \frac{4\beta}{(1+\beta)^2}$, the distinct and real solutions, the repeated and real roots and the complex conjugate roots are respectively above, on and below the curve $\Gamma$. Now we indicate with $b_1$ and $b_2$ the solutions of (9.3) and we study the stability of the intertemporal equilibrium. From the following simple relations

$$b_1 + b_2 = \alpha(1 + \beta);$$

$$b_1 b_2 = \beta\alpha,$$

we deduce that $b_1$ and $b_2$ are positive. Moreover it proves that

- **Case I** If $0 < b_2 < b_1 < 1$ then there is convergence without cycles and $\alpha\beta < 1$; if $1 < b_2 < b_1$ then there is divergence without cycles and $\alpha\beta > 1$ .

- **Case II** If $0 < b < 1$ then there is convergence without cycles and $\alpha\beta < 1$; if $1 < b$ then there is divergence without cycles and $\alpha\beta > 1$.

- **Case III** Let $R = \sqrt{\alpha\beta}$, if $R < 1$ there is convergence with damped oscillations and $\alpha\beta < 1$; if $R > 1$ there is divergence with explosive oscillations and $\alpha\beta > 1$.

Thus only Case III presents endogenous cycles. Geometrically, in the plane $(\beta, \alpha)$, the stable cycles fall in the strip $S$ both below the $\Gamma$-curve and the hyperbola $\{(\beta, \alpha) : \alpha\beta = 1\}$ (See **Figura 1.30**).

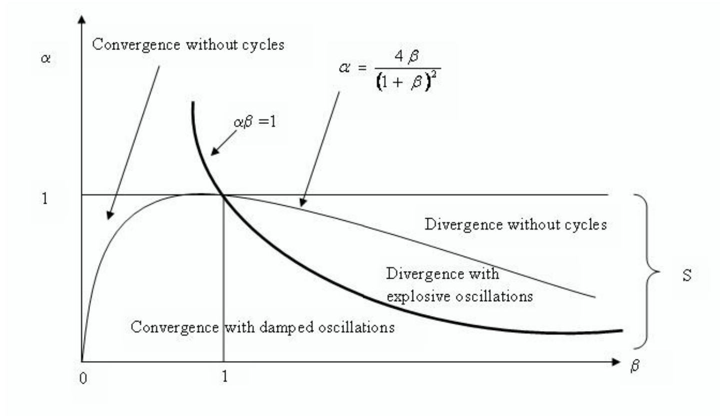


Figure 1.30: *Samuelson's Model*

### 1.13.1 The model of Puu-Sushko in discrete time

About explosive motion generated in the model of the linear accelerator of Samuelson(1939), Puu (1993)[12] observes that "in all physical systems, the source of inspiration for mathematical modelling in economics, linearization holds when the variables remain within reasonable bounds".

Hicks (1950), in order to eliminate the absurdity of explosive cycles, has rearranged the Samuelson's model introducing a lower bound $I_{min}$ and an upper bound $I_{max}$ for investment, that is $I_{min} < I_t < I_{max}$ for all period $t$: $I_{min}$ represents a *floor* for disinvestments and $I_{max}$ is a *ceiling* to investments.

Hicks (1950) modifies nonlinearly the investment using a piecewise function. Later Goodwin (1951) introduced a smooth nonlinearity in the investment function.

Puu (1993, 2000, 2003) simplifies the Samuelson's model ignoring the government deficit spending $g_t$ writing the national income as $Y_t = C_t + I_t$, defines nonlinearly the investment function as $I_t = v(Y_{t-1} - Y_{t-2}) - v(Y_{t-1} - Y_{t-2})^3$, where $v$ is the accelerator and he distributes the consumption over the two previous periods setting $C_t = (1 - s)Y_{t-1} + sY_{t-2}$, where $s$ is the constant saving rate. It gets $(Y_t - Y_{t-1}) = (v - s)(Y_{t-1} - Y_{t-2}) - v(Y_{t-1} - Y_{t-2})^3$. Now if we put $Y_t - Y_{t-1} = Z_t$ we can rewrite the last equation as $Z_t = (v - s)Z_{t-1} - vZ_{t-1}^3$, or,

[12]See also the paper of Marij Lines, *Heterogeneous Agents in a Multiplier-Accelerator Model*, (2005) Universita' di Udine

by a rescaling $((1+v-s)/v))^{\frac{1}{2}}$ of variables, $Z_t = aZ_{t-1} - (a+1)Z_{t-1}^3$, where $a = v - s$.

We consider the family of maps $f_a(Z) = aZ - (a+1)Z^3$. From equation $f_a(Z) = Z$, for all $a > 1$ we have as fixed points $Z = 0$ and $Z_{\pm} = \pm\sqrt{\frac{a-1}{a+1}}$. *The fixed points $Z_{\pm}$ are stable if $a < 2$.* As a matter of fact the condition $|f_a^{'}(Z_{\pm})| < 1$ is equivalent to inequality $|a - 3(a+1)Z_{\pm}^2| = |3 - 2a| < 1$ which is true for all $a < 2$.

We observe that for all $a$ the diagram of map $f_a(Z)$ goes across the points $(-1, 1)$, $(0, 0)$, $(1, -1)$ and if $a < 3$ the diagram of cubic $f_a(Z)$ is contained in the square $[-1, 1]^2$. Puu (1993) observes that:

- the process converges toward the positive fixed point if it starts from the right-hand side of 0 and $a = 1.9$;
- the process presents a 2-cycle if $a = 2.1$, a 4-cycle if $a = 2.25$, and a 8-cycle if $a = 2.295$;
- the period doubling points accumulate and chaos occurs around the parameter value $a = 2.302$ (*Feigenbaum point*).

Puu (1993) and Puu-Sushko (2003) generalize the consumption function given above such that:

$$C_t = (1-s)Y_{t-1} + \epsilon s Y_{t-2}$$

where $0 \leq \epsilon \leq 1$. We observe that for $\epsilon = 1$ we obtain the consumption function of the previous model and for $\epsilon = 0$ the consumption function of Samuelson.

As before we can put the system in the form:

$$Y_t = Y_{t-1} + Z_{t-1},$$

$$Z_t = aZ_{t-1} - (a+1)Z_{t-1}^3 - bY_{t-1},$$

where $b = (1-\epsilon)s$ is the rate of saving.

In the last model we change the notation of the variables: $x = Y$ and $y = Z$. We can view the dynamical system as generated by a family of two-dimensional continuous system non-invertible maps $F : \Re^2 \to \Re^2$ given by

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + y \\ ay - (a+1)y^3 - bx \end{pmatrix},$$

where $a > 0$ and $0 < b < 1$. We observe that $(0, 0)$ is the only fixed point for the map $F$.

We find that the jacobian matrix $DF$ of map $F$ is

$$DF = \begin{pmatrix} 1 & 1 \\ -b & a - 3(a+1)y^3 \end{pmatrix}$$

We write the jacobian matrix $DF$ at the fixed point $(0, 0)$ and we obtain that

$$DF(0,0) = \begin{pmatrix} 1 & 1 \\ -b & a \end{pmatrix}$$

We note that:

- the trace $trDF(0, 0) = 1 + a$;
- the determinant $|DF(0, 0)| = a + b$.

The characteristic equation associated to $DF(0, 0)$ is

$\mu^2 - trDF(0, 0)\mu + \mid DF(0, 0) \mid = 0$, or

$\mu^2 - (1 + a)\mu + (a + b) = 0.$

We set $\Delta = (TrDF(0, 0))^2 - 4 \mid DF(0, 0) \mid$. We note that the eigenvalues of the jacobian matrix $DF(0, 0)$ correspond to the roots of the characteristic equation.

They are real if $\Delta \geq 0$ and they are complex conjugate otherwise. The real eigenvalues of $DF(0,0)$ are given by

$$\mu_{1,2} = (a + 1 \pm \sqrt{(a+1)^2 - 4b})/2.$$

The dynamical system in the $TD$-plane is given by the equations

$$D(a,b) = a + b;$$

$$T(a,b) = 1 + a.$$

Now we apply the sufficient and the necessary conditions for detect the triangle $S$ of stability in the $(b,a)$-plane for the dynamical system and we have that

$$1 + |DF(0,0)| + trDF(0,0) > 0,$$

$$1 + |DF(0,0)| - trDF(0,0) > 0,$$

$$1 + |DF(0,0)| > 0,$$

that is:

$$2 + 2a + 2b > 0,$$

$$b > 0,$$

$$1 - a - b > 0.$$

Since $a > 0$ and $0 < b < 1$, we see easily that

$S = \{(b,a) : 0 < b < 1, 0 < a < 1 - b\}$ and $(0,0) \in S$. We observe that $(\Delta < 0)$ if and only if $(1 - 2\sqrt{b} < a < 1 + 2\sqrt{b})$. The eigenvalues complex conjugate are

$\mu_{1,2} = \alpha \pm i\beta$, where $\alpha = \frac{1}{2}TrDF(0,0) = \frac{1}{2}(1+a)$ and $\beta = \frac{1}{2}\sqrt{4b-(1+a)^2}$. Moreover, since $\alpha^2+\beta^2 = 1$, we can write $\mu_{1,2}$ such that

$\mu_{1,2} = \cos\theta \pm isin\theta$,

where $\cos\theta = \frac{1}{2}(1+a)$ and $\sin\theta = \frac{1}{2}(1+a)\sqrt{4a-(1+a)^2}$.

Moreover the characteristic equation becomes $\mu^2 - 2\alpha\mu + \alpha^2 + \beta^2 = 0$, or, recalling that $\alpha^2+\beta^2 = 1$ and $\alpha = \frac{1}{2}(1+a)$, $\mu^2 - (1+a)\mu + 1 = 0$.

The loss of stability of the system occurs if

- $\mu = 1$ (fold bifurcation);
- $\mu = -1$ (flip bifurcation);
- $modulus(\mu) = 1$ (Neimark-Sacker bifurcation).

If we substitute $\mu = \pm 1$ in the characteristic equation we get respectively $b = 0$ and $2(1+a)+b = 0$, i.e. $a = 0$. Since we supposed positive $a$ and $b$, the fold bifurcation and the flip bifurcation does not occur. Thus the loss of stability would present with the Neimark-Sacker bifurcation.

Since $|DF(0,0)| = 1$, we have $a+b = 1$, or $a = b-1$ (*Neimark-Sacker condition*). If $\theta$ is an irrational multiple of $2\pi$, the bifurcation presents as an invariant curve, otherwise as a periodic cycle. In the latter case we suppose that $\theta = 2\pi\frac{m}{n}$, where $m$ and $n$ are integer such that $(m,n) = 1$, where $(m,n)$ denotes the greatest common divisor between $m$ and $n$. The number $m/n$ is called *rational rotation number*.

From the relation $cos\theta = \frac{1}{2}(1+a)$ we deduce that $a = 2\cos(2\pi\frac{m}{n}) - 1$. We put $m = 1$ and we rewrite the last relation as

$a_n = 2\cos(2\pi\frac{1}{n}) - 1$, $b_n = 1 - a_n$ (9.2.1)

If replacing $n$ respectively with $1, 2, 3, 4, \ldots$ into the (9.2.1) we will have $a_n > 0$ and $0 < b_n < 1$, then we can say that a *resonance* occurs in the dynamical system and we can deduce the existence of Arnold tongues. If the conditions $a_n > 0$ and $0 < b_n < 1$ are verified for $n = 1, 2, 3, 4$ we say that a *strong resonance* occurs. We obtain

- **Case** $n = 1 : a_1 = 2\cos(2\pi) - 1 = 1$, from which, $b_1 = 1 - a_1 = 0$;

- **Case** $n = 2 : a_2 = 2\cos(2\pi\frac{1}{2}) - 1 = 2\cos(\pi) - 1 = -3$, from which $b_2 = 1 - a_2 = 4$;

- **Case** $n = 3 : a_3 = 2\cos(2\pi\frac{1}{3}) - 1 = 2(-\frac{1}{2}) - 1 = -2$, from which $b_3 = 1 - a_3 = 3$;

- **Case** $n = 4$: $a_4 = 2\cos(2\pi\frac{1}{4}) - 1 = -1$, from which $b_4 = 1 - a_4 = 2$.

Thus *the strong resonance does not occur.* The previous cases are called also respectively 1:1, 1:2, 1:3, 1:4. Moreover, substituting $n = 5$ into (9.2.1) (1:5) we have $a_5 = 2\cos(2\pi\frac{1}{5}) - 1 = 2\frac{\sqrt{5}-1}{4} - 1 = \frac{\sqrt{5}-3}{2} < 0$, that we don't accept. Now we study the case $n = 6$. We have $a_6 = 2\cos(2\pi\frac{1}{6}) - 1 = 2(\frac{1}{2}) - 1 = 0$, from which $b = 1 - a = 1$.

We note that *for all $n > 6$ the values of $a_n$ and $b_n$ are admissible.* As a matter of fact, we have that (See **Figura 1.31**)

- $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} 2\cos(2\pi\frac{1}{n}) - 1 = \lim_{y\to 0} 2\cos(y) - 1 = 1$;

- $D[2cos(\frac{2\pi}{x}) - 1] = 4\pi\frac{sin(2\pi/x)}{x^2} > 0$ for all $x > 0$.

The previous relations imply that $a_n$ *converges to* 1 *monotonically increasing and $a_n < 1$ for all $n > 6$.* From which $\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} a_n - 1 = 0$.

The following table illustrates numerically the previous results:

| n | $\cos(2\pi/n)$ | $a_n$ | $b_n$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 2 | -1 | -3 | 4 |
| 3 | -0.5 | -2 | 3 |
| 4 | 6.1257E-17 | -1 | 2 |
| 5 | 0.309016994 | -0.381966011 | 1.38196011 |
| 6 | 0.5 | 0 | 1 |
| 7 | 0.623489802 | 0.24979604 | 0.753020396 |
| 8 | 0.707106781 | 0.414213562 | 0.585786438 |
| 9 | 0.766044443 | 0.532088886 | 0.467911114 |
| 10 | 0.809016994 | 0.618033989 | 0.381966011 |
| 11 | 0.841253533 | 0.682507066 | 0.317492934 |
| 12 | 0.866025404 | 0.732050808 | 0.267949192 |
| 13 | 0.885456026 | 0.770912051 | 0.229087949 |
| 14 | 0.900968868 | 0.801937736 | 0.198062264 |
| 15 | 0.013545458 | 0.827090915 | 0.172909085 |
| 16 | 0.923879533 | 0.847759065 | 0.152240935 |
| 17 | 0.932472229 | 0.864944459 | 0.135055541 |
| 18 | 0.939692621 | 0.879385242 | 0.120614758 |
| 19 | 0.945817242 | 0.891634483 | 0.108365517 |
| 20 | 0.951056516 | 0.902113033 | 0.097886967 |
| 21 | 0.955572806 | 0.911145612 | 0.088854388 |
| 22 | 0.959492974 | 0.918985947 | 0.081014053 |
| 23 | 0.962917287 | 0.925834575 | 0.074165425 |
| 24 | 0.965925826 | 0.931851653 | 0.068148347 |
| 25 | 0.968583161 | 0.937166322 | 0.062833678 |
| 26 | 0.970941817 | 0.941883635 | 0.053910259 |
| 27 | 0.973044871 | 0.946089741 | 0.053910259 |
| 28 | 0.974927912 | 0.949855824 | 0.050144176 |
| 29 | 0.976620556 | 0.953241111 | 0.046758889 |
| 30 | 0.978147601 | 0.956295201 | 0.043704799 |

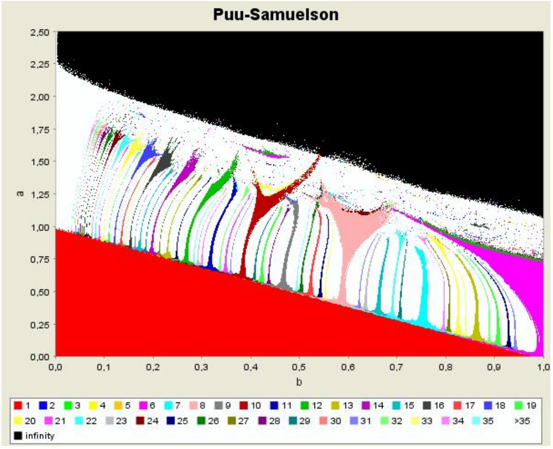


Figure 1.31: *The Arnol'd Tongues in the Puu-Samuelson Model*

## 1.14 Appendix: Basic Concepts on the Family of Logistic Maps

The notion of logistic map plays a central role in many economic dynamic models with chaos, particularly in the Day's model (1982, 1983) (See **Chapter 2**). We define the logistic map setting $f(x) = ax(1-x)$, where $a \geq 0$ and $x \in \Re$, and we find the fixed points of $f(x)$ solving the equation $ax(1-x) = x$. We obtain the product $x[(a-1) - ax] = 0$ that leads to solutions $x = 0$ and $x = (a-1)/a$ $(a \neq 1)$. We observe that $f'(x) = a - 2ax$ and if we evaluate $f'(x)$ at $x = 0$ and $x = (a-1)/a$ we have $f'(0) = a$ and $f'(a-1)/a) = 2 - a$. Thus we deduce that $x = 0$ is stable if $-1 < a < 1$ and $x = (a-1)/a$ is stable if $1 < a < 3$. If we see the logistic map as a dynamical system, i.e. $x_{t+1} = ax_t(1-x_t)$, where $t$ is a discrete time $(t = 0, 1, \ldots)$, we can say that *if* $-1 < a < 1$ *the attractor* $x = 0$ *have as basin of attraction the set of point between* 0 *and* 1. Following Alligood et al. (1996), about the dynamic of growth of populations, the previous result means that *with small reproduction rates, small populations tend to die out.* Instead for $1 < a < 3$ the point $x = 0$ is unstable and $x = (a-1)/a$ is stable and we can say that *small populations grow to steady-state of* $x = (a-1)/a$ (See **Figure 1.32**).

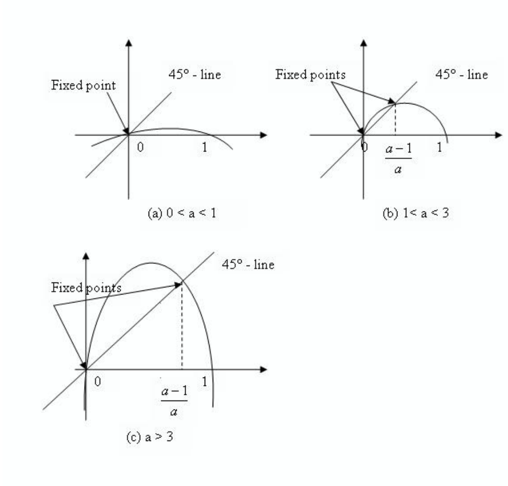


Figure 1.32: *Logistic Map*

We suppose that $x_t \in [0,1]$, $a \in [0,4]$ and we note that :

- $x_{t+1} = ax_t(1-x_t)$ is a concave quadratic function which maps $[0,1]$ onto itself for all $a \in [0,4]$;
- in the $(x_t, x_{t+1})$-plane $x_{t+1} = ax_t(1-x_t)$ represents an example of *unimodal map*, i.e. it has an unique point $x^*$ which maximize $f(x_t, a)$, it is smooth and there are two points $\alpha$ and $\beta$ such that $f(\alpha, a) = 0 = f(\beta, a)$, where $f(x_t, a) = ax_t(1-x_t)$;
- the one-dimensional map $f(x_t, \mu)$ is not invertible because, fixed $x_{t+1}$, exist two points $x_t$ and $x_{t'}$ such that $x_{t+1} = f(x_t, a) = f(x_{t'}, a)$.

From the assumptions on $a$ and $x_t$ we deduce that

- $f^{'}(x_t, a) = a(1-x_t) - ax_t = 0$ if and only if $x^* = \frac{1}{2}$;
- $f(\frac{1}{2}, a) = \frac{a}{4} \leq (4)(\frac{1}{4}) \leq 1$.

The trajectories of dynamical system $x_{t+1}$ depend on the value of $a$. As a matter of fact $x_{t+1}$ presents (R.H. Day, 1982)

- monotonic contraction to 0 if $0 < a \leq 1$;
- monotonic growth converging to $x = \frac{a-1}{a}$ if $1 < a \leq 2$;
- oscillations converging to $x = \frac{a-1}{a}$ if $2 < a \leq 3$;
- continued oscillations if $3 < a \leq 4$.

## 1.15 Appendix: The Li-Yorke Theorem

In 1975 Li and Jorke published a work entitled "*Period three implies chaos* " which has collected favor among economists "because its simplicity as it requires only checking the existence of a period-3 orbit in order to deduce the existence of "chaos"" one-dimensional (Boldrin-Woodford (1990, 1992)). In **Chapter 2** we will develop a model of growth due to R.H. Day which applies the result of Li-Yorke. We simply stating the Li-Yorke theorem and refer to the original work for a demonstration (See **Figure 1.33**).

**Theorem of Li-Yorke** Let $J$ be an interval in $\Re$ and let $f : J \to J$ be a continuous map. We consider the difference equation

$$x_{t+1} = f(x_t) \ (\star)$$

and we admit there exists a point $x \in J$ such that

$$f^3(x) \leq x < f(x) < f^2(x).$$

Then

- For every $k = 1, 2, 3, \ldots$, there exists a $k$-periodic solution such that $x_t \in J$ for all $t$.
- There is a countable set (containing no periodic points) $S \subset J$ for every $x_0 \in J$ the solution path of difference equation $(\star)$ remains in $S$ and
  - for all $x, y \in S$, $x \neq y$,

    $\limsup_{t\to\infty} |f^t(x) - f^t(y)| > 0$, $\liminf_{t\to\infty} |f^t(x) - f^t(y)| = 0$;
  - for all periodic points $x$ and all points $y \in S$,

    $\limsup_{t\to\infty} |f^t(x) - f^t(y)| > 0$.

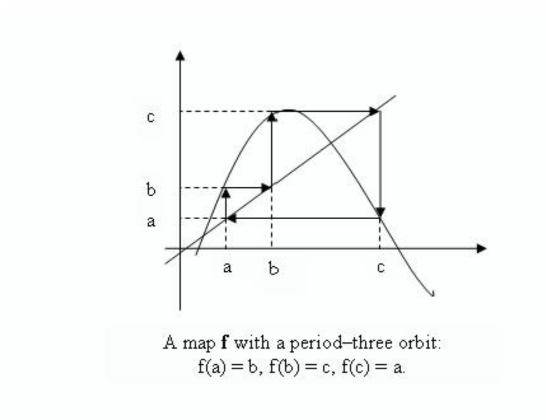


Figure 1.33: *A map with a period three orbit*

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# Chapter 2

## 2.1 Contents

- Introduction
- The Solow Growth Model in Discrete Time
- Complex Dynamics in the Solow Discrete Time Growth Model
- A Two Class one-dimensional Growth Model: The Model of Böhm and Kaas (1999)
- Complex Dynamics in a Pasinetti-Solow Model of Growth and Distribution
- Appendix: The CES Production Function

## 2.2 Introduction

The analysis of the fundamental issues in dynamical macroeconomics usually begins with the study of two (one-sector and one-dimensional) growth models: the Ramsey model (Ramsey, 1928) and the Solow model (Solow, 1956). In the Ramsey model a representative consumer has an infinite horizon of life and optimizes his/her utility. A basic Ramsey model in discrete time requires to find

$max\ W = \sum_{t=0}^{t=\infty}(\frac{1}{1+\varrho})^t u(c_t),$

subject to the constraints $y_t = f(k_t)$, $y_t = y_t + i_t$, $k_{t+1} = k_t + i_t$, where $f(k_t)$ is the production function, $k_t$ is the capital-labor ratio at time $t$, $y_t$ the income over labor at time $t$, $u(c_t)$ an utility function on the consumption per capita $c_t$ at time $t$, $i_t$ the investment over labor at time $t$, $\varrho$ the discount rate, with the following properties $u(c_t) \geq 0$, $u'(c_t) > 0$, $u''(c_t) < 0$, $f(0) = 0$, $f'(0) = 0$, $f'(\infty) = 0$, $f'(k) > 0$, $f''(k) < 0$.

In the Solow model consumption is not optimal the representative agent saves a constant fraction of his income. In the next sections we will describe only the Solow model and the most relevant models for our thesis. We note here that researches in several direction have spanned from the Solow model. For example, the Solow model inspired the works of Shinkay (1960), Meade (1961), Uzawa (1961,1963), Kurz (1963), Srinivasan (1962-1964), on two-sector growth models. Following this line of research, works about two-sector models appeared on the *Review of Economic Studies* in the 1960s (Drandakis (1963), Takajama (1963,1965), Oniki-Uzawa (1965), Hahn (1965), Stiglitz (1967), among others). This line of research has been further developed in the 80s with the introduction of chaos and Overlapping Generations (OLG) into the two-sector model (Galor and Ryder (1989), Galor (1992), Azariadis (1993), Galor and Lin (1994). Recently Karl Farmer and Ronald Wendner (2003) developed two-sector models including overlapping generation (OLG), instead Schmitz (2006) presented a two-sector model in discrete time that exhibits complex dynamics (topological chaos and strange attractors). Another line of research was opened by P.Diamond (1965) which was the first to extend the Solow model including OLG developing a one-sector and one-dimensional model with public debt. R.Farmer (1968) extended the Diamond model to the two-dimension case. Many authors developed model Farmer-type with chaos (Grandmont (1985), B. Jullien (1988), B. Reichlin (1986), A. Medio (1992), C. Azariadis (1993), V. Bohm (1993), A. Medio and G. Negroni (1996), de Vilder (1996), M. Yokoo (2000) ). Moreover, the seminal ideas of Kaldor (1956, 1957), Pasinetti (1962), Samuelson and Modigliani (1966), Chiang (1973) about the influence on the growth path by different savings behaviour of two income group (labor and capital) originated two-class one-dimensional (Böhm and Kaas (2000)) and two-dimensional (Commendatore (2005)) discrete time models. We note that in the two-class extensions of the Solow model, the neoclassical features of the production function, the Inada conditions, are weakened or disappear, and both models present complex dynamics. Following Samuelson-Modigliani (1966) and T. Michl (2005), in the **Chapter 3**, we will develop a two-dimensional and two-class discrete time model which extends the Solow model to OLG and dynasties a lá Barro (1974)[1]. We present a detailed taxonomy of the researches in several directions, originated by the Solow model, in an Appedix of this Chapter.

[1] "*Current generations act effectively as they were infinite-lived when they are connected to future generations by a chain of operative intergenerational transfers.*", R.J.Barro (1974, p.1097). See also the seminal papers of Becker (1965), Burbidge (1983), Weil (1987), Abel (1987) and the recent OLG-model with altruism of Cardia and Michel (2005)

## 2.3 The Solow Growth Model in Discrete Time

Following Hans-Walter Lorenz (1989) and Costas Aziariadis (1993), we will develop a discrete time variant of the growth model due to Solow (1956). We consider *a single good economy, i.e. an economy in which only one good is produced and consumed*. We assume that the time $t$ is *discrete*, that is $t = 0, 1, 2, \ldots$. The symbols $Y_t$, $K_t$, $C_t$, $I_t$, $L_t$, $S_t$ indicate economywide aggregates respectively equal to *income, capital stock, consume, investment, labor force, saving at time* $t$. The capital stock $K_0$ and labor $L_0$ at time 0 are given. The constant $s$ denotes the *marginal savings rate* and the constant $n$ indicates the *growth rate of population*. We consider $s$ and $n$ as given exogenously. The map $F : (K_t, L_t) \to F(K_t, L_t)$ is *the production function*. We assume that:

1. $Y_t = C_t + I_t$: *for all time* $t = 0, 1, \ldots$, *the economy is in equilibrium, i.e. the supply of income* $Y_t$ *is equal to the demand composed of the quantity* $C_t$ *of good to consume plus the stock* $I_t$ *of capital to invest (closed economy like a Robinson Crusoe economy)*;
2. $I_t = K_{t+1}$: *investment at time* $t$ *corresponds to all capital available to produce at time* $t + 1$ *(working capital hypothesis)*;
3. $S_t = Y_t - C_t = sY_t$ $(0 < s < 1)$: *saving is a share of income*;
4. $Y_t = F(K_t, L_t)$, i.e. *at time* $t$ *all income is equal to the output obtained by the inputs capital and labor*;
5. $L_t = (1 + n)^t L_0$ $(n > 0)$: *the labor force grows as a geometric progression at the rate* $(1 + n)$.

From the first (3.) we deduce that in a short run equilibrium $Y_t = C_t + S_t$, which, after a comparison with (1.), gives $I_t = S_t$. Thus, applying (2.) and (3.), we have $K_{t+1} = sY_t$. Finally, from (4.) we obtain $K_{t+1} = sF(K_t, L_t)$.

From the later expression, $\frac{K_{t+1}}{L_t} = \frac{sF(K_t,L_t)}{L_t}$.

If $F$ is *linear-homogeneous* (or it tells that $F$ *exhibits constant returns to scale)*, i.e.

6. $F(\lambda K, \lambda L) = \lambda F(K, L)$ (for all $\lambda > 0$),

then we have $\frac{K_{t+1}}{\frac{L_t}{1+n}} = \frac{sL_t F(\frac{K_t}{L_t},1)}{L_{t+1}}$.

We set $k_t = \frac{K_t}{L_t}$ (*capital-labor ratio or capital per worker*) and $f(k_t) = f(\frac{K_t}{L_t}, 1)$. We call output *per worker the ratio* $y_t = \frac{Y_t}{L_t}$.

Therefore we get the equation of accumulation *for the Solow model in discrete time with the working capital hypothesis*:

$k_{t+1}(1+n) = sf(k_t)$ (1.1)

If we assume that *capital depreciates at the rate* $0 \leq \delta \leq 1$*(fixed capital hypothesis)*, the capital available at time $t+1$ corresponds to $K_{t+1} = K_t - \delta K_t + I_t$, from which $K_{t+1} = sF(K_t, L_t) + (1-\delta)K_t$.

As before we get the following time-map for capital accumulation

$k_{t+1}(1+n) = sf(k_t) + (1-\delta)k_t$ (1.2)

or

$k_{t+1} = h(k_t)$,

where $h(k_t) = \frac{1}{1+n}[sf(k_t) + (1-\delta)k_t]$.

We notice that $I_t$ is the *gross investment* while $K_{t+1} - K_t = I_t - \delta K_t$ is *the net investment.*

Costas Azariadis (1993, *p*.4) tells us that *this model captures explicitly a simple idea that is missing in static formulations: there is a tradeoff between consumption and investment or between current and future consumption. The implications of this ever-present competition for resources between today and tomorrow are central to macroeconomics and can be explored only in a dynamic framework. Time is clearly of the essence.*

If $f(k_t)$ is a concave production function, for example, a Cobb-Douglas function $f(k_t) = Bk_t^{\beta}$ ($B > 0$, $0 < \beta < 1$, $k \leq 0$), then the equation (1.1) becomes

$k_{t+1} = \frac{sBk_t^\beta}{1+n}$. Setting $h(k_t) = \frac{sBk_t^\beta}{1+n}$, we notice that $h(k_t)$ is monotonically increasing and concave for all $k < 0$:

$\frac{df(k)}{dk} = \frac{s}{1+n}\beta Bk^{\beta-1} > 0$ and $\frac{d^2f(k)}{dk^2} = \frac{s}{1+n}B\beta(\beta-1)k^{\beta-2} < 0$.

**Remark 2.3.1** About the Cobb-Douglas, we observe that the assumption $0 < \beta < 1$ implies the concavity of $f(k)$. Moreover in the plane $(k_t, k_{t+1})$ *the graph of the Cobb-Douglas is below the* $45°$*-line if* $f(k_t) < k_t$*, from which* $k_t < (1/B)^{\frac{1}{B-1}}$.

**Remark 2.3.2** About the Cobb-Douglas, we have also

$f^{'}(k) < 1$ if $k > (B\beta)^{\frac{1}{1-\beta}}$. As a matter of fact

$f^{'}(k) < 1 \Leftrightarrow B\beta k^{\beta-1} < 1 \Leftrightarrow k^{\beta-1} < \frac{1}{B\beta} \Leftrightarrow (k^{-1})^{1-\beta} < (B\beta)^{-1}$

$\Leftrightarrow k^{-1} < (B\beta)^{-\frac{1}{1-\beta}}$.Q.E.D.

For example, let $B = 0.2$ be and let $\beta = 0.7$ be, it needs that $k > 0.001425$.

Moreover the dynamical system $k_{t+1} = h(k_t)$ has two steady-states: the first, at $k = 0$, is *a trivial and repelling (or instable) fixed point*, while the second, at $k^* = [\frac{Bs}{1+n}]^{\frac{1}{\beta-1}}$, is *interior* and *asymptotically stable.*

## 2.4 Complex dynamics in the Solow Discrete Time Growth Model

R.H. Day (1982,1983) first has noticed that *complex dynamics can emerge from simple economic strutures* as, for example, the neoclassical theory of capital accumulation. In particulary Day argues that the nonlinearity of the $h(k_t)$ map and the lag present in (1.1) are not sufficient to lead to chaos. Instead making changes in (1.1) in the production function or thinking the saving propensity $s$

as a function of $k_t$, i.e. $s = s(k_t)$, he obtains a *robust* result (Michele Boldrin and Michael Woodford, 1990).

In the former case he defines

$$f(k_t) = \begin{cases} Bk_t^{\beta}(m-k_t)^{\gamma}, & \text{if } k_t < m; \\ 0, & \text{otherwise,} \end{cases}$$

where $m$ is a positive constant, $0 < \beta < 1$, $0 < \gamma < 1$ and $B > 0$.

In the latter case he sets $f(k_t) = Bk_t^{\beta}$ ($B \geq 2$, $0 < \beta < 1$) and he replaces the constant $s$ with the saving function

$$s(k_t) = a(1 - \tfrac{b}{r})\tfrac{k_t}{y_t},$$

where $r = f^{'}(k_t) = \beta\frac{y_t}{k_t}$, $a > 0, b > 0$.

Thus from the equation (1.1) we deduce respectively the equations

$$k_{t+1} = \tfrac{1}{1+n} sBk_t^{\beta}(m-k_t)^{\gamma} \text{ (4.1)}$$

and

$$k_{t+1} = \tfrac{a}{1+n} k_t[1 - (\tfrac{b}{\beta B})k_t^{1-\beta}] \text{ (4.2).}$$

It is very simple to solve the equation (4.1) when $m = \gamma = \beta = 1$. As a matter of fact we can rewrite it like this

$$k_{t+1} = \tfrac{1}{1+n} sBk_t(1-k_t) \text{ (4.3).}$$

If we set $\mu = \frac{sB}{1+n}$ then the (4.3) becomes the well-known logistic equation

$$k_{t+1} = \mu k_t(1-k_t).$$

We can use the Li-Yorke Theorem (see **Chapter 1**). Following Day (1982, 1983), first we observe that the right-hand side $h(k_t) = \frac{1}{1+n} sBk_t^{\beta}(m-k_t)^{\gamma}$ of equation (3.1) is a map concave, one-humped shaped, has a range equal to the interval $[0, h(k^c)]$, where $k^c$ is the unique value of $k_t$ which maximizes the map $h(k_t)$. Moreover fixing the parameters $\beta, \gamma$ and $m$, the graph of $h(k_t)$ stretches upwards as $B$ is increased and at same time the position of $k^c$ doesn't changes because in the expression of $k^c$ the parameter $B$ don't appear while the maximum $h(k^c)$ depends linearly on $B$ (See **Figure 2.1** and **Figure 2.2**).

As a matter of fact, from the equation

$$\frac{dk_{t+1}}{dk_t} = \frac{sB}{1+n}(\beta k_t^{\beta-1}(m-k_t)^{\gamma} - k_t^{\beta}\gamma(m-k_t)^{\gamma-1}) = 0,$$

we get $k^c = \frac{\beta m}{\gamma+\beta}$ and $h(k_t^c) = \frac{Bs}{1+n}\beta^{\beta}\gamma^{\gamma}(\frac{m}{\beta+\gamma})^{\beta+\gamma}$.

Moreover we assume that $k^b$ is the backward iteration of $k^c$, i.e. $k^b = h^{-1}(k^c)$, $k^m$ is the forward of $k^c$, i.e. $h(k^c) = k^m$ and $k^m$ is the maximum $k$ such that $h(k) = 0$. Thus $h(k^m) = 0$, $k^c = h(k^b)$, $k^m = h(k^c) = h(h(k^b))$, $h(k^m) = h(h(h(k^b))) = 0$. If $B$ is large enough, $k^c$ lies to left of the fixed point $k^*$, from which it follows that $k^b < k^c$.

The previous conditions

$$0 < k^b < k^c < k^m,$$

imply that

$$h(k^m) < k^b < h(k^b) < h(k^c),$$

which are equivalent to the inequalities

$$h^3(k^b) < k^b < h(k^b) < h^2(k^b).$$

Therefore the hypotheses of Li-Yorke theorem are satisfied.

From (4.2) we get

$$\frac{dk_{t+1}}{dk_t} = \frac{a}{1+n}\{[1-\frac{b}{\beta B}k_t^{1-\beta}] + k_t[(-\frac{b}{\beta B})(1-\beta)k_t^{-\beta}]\}$$

$$= \frac{a}{1+n}[1-(2-\beta)\frac{b}{\beta B}k_t^{1-\beta}] = 0$$

if and only if $k^* = [\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}$.

If we call $\psi(k_t)$ the right-hand side of (4.2) we have

$$\psi(k^*) = \frac{a}{1+n}[\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}\frac{1-\beta}{2-\beta}.$$

Let $k_c$ the smaller root of the equation

$\psi(k_t) = x^*$ (4.4),

that is $\frac{a}{1+n}k_t[1-(\frac{b}{\beta B})k_t^{1-\beta}] = [\frac{\beta B}{b(2-\beta)}]^{\frac{1}{1-\beta}}$ (4.5).

As above conditions of the of Li-Yorke Theorem are satisfied.

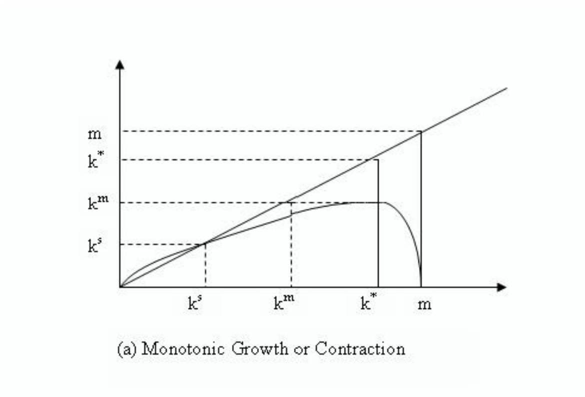
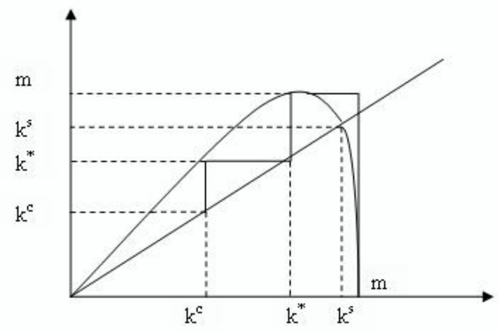



Figure 2.1:




(b) Sufficient Conditions for Chaos

Figure 2.2:

## 2.5 A Two Class Growth Model: A Model of Böhm and Kaas

In the model of Böhm and Kaas (1999) there are two types of agents (*two class model*), called workers and shareholders, and only one good (or commodity) is produced which is consumed or invested (*one sector model*). Like Kaldor (1956,1957) and Pasinetti (1962), the workers and shareholders have constant savings propensities, denoted respectively with $s_w$ and $s_r$ ($0 \leq s \leq 1$ and $0 \leq s \leq 1$). The output is produced with two factors: labor and capital. We consider that the capital depreciates at a rate $0 < \delta \leq 1$ and the labor grows at rate $n \geq 0$. We write the production function $f : \Re \rightarrow \Re$ in intensive form (i.e. it is maps capital per worker $k$ into output per worker $y$), and suppose that $f$ satisfies the following conditions :

- $f$ is $C^2$;
- $f(\lambda k) = \lambda f(k)$ (*constant returns to capital*);
- $f$ is monotonically increasing and strictly concave (i.e. $f^{'}(k) > 0$ and $f^{''}(k) > 0$ for all $k > 0$);
- $\lim_{k\rightarrow\infty} f(k) = \infty$;
- (a) $\lim_{k\rightarrow 0} \frac{f(k)}{k} = \infty$ and (b) $\lim_{k\rightarrow\infty} \frac{f(k)}{k} = 0$ (*weak Inada conditions (WIC)*)

**Remark 2.7.1** Following Böhm et al. (2007), we now introduce two families of production functions that violate the WIC: the *linear production functions* and the *Leontief production functions* given by $f(k) = a + bk$, $(a, b > 0)$ and $g(k) = min\{a, bk\}$ $(a > 0, b > 0)$ respectively.

Since

$\lim_{k\to 0} \frac{f(k)}{k} = \infty$ and $\lim_{k\to\infty} \frac{f(k)}{k} = b$,

$f$ violates property (b) of WIC. Instead since

$\lim_{k\to 0} \frac{g(k)}{k} = b$ and $\lim_{k\to\infty} \frac{f(k)}{k} = 0$,

$g$ does not satisfy property (a) of WIC. We conclude this remark offering an example of production functions that satisfy WIC: the *isoelastic production functions of the form*

$h(k) = Ak^{\alpha}$, $A > 0$, $0 < \alpha < 1$.

It easy verify that $h(k)$ satisfies WIC.

**Remark 2.7.2** We observe that, *for any differentiable function* $f : \Re_+ \to \Re_+$, *the Inada conditions*

$(\alpha)$ $\lim_{k\to 0} f'(k) = \infty$ *and* $(\beta)$ $\lim_{k\to\infty} f'(k) = 0$,

*imply WIC* . As a matter of fact, since

$\lim_{k\to 0} f(k) = 0$ and $\lim_{k\to\infty} f(k) = \infty$,

by l'Hôpital's rule,

$\lim_{k\to 0} f'(k) = \lim_{k\to 0} \frac{f(k)}{k}$ and $\lim_{k\to\infty} f'(k) = \lim_{k\to\infty} \frac{f(k)}{k}$.

If we assume that the market is competitive then the wage rate $w(k)$ is coincident with the marginal product of labor, i.e. $w(k) = f(k) - kf'(k)$, and the interest rate (or investment rate) $r$ is equal to the marginal product of capital, i.e. $r = f'(k)$. We suppose that $f(0)$ generally is not equal to 0. We observe that the total capital income per worker is $kf'(k)$. Moreover from WIC we deduce that:

- $w(k) \geq 0$;
- $w^{'}(k) = -kf^{''}(k) > 0$ ($w(k)$ is strictly monotonically increasing);
- $0 \leq kf^{'}(k) \leq f(k) - f(0)$;
- $\lim_{k \to 0} kf^{'}(k) = 0$.

**Remark 2.7.3** There are several ways to obtain the inequality $0 \leq kf^{'}(k) \leq f(k) - f(0)$. The first way is the following. We recall that $f$ *is concave in* $[0, +\infty[$ *if and only if* $f(k_1) \leq f(k_0) + f^{'}(k_0)(k_1 - k_0)$, *for all* $k_0, k_1 \geq 0$. In particulary, if $k_0 = k$ and $k_1 = 0$, we have $f(0) \leq f(k) + f^{'}(k)(0 - k)$, from which $0 \leq kf^{'}(k) \leq f(k) - f(0)$.

Alternately, if $f^{'}(0) < \infty$, by the inequality $w(0) \leq w(k)$ for all $k \geq 0$, we have $f(0) - 0 \cdot f^{'}(0) \leq f(k) - kf^{'}(k)$, from which $0 \leq kf^{'}(k) \leq f(k) - f(0)$.

Finally, consider the graph of a monotonically strictly increasing and concave function $f$ with $f(0) > 0$. Geometrically we may intuit the inequality drawing in the plane $(k, f(k))$ the line which goes across the points $(0, f(0))$ and $(k, f(k))$ and the tangent line in the point $(k, f(k))$: the slope of the first line, $\frac{f(k)-f(0)}{k}$, will appear greater or equal to the slope $f^{'}(k)$ of the second line. By continuity of $f(k)$ on $k = 0$, we obtain the $\lim_{k \to 0} f(k) = f(0)$. Thus, from the previous inequality, $\lim_{k \to 0} kf^{'}(k) \leq \lim_{k \to 0}(f(k) - f(0)) = f(0) - f(0) = 0$.

Similarly to the Solow model we obtain that the time-one map of capital accumulation is

$$k_{t+1} = G(k_t) = \tfrac{1}{1+n}((1 - \delta)k_t + s_w w(k_t) + s_r k_t f^{'}(k_t)).$$

**Proposition 1** Given $n \geq 0$ and $0 \leq \delta \leq 1$, let $f(k)$ be a production function which satisfies the WIC. If the workers do not save less than shareholders (i.e. $s_w \geq s_r$) or $e_{f'}(k) \geq -1$ then $G$ is monotonically increasing in $k$.

**Proof** We observe that $\frac{dG(k_t)}{dk_t} = \frac{1}{1+n}((1-\delta) - s_w kf''(k) + s_r(f'(k_t) + k_t f''(k_t)))$. Thus $\frac{dG(k_t)}{dk_t} \geq 0$ is equivalent to inequality $(s_w - s_r)kf''(k) \leq 1 - \delta + s_r f'(k)$. From the assumptions $f'(k) > 0$, $1 - \delta \geq 0$ and $s_r > 0$, we deduce that $(1 - \delta + s_r f'(k) > 0)$. Being $f''(k) < 0$, if $s_w \geq s_r$, the left-hand side of inequality is negative and the inequality is satisfied trivially. Otherwise, rewriting the inequality in the following manner $s_w kf''(k) \leq (1 - \delta) + s_r(kf^{''}(k) + f'(k))$, we notice that it is true if $(kf''(k) + f'(k) \geq 0)$, i.e. $e_{f'}(k) \geq -1$.

The following proposition investigates *the existence and the uniqueness of steady states.*

**Proposition 2** Consider $n$ and $\delta$ fixed and let $f(k)$ be a production function which satisfies the WIC. The following conditions hold:

- $k = 0$ if and only if $s_w = 0$ or $f(0) = 0$.
- There exists al least one positive steady state if ($s_r > 0$ and $\lim_{k\to 0} f'(k) = 0$) or if ($s_w > 0$ and $f'(0) < \infty$).
- There exists at most one positive steady state if ($s_r \geq s_w$).

**Proof** We observe that $k$ is a steady state if and only if $k = G(k)$, that is

$$s_w w(k) + s_r k f'(k) = (n + \delta)k.$$

Thus $0 = G(0)$ if and only if $(s_w(f(0) - \lim_{k\to 0} kf'(k)) + s_r \lim_{k\to 0} kf'(k) = 0)$.

By a previous observation we have that $\lim_{k\to 0} kf'(k) = 0$, therefore $k = 0$ is a steady state if and only if $s_w f(0) = 0$.

Moreover the existence of a positive steady state $k$ is equivalent to

$$s_w(\frac{f(k)}{k} - f'(k)) + s_r f'(k) = n + \delta.$$

We set $H(k) = s_w(\frac{f(k)}{k} - f'(k)) + s_r f'(k)$. By Bolzano's Theorem, being $H(k)$ continuous in interval $]0, +\infty[$, the range $J$ of $H(k)$ is an interval. We notice that $J = ]0, +\infty[$. As a matter of fact, if suppose that $\lim_{k\to\infty} f'(k) = +\infty$, we may apply the Hôpital's Rule to the first of the conditions denoted above with $(I)$, and we have $0 = \lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k)$, from which $\lim_{k\to+\infty} H(k) = 0$. From the second relation of $(I)$ and setting $f'(0) < +\infty$, we obtain that $\lim_{k\to 0} H(k) = +\infty$. Therefore, the equation $H(k) = n + \delta$ accepts at least one positive solution. Being $\frac{dH(k)}{dk} = s_w(\frac{kf'(k) - f(k)}{k^2} - f''(k)) + s_r f''(k)$ $= s_w(\frac{kf^{'}(k) - f(k)}{k^2}) + (s_r - s_w)f^{''}(k)$ and since $kf^{'}(k) - f(k) = -w(k) < 0$, if we suppose $s_r \geq s_w$, we deduce that $\frac{dH(k)}{dk} \leq 0$. Thus $H(k)$ is strictly monotonically decreasing and the equation $H(k) = n + \delta$ admits only one root.

**Proposition 3** $k^\star$ is a steady state of Pasinetti-Kaldor iff, for given $n$ and $\delta$, the pairs $(s_r, s_w)$ of savings rate describe the line $s_r + \frac{1-e_f(k^\star)}{e_f(k^\star)} s_w = 1$ in the $(s_r, s_w)$-diagram, where $e_f(k) = \frac{kf^{'}(k)}{f(k)}$.

**Proof** We observe that the total consumption per worker is $c(k) = f(k) - sw(k) - skf^{'}(k)$. If $k^\star$ is a steady state then $c(k^\star) = f(k^\star) - (n+\delta)k^\star$. We want the steady state $k^\star$, with different savings rate, which maximize $c(k^\star)$. Thus, setting $\frac{dc(k^\star)}{dk^\star} = 0$, we find $f^{'}(k^\star) = (n+\delta)$, that is $k^\star = f^{-1}((n+\delta))$. We call *Kaldor-Pasinetti equilibrium* the optimal steady state consumption (or the *golden rule for capital stock*). Replacing $(n+\delta)$ with $f^{'}(k^\star)$ in the right-hand side of the steady state condition $s_w w(k^\star) + s_r k^\star f^{'}(k^\star) = (n+\delta)k^\star$, we obtain $s_w w(k^\star) + s_r k^\star f^{'}(k^\star) = k^\star f^{'}(k^\star)$, that is $s_w(f(k^\star) - k^\star f^{'}(k^\star)) + s_r k^\star f^{'}(k^\star) = k^\star f^{'}(k^\star)$. Dividing both sides of the previous equation by $f(k^\star)$ and recalling the definition of $e_f(k)$, we have $s_r + \frac{1-e_f(k^\star)}{e_f(k^\star)} s_w = 1$. We notice that in the $(s_r, s_w)$-plane the last equation can be viewed as a line that

- has negative slope;
- goes across the point $(s_r, s_w) = (1, 0)$;
- is below or above the 45°-line $s_w = s_r$ depending on $e_f(k^\star)$ is less or greater than $\frac{1}{2}$.

The $(s_r, s_w)$-plane is coincident with the square $[0,1]^2$.

### 2.5.1 The dynamics with fixed proportions

We consider the Leontief technology

$f_L(k) = min\{ak, b\} + c$, $a, b, c > 0$.

Let $k^\star = b/a$ be. We have

$$f_L(k) = \begin{cases} ak + c, & \text{if } k \le k^\star, \\ b + c, & \text{if } k > k^\star; \end{cases} \text{ and } f_L^{'}(k) = \begin{cases} a, & \text{if } k \le k^\star, \\ 0, & \text{if } k > k^\star. \end{cases}$$

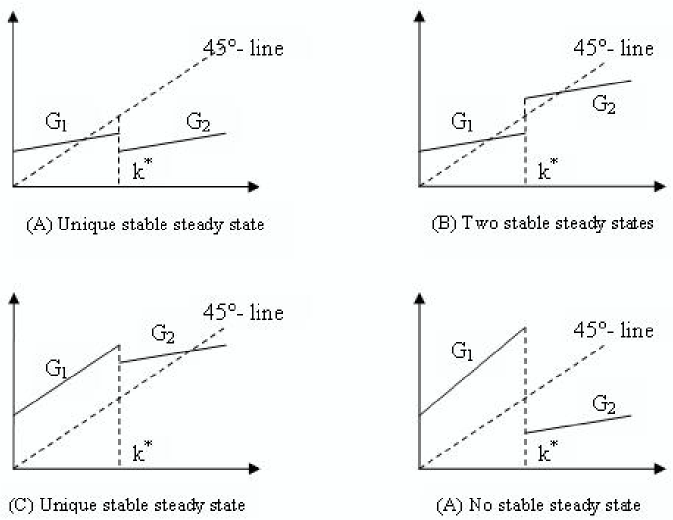
The map $G$ becomes

$$G_L(k) = \begin{cases} G_1(k) = \frac{1}{1+n}((1-\delta+s_r a)k + s_w c), & \text{if } k \leq k^\star, \\ G_2(k) = \frac{1}{1+n}((1-\delta)k + (b+c)s_w), & \text{if } k > k^\star. \end{cases}$$

We may say that:

- $G_1$ and $G_2$ are affine-linear maps strictly monotonically increasing;
- $G_1' = \frac{1}{1+n}(1-\delta+s_r a) > G_2' = \frac{1}{1+n}(1-\delta)$;
- $G_2' < 1$: the map $G_2'$ has always a fixed point $k_2$;
- $G_1$ has the fixed point $k_1$ if and only if $G_1' < 1$, that is $n+\delta-s_r a > 0$;
- $G_1(0) = \frac{1}{1+n}s_w c < G_2(0) = \frac{1}{1+n}(b+c)s_w$.

Let $k_1$ be the fixed point for $G_1$. Then $k_1$ *is a fixed point also for $G$ if and only if $k_1 < k^\star$*. Analogously, found the fixed point $k_2$ for $G_2$, we have that $k_2$ *is a fixed point also for $G$ if and only if $k^\star < k_2$* (See **Figure 2.3**).




**Proposition 1** Let $G_1' < 1$ be. We obtain that:

(i) the fixed point $k_1$ for $G_1$ is equal to $\frac{cs_w}{n+\delta-as_r}$;

(ii) $k_1$ is a fixed point also for $G$ if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$;

(iii) $G_1(k^\star) < k^\star$ if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$.

**Proof** We solve the equation $G_1(k) = k$. We get

$\frac{1}{1+n}((1-\delta+s_r a)k + s_w c) = k$, from which

$(s_r a - n - \delta)k = -s_w c$. Thus $k_1 = \frac{cs_w}{n+\delta-as_r}$.

Moreover $k_1 < k^\star$ if and only if $\frac{cs_w}{n+\delta-as_r} < \frac{b}{a}$. From the assumption $G_1^{'} < 1$ we deduce $n+\delta-s_r a > 0$. Therefore $cs_r < -bs_w + (n+\delta)\frac{b}{a}$, from which $bs_r + cs_w < (n+\delta)\frac{b}{a}$.

The inequality $G_1(k^\star) < k^\star$ is equivalent to the following $\frac{1}{1+n}((1-\delta+s_r a)k^\star + s_w c) < k^\star$. We get before $(as_r - n - \delta)k^\star < -s_w c$, and after $s_r a k^\star - (n+\delta)k^\star < -s_w c$. We deduce the relation (iii). $(i)$ and $(ii)$ are equivalent.

**Proposition 2** We get

(i) the fixed point of $G_2$ is $k_2 = \frac{(b+c)s_w}{n+\delta}$;

(ii) $k_2$ is the fixed point also for $G$ if and only if $s_w > \frac{(n+\delta)b}{(b+c)a}$;

(iii) $G_2(k^\star) > k^\star$ if and only if $s_w > \frac{(n+\delta)b}{(b+c)a}$.

**Proof** Solving the equation $G_2(k) = k$, we obtain the following equivalent relations:

$\frac{1}{1+n}((1-\delta)k + (b+c)s_w) = k$,

$(1-\delta)k-(1+n)k=-(b+c)s_w$,

$-(n+\delta)k=-(b+c)s_w$, from which $k_2=\frac{(b+c)s_w}{n+\delta}$.

Moreover $k_2 > k^\star$ if and only if $\frac{(b+c)s_w}{n+\delta} > \frac{b}{a}$, from which $s_w > \frac{(n+\delta)b}{(b+c)a}$. (iii) trivial. Obviously (*ii*) and (*iii*) are equivalent (See **Figure 2.4**).

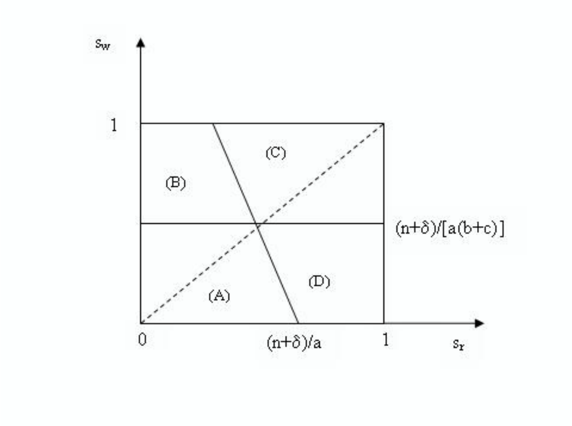



Figure 2.4: *Stability regions for the Leontief technology*

**Remark** $G_L$ has two fixed point if and only if $G_1(k^\star) < k^\star < G_2(k^\star)$, from which $G_1(k^\star) < G_2(k^\star)$. Then $\frac{1}{1+n}((1-\delta+s_ra)k^\star+s_wc) < \frac{1}{1+n}((1-\delta)k^\star+(b+c)s_w$. Thus $s_r < s_w$.

(A) $G_L$ has only one fixed point: the fixed point of $G_1$, that is it holds the system

$$\begin{cases} bs_r+cs_w < (n+\delta)\frac{b}{a}, \\ s_w < \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(B) $G_L$ has two fixed points: the fixed point of $G_1$ and the fixed point of $G_2$, that is it holds the system

$$\begin{cases} bs_r + cs_w < (n+\delta)\frac{b}{a}, \\ s_w > \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(C) $G_L$ has only one fixed point: the fixed point of $G_2$, that is it holds the system

$$\begin{cases} bs_r + cs_w > (n+\delta)\frac{b}{a}, \\ s_w > \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

(D) $G_L$ don't has fixed point, that is it holds the system

$$\begin{cases} bs_r + cs_w > (n+\delta)\frac{b}{a}, \\ s_w < \frac{(n+\delta)b}{(b+c)a}. \end{cases}$$

**Remark** Now consider the case (B). Since $G_1(k^\star) < k^\star < G_2(k^\star)$, we get

$$G_1(k_1) < G_1(k^\star) < k^\star < G_2(k^\star) < G_2(k_2),$$

from which

$G_1(k_1) < G_2(k_2)$ for all pairs $(k_1, k_2)$ such that $0 \le k_1 \le k^\star$ and $k_2 > k^\star$.

Thus $G_L$ *is strictly monotonically increasing (and therefore injective) in the case (B).*

**Remark** Look at case $(D)$, that is $G_2(k^\star) < k^\star < G_1(k^\star)$. Then $G_L(G_2(k^\star)) = G_1(G_2(k^\star))$ and $G_L(G_1(k^\star)) = G_2(G_1(k^\star))$. Moreover, by relations

$$G_1(G_2(k^\star)) = \frac{(1-\delta+s_r a)(1-\delta)}{(1+n)^2}k^\star + \frac{(1-\delta+s_r a)(b+c)s_w}{(1+n)^2} + \frac{cs_w}{(1+n)},$$

$$G_2(G_1(k^\star)) = \frac{(1-\delta+s_r a)(1-\delta)}{(1+n)^2}k^\star + \frac{(1-\delta)cs_w}{(1+n)^2} + \frac{(b+c)s_w}{(1+n)},$$

we will show that $G_1(G_2(k^\star)) > G_2(G_1(k^\star))$, and thinking as before,

we may deduce that $G_L$ *is injective on the interval* $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$.

As a matter of fact, we can write $G_1$ and $G_2$ such that:

$G_1(k^\star) = m_1k^\star + n_1$ and $G_2(k^\star) = m_2k^\star + n_2$, where $m_1 \geq 1 > m_2 > 0$ and $n_2 > n_1 > 0$.

We have

$$G_1(G_2(k^\star)) = m_1(m_2k^\star + n_2) + n_1 = m_1m_2k^\star + m_1n_2 + n_1,$$

$$G_2(G_1(k^\star)) = m_2(m_1k^\star + n_1) + n_2 = m_1m_2k^\star + m_2n_1 + n_2.$$

Let $n_2 = n_1 + \epsilon$ be, where $\epsilon > 0$. Then we may conclude observing that $m_1n_2+n_1 = m_1(n_1+\epsilon)+n_1 = m_1n_1+m_1\epsilon+n_1 > m_2n_1+n_2 = m_2n_1+n_1+\epsilon$.

**Proposition 3** We consider the case (D), i.e. $G_2(k^\star) < k^\star < G_1(k^\star)$. Let $K_\tau = (k_s)_{s=1,\dots,\tau}$ be a cycle of order $\tau$ for $G_L$ such that $k_s \neq k^\star$ for all $s = 1, \dots, \tau$. Then $K_\tau$ is globally stable.

**Proof** By recurrence it proves that on the interval $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$

- each $s$th iterate $G_L^s$ is injective;
- the $\tau$th iterate $G_L^\tau$, presents a discontinuity either at $k^\star$ or at $G_L^{-s}(k^\star)$, $s = 1, \dots, \tau - 1$.

Thus $G_L^\tau$ shows at most $\tau$ discontinuities and we may find a partition $\{I_1, \dots, I_m\}$ of $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$ into $m$ intervals $I_s$ ($s = 1, \dots, m$ and $m \leq \tau + 1$) such that $G_L^\tau(k) = A_s + B_sk$, $s \in I_s$, where $A_s$ and $B_s$ are positive constants.

Let $(k_s)_{s=1,\dots,\tau}$ be a cycle of order $\tau$. If we assume that $k_s \in I_s$ ($s = 1, \dots, \tau$), we obtain that $B_s < 1$. As a matter of fact, imposing $k_s = A_s + k_sB_s$, we have $(1-B_s)k_s = A_s$. Being $k_s$ and $A_s$ positive, we deduce that $1-B_s > 0$.Therefore we may say that *each trajectory starting in* $[G_2(G_1(k^\star)), G_1(G_2(k^\star))]$ *converges to* $K_\tau$.

## 2.6 Complex Dynamics in a Pasinetti-Solow Model of Growth and Distribution: a Model of P.Commendatore

### 2.6.1 Introduction

Similarly to the paper of Böhm and Kaas (1999), the model of Commendatore (2005)

- is a two-class model, that is two distinct group of economic agents (workers and capitalists) exist, with constant propensities to save (Kaldor, 1956);
- labor and capital markets are perfectly competitive;
- the income sources of workers are wages and profits and the income of capitalists is only profits (Pasinetti, 1962);
- the time is discrete;
- there is a single good in the economy (one sector model).

Commendatore's model differs from the model of Böhm and Kaas in some assumptions:

- following Chiang (1973), workers not save in same proportions out of labor and income of capital;
- the production function is not with fixed proportions (Leontief technology) but it is a CES production function;
- likewise Samuelson-Modigliani (1966) that, following Pasinetti (1962), extend the Solow growth model (1956) to two-dimensions, the map that describes the accumulation of capital in discrete time is two-dimensional because it considers not only the different saving behaviour of two-classes but also their respective wealth (capital) accumulation.

### 2.6.2 The model: the economy, short-run equilibrium, steady growth equilibrium

Let $f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}}$ be the CES production function in intensive form, where $k$ is the capital/labor ratio, $0 < \alpha < 1$ is the distribution coefficient,

$-\infty < \rho < 1$ $(\rho \neq 0)$, $\eta = \frac{1}{1-\rho}$ is the constant elasticity of substitution. We consider $f(k) > 0$. Therefore $f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}} = [\alpha k^{-\rho} + (1-\alpha)]^{\frac{1}{\rho}} k$. The terms $k_w$ and $k_c$ denote, respectively, workers' and capitalists' capital per worker, where $0 \leq k_w \leq k$, $0 \leq k_c \leq k$, $k = k_w + k_c$. The workers' saving out of wages are represented by $s_{ww}(f(k) - kf^{'}(k))$ and the workers' saving out of capital revenues consist in $s_{wP}f^{'}(k)k_w$, where $0 \leq s_{ww} \leq 1$, $0 \leq s_{wP} \leq 1$. Instead the capitalists' savings are $s_c f^{'}(k)k_c$, where $0 \leq s_c \leq 1$. We assume $s_c > max\{s_{ww}, s_{wP}\}$. Thus the aggregate savings correspond to

$$s(k_c, k_w) = s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w + s_c f^{'}(k_c).$$

Let $n$ be the constant rate of growth of labor force, the following map

$$G(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k + i]$$

describes the rule of capital accumulation per worker, where $i$ indicates gross investment per worker and $0 < \delta < 1$ is the constant rate of capital depreciation. In a short-run equilibrium $G$ becomes

$$G(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k + s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w + s_c f^{'}(k_c)],$$

from which we deduce the capitalist' process of capital accumulation

$$G_w(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k_w + s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w]$$

and the capitalist's rule of capital accumulation

$$G_c(k_w, k_c) = \frac{1}{1+n}[(1-\delta)k_c + s_c f^{'}(k)k_c].$$

In order to obtain the steady states of $G_w$ and $G_c$, we imposing

$G_w(k_w, k_c) = k_w$ and $G_c(k_w, k_c) = k_c$.

We get

$(n+\delta)k_w = s_{ww}(f(k) - f^{'}(k)k) + s_{wP}f^{'}(k)k_w$, $(\star)$

$(n+\delta)k_c = s_c f^{'}(k_c)$ $(\star\star)$

We find three types of equilibria: *Pasinetti equilibrium* (capitalists own positive share of capital), *dual equilibrium* (only workers own capital) and *trivial equilibrium* (the overall capital is zero).

**Pasinetti equilibrium**

Now we indicate a Pasinetti equilibrium with $(k_w^P, k_c^P)$,

where, by definition, $k^P = k_w^P + k_c^P$. We prove the following

**Proposition 7.2.1.1** For the Pasinetti Equilibrium the following conditions hold:

- $f^{'}(k^P) = \frac{n+\delta}{s_c}$,
- $k_w^P = \frac{s_{ww}}{s_c - s_{wP}} \frac{1-e_f(k^P)}{e_f(k^P)} k^P$,
- $k_c^P = (1 - \frac{s_{ww}}{s_c - s_{wP}} \frac{1-e_f(k^P)}{e_f(k^P)})k^P$.

**Proof** We start by the relation $(\star\star)$. Since $k_c \neq 0$ then $(n+\delta) = s_c f^{'}(k)$, from which $f^{'}(k^P) = \frac{n+\delta}{s_c}$. In the left-hand side of $(\star)$, we replace $(n+\delta)$ with $s_c f^{'}(k)$. We get

$$s_c f^{'}(k)k_w - s_{wp}f^{'}(k)k_w = s_{ww}(f(k) - f^{'}(k)k),$$

$$k_w f^{'}(k)(s_c - s_{wp}) = s_{ww}(f(k) - f^{'}(k)k),$$

$$k_w f^{'}(k)(s_c - s_{wp}) = s_{ww}f(k)[1 - \frac{f^{'}(k)k}{f(k)}],$$

$k_w f^{'}(k)k(s_c - s_{wp}) = s_{ww} f(k)[1 - \frac{f^{'}(k)k}{f(k)}]k,$

$k_w \frac{f^{'}(k)k}{f(k)}(s_c - s_{wp}) = s_{ww}[1 - \frac{f^{'}(k)k}{f(k)}]k,$

$k_w e_f(k)(s_c - s_{wp}) = s_{ww}(1 - e_f(k))k,$

$k_w^P = \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)} k^P.$

Since $k^P = k_w + k_c$, we have $k_c = k^P - k_w$, from which

$k_c^P = k^P - \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)} k^P = [1 - \frac{s_{ww}}{s_c - s_{wp}} \frac{1-e_f(k)}{e_f(k)}]k^P.$

**Dual equilibrium**

We indicate the dual equilibrium with $(k_w^D, k_c^D)$, where $k^D = k_w^D + k_c^D$.

We prove the following

**Proposition 7.2.2.1** The dual equilibria are given by the relations

$\frac{f(k^D)}{k^D} = \frac{n+\delta}{s_{ww}(1-e_f(k^D)) + s_{wP} e_f(k^D)}$, $k_w^D = k^D$ and $k_c^D = 0$

**Proof** We rewrite the relation $(\star)$ replacing $k_w^D$ with $k^D$ and $k$ with $k^D$.

We get

$(n+\delta)k^D = s_{ww}(f(k^D) - f^{'}(k^D)k^D) + s_{wp} f^{'}(k^D)k^D,$

from which

$(n+\delta)k^D = s_{ww} f(k^D)(1 - \frac{f^{'}(k^D)k^D}{f(k^D)}) + s_{wp} \frac{f^{'}(k^D)k^D}{f(k^D)},$

$(n+\delta)\frac{k^D}{f(k^D)} = s_{ww}(1 - e_f(k^D)) + s_{wp} e_f(k^D),$

$\frac{f(k^D)}{k^D} = \frac{n+\delta}{s_{ww}(1-e_f(k^D)+s_{wp}e_f(k^D}.$

**Trivial equilibrium**

$(k_w^0, k_c^0)$ and $k^0 = k_w^0 + k_c^0$ where $k^0 = k_w^0 = k_c^0 = 0.$

**Output elasticity**

We see immediately that

$e_f(k) = \frac{kf^{'}(k)}{f(k)} = (1-\alpha)(\alpha k^{-\rho} + 1 - \alpha)^{-1},$

$0 < e_f(k) \leq 1.$

### 2.6.3 Meade's Relation For Pasinetti Equilibria

We introduce the *Meade's relation* for Pasinetti equilibria

$\frac{f(k)}{k} = \varphi(e_f(k)),$

where $\varphi(x) = (\frac{1-\alpha}{x})^{\frac{1}{\rho}}.$

We notice that for $\varphi(x)$ occurs:

- $\varphi^{'}(x) = \frac{(1-\alpha)}{\rho}(\frac{1-\alpha}{x})^{\frac{1}{\rho}-1}(-\frac{1}{x^2}) = -\frac{(1-\alpha)}{\rho}\frac{1}{x^2}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$
- $\varphi^{''}(x) = -\frac{(1-\alpha)}{\rho}\{-2x^{-3}(\frac{1-\alpha}{\rho})^{\frac{1-\rho}{\rho}} + x^{-2}(\frac{1-\rho}{\rho})(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}-1}(1-\alpha)(-x^{-2})\}$

  $= \frac{(1-\alpha)}{\rho}x^{-3}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}(2 + \frac{1-\rho}{\rho})$

  $= (1+\rho)\frac{(1-\alpha)}{\rho^2}x^{-3}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$

The former features of $\varphi(x)$ lead us to state that (See **Figure 2.5**)

**Proposition 7.3.1** For the function $\varphi(x)$ is true that:

- it is strictly monotonic for all $\rho < 1$ and $\rho \neq 0$;
- it is strictly convex for all $0 < \rho < 1$ and strictly concave for all $\rho < -1$;
- it becomes the line $\varphi(x) = \frac{x}{1-\alpha}$ if $\rho = -1$.
- $\lim_{x \to 0} \varphi(x) = +\infty$ if $0 < \rho < 1$.

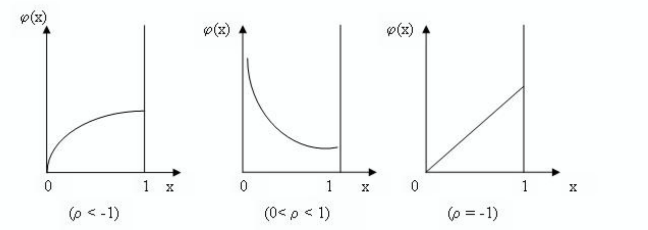



Figure 2.5: *The diagram of $\varphi$ for different $\rho$.*

**Proposition 7.3.2** Both workers and capitalists own a positive share of capital if and only if

$$0 < e_f^T < e_f(k^P) < 1,$$

where $e_f^T = \frac{s_{ww}}{s_c - (s_{wP} - s_{ww})}$.

**Proof** We observe that $k_w^P > 0$ is equivalent to say that ($e_f < 1$ and $s_c > s_{wp}$) or ($e_f > 1$ and $s_c < s_{wp}$).

We don't accept the second condition because the CES don't satisfies the inequality $e_f > 1$.

Moreover the inequality $k_c^P > 0$ holds iff $\frac{1-e_f}{e_f}\frac{s_{ww}}{s_c - s_{wP}} < 1$.

Thus is true that

$$\frac{1-e_f}{e_f} < \frac{s_c - s_{wP}}{s_{ww}},$$

from which

$$\frac{1}{e_f} < 1 + \frac{s_c - s_{wP}}{s_{ww}},\ \frac{1}{e_f} < \frac{s_c - (s_{wP} - s_{ww})}{s_{ww}}.\ \text{Q.E.D.}$$

Observed that

- Case (a): $s_{ww} = s_c$. Then $e_f^T = \frac{s_{ww}}{s_c}$;
- Case (b: $s_{ww} < s_c$. Then $s_c - (s_{wP} - s_{ww}) < s_c$;
- Case (c): $s_{ww} > s_c$. Then $s_c - (s_{wP} - s_{ww}) > s_c$;

we deduce that

$e_f^T(Case(c)) < e_f^T(Case(a)) < e_f^T(Case(b))$.

**Proposition 7.3.3** We have $e_f(k^P) = (1-\alpha)^{\frac{1}{1-\rho}}\ (\frac{n+\delta}{s_c})^{\frac{\rho}{\rho-1}}$

**Proof** From definition of $e_f$ we obtain that $\frac{f(k)}{k} = \frac{f'(k)}{e_f(k)}$ and by Meade's relation $\frac{f(k)}{k} = \varphi(e_f(k))$ we get $\varphi(e_f(k^P)) = \frac{f'(k^P)}{e_f(k^P)} = \frac{n+\delta}{s_c}\frac{1}{e_f(k^P)}$: *the intersection between the arc of hyperbola* $\Gamma : \frac{n+\delta}{s_c}\frac{1}{e_f(k^P)}$ *and the curve* $\varphi(e_f(k^P))$ *identifies the unique Pasinetti equilibrium.*

From $e_f(k^P) = \frac{f'(k^P)}{\varphi(e_f(k^P))}$ and by definition of $\varphi(k)$ we have $(\frac{n+\delta}{s_c})\ (\frac{e_f(k^P)}{1-\alpha})^{\frac{1}{\rho}} = e_f(k^P)$. We obtain

$(\frac{n+\delta}{s_c})^{\rho}(\frac{e_f(k^P)}{1-\alpha}) = (e_f(k^P))^{\rho}$,

$(e_f(k^P))^{\rho-1} = \frac{1}{1-\alpha}(\frac{n+\delta}{s_c})^{\rho}$. Q.E.D.

Commendatore (2005), generalizing a relation of Samuelson-Modigliani (1966) and Miyazaki (1991), shows that

**Proposition 7.3.4** We assume that:

- $f^{'}(k)$ is monotonically increasing,
- $e_f(k) < 1$,
- $s_{ww} \leq s_{wP}$,
- $k^D > k^P$.

Then is true that
$$e_f^T > e_f(k^P),$$

where $e_f^T = \frac{s_{ww}}{s_c - (s_{wP} - s_{ww})}$ and $e_f(k) = \frac{kf^{'}(k)}{f(k)}$.

**Proof** We observe that *a CES production function satisfies the former two assumptions of proposition* first, then we prove that $\frac{f(k)}{k}$ *is monotonically decreasing if and only if* $f^{'}(k) < \frac{f(k)}{k}$ . As a matter of fact, let $g(k) = \frac{f(k)}{k}$ be. We have that $g^{'}(k) = \frac{f^{'}(k)k - f(k)}{k^2} < 0$ if and only if $f'(k)k < f(k)$. Since $e_f(k) = \frac{f^{'}(k)k}{f(k)} < 1$ then the previous inequality is satisfied. Thus from the assumption $k^P < k^D$ we deduce $\frac{f(k^P)}{k^P} > \frac{f(k^D)}{k^D}$.

Moreover the *dual equilibrium* can be rewritten as follows

$$(n+\delta)k^D = s_{ww}(f(k^D) - f^{'}(k^D)k^D) + s_{wP}f^{'}(k^D)k^D,$$

$$(n+\delta)k^D = s_{ww}f(k^D) - s_{ww}f^{'}(k^D)k^D + s_{wP}f^{'}(k^D)k^D,$$

$$(n+\delta)k^D = s_{ww}f(k^D) + (s_{wP} - s_{ww})f^{'}(k^D)k^D,$$

$$(n+\delta) = s_{ww}\frac{f(k^D)}{k^D} + (s_{wP} - s_{ww})f^{'}(k^D),$$

$\frac{(n+\delta)}{s_{ww}} = \frac{f(k^D)}{k^D} + \frac{s_{wP}-s_{ww}}{s_{ww}} f^{'}(k^D)$,

$\frac{f(k^D)}{k^D} = \frac{(n+\delta)}{s_{ww}} - \frac{s_{wP}-s_{ww}}{s_{ww}} f^{'}(k^D)$.

Therefore $\frac{f(k^P)}{k^P} > \frac{(n+\delta)}{s_{ww}} - \frac{s_{wP}-s_{ww}}{s_{ww}} f^{'}(k^D)$.

Then, recalling that $s_{ww} \leq s_{wP}$ and $f^{'}(k^P) = \frac{n+\delta}{s_c}$ , we have

$s_{ww}\frac{f(k^P)}{k^P} > (n+\delta) - (s_{wP} - s_{ww})f^{'}(k^D) = s_c f'(k^P) - (s_{wP} - s_{ww})f^{'}(k^D)$,

and, observing that from the strict monotonicity of $f^{'}(k)$, the inequality $k^D > k^P$ implies $f^{'}(k^D) > f^{'}(k^D)$, we get

$s_{ww}\frac{f(k^P)}{k^P} > [s_c - (s_{wP} - s_{ww})]f^{'}(k^P)$. Q.E.D.

### 2.6.4 Meade's Relation For Dual Equilibria

In order to detect geometrically the dual equilibria we will use the following *Meade's relation* for dual equilibria

$\frac{f(k)}{k} = \theta(e_f(k))$,

where $\theta(x) = \frac{n+\delta}{s_{ww}(1-x)+s_{wP}x}$.

We observe that

- $\theta : [0,1] \rightarrow [0,1]$ and $\theta(x) > 0$ for all $x \in [0,1]$;
- $\theta(0) = \frac{n+\delta}{s_{ww}} > 0$ and $\theta(1) = \frac{n+\delta}{s_{wP}} > 0$;
- $\theta(x)$ is a continuous function in [0,1 ];

- $\theta^{'}(x) = (s_{ww} - s_{wP})\frac{\theta(x)^2}{n+\delta}$;
- $\theta^{''}(x) = \frac{2(s_{ww}-s_{wP})^2}{(n+\delta)^2}\theta(x)^3 \geq 0$;

Thus $\theta(x)$ is (See **Figure 2.6**)

- constant if $s_{ww} = s_{wP}$;
- strictly monotonically increasing if $s_{ww} > s_{wP}$;
- strictly monotonically decreasing if $s_{ww} < s_{wP}$;
- strictly convex if $s_{ww} \neq s_{wP}$.

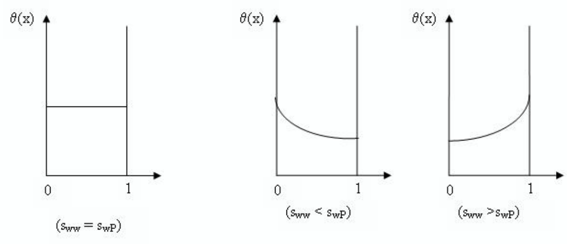



Figure 2.6: *The diagram of $\theta$ for different comparisons of $s_{ww}$ with $s_{wP}$.*

**Proposition 7.4.1** The dual equilibria are given by the set

$\{x \in [0, 1] : \varphi(x) = \theta(x)\}$.

**Proof** We distinguish the following two cases:

- Case I: $\rho = -1$. Then $\varphi(x)$ becomes $(\frac{1-\alpha}{x})^{-1}$. Thus we must solve the equation (See **Figure 2.7**)

$\frac{x}{1-\alpha} = \frac{n+\delta}{s_{ww}(1-x)+s_{wp}x}$.

If $s_{ww} = s_{wP}$ then the equation $\varphi(x) = \theta(x)$ is equivalent to relation

$\frac{x}{1-\alpha} = \frac{n+\delta}{s_{ww}}$,

from which, trivially, it follows the solution $x = \frac{n+\delta}{s_{ww}}(1-\alpha)$. We notice that $x$ is acceptable iff $x \in [0,1]$.

If $s_{ww} \neq s_{wp}$, from the relation

$x[s_{ww}(1-x) + s_{wP}x] = (n+\delta)(1-\alpha)$,

we obtain that

$-s_{ww}x^2 + (s_{ww} + s_{wP})x = (n+\delta)(1-\alpha)$.

Thus

$s_{ww}x^2 - (s_{ww} + s_{wP})x + (n+\delta)(1-\alpha) = 0$.

We set

$A = s_{ww}$, $B = -(s_{ww} + s_{wP})$, $C = (n+\delta)(1-\alpha)$, $\Delta = B^2 - 4AC$.

We may conclude that *if* $\Delta \geq 0$ *then dual equilibria exist (two real repeated equilibria or two real distinct equilibria).*

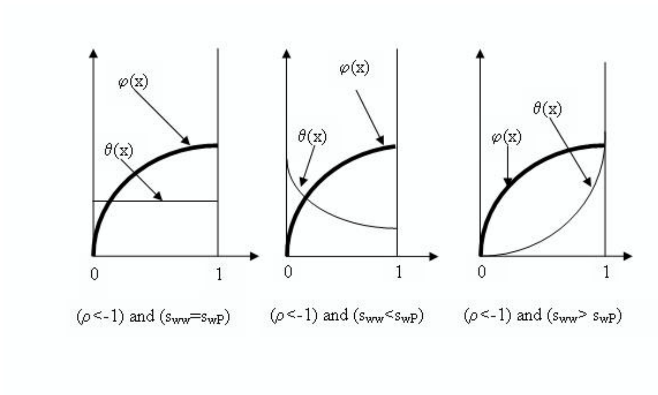



Figure 2.7: *The diagram of* $\varphi$ *for* $\rho = -1$ *and the different diagrams of* $\theta$.

- Case II: $(\rho < -1) \vee (0 < \rho < 1)$.

  We find the solutions of the equation (See **Figure 2.8** and **Figure 2.9**)

  $(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}(1-x)+s_{ww}x}$.

  We may rewrite the previous equation such that (for details, see **Remark 1**)

  $\frac{1-\alpha}{(n+\delta)^{\rho}} = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}$.

  Now we set $g(x) = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}$.

  After some transformations (see **Remark 2**) we get

  $g^{'}(x) = \frac{s_{ww}+(1-\rho)(s_{wP}-s_{ww})x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho+1}}$.

  If $s_{wP} \geq s_{ww}$ then $g(x)$ *is strictly monotonically increasing in* $[0, 1]$ and the range of $g(x)$ is

  $$[0, \frac{1}{[s_{ww}+(s_{wP}-s_{ww})]^{\rho}}].$$

  By *Bolzano's Theorem* and by the strictly monotonicity of $g(x)$ exists an unique solution of equation

  $$g(x) = \frac{1-\alpha}{(n+\delta)^{\rho}}.$$

  If $s_{ww} < s_{wp}$ then $g(x)$ can be monotonically decreasing and exists an unique dual equilibrium.

  Notice that $g^{'}(x) = 0$ iff $s_{ww}+(1-\rho)(s_{wP}-s_{ww})x$, i.e., $x = -\frac{s_{ww}}{(1-\rho)(s_{wp}-s_{ww})}$.

  Therefore the point $x^{\star} = \frac{s_{ww}}{(1-\rho)(s_{wP}-s_{ww})}$ may be the maximum or minimum for $g(x)$.

  Observed that $g(x)$ is strictly concave (or strictly convex), also by Bolzano's Theorem, we obtain one or two dual equilibrium if and only if $\frac{1-\alpha}{(n+\delta)^{\rho}} \leq g(x^{\star})$.

We can say that an unique dual equilibrium exists if the line $y = \frac{1-\alpha}{(n+\delta)^\rho}$ intersects the graph of function $g(x)$ at $(x^\star, g(x^\star))$, being $g(x^\star)$ the maximum of $g(x)$.

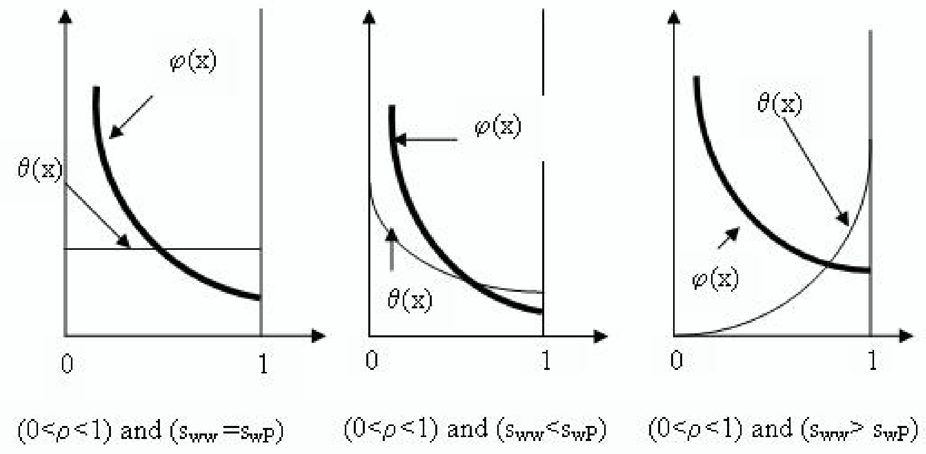



Figure 2.8: *The diagram of $\varphi$ for $\rho < -1$ and the different diagrams of $\theta$.*

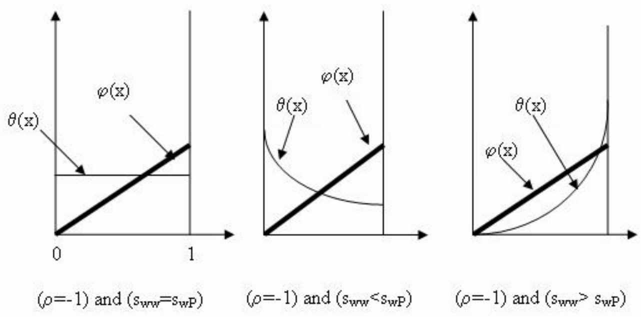



Figure 2.9: *The diagram of $\varphi$ for $0 < \rho < 1$ and the different diagrams of $\theta$.*

In the figures **2.10, 2.11, 2.12** we identify the steady-growth equilibria (*Pasinetti, Dual and Trivial*) for the cases (a)$s_{ww} = s_{wP}$, (b) $s_{ww} < s_{wP}$ and (c)$s_{ww} > s_{wP}$:

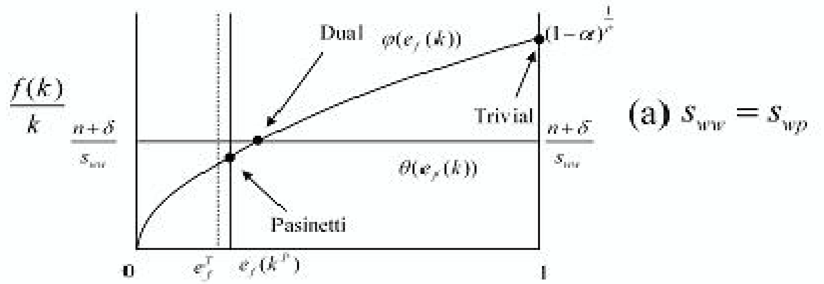



Figure 2.10: *Steady-growth equilibria identified for the case* $s_{ww} = s_{wP}$.

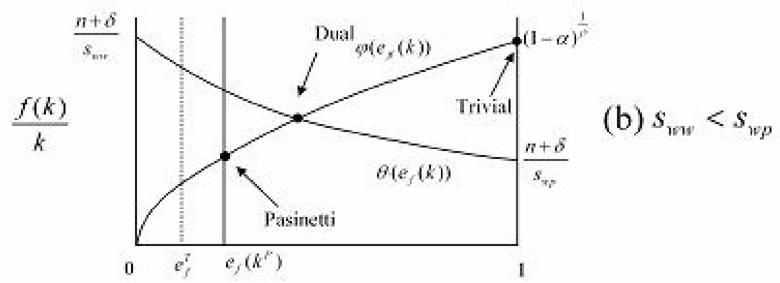



Figure 2.11: *Steady-growth equilibria identified for the case* $s_{ww} < s_{wP}$.

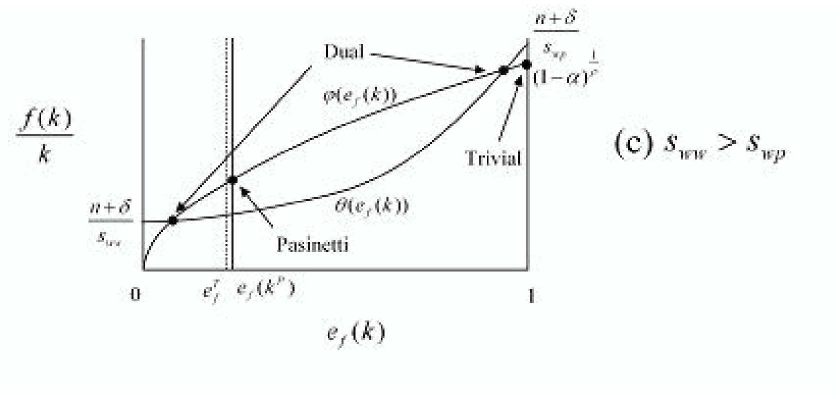



Figure 2.12: *Steady-growth equilibria identified for the case* $s_{ww} > s_{wP}$.

**Remark 1**

$$(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}+(s_{wP}-s_{ww})x},$$

$$(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{n+\delta}{s_{ww}-s_{ww}x+s_{wP}x},$$

$$\frac{(1-\alpha)^{\frac{1}{\rho}}}{x^{\frac{1}{\rho}}} = \frac{n+\delta}{s_{ww}-s_{ww}x+s_{wP}x},$$

$$\frac{1-\alpha}{(n+\delta)^{\rho}} = \frac{x}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}}$$

**Remark 2**

$$g^{'}(x) = \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}-\rho x(s_{wP}-s_{ww})[s_{ww}+(s_{wP}-s_{ww})x]^{\rho-1}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}\{1-\rho x(s_{wP}-s_{ww})[s_{ww}+(s_{wP}-s_{ww})x]^{-1}\}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho}\{1-\frac{\rho(s_{ww}-s_{wP})x}{s_{ww}+(s_{wP}-s_{ww})x}\}}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho}}$$

$$= \frac{s_{ww}+(s_{wP}-s_{ww})x-\rho x(s_{wP}-s_{ww})}{[s_{ww}+(s_{wP}-s_{ww})x]^{2\rho-\rho+1}} = \frac{s_{ww}+(1-\rho)x(s_{wP}-s_{ww}))}{[s_{ww}+(s_{wP}-s_{ww})x]^{\rho+1}}.$$

We note that $e_f(k=0) = (1-\alpha)(1-\alpha)^{-1} = 1$, from which $\varphi(e_f(0)) = \varphi(1) = (1-\alpha)^{\frac{1}{\rho}}$. Thus *the intersection between the curve $\varphi(e_f(k))$ and the vertical line at 1 identifies the trivial equilibrium.*

### 2.6.5 Local stability analysis

#### 1.7.5.1 The Jacobian evaluated at a Pasinetti equilibrium

In order to determine the local stability of the fixed points of our dynamical system we will linear approximate it with **the Hartman-Grobman Theorem**. We begin with the Jacobian matrix of the dynamical system evaluated at a Pasinetti-equilibrium:

$$J(k_w^P, k_c^P) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{11} = \frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^P)k^P + s_{wP}(f^{'}(k^P) - f^{''}(k^P)k_c^P)],$$

$$J_{12} = \frac{1}{1+n}[(s_{wP}-s_{ww})f^{''}(k^P)k^P - s_{wP}f^{''}(k^P)k_c^P],$$

$$J_{21} = \frac{1}{1+n}[s_c f^{''}(k^P)k_c^P],$$

$$J_{22} = \frac{1}{1+n}[1-\delta+s_c(f^{'}(k^P) + f^{''}(k^P)k_c^P)].$$

After some transformations we obtain the *trace* of the Jacobian matrix at the Pasinetti-equilibrium

$$T(k_w^P, k_c^P) = \frac{n+\delta}{1+n}[\frac{2(1-\delta)}{n+\delta} + 1 + e_{f'}(k^P) + (\frac{s_{wP}e_f(k^P) - s_{ww}e_{f'}(k^P)}{s_c e_f(k^P)})],$$

and the *determinant* of the Jacobian matrix at the Pasinetti-equilibrium

$$D(k_w^P, k_c^P) = T(k_w^P, k_c^P)(\frac{1-\delta}{1+n}) - (\frac{1-\delta}{1+n})^2 + \frac{e_{f'}(k^P)(s_{wP} - s_{ww}) + s_{wP}}{s_c}(\frac{n+\delta}{1+n})^2.$$

For two-dimensional discrete time maps, to search the region of stability of Pasinetti-equilibrium and to study how here frontier is crossed, we will apply the following three conditions:

(1) $1 + T(k_w^P, k_c^P) + D(k_w^P, k_c^P) > 0$;

(2) $1 - T(k_w^P, k_c^P) + D(k_w^P, k_c^P) > 0$;

(3) $1 - D(k_w^P, k_c^P) > 0$.

The previous relations in the plane *trace-determinant* lead to *the triangle of stability* and they guarantee that the modulus of each eigenvalue of the Jacobian matrix, calculated at the Pasinetti-equilibrium, is less than one (see **Chapter 1**). From the characteristic equation we derive the eigenvalues of the Jacobian matrix evaluated at an equilibrium point. For the Pasinetti-equilibrium we have:

$\lambda_i^P = \frac{1}{2}(T(k_w^P, k_c^P) \pm \sqrt{(T(k_w^P, k_c^P))^2 - 4D(k_w^P, k_c^P)})$, where $i = 1, 2$.

Commendatore (2005), rewriting the stability conditions in terms of $e_f(k)$ and $e_{f'}(k)$, deduces very interesting relations.

Setting

$$e_{f'}^F = -2(\frac{1+n}{n+\delta})\frac{(n+2-\delta)s_c + (n+\delta)s_{wP}}{(n+2-\delta)(s_c - s_{ww}\frac{1}{e_f(k)}) + (n+\delta)(s_{wP} - s_{ww})},$$

and

$$\bar{e}_f = \frac{s_{ww}(n+2-\delta) - (s_{wp} - s_{ww})(n+\delta)}{s_c(n+2-\delta)},$$

from (1), after some transformations, we obtains the first relations:

- $e_{f'}(k) > e^F_{f'}$ if $e_f(k) > \overline{e}_f$;
- $e_{f'}(k) < e^F_{f'}$ if $e_f(k) < \overline{e}_f$.

In the $(e_f(k)), e_{f'}(k))$-plane the former inequality is satisfied by points which are above the diagram of $e^F_{f'}$ and at left of the right-line $e_f(k) = \overline{e}_f$. Analogously we will think for the last inequality. Moreover the condition (2) always holds if $e_f(k) < \overline{e}_f$ and it reduces to relation $e_f > e^T_f$.

We pose

$$e^N_{f'} = \frac{(s_c - s_{wP})(1+n)}{(s_{wP} - s_{ww})(n+\delta) + (1-\delta)(s_c - s_{ww}\frac{1}{e_f(k)})},$$

and

$$\overline{\overline{e}}_f = \frac{s_{ww}}{s_c + (s_{wP} - s_{ww})\frac{n+\delta}{1-\delta}}.$$

We have that the condition (3) is equivalent to the inequalities

- $e_{f'}(k) < e^N_{f'}$ for $e_f(k) > \overline{\overline{e}}_f$;
- $e_{f'}(k) > e^N_{f'}$ for $e_f(k) < \overline{\overline{e}}_f$.

We note that:

- $e^F_{f'}$ depends on $e_f \neq e_0$, where $e_0 = \frac{(n+2-\delta)s_{ww}}{(n+\delta)(s_{wP} - s_{ww}) + (n+2-\delta)s_c}$;
- $e^F_{f'}$ is continuous and monotonically strictly increasing in $X = ]0, e_0[\cup]e_0, 1]$;
- $e^F_{f'}$ is never vanish in $X$;
- $\lim_{e_f \to e_0} e^F_{f'} = \infty$: in the $(e_f, e^F_{f'})$-plane the straight-line $e_f = e_0$ is an asymptote for $e^F_{f'}$;
- $\lim_{e_f \to 0} e^F_{f'} = 0$;

- $\lim_{e_f \to 1} e^F_{f'} = -2(\frac{1+n}{n+\delta})\frac{(n+2-\delta)s_c+(n+\delta)s_{wP}}{(n+2-\delta)(s_c-s_{ww})+(n+\delta)(s_{wP}-s_{ww})}$;
- $\lim_{e_f \to e^T_f} e^F_{f'} = -\frac{(n+2-\delta)s_c+(n+\delta)s_{wP}}{(n+\delta)(s_{wP}-s_{ww})} \begin{cases} <0 & \text{if } s_{wP} > s_{ww}, \\ >0 & \text{if } s_{wP} < s_{ww}; \end{cases}$
- by the **Theorem about Sign Permanence** the function $e^F_{f'}$ has constant sign on both convexes $]0, e_0[$ and $]e_0, 1]$, particularly $e^F_{f'}$ is positive on the left of $e_0$ and negative on the right of $e_0$. Moreover the *test-point* $e^T_f$ lies on the left of $e_0$ if $s_{wP} < s_{ww}$ and on the right of $e_0$ if $s_{wP} > s_{ww}$.

Analogously for $e^N_{f'}$ we may say that:

- $e^N_{f'}$ depends on $e_f \neq e_{00}$, where $e_{00} = \frac{(1-\delta)s_{ww}}{(n+\delta)(s_{wP}-s_{ww})+(1-\delta)s_c}$;
- $e^N_{f'}$ is continuous and monotonically strictly decreasing in $X = ]0, e_{00}[ \cup ]e_{00}, 1]$;
- $e^N_{f'}$ is never vanish in $X$;
- $\lim_{e_f \to e_0} e^N_{f'} = \infty$: in the $(e_f, e^N_{f'})$-plane the straight-line $e_f = e_{00}$ is an asymptote for $e^N_{f'}$;
- $\lim_{e_f \to 0} e^N_{f'} = 0$;
- $\lim_{e_f \to 1} e^N_{f'} = \frac{(s_c-s_{wP})(1+n)}{(s_{wP}-s_{ww})(n+\delta)+(1-\delta)(s_c-s_{ww})}$;
- $\lim_{e_f \to e^T_f} e^N_{f'} = \frac{s_c-s_{ww}}{s_{wP}-s_{ww}} \begin{cases} <0 & \text{if } s_{wP} < s_{ww}, \\ >0 & \text{if } s_{wP} > s_{ww}; \end{cases}$
- by the **Theorem about Sign Permanence** the function $e^N_{f'}$ has constant sign on both convexes $]0, e_{00}[$ and $]e_{00}, 1]$, particularly $e^N_{f'}$ is negative on the left of $e_{00}$ and positive on the right of $e_{00}$. Moreover the *test-point* $e^T_f$ lies on the left of $e_{00}$ if $s_{wP} < s_{ww}$ and on the right of $e_{00}$ if $s_{wP} > s_{ww}$.

#### 1.7.5.2 The Jacobian matrix evaluated at a dual equilibrium

Setting $k^D_c = 0$ we calculate the Jacobian matrix at a dual equilibrium we obtain

$$J(k^D_w, k^D_c) = \begin{pmatrix} \frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D)] & \frac{1}{1+n}(s_{wP}-s_{ww})f^{''}(k^D)k^D \\ 0 & \frac{1}{1+n}(1-\delta+s_cf^{'}(k^D)) \end{pmatrix}.$$

Since the Jacobian matrix $J(k_w^D, k_c^D)$ is a diagonal matrix on $\Re$, then *the eigenvalues* $\lambda_1^D$ *and* $\lambda_2^D$ *are real and they correspond to diagonal elements of the matrix* $J(k_w^D, k_c^D)$. Therefore *the dual equilibrium can't lose stability through a Neimark-Saker bifurcation.* We recall that the dual equilibrium is stable if $-1 < \lambda_1^D < 1$ and $-1 < \lambda_2^D < 1$. The expression of eigenvalues depends on saving propensities $s_{ww}$ and $s_{wp}$ and that lead us to distinguish three cases:

- **Case I**: $s_{ww} = s_{wp}$. The eigenvalues become $\lambda_1^D = \frac{1}{1+n}[1-\delta+s_{wp}f^{'}(k^D)]$ and $\lambda_2^D = \frac{1}{1+n}[1-\delta+s_c f^{'}(k^D)]$. Since $f^{'}(k^D) > 0$ we deduce that *both eigenvalues are positive.* By the assumption $s_{wp} < s_c$ we obtain that $\lambda_1^D < \lambda_2^D$. Thus the stability conditions for dual equilibrium reduces to relation $\lambda_2^D < 1$, which holds for $k^D > k^P$. As a matter of fact, the inequality $\lambda_2^D < 1$ is equivalent to relation $\frac{1}{1+n}[1-\delta+s_{wp}f^{'}(k^D)] < 1$, from which we have firstly $f'(k^D) < \frac{n+\delta}{s_c}$ and secondly, by $f^{'}(k^P) = \frac{n+\delta}{s_c}$, $f^{'}(k^D) < f^{'}(k^P)$. Finally, by the property $f^{''}(k) < 0$ of CES production function, we deduce $k^D > k^P$. Commendatore (2005) explains the last inequality saying that *a stability loss involves a transcritical bifurcation which goes in the opposite direction to the one that concerns the Pasinetti equilibrium. Now, it is the dual equilibrium which loses stability and the Pasinetti equilibrium, already existing, that gains stability.*

- **Case II**: $s_{ww} < s_{wp}$. Since $f^{''}(k^D) < 0$ we notice that the term $(s_{wP} - s_{ww})f^{''}(k^D)k^D$ of eigenvalue $\lambda_1^D$ is negative and $\lambda_1^D$ could be itself negative. Everyone $\lambda_2^D > 0$ and $\lambda_2^D > max\{\lambda_1^D, 0\}$. Thinking as above, we deduce that $\lambda_2^D < 1$ for $k^P > k^D$. Moreover from inequality $\lambda_1^D > -1$ we obtain the following equivalent relations

  $$\frac{1}{1+n}[1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D)] > -1,$$

  $$1-\delta+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D) > -1-n,$$

  $$(2+n-\delta)+(s_{wP}-s_{ww})f^{''}(k^D)k^D + s_{wP}f^{'}(k^D) > 0,$$

  $$\frac{2+n-\delta}{f^{'}(k^D)} + (s_{wP}-s_{ww})\frac{f^{''}(k^D)k^D}{f^{'}(k^D)} + s_{wP} > 0,$$

  $$\frac{s_{wP}+\frac{2+n-\delta}{f^{'}(k^D)}}{s_{wP}-s_{ww}} + e_{f'}(k^D) > 0,$$

  $$e_{f'}(k^D) > \epsilon_F < -1,$$

  where $\epsilon_F = -\frac{s_{wP}+\frac{2+n-\delta}{f^{'}(k^D)}}{s_{wP}-s_{ww}}$.

We observe that *the stability of dual equilibrium may be lost through a transcritical bifurcation when* $\lambda_2^D$ *crosses 1 or through a flip bifurcation when* $\lambda_1^D$*crosses* $-1$.

- **Case III**: $s_{ww} > s_{wp}$. We notice immediately that both eigenvalues are positive. As a matter of fact is sufficient to observe that the term $(s_{wP} - s_{ww})f^{''}(k^D)k^D$ of $\lambda_1^D$ is positive. Moreover $\lambda_2^D < 1$ for $k^D > k^P$ and $\lambda_2^D < 1$ for $e_{f'}(k^D) > \epsilon^S < 0$, where

$$\epsilon^S = -\frac{\frac{n+\delta}{f'(k^D)} - s_{wP}}{s_{ww} - s_{wP}}.$$

We conclude that *the dual equilibrium may lose stability through a saddle-node (fold or tangent) bifurcation and two equilibria of dual type are created, one stable and the other unstable.*

#### 1.7.5.3 The Jacobian matrix evaluated at a trivial equilibrium

We recall that if $f(k)$ is the $CES$ production function then $f^{'}(0) = (1-\alpha)^{\frac{1}{\rho}}$, where $0 < \alpha < 1$ and $\rho < 1$ ($\rho \neq 0$), i.e. $0 < f^{'}(0) < \infty$. By definition of trivial equilibrium we have

$$J(k_w^0, k_c^0) = \begin{pmatrix} \frac{1}{1+n}(1-\delta+s_{wP}f^{'}(0)) & 0 \\ 0 & \frac{1}{1+n}(1-\delta+s_c f^{'}(0)) \end{pmatrix}.$$

Since the Jacobian matrix $J(k_w^0, k_c^0)$ is an upper triangular matrix on $\Re$, then the eigenvalues $\lambda_1^0$ and $\lambda_2^0$ are real and lie along the principal diagonal of the matrix $J(k_w^0, k_c^0)$. If we assume $s_{wp} < s_c$, we get $0 < \lambda_1^0 < \lambda_2^0$. Therefore the stability of trivial equilibrium depends on the inequality $\lambda_2^0 < 1$, i.e. $f^{'}(0) < \frac{n+\delta}{s_c}$. We recall that $f^{'}(k^P) = \frac{n+\delta}{s_c}$ and $f^{''}(k) < 0$. Then we derive the relation $k^P < 0 = k^0$, that can't occur. Thus *the trivial equilibrium is never stable.*

## 2.7 Appendix: A CES Production Function

We define *CES Production Function* , where the term *CES* stands for *Constant Elasticity of Substitution*, the following function

$f(k) = [\alpha + (1-\alpha)k^\rho]^{\frac{1}{\rho}}$,

being $k$ the capital/labor ratio, $0 < \alpha < 1$ a constant, $-\infty < \rho < 1$ and $\rho \neq 0$ a parameter.

The main features of CES production function $f(k)$ are:

1. $f'(k) > 0$ for all $k \geq 0$ (*i.e. f(k) is increasing*);
2. $f''(k) < 0$ for all $k \geq 0$ (*i.e. f(k) is concave*);
3. $\lim_{\rho \to 0} f(k) = k^{1-\alpha}$ (*i.e. when* $\rho$ *tends towards* 0 *the CES behaves as a Cobb-Douglas*);
4. $\lim_{\rho \to -\infty} f(k) = min\{1, k\} = \begin{cases} k, & \text{if } 0 < k < 1 \\ 1, & \text{if } k \geq 1 \end{cases}$;
5. $\lim_{\rho \to 1} f(k) = \alpha + (\alpha - 1)k$;
6. $0 < f'(0) < \infty$.

As a matter of fact:

- $f'(k) = \frac{1}{\rho}[\alpha + (1-\alpha)k^\rho]^{\frac{1}{\rho}-1}\rho(1-\alpha)k^{\rho-1}$

  $= (1-\alpha)k^{\rho-1}[\alpha + (1-\alpha)k^\rho]^{\frac{1}{\rho}-1}$

  $= (1-\alpha)k^{\rho-1}k^{1-\rho}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}}$

  $= (1-\alpha)[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}} > 0$;

- $f''(k) = (1-\alpha)\frac{1-\rho}{\rho}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-\rho}{\rho}-1}(-\rho\alpha k^{-\rho-1})$

  $= \alpha(1-\alpha)(\rho-1)k^{-\rho-1}[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1-2\rho}{\rho}} < 0$;

- $\lim_{\rho\to 0} f(k) = \lim_{\rho\to 0} e^{\frac{\ln[\alpha+(1-\alpha)k^{\rho}]}{\rho}} = \lim_{\rho\to 0} e^{\frac{(1-\alpha)k^{\rho}\ln k}{\alpha+(1-\alpha)k^{\rho}}}$

  $= \lim_{\rho\to 0} e^{\ln k^{1-\alpha}} = k^{1-\alpha}$;

- Because $\lim_{\rho\to -\infty} k^{\rho}$ is equal to 0 if $k > 1$ and it is equal to $\infty$ if $0 < k < 1$, then

  $\lim_{\rho\to -\infty} f(k) = \lim_{\rho\to -\infty} e^{\frac{\ln[\alpha+(1-\alpha)k^{\rho}]}{\rho}}$

  is equal to $e^0 = 1$ if $k > 1$ while it is equal to $e^{\ln k} = k$ if $0 < k < 1$.

Let $f(k)$ be a production function in intensive form. We set $e_f(k) = \frac{kf'(k)}{f(k)}$ and $e_{f'}(k) = \frac{kf''(k)}{f'(k)}$. If $f(k)$ is a *CES* production function we obtain that $e_f(k) = (1-\alpha)(\alpha k^{-\rho} + 1 - \alpha)^{-1}$ and $e_{f'}(k) = \alpha(\rho-1)[\alpha + (1-\alpha)k^{\rho}]^{-1}$. As a matter of fact

- $e_f(k) = \frac{f'(k)k}{f(k)} = \frac{(1-\alpha)[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-\rho}{\rho}}k}{[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1}{\rho}}k} = (1-\alpha)[\alpha k^{-\rho} + (1-\alpha)]^{-1}$;

- $e_{f'}(k) = \frac{kf''(k)}{f'(k)} = \frac{\alpha(1-\alpha)(\rho-1)k^{-\rho-1}[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-2\rho}{\rho}}k}{(1-\alpha)[\alpha k^{-\rho}+(1-\alpha)]^{\frac{1-\rho}{\rho}}}$

  $= \alpha(\rho-1)k^{-\rho}[\alpha k^{-\rho} + (1-\alpha)]^{-1}$

  $= \alpha(\rho-1)k^{-\rho}k^{\rho}[\alpha + (1-\alpha)k^{\rho}]^{-1}$

  $= \alpha(\rho-1)[\alpha + (1-\alpha)k^{\rho}]^{-1}$.

Obviously, $e_{f'}(k) < 0$ for all $\rho < 1$ ($\rho \neq 0$) and for all $k \geq 0$.

Developing an observation of Commendatore (2005, p.16) we establish that (See **Figure 2.13** and **Figure 2.14**)

**Proposition 5.1** If $f(k)$ is the CES production function then the inequality

$$e_{f'}(k) > -1$$

is true always for all $0 < \rho < 1$ and for all $k \geq 0$; while if $\rho < 0$ the inequality is verified only for those $k \in ]0, k^{\star}[$, where $k^{\star} = (\frac{\alpha\rho}{\alpha-1})^{\frac{1}{\rho}}$ and $e_{f'}(k^{\star}) = -1$.

**Proof** Let $0 < \alpha < 1$ be. We observe that:

- $\frac{de_{f'}(k)}{dk} = \frac{\alpha\rho(\rho-1)(\alpha-1)k^{\rho-1}}{[\alpha+(1-\alpha)k^{\rho}]^2}$;

- $e_{f'}(k)$ is strictly increasing if $0 < \rho < 1$ and is strictly decreasing if $\rho < 0$;

- $\lim_{k\to 0} e_{f'}(k) = \begin{cases} (\rho-1) & \text{if } 0 < \rho < 1, \\ 0 & \text{if } \rho < 0; \end{cases}$

- $\lim_{k\to +\infty} e_{f'}(k) = \begin{cases} 0 & \text{if } 0 < \rho < 1, \\ (\rho-1) & \text{if } \rho < 0. \end{cases}$

Being $e_{f'}(k)$ continuous on the interval $]0, +\infty[$, by *Bolzano's Theorem* [2], the range $J$ of $e_{f'}(k)$ is an interval, and, by *Theorem about limits of monotonically functions*[3], $J$ is equal to $](\rho-1), 0[$ for all $\rho < 1$ ($\varrho \neq 0$).

Now we consider $0 < \rho < 1$. Since $-1 < \rho - 1 = inf\{e_{f'}(k) : k \geq 0\} \leq e_{f'}(k)$, we obtain that $e_{f'}(k) > -1$.

After we fix $\rho < 0$ and we solve the equation $e_{f'}(k) = -1$. We have as an unique solution $k^{\star} = (\frac{\alpha\rho}{\alpha-1})^{\frac{1}{\rho}}$. Being $e_{f'}(k)$ strictly decreasing, for all $0 < k < k^{\star}$, $e_{f'}(k) > e_{f'}(k^{\star}) = -1$. Q.E.D.

[2]Let $g : X \subseteq \Re \to \Re$ be. If $g$ is continuous on $X$ and $X$ is an interval, then $g(X)$ is an interval.(For a proof of the Bolzano's Theorem see Vincenzo Aversa (2006))

[3]Let $g : X \subseteq \Re \to \Re$ be. We suppose that $infX$ and $supX$ are points of accumulation for $X$. Then,

- for $x \to infX$, $g(x) \to inf(g(X))$ if $g$ is monotonically increasing, otherwise $g(x) \to sup(g(X))$ if $g$ is monotonically decreasing;

- for $x \to supX$, $g(x) \to sup(g(X))$ if $g$ is monotonically increasing, otherwise $g(x) \to inf(g(X))$ if $g$ is monotonically decreasing.

(See Vincenzo Aversa (2006))

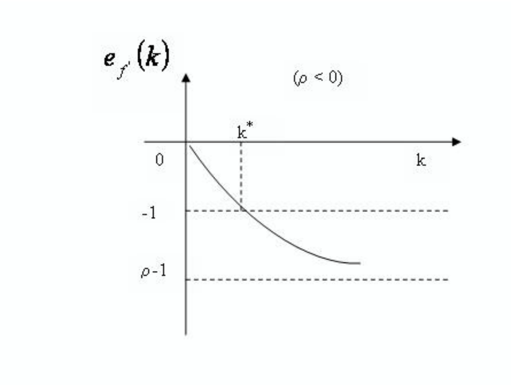



Figure 2.13: *The case* $\rho < 0$

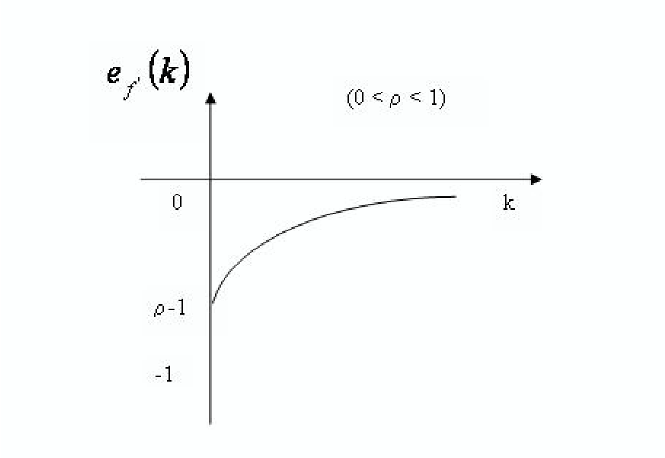
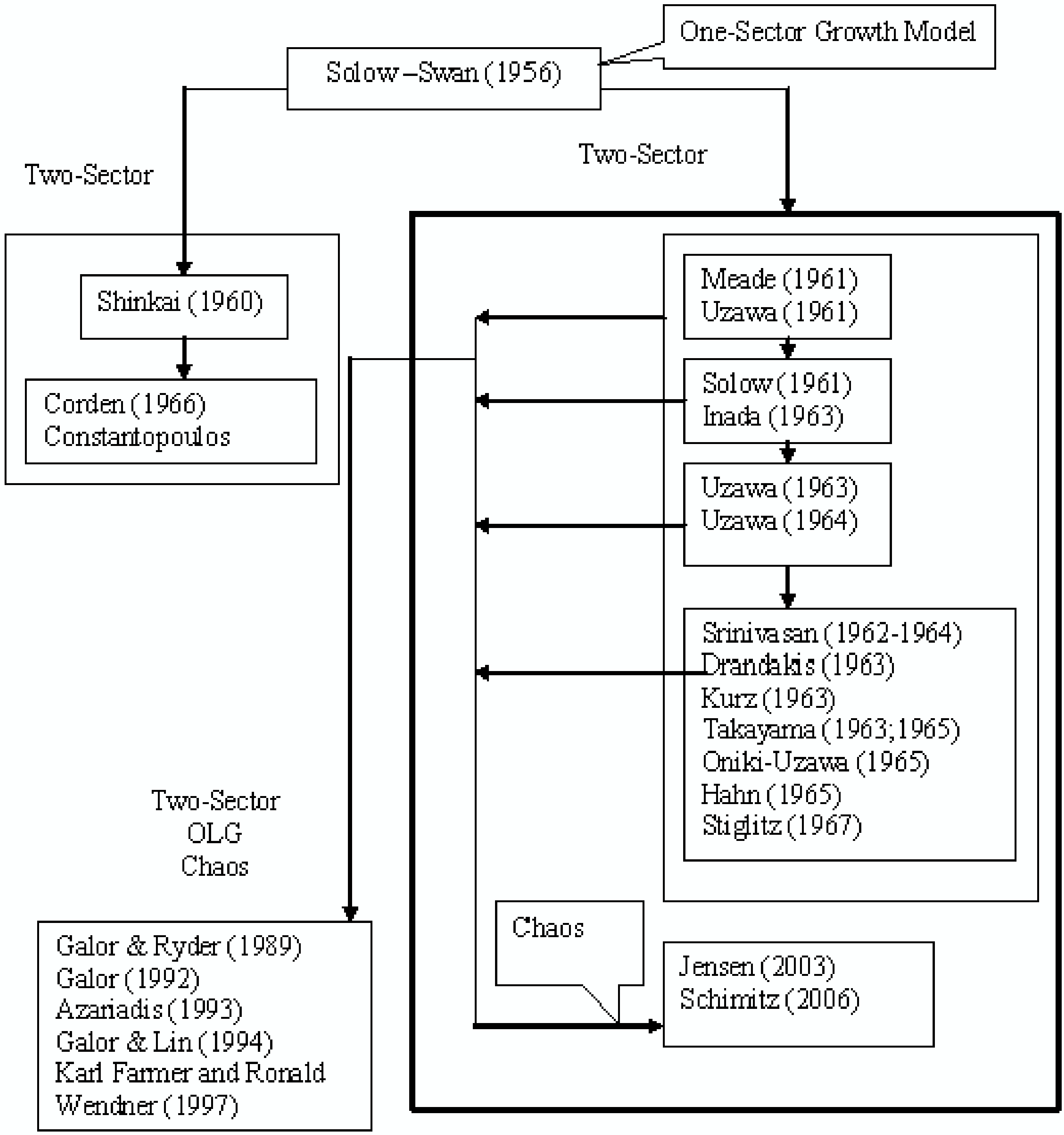
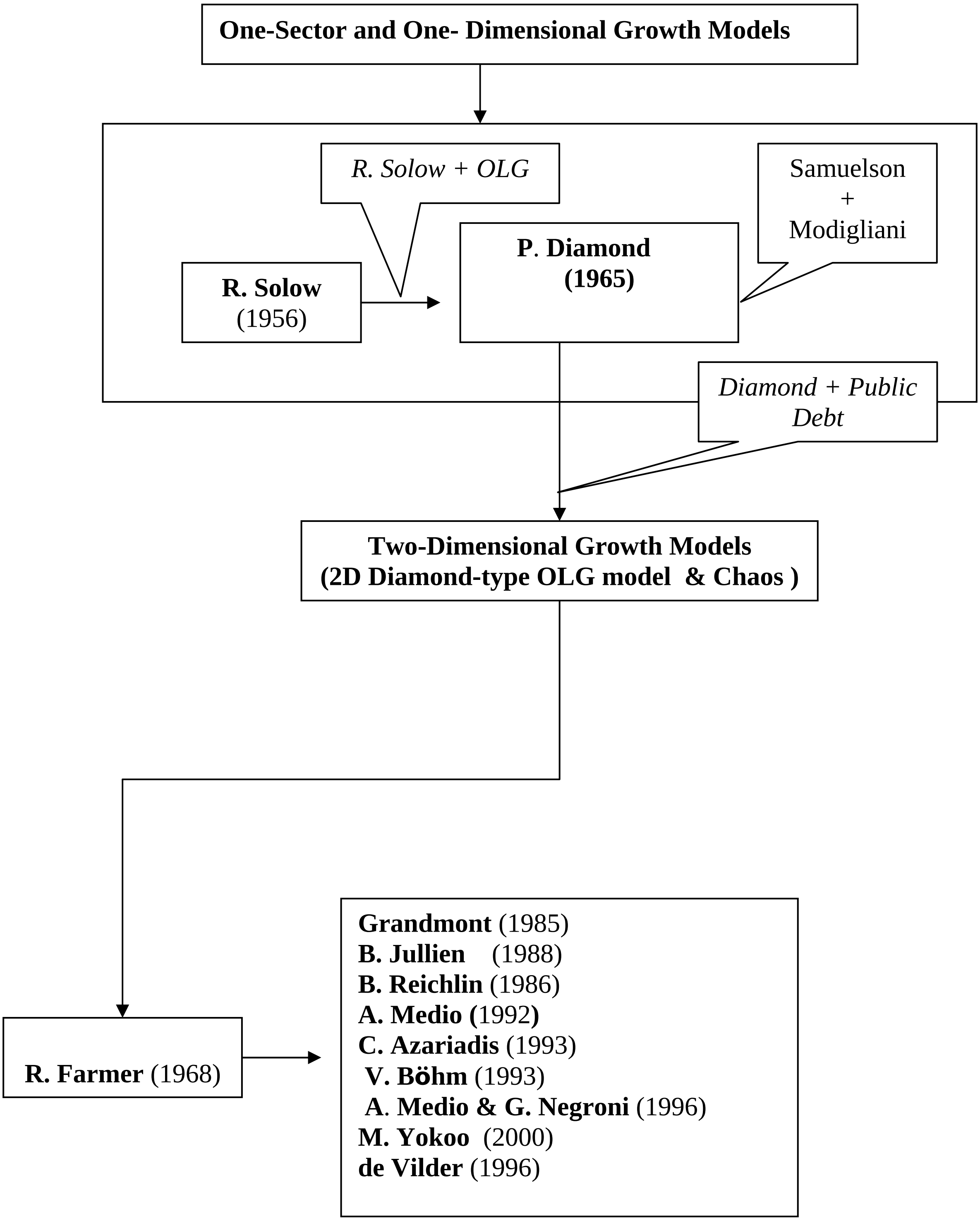



Figure 2.14: *The case* $\rho < 1$

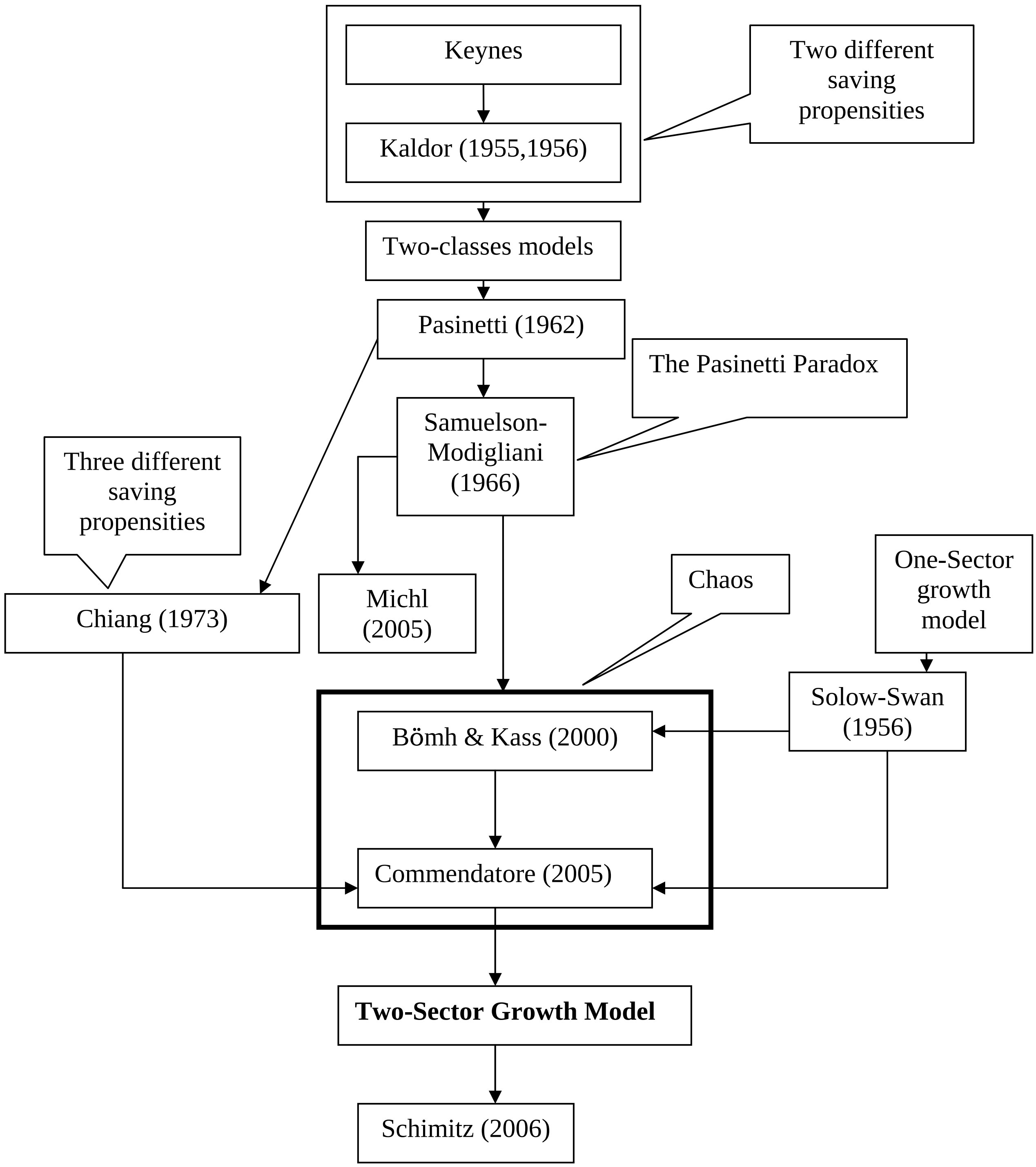
Appendix: Literature on the growth models





One-Sector and One- Dimensional Growth Models
R. Solow + OLG
Samuelson + Modigliani
P. Diamond (1965)
R. Solow (1956)
Diamond + Public Debt
Two-Dimensional Growth Models (2D Diamond-type OLG model & Chaos )
R. Farmer (1968)
Grandmont (1985)
B. Jullien (1988)
B. Reichlin (1986)
A. Medio (1992)
C. Azariadis (1993)
V. Böhm (1993)
A. Medio & G. Negroni (1996)
M. Yokoo (2000)
de Vilder (1996)

**Two-classes models**



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# Chapter 3

## 3.1 Introduction

In this chapter, we develop a two-class growth model in discrete time optimal consumption choices, which is able to generate chaotic dynamics. The model elaborated by Commendatore, presented in the previous Chapter, representing a discrete time version of the growth and distribution models proposed by Pasinetti (1962), Samuelson and Modigliani (1966) and Chiang (1973), does not assume optimal saving behaviour, eventhough it is able to generate complex behaviour. In order to model capitalists and workers saving behaviour we follow Michl (2004, 2006). This author uses a hybrid optimization model, that combines the assumption of overlapping generations in order to describe the consumption behaviour of the "workers'" class with the assumption of an eternal dynasty (introduced in Barro (1974)) in order to describe the consumption behaviour of the "capitalists'" class. Our model represents a discrete time, microfounded version of the Pasinetti and Samuelson-Modigliani growth models.

## 3.2 The model

### 3.2.1 The Capitalists' Optimization

Each generation of "capitalists" cares about its offspring and saves for a bequest motive. It behaves like one infinitely-lived household. Thus the capitalists have an infinite time horizon $t = 0, 1, \ldots$ and behave as a dynasty (Barro, 1974). At the beginning of period 0 each generation has an endowment of positive wealth $K_{c,0}$ and it invests $K_{c,0}$ for one period at the gross market interest rate $r$. At the end of period 0 the wealth of generation will become $(1 + r - \delta)K_{c,0}$, being

$\delta$ the constant rate of depreciation of capital. The rates $r$ and $\delta$ are given. At the end of period 0 capitalists consume the sum $C_0^c$ and they can accumulate the capital $K_{c,1}$ with a budget constraint $C_{c,0} + K_{c,1} \leq (1 + r - \delta)K_{c,0}$. The same will happen in the next periods $1, 2, \ldots$ . In summary, the dynasty has to make a sequence of decision $C_{c,0}, C_{c,1}, \ldots$ about the consumption and saving subjected to following budget constraints:

$C_{c,0} + K_{c,1} \leq (1 + r - \delta)K_{c,0},$

$C_{c,1} + K_{c,2} \leq (1 + r - \delta)K_{c,1},$

$\ldots$

$C_{c,t} + K_{c,t+1} \leq (1 + r - \delta)K_{c,t}.$

We assume that capitalists choose $C_{c,0}, C_{c,1}, \ldots$ in order to maximize the following utility from their consumption

$U = (1 - \beta_c) \sum_{t=0}^{\infty} \beta_c^t \ln C_t$

where $\beta_c$ is the discounting factor ($0 < \beta_c < 1$).

The solution of the infinite-horizon problem is (See below the **Remark 3.2.1.1**)

$C_{c,t} = (1 - \beta_c)(1 - \delta + r)K_{c,t},$

which, replaced in to the last constraint $C_{c,t} + K_{c,t+1} \leq (1 + r - \delta)K_{c,t}$, gives the following relation

$K_{c,t+1} = \beta_c(1 - \delta + r)K_{c,t}.$

**Remark 3.2.1.1** (About the solution of capitalists' infinite-horizon problem)

First we begin by writing the Lagrangian function for the capitalist's planning problem:

$L = (1-\beta_c)\sum_{t=0}^{\infty}\beta_c^t \ln C_{c,t} - \sum_{t=0}^{\infty}\lambda_t(C_{c,t} + K_{c,t+1} - (1+r-\delta)K_{c,t})$

$= (1-\beta_c)\sum_{t=0}^{\infty}\beta_c^t \ln C_{c,t} \; - \sum_{t=0}^{\infty}\lambda_t C_{c,t} +$

$+ \sum_{t=0}^{\infty}\lambda_t(K_{c,t+1} - (1+r-\delta)K_{c,t}),$

where $\lambda_t$ $(t = 0, 1, \ldots)$ is the shadow price for each period's budget constraint.

Expanding the last sum, we obtain

$\sum_{t=0}^{\infty}\lambda_t(K_{c,t+1} - (1+r-\delta)K_{c,t})$

$= \lambda_0 - \lambda_0(1+r-\delta)K_{c,0} + \lambda_1 - \lambda_1(1+r-\delta)K_{c,1} + \lambda_2 - \lambda_2(1+r-\delta)K_{c,2} + \ldots$

from which

$= -\lambda_0(1+r-\delta)K_{c,0} + \lambda_0 K_{c,1} - \lambda_1(1+r-\delta)K_{c,1} +$

$+\lambda_1 K_{c,2} - \lambda_2(1+r-\delta)K_{c,2} + \lambda_2 K_{c,3} - \lambda_3(1+r-\delta)K_{c,3} + \ldots$

$= -\lambda_0(1+r-\delta)K_{c,0} + (\lambda_0 - \lambda_1(1+r-\delta))K_{c,1} +$

$+(\lambda_1 - \lambda_2(1+r-\delta))K_{c,2} + (\lambda_2 - \lambda_3(1+r-\delta))K_{c,3} + \ldots$

$= -\lambda_0(1+r-\delta)K_{c,0} + \sum_{t=0}^{\infty}(\lambda_t - \lambda_{t+1}(1+r-\delta))K_{c,t+1}.$

Rewriting the Lagrangian function, we have

$L = (1-\beta_c)\sum_{t=0}^{\infty}\beta_c^t \ln C_{c,t} - \sum_{t=0}^{\infty}\lambda_t(C_{c,t} + K_{c,t+1} - (1+r-\delta)K_{c,t})$

$= (1-\beta_c)\sum_{t=0}^{\infty}\beta_c^t \ln C_{c,t} \; - \sum_{t=0}^{\infty}\lambda_t C_{c,t} +$

$= +\lambda_0(1+r-\delta)K_{c,0} - \sum_{t=0}^{\infty}(\lambda_t - \lambda_{t+1}(1+r-\delta))K_{c,t+1}$.

We observe that $L = L(C_{c,t}, K_{t+1}, \lambda_t)$ and we recall that

$\frac{d \ln C_{c,t}}{dC_{c,t}} = \frac{1}{C_{c,t}}$ $(t = 0, 1, \ldots)$. We have, for all $t = 0, 1, \ldots$, the first-order conditions:

$\frac{\partial L}{\partial C_{c,t}} = \frac{(1-\beta_c)\beta_c^t}{C_{c,t}} - \lambda_t \leq 0$ $(= 0$ if $C_{c,t} > 0)$,

$\frac{\partial L}{\partial K_{c,t+1}} = -\lambda_t + (1+r-\delta)\lambda_{t+1} \leq 0$, $(= 0$ if $K_{c,t+1} > 0)$,

$\frac{\partial L}{\partial \lambda_t} = -(C_{c,t} + K_{c,t+1} - (1+r-\delta)K_{c,t}) \geq 0$ $(= 0)$ if $\lambda_t > 0$.

The value of *the penalty function* is equal to 0 at the saddle-point, that is

$\sum_{t=0}^{\infty} \lambda_t C_{c,t} = \sum_{t=0}^{\infty}(-\lambda_t + \lambda_{t+1}(1+r-\delta))K_{c,t+1} + \lambda_0(1+r-\delta)K_{c,0}$.

From first-order conditions $\sum_{t=0}^{\infty} \lambda_t C_{c,t} = \sum_{t=0}^{\infty}(1-\beta_c)\beta_c^t = 1$, because

$(1-\beta_c)\sum_{t=0}^{\infty} \beta_c^t = 1$.

Again from the first-order conditions

$\sum_{t=0}^{\infty} K_{c,t+1}(-\lambda_t + (1+r-\delta)\lambda_{t+1}) = 0$. Thus

$\lambda_0 = \frac{1}{(1+r-\delta)K_{c,0}}$. We consider the first-order condition $\frac{\partial L}{\partial C_{c,t}} = 0$ for $t = 0$,

that is $C_{c,0} = \frac{1-\beta_c}{\lambda_0}$, we obtain $C_{c,0} = (1-\beta_c)(1+r-\delta)K_{c,0}$.

The last relation is also true for all $t = 1, 2, \ldots$. From which

$K_{c,t+1} = \beta_c(1+r-\delta)K_{c,t}$.

### 3.2.2 The Workers' Optimization

We assume that each generation of "workers" has a finite time horizon because lives two periods. In his/her first period of life we call "young" an individual worker while we will consider "old" the worker which lives in his/her second period of life. Each individual is active, that is, he works and is able to earn money only as young, while he is in retirement as old. Each young supplies one inelastic unit of labor-power for the wage $w$, where $w$ is exogenous. We indicate respectively with $C^w$ and $C^r$ the consumption as young and as old, and we call $S^w$ its saving in the first period. The worker invests $S^w$ at the constant gross return rate $r$ for one period and at beginning of the second period he has the wealth $(1 + r - \delta)S^w$, where $\delta$ is the depreciation rate of capital and $r - \delta$ is the net profit rate. In contrast with the capitalists, the workers save only for to consume the whole wealth and income during the retirement, that is for the life-cycle motive. Then we have the following budget constraints:

$C^w + S^w = w$ (first period),

$C^r = (1 + r - \delta)S^w$ (second period).

The previous constraints can be combined into a single household constraint:

$C^w + \frac{C^r}{(1+r-\delta)} = w.$

Given the wage $w$ and subject to the previous budget constraint, the household wants to choose the consumption $C^w$ so as maximize the utility

$U = U(C^w, C^r) = (1 - \beta_w)\ln(C^w) + \beta_w \ln(C^r),$

where $\beta_w$ is the discount rate of utility of the workers.

It is easy to see that $C^w = (1 - \beta_w)w$ (See **Remark** 3.2.2.1).

Therefore the individual worker saving is

$S^w = w - (1 - \beta_w)w = \beta_w w.$

**Remark 3.2.2.1** (About worker's consumption) In order to derive the expression of worker's consumption, we begin from the budget constraint and we observe that the utility function can be so rewritten:

$$U = (1 - \beta_w)\ln(C^w) + \beta_w \ln[(w - C^w)(1 + r - \delta)].$$

Thus

$$\frac{dU}{dC^w} = (1 - \beta_w)\frac{1}{C^w} + \beta_w \frac{1}{(w-C^w)(1+r-\delta)}[-(1 + r - \delta)]$$

$$= (1 - \beta_w)\frac{1}{C^w} - \beta_w \frac{1}{w-C^w} = 0$$

if and only if $\frac{1-\beta_w}{C^w} = \frac{\beta_w}{w-C^w}$, $\frac{C^w}{1-\beta_w} = \frac{w-C^w}{\beta_w}$, $\beta_w C^w = (1 - \beta_w)w - C^w$,

$C^w = (1 - \beta_w)w.$

### 3.2.3 Capitalists' and Workers' Processes of Capital Accumulation

Suppose that for the production function we have

$f(k) \geq 0, f'(k) \geq 0, f''(k) \leq 0$, where $(0 \leq k \leq \infty)$ and $(k = k_c + k_w)$.

Capitalist's capital accumulation law corresponds to

$$K_{c,t+1} = I_{c,t} + (1 - \delta)K_{c,t}.$$

We need to find $I_{c,t}$ (that in a short-run equilibrium is equal to $S_{c,t}$).

From the capitalists' budget constraint

$$rK_{c,t} = C_{c,t} + S_{c,t} = C_{c,t} + I_{c,t}.$$

It follows

$$I_{c,t} = rK_{c,t} - C_{c,t}.$$

From the solution of the maximization problem:

$$C_{c,t} = (1 - \beta_c)(1 - \delta + r)K_{c,t},$$

we get

$$I_{c,t} = \beta_c(1 - \delta + r)K_{c,t} - (1 - \delta)K_{c,t}.$$

Finally we obtain

$$K_{c,t+1} = \beta_c(1 - \delta + r)K_{c,t},$$

with a neoclassical production function $f(K_t, L_t)$ and equilibrium in the capital market $r = f_K(K_t, L_t)$ (marginal productivity of capital). The accumulation law of capitalists' capital is

$$K_{c,t+1} = \beta_c(1 - \delta + f_K(K_t, L_t))K_{c,t},$$

or, in terms of quantities per worker, assuming that $L_{t+1} = (1 + n)L_t$:

$$k_{c,t+1} = \frac{1}{1+\mathrm{n}}\beta_c(1 - \delta + f'(k_t))k_{c,t}.$$

The accumulation law of workers is

$$k_{w,t+1} = \frac{1}{1+\mathrm{n}}\beta_w w = \frac{1}{1+\mathrm{n}}\beta_w[f(k) - f'(k)k].$$

### 3.2.4 Steady Growth Equilibrium

The map

$G_w(k_w, k_c) = \frac{1}{1+n}\beta_w w = \frac{1}{1+n}\beta_w[f(k) - f'(k)k]$

denotes workers's accumulation law and the following dynamic map

$G_c(k_w, k_c) = \frac{1}{1+n}\beta_c[1 - \delta + f'(k)]k_c$

denotes the capitalists'accumulation law, where $k = k_w + k_c$.

$G(k_w, k_c) = (\frac{1}{1+n}\beta_w[f(k) - f'(k)k], \frac{1}{1+n}\beta_c[1 - \delta + f'(k)]k_c)$

The steady growth solutions are obtained by imposing

$G_w(k_w, k_c) = k_w$ and $G_c(k_w, k_c) = k_c$

and solving the following equations

$k_w = \frac{1}{1+n}\beta_w[f(k) - f'(k)k]$ $(\star)$

$k_c = \frac{1}{1+n}\beta_c[1 - \delta + f'(k)]k_c$ $(\star\star)$

## 3.3 Local Dynamics

There exist three different types of equilibria: a Pasinetti equilibrium involves capitalists owing a positive share of capital; a dual equilibrium, instead, allows only workers to own capital; finally, in a trivial equilibrium, the overall capital is zero.

### 3.3.1 Pasinetti Equilibrium

Suppose $k_c \neq 0$. Using $(\star\star)$ we obtain:

$(1-\delta) + f'(k) = \frac{1+n}{\beta_c}$

from which

$f'(k) = \frac{1+n}{\beta_c} - (1-\delta) = \frac{1}{\beta_c}[1+n-(1-\delta)\beta_c]$

Moreover, also from $(\star)$, we have

$1+n = \beta_c[(1-\delta) + f'(k)]$.

Substituting $(1+n)$ in $(\star\star)$ it has:

$k_w = \frac{\beta_w}{\beta_c[(1-\delta)+f'(k)]} = [f(k) - f'(k)k] = (\star\star\star)$

$= \frac{\beta_w}{\beta_c} \frac{1-\frac{f'(k)k}{f(k)}}{\frac{1-\delta}{f(k)} + \frac{e_f(k)}{k}} = \frac{\beta_w}{\beta_c} \frac{1-e_f(k)}{\frac{1-\delta}{f(k)} + e_f(k)} k.$

By the identity $k = k_w + k_c$, we obtain

$k_c = k - k_w = k - \frac{\beta_w}{\beta_c} \frac{1-e_f(k)}{\frac{1-\delta}{f(k)} + e_f(k)} k = [1 - \frac{\beta_w}{\beta_c} \frac{1-e_f(k)}{\frac{(1-\delta)k}{f(k)} + e_f(k)}]k$ $(\star\star\star\star)$

### 3.3.2 Dual Equilibrium

Suppose $k_c = 0$. We have $k = k_w$ and, from $(\star)$ and $(\star\star\star\star)$,

$k = \frac{1}{1+n}\beta_w[f(k) - f'(k)k]$, from which

$\frac{k}{f(k)} = \frac{\beta_w}{1+n}[1 - \frac{f'(k)k}{f(k)}] = \frac{\beta_w}{1+n}[1 - e_f(k)]$.

Therefore, if $1 - e_f(k) \neq 0$,

$\frac{f(k)}{k} = \frac{1+n}{\beta_w[1-e_f(k)]}$. (4.2.1)

The (4.2.1) says that we can write $f(k)/k$ as a function of $e_f(k)$, i.e.,

$\frac{f(k)}{k} = \theta(e_f(k))$,

where, by definition, $\theta(x) = \frac{1+n}{\beta_w(1-x)}$, $(0 \leq x < 1)$.

**Remark 3.3.2.1** (About convexity of $\theta(x)$) For all $(0 \leq x < 1)$, notice that

- $\theta'(x) = \frac{1+n}{\beta_w(1-x)^2}$;
- $\theta''(x) = \frac{2(1+n)}{\beta_w(1-x)^3}$.

Therefore $\theta(x)$ is monotonically increasing and convex for all $0 \leq x < 1$.

### 3.3.3 Trivial Equilibrium

We impose that $k_c = k_w = 0$. We have that $k = 0$ and $f(0) = 0$.

**Remark 3.3.3.1 (On Meade's diagrammatic approach to find equilibria)**

We notice that, if $\varphi(x) = (\frac{1-\alpha}{x})^{\frac{1}{\rho}}$,

- $\varphi(1) = (1-\alpha)^{\frac{1}{\rho}}$;

- $\lim_{x\to 0}\varphi(x) = \begin{cases} 0, & \text{if } \rho < 0 \\ +\infty, & \text{if } 0 < \rho < 1 \end{cases}$ ;

- $\varphi^{'}(x) = \frac{1}{\rho}(\frac{1-\alpha}{x})^{\frac{1}{\rho}-1}(1-\alpha)(-\frac{1}{x^2})$

  $= (-\frac{1-\alpha}{\rho})(\frac{1}{x^2})(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$;

- $\varphi^{''}(x) = (-\frac{1-\alpha}{\rho})\ [-\frac{2}{x^3}(\frac{1-\alpha}{x})^{\frac{\rho}{1-\rho}} + \frac{1}{x^2}\frac{1-\rho}{\rho}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}-1}\ (1-\alpha)(-\frac{1}{x^2})]$

  $= (-\frac{1-\alpha}{\rho})[-\frac{1}{x^2}(\frac{1-\alpha}{x})^{\frac{\rho}{1-\rho}}]\ [\frac{2}{x} + \frac{1-\rho}{\rho}(\frac{1-\alpha}{x})^{-1}(1-\alpha)\frac{1}{x^2}]$

  $= (\frac{1-\alpha}{\rho})[\frac{1}{x^2}(\frac{1-\alpha}{x})^{\frac{\rho}{1-\rho}}]\ [\frac{2}{x} + \frac{1-\rho}{\rho}\frac{1}{x}]$

  $= \frac{\rho+1}{\rho^2}(1-\alpha)\frac{1}{x^3}(\frac{1-\alpha}{x})^{\frac{1-\rho}{\rho}}$;

- if $\rho = -1$, we have $\varphi(x) = \frac{x}{1-\alpha}$, i.e. the graph of $\varphi$ is a line with slope equal to $\frac{1}{1-\alpha} > 0$.

Thus we deduce that

- $\varphi(x)$ is strictly increasing if $\rho < 0$ and strictly decreasing if $0 < \rho < 1$;
- $\varphi(x)$ is convex if $\rho < -1$ and concave if $(-1 < \rho < 0)$ or $(0 < \rho < 1)$.

In order to find graphically the equilibria of dynamic system Commendatore (2005), following Meade(1966), has stated that

**Proposition 3.3.3.2** If $f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}} = [\alpha k^{-\rho} + (1-\alpha)]^{\frac{1}{\rho}}k$ $(k > 0)$

is a CES production function $(0 < \alpha < 1,\ \rho < 1,\ \rho \neq 0)$, then $\frac{f(k)}{k}$ depends on $e_f(k)$, i.e. $\frac{f(k)}{k} = \varphi(e_f(k))$, where $e_f(k) = \frac{kf^{'}(k)}{f(k)}$.

**Proof** By definitions of $f(k)$ and $e_f(k)$, we observe that

$\frac{f(k)}{k} = [\alpha k^{-\rho} + (1-\alpha)]^{\frac{1}{\rho}}$ and $e_f(k) = \frac{1-\alpha}{\alpha k^{-\rho}+(1-\alpha)}$.

From last relation we obtain that $\alpha k^{-\rho} + (1-\alpha) = \frac{1-\alpha}{e_f(k)}$,

from which $[\alpha k^{-\rho} + (1-\alpha)]^{\frac{1}{\rho}} = (\frac{1-\alpha}{e_f(k)})^{\frac{1}{\rho}} = \varphi(e_f(k))$.

Generally $\frac{f(k)}{k} = \frac{f'(k)}{e_f(k)}$.

**Proposition 3.3.3.3** About the existence of dual equilibria we may distinguish three cases:

- Case I: $\rho = -1$. There are two dual equilibria (real and coincident or real and distinct) if and only if $\frac{(1+n)(1-\alpha)}{\beta_w} \leq \frac{1}{4}$.
- Case II: $0 < \rho < 1$. There is one dual equilibrium.
- Case II: $(-\infty < \rho < -1) \vee (-1 < \rho < 0)$. There is one or two real and distinct dual equilibria if and only if $(1-\alpha)(\frac{1+n}{\beta_w})^{-\rho} \leq M$, where $M$ is the maximum of function $\frac{x}{(1-x)^\rho}$.

**Proof** We solve the equation $\varphi(x) = \theta(x)$, i.e. $(\frac{1-\alpha}{x})^{\frac{1}{\rho}} = \frac{1+n}{\beta_w(1-x)}$.

If $\rho = -1$ the equation becomes $\frac{x}{1-\alpha} = \frac{1+n}{\beta_w(1-x)}$,

which is equivalent to relation $x^2 - x + \frac{(1+n)(1-\alpha)}{\beta_w} = 0$.

Setting $A = 1$, $B = -1$, $C = \frac{(1+n)(1-\alpha)}{\beta_w}$, $\Delta = B^2 - 4AC$, we notice that

$\Delta \geq 0 \Leftrightarrow \frac{(1+n)(1-\alpha)}{\beta_w} \leq \frac{1}{4}$.

If $\rho < 1$ and $\rho \neq -1$ ($\rho \neq 0$) the equation reduces to

$\frac{x}{(1-x)^\rho} = (1-\alpha)(\frac{1+n}{\beta_w})^{-\rho}$.

We pose $h(x) = \frac{x}{(1-x)^\rho}$ and we observe that:

- $h(x)$ is positive and continuous in the interval $0 < x < 1$;
- $\lim_{x\to 0} h(x) = 0$;
- $\lim_{x\to 1} h(x) = \begin{cases} +\infty, & \text{if } 0 < \rho < 1 \\ 0, & \text{if } \rho < 0 \end{cases}$;
- $h^{'}(x) = \frac{1}{(1-x)^{\rho+1}}[1-(1-\rho)x]$;
- $h^{''}(x) = \rho(1-x)^{-\rho-2}(x(\rho-1)+2)$.

We have $h^{'}(x) > 0 \Leftrightarrow x < \frac{1}{1-\rho}$. We consider now $0 < \rho < 1$. Then $\frac{1}{1-\rho} > 1$. Since $x < 1$ we deduce that $h(x)$ is strictly increasing and convex for all $0 < x < 1$ and the range of function $h(x)$ is $]0, +\infty[$. Therefore, by Bolzano's Theorem, there is a unique $x$ for which holds the equation $h(x) = (1-\alpha)(\frac{1+n}{\beta_w})^{-\rho}$. If $\rho < 0$, then the point $x = \frac{1}{1-\rho}$ maximizes the function $h(x)$ because $h(x)$ is strictly increasing for all $x < \frac{1}{1-\rho}$ and $h(x)$ is strictly decreasing for all $x > \frac{1}{1-\rho}$. Moreover $h(x)$ is concave in $]0,1[$ and the range of $h(x)$ is $]0, h(\frac{1}{1-\rho})]$. By Bolzano's Theorem, if $(1-\alpha)(\frac{1+n}{\beta_w})^{-\rho} \leq h(\frac{1}{1-\rho}) = (1-\rho)^{\rho-1}(\rho)^{-\rho}$ there is at least one dual equilibrium.

### 3.3.4 The Jacobian matrix of the G map

We have

$\frac{\partial G_w}{\partial k_w} = \frac{1}{1+n}\beta_w[f'(k) - f''(k)k - f'(k)] = -\frac{1}{1+n}\beta_w f''(k)k,$

and $\frac{\partial G_w}{\partial k_c} = \frac{\partial G_w}{\partial k_w}$. Moreover $\frac{\partial G_c}{\partial k_w} = \frac{1}{1+n}\beta_c[f''(k)k_c]$ and

$\frac{\partial G_c}{\partial k_c} = \frac{1}{1+n}\beta_c[f''(k)k_c + 1 - \delta + f'(k)].$

The jacobian matrix $J$ evaluated at $(k_w, k_c)$ is

$$J(k_w, k_c) = \begin{pmatrix} \frac{\partial G_w}{\partial k_w} & \frac{\partial G_w}{\partial k_c} \\ \frac{\partial G_c}{\partial k_w} & \frac{\partial G_c}{\partial k_c} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{1+n}\beta_w f''(k)k & -\frac{1}{1+n}\beta_w f''(k)k \\ \frac{1}{1+n}\beta_c f''(k)k_c & \frac{1}{1+n}\beta_c[f''(k)k_c + 1 - \delta + f'(k)] \end{pmatrix}.$$

The trace $T$ of the jacobian $J$ at the point $(k_w, k_c)$ is:

$T(k_w, k_c) = -\frac{1}{1+n}\beta_w f''(k)k + \frac{1}{1+n}\beta_c[f''(k)k_c + 1 - \delta + f'(k)]$ (3.3.4.1)

The determinant $Det(k_w, k_c)$ of jacobian $J$ is:

$Det(k_w, k_c) = (\frac{1}{1+n})^2 \beta_w \beta_c e_{f'}(k) f'(k)[\delta - 1 - f'(k)]$. (3.3.4.2)

From $(\star)$ and $(\star\star)$ we can rewrite *Det* and *Trace* for Pasinetti equilibrium :

$Det(k^P) = -\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} e_{f'}(k^P)$, (3.3.4.3)

$Trace(k^P) = (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)})\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c + 1$, (3.3.4.4)

where:

$\sigma_w = 1 + n - (1 - \delta)\beta_w$,

$\sigma_c = 1 + n - (1 - \delta)\beta_c$.

### 3.3.5 Eigenvalues of jacobian matrix for Pasinetti equilibrium

Following Azariadis(1993) and M.W.Hirsch, S.Smale, R.L.Devaney(2003), the eigenvalues $\lambda_+$ and $\lambda_-$ are solutions of *the characteristic equation*:

$$p(\lambda) = |J - \lambda I| = \begin{vmatrix} \frac{\partial G_w}{\partial k_w} - \lambda & \frac{\partial G_w}{\partial k_c} \\ \frac{\partial G_c}{\partial k_w} & \frac{\partial G_c}{\partial k_c} - \lambda \end{vmatrix} =$$

$$= (\tfrac{\partial G_w}{\partial k_w} - \lambda)(\tfrac{\partial G_c}{\partial k_c} - \lambda) - \tfrac{\partial G_w}{\partial k_c}\tfrac{\partial G_c}{\partial k_w} =$$

$$= \lambda^2 - (\tfrac{\partial G_w}{\partial k_w} + \tfrac{\partial G_c}{\partial k_c})\lambda + \tfrac{\partial G_w}{\partial k_w}\tfrac{\partial G_c}{\partial k_c} - \tfrac{\partial G_w}{\partial k_c}\tfrac{\partial G_c}{\partial k_w} =$$

$$= \lambda^2 - (TraceJ)\lambda + detJ = 0,$$

where $I$ is the identity matrix.

It notices that $\lambda_+ + \lambda_- = TraceJ$ and $\lambda_+\lambda_- = DetJ$.

Moreover from the sign of the discriminant $\Delta = T^2 - 4D$ it deduces that the eigenvalues are

1. complex with non zero imaginary part if $\Delta < 0$;
2. real and distinct if $\Delta > 0$;
3. real and repeated if $\Delta = 0$.

and are given by

$$\lambda_\pm = \tfrac{TraceJ + \sqrt{(TraceJ)^2 - 4DetJ}}{2}.$$

If the eigenvalues are real, they are given by

$\lambda_{\pm} = \frac{TraceJ + \sqrt{(TraceJ)^2 - 4DetJ}}{2}$.

Instead, if the eigenvalues are complex, they are given by

$\lambda_{\pm} = \frac{TraceJ}{2} + \frac{\sqrt{4DetJ - (TraceJ)^2}}{2} i,$

where $i$ is the imaginary unit.

In the last case, it observes that the square of the modulus of the each eigenvalue is $DetJ$. As a matter of fact

$(\frac{TraceJ}{2})^2 + (\frac{\sqrt{4DetJ - (TraceJ)^2}}{2})^2 =$

$= \frac{(TraceJ)^2 + 4DetJ - (TraceJ)^2}{4} = detJ.$

It says *trace - determinant plane* ($TD$-plane) the Cartesian plane which has $T = TraceJ$ as the horizontal axis and $D = detJ$ as the vertical axis.

In the $TD$-plane, the matrix $J$ with trace $T$ and determinant $D$ corresponds to the point $(T, D)$ and the location of point $(T, D)$ determines the geometry of phase portrait of the dynamic map G.

In the $TD$-plane, the equation $T^2 - 4D = 0$ describe a parabola with a minimum at the origin $O(0, 0)$: the region above the parabola is associated to the complex eigenvalues, instead, the region below the parabola and the parabola itself are associated to the real eigenvalues.

From (3.3.4.3) and (3.3.4.4) it obtains:

$\Delta = [T(k^P)]^2 - 4Det(k^P) =$

$= [(\sigma_w - \frac{\beta_w}{\beta_c} \frac{\sigma_c}{e_f(k^P)}) \frac{e_{f'}(k^P)}{(1+n)^2} \sigma_c + 1]^2 + 4 \frac{\sigma_c}{1+n} \frac{\beta_w}{\beta_c} e_{f'}(k^P).$

We set

$\beta = \frac{\beta_w}{\beta_c}$, $\overline{e} = e_{f'}(k)$, $e = e_f(k)$, $\overline{n} = 1 + n$.

In order to state if the eigenvalues of the Jacobian $J$ are real, we'll derive some helpful relations.

**Proposition 3.3.5.1**

*(1) If* $e = \beta\frac{\sigma_c}{\sigma_w}$ *the eigenvalues are real when* $\overline{e} \geq -\frac{\overline{n}}{4\sigma_c\beta}$*;*

*(2) if* $e \neq \beta\frac{\sigma_c}{\sigma_w}$ *the eigenvalues are real when* $e \geq \frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}$*.*

*Proof*

The condition $\Delta \geq 0$ holds if

$[(\sigma_w - \beta\frac{\sigma_c}{\overline{e}})\frac{\overline{e}}{\overline{n}^2}\sigma_c + 1]^2 + 4\frac{\sigma_c}{\overline{n}}\beta\overline{e} \geq 0.$

The last inequality is equivalent to the following:

$\frac{\sigma_c^2}{\overline{n}^4}(\sigma_w - \beta\frac{\sigma_c}{e})^2\overline{e}^2 + [2(\sigma_w - \beta\frac{\sigma_c}{e})\frac{\sigma_c}{\overline{n}^2} + 4\frac{\sigma_c\beta}{\overline{n}}]\overline{e} + 1 \geq 0,$

which can be seen like an inequality of second degree in $\overline{e}$.

Setting

$A = \frac{\sigma_c^2}{\overline{n}^4}(\sigma_w - \beta\frac{\sigma_c}{e})^2$, $B = [2(\sigma_w - \beta\frac{\sigma_c}{e})\frac{\sigma_c}{\overline{n}^2} + 4\frac{\sigma_c\beta}{\overline{n}}]$, and $C = 1$.

If $A = 0$ then

$e = \beta\frac{\sigma_c}{\sigma_w}$ and the inequality becomes

$(4\frac{\sigma_c\beta}{\overline{n}})\overline{e} + 1 \geq 0$, i.e. $\overline{e} \geq -\frac{\overline{n}}{4\sigma_c\beta}$.

If $A \neq 0$ then it notices that $A > 0$ and let $\Delta' = B^2 - 4AC$.

Let $\tau = (\sigma_w - \beta\frac{\sigma_c}{e})\frac{\sigma_c}{\overline{n}^2}$ be, then $A = \tau^2$, $B = 2\tau + 4\frac{\sigma_c\beta}{\overline{n}}$, $C = 1$

and the condition $\Delta' \geq 0$ is equivalent to the following inequalities

$4\tau^2 + 16\frac{\sigma_c^2\beta^2}{\overline{n}^2} + 16\tau\frac{\sigma_c\beta}{\overline{n}} - 4\tau^2 \geq 0,$

$16\frac{\sigma_c^2\beta^2}{\overline{n}^2} + 16\tau\frac{\sigma_c\beta}{\overline{n}} \geq 0,\ \frac{\sigma_c\beta}{\overline{n}} + \tau \geq 0$

$\tau \geq -\frac{\sigma_c\beta}{\overline{n}}.$

Remembering the meaning of $\tau$ it has

$(\sigma_w - \beta\frac{\sigma_c}{e})\frac{\sigma_c}{\overline{n}^2} \geq -\frac{\sigma_c\beta}{\overline{n}},\ (\sigma_w - \beta\frac{\sigma_c}{e})\frac{1}{\overline{n}} \geq -\beta,\ \sigma_w - \beta\frac{\sigma_c}{e} \geq -\beta\overline{n},$

$-\beta\frac{\sigma_c}{e} \geq -\beta\overline{n} - \sigma_w$

$\frac{1}{e} \leq \frac{\beta\overline{n}+\sigma_w}{\beta\sigma_c}$, Q.E.D..

### 3.3.6 Local stability and triangle stability for Pasinetti equilibrium

**The conditions of local stability of dynamical system in terms of $e_f$ or $e_{f'}$**

It is well known that the necessary and sufficient conditions for the local stability of the dynamical system can be written as

(1) $1 - trJ + detJ > 0$ (**T**ranscritical bifurcations)

(2) $detJ < 1$ (**N**eimark-**S**acker bifurcations)

(3) $1 + trJ + detJ > 0$ (**F**lip bifurcations)

which are, in the *trace-determinant plane*, a triangle, "*the stability triangle*".

In order to draw the stability triangle in the $e_f$-$e_{f'}$ plane, we propose to rewrite the previous conditions in terms of $e_f$ or $e_{f'}$.

- *The condition (1) corresponds to the inequality* $e > \frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}$. As a matter of fact, from the following equivalent inequalities we obtain

  $1 - (\sigma_w - \beta\frac{\sigma_c}{e})\frac{\overline{e}}{\overline{n}^2}\sigma_c - 1 - \frac{\sigma_c}{\overline{n}}\beta\overline{e} > 0,$

  $\overline{e}[-(\sigma_w - \beta\frac{\sigma_c}{e})\frac{1}{\overline{n}} - \beta] > 0,$

  $-(\sigma_w - \beta\frac{\sigma_c}{e})\frac{1}{\overline{n}} < \beta,$

  $\sigma_w - \beta\frac{\sigma_c}{e} > -\beta\overline{n},$

  $-\beta\frac{\sigma_c}{e} > -\beta\overline{n} - \sigma_w,$

  $\beta\frac{\sigma_c}{e} < \beta\overline{n} + \sigma_w,$

  $e > \frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}.$

- *The condition (2) corresponds to inequality*

  $-\frac{\sigma_c}{\overline{n}}\beta\overline{e} < 1.$ Thus

  $\overline{e} > -\frac{\overline{n}}{\sigma_c\beta}.$

- *The condition (3) corresponds to the following equivalent inequalities*

  $1+(\sigma_w-\beta\frac{\sigma_c}{e})\frac{\overline{e}}{\overline{n}^2}\sigma_c+1-\frac{\sigma_c}{\overline{n}}\beta\overline{e}>0,$

  $[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]\overline{e}+2>0,$

  $[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]\overline{e}>-2.$

  If $[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]>0$ it has $\overline{e}>-\frac{2}{[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]}$.

  If $[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]<0$, because $\overline{e}<0$ the inequality $\overline{e}<-\frac{2}{[\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta]}$ is always true.

**Remark** (About $\frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}$) We'll shaw that *the expression* $\frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}$ *lies between* 0 *and* 1. As a matter of fact, from inequality $\sigma_c<\overline{n}$ we obtain

$\beta\sigma_c<\beta\overline{n}$, $\beta\sigma_c<\beta\overline{n}+\sigma_w$, $0<\frac{\beta\sigma_c}{\beta\overline{n}+\sigma_w}<1$.

**Remark** (About $-\frac{\overline{n}}{\sigma_c\beta}$) We propose to prove that $-1$ *is a lower-bound of* $-\frac{\overline{n}}{\sigma_c\beta}$. As a matter of fact, from the inequality

$\beta\sigma_c<\overline{n}$ we have $-\frac{\overline{n}}{\beta\sigma_c}>-1$.

**Remark** It notices that the following inequalities are equivalent

$\frac{\sigma_c}{\overline{n}^2}(\sigma_w-\beta\frac{\sigma_c}{e})-\frac{\sigma_c}{\overline{n}}\beta>0,$

$\frac{1}{\overline{n}}(\sigma_w-\beta\frac{\sigma_c}{e})-\beta>0,$

$\sigma_w-\beta\frac{\sigma_c}{e}>\beta\overline{n},$

$-\beta\frac{\sigma_c}{e}>\beta\overline{n}-\sigma_w,$

$\beta\frac{\sigma_c}{e}<\sigma_w-\beta\overline{n},$

$\frac{1}{e} < \frac{\sigma_w - \beta\overline{n}}{\beta\sigma_c}$ (if $\sigma_w - \beta\overline{n} > 0$ because $e > 0$),

$e > \frac{\beta\sigma_c}{\sigma_w - \beta\overline{n}}$.

**Remark** We note that $\sigma_c + \beta\overline{n} > \sigma_c - \beta\overline{n}$.

If $\beta$ is such that $\sigma_c - \beta\overline{n} > 0$, we obtain that

$\frac{1}{\sigma_c + \beta\overline{n}} < \frac{1}{\sigma_c - \beta\overline{n}}$,

from which

$\frac{\beta\sigma_c}{\sigma_c + \beta\overline{n}} < \frac{\beta\sigma_c}{\sigma_c - \beta\overline{n}}$.

**The boundary of triangle stability in terms of $e_f$ or $e_{f'}$**

From the conditions for the local stability of dynamical system rewritten in terms of $e_f$ or $e_{f'}$, easy we get the following correspondent conditions for the boundary of triangle stability:

- *Neimark-Sacker bifurcation curve*, defined by the condition $Det(k^P) = 1$, corresponds to

  $e_{f'}(k) = -\frac{(1+n)\beta_c}{\beta_w \sigma_c}$, denoted by **N**.

- The *Transcritical bifurcation curve* **T**, defined by the condition $Det(k^P) - Trace(k^P) + 1 = 0$, corresponds to

  $e = \frac{\beta\sigma_c}{\beta\overline{n} + \sigma_w}$.

- The *Flip bifurcation curve* **F** defined by $Det(k^P) + Trace(k^P) + 1 = 0$, corresponds to

  $e_{f'}(k) = -\frac{2}{-\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} + (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2}} =$

  $= \frac{2}{\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} - (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2}}$.

In order to describe in details the diagram of Flip bifurcation curve, we set

$$g(e) = \frac{2}{\frac{\sigma_c}{\overline{n}}\beta - (\sigma_c - \beta\frac{\sigma_c}{e})\frac{\sigma_c}{\overline{n}^2}}, \quad (e \neq \frac{\beta\sigma_c}{\sigma_w - \beta\overline{n}}).$$

We note that:

- *the straight line* $e = \frac{\beta\sigma_c}{\sigma_w - \beta\overline{n}}$ *is a vertical asymptote of* $g(e)$;
- *the straight line* $\overline{e} = \frac{\sigma_c}{\overline{n}}\beta - \frac{\sigma_c^2}{\overline{n}^2}$ *is an horizontal asymptote of* $g(e)$;
- *the map* $g(e)$ *is not monotonically decreasing.* As a matter of fact

$$g'(e) = \frac{2\beta^2\sigma_c^3}{\overline{n}^2}\frac{1}{[\frac{\sigma_c}{\overline{n}^2}(\sigma_w - \beta\frac{\sigma_c}{e}) - \frac{\sigma_c}{\overline{n}}\beta]^2}\frac{1}{e^2} > 0.$$

We recall that the Neimarck-Sacker boundary is, in the TD-plane, the segment of point $(T, D)$ such that $|T| \leq 2$ and $D = 1$. We will analyze the behaviour of $(T, D)$ when it is on this segment and varies some parameter.

If $D = 1$ then $\overline{e} = -\frac{\overline{n}}{\sigma_c}\frac{\beta_c}{\beta_w}$, from which $T = (\sigma_w - \frac{\beta_c}{\beta_w}\frac{\sigma_c}{e})(-\frac{1+n}{\sigma_c})\frac{1}{(1+n)^2}\sigma_c + 1 =$

$= (\sigma_w - \frac{\beta_c}{\beta_w}\frac{\sigma_c}{e})(-\frac{1}{1+n}) + 1.$

**Remark** If $f(k)$ is a CES then $0 < e \leq 1$. Thus the last case become $e = 1$, from which $T \to (\sigma_w - \frac{\beta_w}{\beta_c}\sigma_c)(-\frac{1}{1+n}) + 1.$

## 3.4 Study of the dynamical system in dependence on a single parameter

We propose to study the dynamical system when it depends on worker's discount rate, on capitalists' discount rate and on parameter $\rho$ of CES production function.

### 3.4.1 Dynamical system and workers' discount rate

We observe that when the workers' discount rate $\beta_w$ moves in $[0,1]$ in TD-plane the couple $(T(\beta_w), D(\beta_w))$ describe a segment which starts by $(T^\star, 0) = (T(0), D(0))$ and ends at $(T^{\star\star}, D^{\star\star}) = (T(1), D(1))$, where

$T(0) = \frac{e_{f'}(k^P)}{1+n}\sigma_c + 1,\ D(0) = 0,$

$T(1) = \{[1 + n - (1 - \delta)] - \frac{1}{\beta_c}\frac{\sigma_c}{e_f(k^P)}\}\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c + 1,$

$D(1) = -\frac{\sigma_c}{1+n}\frac{1}{\beta_c}e_{f'}(k^P).$

Moreover $D(\beta_w) \geq 0$ for all $\beta_w \in [0,1]$.

**Proposition 1** *The slope of above segment is positive and it is equal to*

$m = \frac{1}{\frac{\beta_c}{1+n}[(1-\delta)+\frac{\sigma_c}{\beta_c e_f(k^P)}]}$

**Proof** As a matter of fact

$m = \frac{D'(\beta_w)}{T'(\beta_w)} = \frac{\frac{\partial D(\beta_w)}{\partial\beta_w}}{\frac{\partial T(\beta_w)}{\partial\beta_w}} = \frac{-\frac{\sigma_c}{1+n}\frac{1}{\beta_c}e_{f'}(k^P)}{\{-(1-\delta)-\frac{\sigma_c}{\beta_c e_f(k^P)}\}}\frac{e_{f'}(k^P)\sigma_c}{(1+n)^2} = \frac{1}{\frac{\beta_c}{1+n}[(1-\delta)+\frac{\sigma_c}{\beta_c e_f(k^P)}]}$

We recall that:

- in the TD-plane the inside of $ABC$-triangle (where $A(0,1)$, $B(1,0)$ and $C(-1,0)$), which sides have respectively the following equations:

  $AC : D = T - 1, 0 \leq T \leq 2$, and $slope(AB) = 1$,

  $BC : D = 1, |T| \leq 2$; and $slope(BC) = 0$

$AB : D = -T - 1, -2 \leq T \leq 0$, and $slope(AB) = -1$,

gives the *stability region of Pasinetti's equilibria*;

- when a point (T,D) of dynamical system moves from inside (resp. outer) of $ABC$-triangle toward the outer (resp. inside) of triangle crossing one side or two sides of $ABC$-triangle, the Pasinetti's equilibria lose (resp. obtain) stability and they show bifurcations.

We observe that *the equation of family of segments which start by* $(T^{\star}, 0)$ *and which have slope* $m > 0$ *is* $D = m(T - T^{*})$ $(D > 0)$.

**Proposition 2**

**Case 1**: $-1 < T^{\star} < 1$. For the segment which starts from $(T^{\star}, 0)$ happens that (See **Figure 3.1**):

- it meets the $BC$-side if $m \geq 1$;
- it cuts the $AC$-side if $0 < m < -\frac{1}{T^{\star}-2}$;
- it is parallel to $AC$-side if $m = 1$;
- it never meets the $AB$-side;
- it goes through $C$-vertex if $\frac{1}{3} < m < 1$.

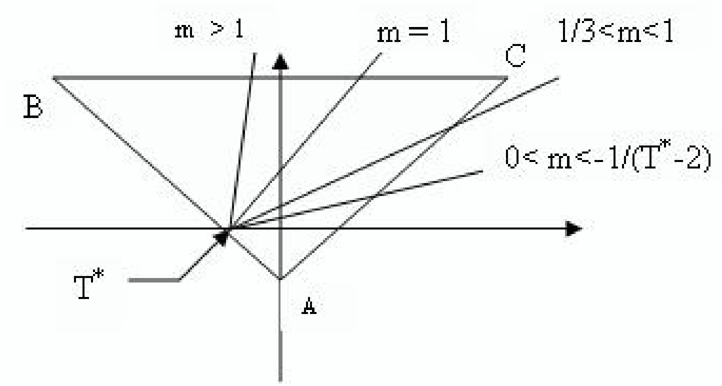


Figure 3.1: *The behaviour of the dynamical system as starting from* $(T^{\star}, 0)$*, where* $(-1 < T^{\star} < 1)$*.*

**Remark 1** It's easy to show that if $-1 < T^\star < 1$ then $\frac{1}{3} < -\frac{1}{T^\star - 2} < 1$.

**Remark 2** If $m = \frac{1}{3}$ then the segment which starts from $(T^\star, 0)$ meets $AC$-side.

**Remark 3** We observe that

$$m \geq 1 \Leftrightarrow \frac{\beta_c}{1+n}[(1-\delta) + \frac{\sigma_c}{\beta_c e_f(k^P)}] < 1 \Leftrightarrow < \frac{\sigma_c}{\beta_c e_f(k^P)} \frac{1+n}{\beta_c} - (1-\delta) = \frac{\sigma_c}{\beta_c}$$

$$\Leftrightarrow \frac{1}{e_f(k^P)} < 1.$$

**Case 2**: $T^\star = -2$. For the segment which starts from $(T^\star, 0)$ happens (See **Figure 3.2**):

- it meets the $AB$-side if $m > 0$;
- it cuts the $BC$-side if $m > \frac{1}{4}$;
- it meets the $AC$-side if $0 < m < \frac{1}{4}$;
- it goes through $C$-vertex if $m = \frac{1}{4}$.

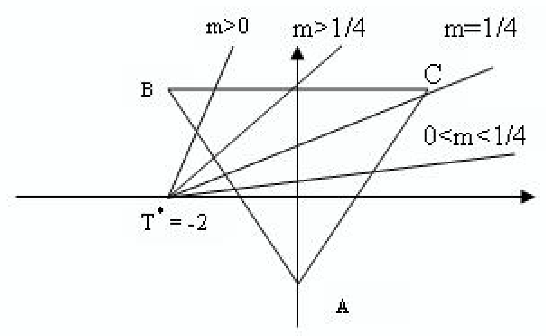


Figure 3.2: *The behaviour of the dynamical system as starting from* $(T^\star, 0)$, *where* $T^\star = -2$.

**Case 3**: $-2 < T^\star < -1$. For the segment which starts from $(T^\star, 0)$ happens (See **Figure 3.3**):

- it meets the $AB$-side if $m > 0$;
- it cuts the $BC$-side if $m > -1/(T^* - 2)$;
- it intersects the $AC$-side if $0 < m < -\frac{1}{T^\star - 2}$;
- it goes through the $C$-vertex if $\frac{1}{4} < m < \frac{1}{3}$.

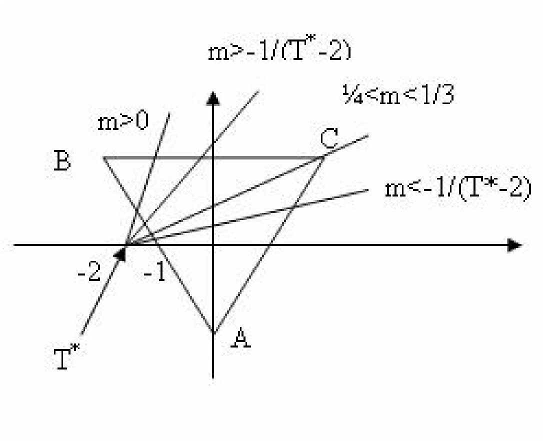


Figure 3.3: *The behaviour of the dynamical system as starting from* $(T^\star, 0)$*, where* $-2 < T^\star < -1$.

**Remark 4** It is to see that if $-2 < T^\star < -1$ then $\frac{1}{4} < -\frac{1}{T^\star - 2} < \frac{1}{3}$.

**Case 4**: $T^\star < -2$. For the segment which starts from $(T^\star, 0)$ happens (See **Figure 3.4**):

- it meets $AC$-side if $0 < m < -\frac{1}{T^\star}$;
- it cuts $BC$-side if $-\frac{1}{T^\star - 2} < m < \frac{1}{T^\star + 2}$;
- it intersects $AC$-side if $0 < m < 1$ and $m \neq -\frac{1}{T^\star - 2}$;
- it goes through $C$-vertex if $0 < m < \frac{1}{4}$.

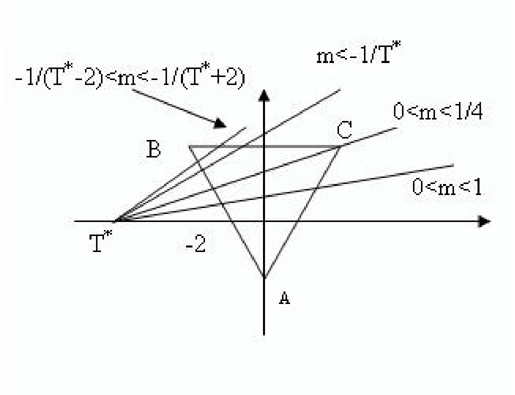


Figure 3.4: *The behaviour of the dynamical system as starting from* $(T^\star, 0)$*, where* $T^\star < -2$.

**Remark 5** It's easy to proof that if $T^\star < -2$ then $-\frac{1}{T^\star - 2} < \frac{1}{4}$.

**Case 5**: $T^\star > 1$. The segment which starts from $(T^\star, 0)$ never meets the $ABC$-triangle.

**Proof**

**Case 1** For brevity we'll prove only statement "*the family of segments which start from* $(T^\star, 0)$ *and which have slope* $m > 0$ *cut the* $AC$*-side only if* $m$ *is less than* $-\frac{1}{T^\star - 2}$". We assume $-1 < T^* < 1$. We start with solving the system

$$\begin{cases} D = T - 1, \\ D = m(T - T^*), \end{cases}$$

where the former item is the equation of $AC$-side and last item is the equation of the family of segments $S$ which start from $(T^*, 0)$ and which have slope equal to $m > 0$. We obtain $T = \frac{mT^* - 1}{m - 1}$. Obviously $m - 1 < 0$. Since $D > 0$, we observe that $S$ intersects $AB$-side only if $1 < T < 2$, thus we have $1 < \frac{mT^* - 1}{m - 1} < 2$. From the inequality $\frac{mT^* - 1}{m - 1} > 1$ we deduce $T^* < 1$ for all $m > 0$. Instead from the inequality $\frac{mT^* - 1}{m - 1} < 2$ we get $m(T^* - 2) > -1$. The case $T^* - 2 > 0$, i.e. $T^* > 2$ is impossible by the assumption $-1 < T^* < 1$. Therefore $m < -\frac{1}{T^* - 2}$. The meaning of statement is that *the dynamical system which starts by a stable point of Pasinetti equilibrium* $(T^*, 0)$ $(-1 < T^* < 1)$ *can cross the boundary* $AC$ *of the flip bifurcations only if* $m < -1/(T^* - 2)$.

**Case 2** We'll prove only statement "*the family of segments which start from* $(-2, 0)$ *and which have slope* $m > 0$ *cut the* $BC$*-side only if* $m > 1/4$". We assume $T^* = -2$ and $-2 < T < 2$. We consider the system

$$\begin{cases} D = m(T+2), \\ D = 1, \end{cases}$$

where the former item is the equation of the family of segments $S$ which start from $(-2, 0)$ and which have slope equal to $m > 0$ and the last item is the equation of $BC$-side. The solution of the system is $T = (1-2m)/m$ and by assumption $-2 < T < 2$ we deduce that $-2 < \frac{1-2m}{m} < 2$. From inequality $\frac{1-2m}{m} < 2$ we obtain $m > 1/4$, instead from inequality $\frac{1-2m}{m} > -2$ we have $m > 0$, that is the conclusion. The meaning of statement is that *the dynamical system which starts by a unstable point of Pasinetti equilibrium* $(-2, 0)$ *can cross the boundary* $BC$ *of the Neimark-Sacker bifurcations only if* $m > 1/4$.

**Case 3** We'll prove only statement "*the family of segments which start from* $(T^*, 0)$ $(-2 < T^* < -1)$ *and which have slope* $m > 0$ *cut the* $BC$*-side only if* $m > -1/(T^* - 2)$". Solve the system

$$\begin{cases} D = m(T - T^*), \\ D = 1, \end{cases}$$

where the former item is the equation of the family of segments $S$ which start from $(T^*, 0)$ $(-2 < T^* < -1)$ and which have slope equal to $m > 0$ and the last item is the equation of $BC$-side $(-2 < T < 2)$. We have $-2 < \frac{1}{m} + T^* < 2$. We solve the inequality $\frac{1}{m} + T^* > -2$. We have $m(T^* + 2) > -1$. By the assumption $(-2 < T^* < -1)$ we obtain $m > -\frac{1}{T^*+2}$. Since $m > 0$ the previous inequality is always true. Instead the inequality $\frac{1}{m} + T^* < 2$ is equivalent to the inequality $m(T^* - 2) < -2$, from which $m > -\frac{1}{T^*-2}$. In terms of bifurcations we can say that *the dynamical system which starts by a unstable point of Pasinetti equilibrium* $(T^*, 0)$ $(-2 < T^* < -1)$ *can cross the boundary* $BC$ *of the Neimark-Sacker bifurcations only if* $m > -1/(T^* - 2)$.

**Case 4** We'll prove only statement "*the family of segments which start from* $(T^*, 0)$ $(T^* < -2)$ *and which have slope* $m > 0$ *cut the* $BC$*-side only if* $-1/(T^* - 2) < m < 1/(T^* + 2)$". Consider the system

$$\begin{cases} D = m(T - T^*), \\ D = 1, \end{cases}$$

where the former item is the equation of the family of segments $S$ which start from $(T^*, 0)$ $(T^* < -2)$ and which have slope equal to $m > 0$ and the last item is the equation of $BC$-side $(-2 < T < 2)$. As before we have $-2 <$

$\frac{1}{m} + T^* < 2$ $(*)$. Since $T^* < -2$, then we get $m < -\frac{1}{T^*+2}$ from the left hand side of $(*)$ and $m > -\frac{1}{T*-2}$ from the right hand side of $(*)$. We conclude saying that *the dynamical system which starts by a unstable point of Pasinetti equilibrium* $(T^*, 0)$ *(*$T^* < -2$*) can cross the boundary BC of the Neimark-Sacker bifurcations only if* $-1/(T^* - 2) < m < -1/(T^* + 2)$.

**Case 5** Very easy to prove.

### 3.4.2 Dynamical system and capitalists' discount rate

We observe that when the capitalists' discount rate $\beta_c$ moves in $]0, 1]$ in TD-plane the couples $(T(\beta_c), D(\beta_c))$ describe an open curve $\Gamma$ which starts by $AA = (\lim_{\beta_c \to 0} T(\beta_c), \lim_{\beta_c \to 0} D(\beta_c)$ and ends at $BB = (T(1), D(1))$, where

$$T(1) = [\sigma_w - \frac{\beta_w}{e_f(k^P)}(n + \delta)](n + \delta)\frac{e_{f'}(k^P)}{(1+n)^2} + 1,$$

$$D(1) = -\frac{(n+\delta)\beta_w e_{f'}(k^P)}{1+n}.$$

We consider, as above, both positive $\sigma_w > 0$ and $\sigma_c > 0$ and we observe that $D > 0$ for all $\beta_c \in ]0, 1]$. From the dynamical equations systems we have

**Proposition 1** If $e_{f'}(k^P) < -\frac{1+n}{\sigma_w \sigma_c}$ then $T > 0$ if and only if

$$\beta_c < \frac{\beta_w \sigma_c^2 e_{f'}(k^P)}{(1+n)^2 e_f(k^P) + \sigma_w \sigma_c e_f(k^P) e_{f'}(k^P)}.$$

**Proof** $T > 0 \Leftrightarrow (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)})\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c > -1$

$$\Leftrightarrow \sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)} < -\frac{(1+n)^2}{e_{f'}(k^P)\sigma_c}$$

$$\Leftrightarrow \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)} > \frac{(1+n)^2}{e_{f'}(k^P)\sigma_c} + \sigma_w$$

$\Leftrightarrow \frac{1}{\beta_c} > \frac{(1+n)^2 e_f(k^P)+\sigma_w\sigma_c e_f(k^P)e_{f'}(k^P)}{\beta_w\sigma_c^2 e_{f'}(k^P)}$.

We observing that

$$\frac{\partial}{\partial\beta_c}(\frac{1}{\beta_c}) = -\frac{1}{\beta_c^2};$$

$$\frac{\partial}{\partial\beta_c}(\sigma_c) = \frac{\partial}{\partial\beta_c}[(1+n)-(1-\delta)\beta_c] = \delta - 1;$$

$$\frac{\partial}{\partial\beta_c}(\frac{\sigma_c}{\beta_c}) = \frac{(\delta-1)\beta_c-\sigma_c}{\beta_c^2} = -\frac{1+n}{\beta_c}.$$

$$\frac{\partial}{\partial\beta_c}(\sigma_w) = 0.$$

From the equation $f^{'}(k^P) = \frac{1+n}{\beta_c} - (1-\delta)$ and if $f^{'}$ has inverse $(f^{'})^{-1}$ we obtain that

$$k^P = (f^{'})^{-1}[\frac{1+n}{\beta_c} - (1-\delta)].$$

Therefore we deduce that $k^P$ depends on $\beta_c$, that is $k^P = k^P(\beta_c)$.

Thus

$$\frac{\partial k^P}{\partial\beta_c} = \frac{\partial}{\partial\beta_c}(f^{'})^{-1}[\frac{1+n}{\beta_c} - (1-\delta)] = \frac{1}{f^{''}[(f^{'})^{-1}(k^P)]}(-\frac{1+n}{\beta_c^2}).$$

Moreover

$$\frac{\partial}{\partial\beta_c}D(\beta_c) = -\frac{\beta_w}{1+n}\frac{\partial}{\partial\beta_c}[\frac{\sigma_c}{\beta_c}e_{f'}(k^P)] = -\frac{\beta_w}{1+n}\{(\frac{\partial}{\partial\beta_c}(\frac{\sigma_c}{\beta_c}))e_{f'}(k^P) + \frac{\sigma_c}{\beta_c}(\frac{\partial}{\beta_c}e_{f'}(k^P))\}$$

$$= -\frac{\beta_w}{1+n}\ \{(-\frac{1+n}{\beta_c^2})e_{f'}(k^P) + \frac{\sigma_c}{\beta_c}\ \frac{\partial}{\partial k^P}e_{f'}(k^P)\frac{\partial}{\partial\beta_c}k^P\}.$$

We recall that

$T(\beta_c) = [\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)}]\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c + 1.$ Then

$$\frac{\partial}{\partial\beta_c}T(\beta_c) = -\beta_w\frac{\partial}{\partial\beta_c}(\frac{\sigma_c}{\beta_c}\frac{1}{e_f(k^P)})\ \frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c + [\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)}]\ \frac{1}{(1+n)^2}\frac{\partial}{\partial\beta_c}[e_{f'}(k^P)\sigma_c]$$

$$= -\beta_w[(\frac{\partial}{\partial\beta_c}(\frac{\sigma_c}{\beta_c}))\frac{1}{e_f(k^P)} + \frac{\sigma_c}{\beta_c}\frac{\partial}{\partial\beta_c}\frac{1}{e_f(k^P)}]\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c +$$

$$+[\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)}]\ \frac{1}{(1+n)^2}\ [(\frac{\partial}{\partial\beta_c}e_{f'}(k^P))\sigma_c + e_{f'}(k^P)\frac{\partial}{\partial\beta_c}\sigma_c]$$

$$= -\beta_w[(-\frac{1+n}{\beta_c^2})\frac{1}{e_f(k^P)}\ + \frac{\sigma_c}{\beta_c}\frac{\partial}{\partial k^P}\frac{1}{e_{f'}(k^P)}\frac{\partial}{\partial\beta_c}k^P]\ \frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c +$$

$$+[\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)}]\ \frac{1}{(1+n)^2}\ [(\frac{\partial}{\partial k^P}e_{f'}(k^P)\frac{\partial}{\partial\beta_c}k^P)\sigma_c + \ e_{f'}(k^P)\frac{\partial}{\partial\beta_c}\sigma_c].$$

In order to illustrate the behaviour of dynamical system we present some interesting simulations:

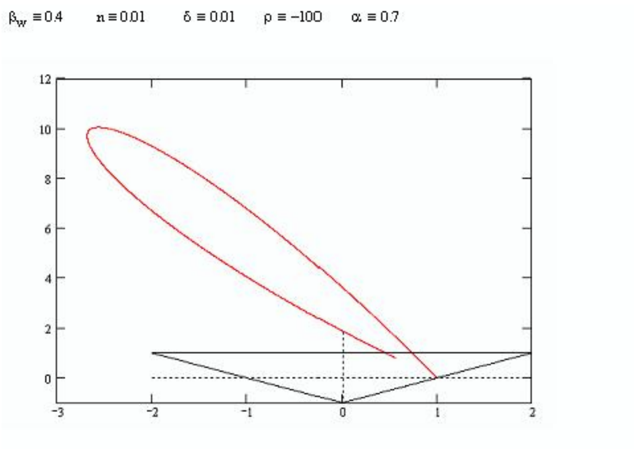


Figure 3.5: $\beta = 0.4$, $n = 0.01$, $\delta = 0.01$, $\rho = -100$, $\alpha = 0.7$

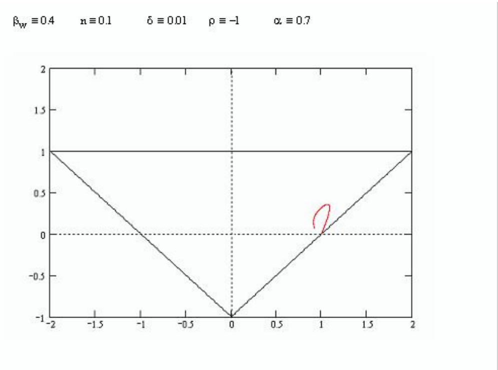


Figure 3.6: $\beta = 0.4,\ n = 0.1,\ \delta = 0.01,\ \rho = -1,\ \alpha = 0.7$

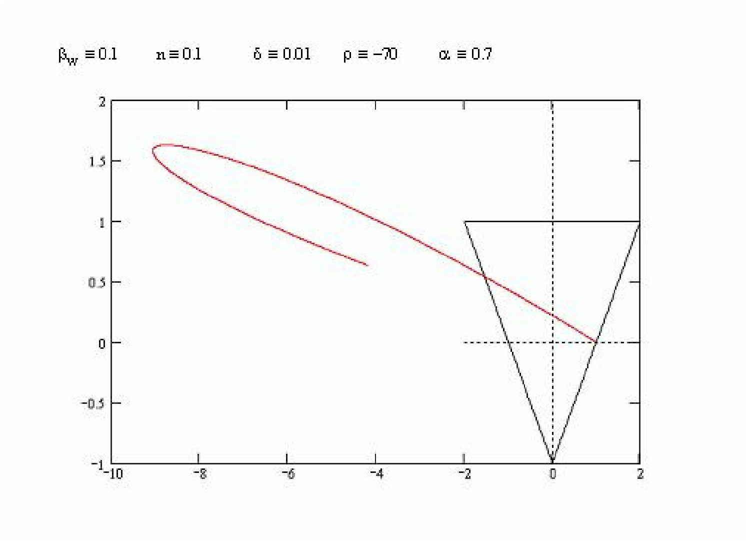


Figure 3.7: $\beta = 0.1,\ n = 0.1,\ \delta = 0.01,\ \rho = -70,\ \alpha = 0.7$

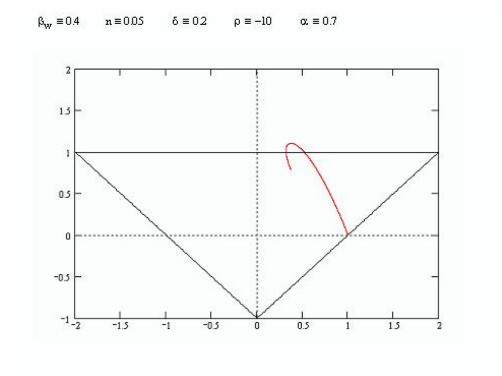


Figure 3.8: $\beta = 0.4,\ n = 0.5,\ \delta = 0.2,\ \rho = -10,\ \alpha = 0.7$

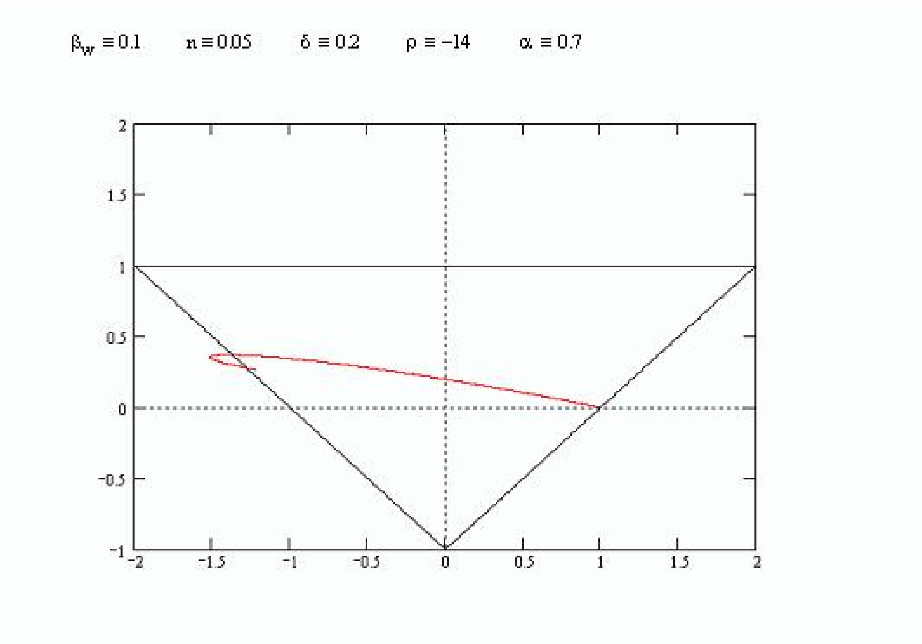


Figure 3.9: $\beta = 0.1,\ n = 0.5,\ \delta = 0.2,\ \rho = -14,\ \alpha = 0.7$

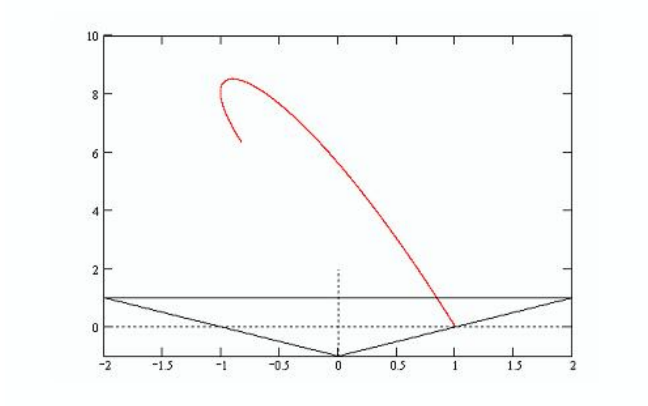


Figure 3.10: $\beta = 0.5,\ n = 0.05,\ \delta = 0.2,\ \rho = -70,\ \alpha = 0.7$

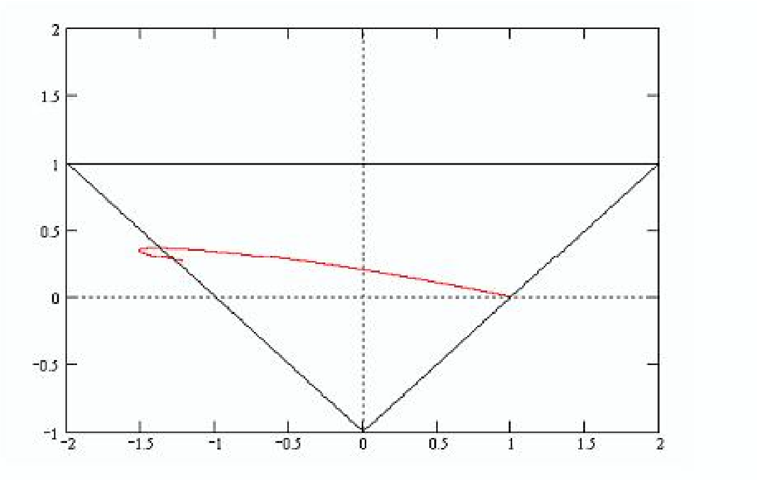


Figure 3.11: $\beta = 0.1,\ n = 0.05,\ \delta = 0.2,\ \rho = -14,\ \alpha = 0.7$

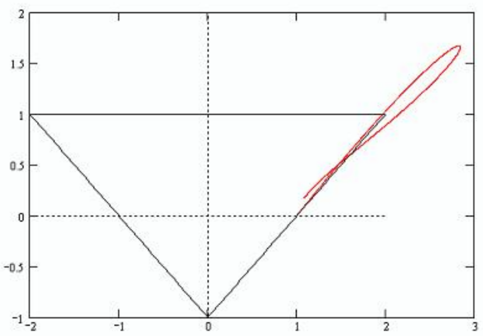


Figure 3.12: $\beta = 1,\ n = 0.05,\ \delta = 0.1,\ \rho = -0.5,\ \alpha = 0.7$

### 3.4.3 Dynamical system and a CES production function

We start from the CES production function

$f(k) = [\alpha + (1-\alpha)k^{\rho}]^{\frac{1}{\rho}}.$

where $-\infty < \rho < 1$, $\rho \neq 0$, $0 < \alpha < 1$.

We have

$e_f(k) = (1-\alpha)(\alpha k^{-\rho} + 1 - \alpha)^{-1};$

$e_{f'}(k) = \alpha(\rho - 1)[\alpha + (1-\alpha)k^{\rho}]^{-1};$

from which the dynamical system

$D(k^P) = -\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} e_{f'}(k^P),$

$T(k^P) = [\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k^P)}]\frac{e_{f'}(k^P)}{(1+n)^2}\sigma_c + 1,$

becomes

$D(\rho) = -\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c}\alpha(\rho-1)[\alpha+(1-\alpha)k^\rho]^{-1};$

$T(\rho) = \frac{\alpha\sigma_c}{(1+n)^2}[\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{\alpha-1}(\alpha k^{-\rho}+1-\alpha)]\ (\rho-1)[\alpha+(1-\alpha)k^\rho]^{-1}+1.$

We recall that

$\lim_{\rho\to-\infty} k^{-\rho} = \begin{cases} +\infty, & \text{if } k>1 \\ 0, & \text{if } 0<k<1 \end{cases};$

$\lim_{\rho\to-\infty} k^{\rho} = \begin{cases} 0, & \text{if } k>1 \\ +\infty, & \text{if } 0<k<1 \end{cases}.$

From the previous results we deduce that:

- $\lim_{\rho\to-\infty} T(\rho) = \begin{cases} +\infty, & \text{if } k>1 \\ 1, & \text{if } 0<k<1 \end{cases};$
- $\lim_{\rho\to-\infty} D(\rho) = \begin{cases} +\infty, & k>1 \\ 0, & 0<k<1 \end{cases};$
- for all $k>0$, $\lim_{\rho\to1} T(\rho) = 1$ and $lim_{\rho\to1} D(\rho) = 0;$
- for all $k>0$, $\lim_{\rho\to0} D(\rho) = \frac{\sigma_c\beta_w\alpha}{(1+n)\beta_c} > 0;$
- for all $k>0$, $\lim_{\rho\to0} T(\rho) = 1 - \frac{\alpha\sigma_c}{(1+n)^2}(\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{\alpha-1})$

  $= 1 - \frac{\alpha\sigma_c}{(1+n)^2}(\sigma_w + \frac{\beta_w}{\beta_c}\frac{\sigma_c}{1-\alpha}).$

Thus we can observe that when $\rho$ moves in $]-\infty, 1[$ ($\rho \neq 0$) and if $k > 1$, in TD-plane *the couples* $(T(\rho), D(\rho))$ *describe an open curve which starts by* $(+\infty, +\infty)$ *and ends at* $(1, 0)$ .

**Remark 1** If $0 < k < 1$, then, by *the Hôpital's Rule*, we get

$\lim_{\rho\to-\infty} \frac{\rho-1}{\alpha+(1-\alpha)k^\rho} = \lim_{\rho\to-\infty} \frac{1}{(1-\alpha)k^\rho \ln k} = 0.$

For all $\rho < 1$ ($\rho \neq 0$) and for all $k > 0$ ($k \neq 1$), we set $\psi(\rho) = \alpha + (1-\alpha)k^\rho + (1-\rho)(1-\alpha)k^\rho \ln k$.

**Proposition 1** For all $-\infty < \rho < 1$ we get:

(A) for all $k > 1$, $\frac{\partial D(\rho)}{\partial \rho} < 0$;

(B) for all $0 < k < 1$, $\frac{\partial D(\rho)}{\partial \rho} \begin{cases} > 0, & \text{if } \rho < \rho_0 \\ < 0, & \text{if } \rho_0 < \rho < 1 \end{cases}$,

where $\rho_0$ is the unique zero of $\psi(\rho)$.

**Proof** We have

$\frac{\partial}{\partial \rho} D(\rho) = [-\frac{\alpha\sigma_c\beta_w}{(1+n)\beta_c}]\ \{[\alpha + (1-\alpha)k^\rho]^{-1} +$

$+(\rho - 1)(-1)[\alpha + (1-\alpha)k^\rho]^{-2}(1-\alpha)k^\rho \ln k\}$

$= [-\frac{\alpha\sigma_c\beta_w}{(1+n)\beta_c}]\ [\alpha + (1-\alpha)k^\rho]^{-2}\psi(\rho).$

We notice that the sign of $\frac{\partial D(\rho)}{\partial \rho}$ depends on factor $\psi(\rho)$.

Therefore we consider two cases.

**Case 1**: $k > 1$. Since $\ln k > 0$, then $\psi(\rho) > 0$.

Thus, for all $k > 1$, $\frac{\partial D(\rho)}{\partial \rho} < 0$.

**Case 2**: $0 < k < 1$. Then $\ln k < 0$. Moreover

- $\lim_{\rho \to -\infty} \psi(\rho) = -\infty < 0$;
- $\lim_{\rho \to 1} \psi(\rho) = \alpha + (1 - \alpha)k > 0$;
- $\lim_{\rho \to 0} \psi(\rho) = 1 + (1 - \alpha) \ln k$;
- $1 + (1 - \alpha) \ln k > 0$ if and only if $k > e^{-\frac{1}{1-\alpha}}$ and, being $-\frac{1}{1-\alpha} < 0$, $e^{-\frac{1}{1-\alpha}} < 1$;
- $\frac{\partial \psi(\rho)}{\partial \rho} = = (1 - \rho)(1 - \alpha)k^{\rho}(\ln k)^2 > 0$.

Thus, being $\psi(\rho)$ continuous and strictly increasing in $]-\infty, 1[$, the *Intermediate Value Theorem* guarantees that there is a unique point $\rho_0$ in which $\psi(\rho_0) = 0$, and $\psi(\rho) < 0$ for all $\rho < \rho_0$ and $\psi(\rho) > 0$ for all $\rho > \rho_0$, Q.E.D..

From definition of $D(\rho)$ and from Proposition 1 we deduce that

**Proposition 2** The function $D(\rho)$ is positive for all $\rho < 1$ and for all $k > 0$ ($k \neq$). About monotonicity of $D(\rho)$ we can say that:

(A) for all $k > 1$ and for all $\rho < 1$, it is strictly decreasing;

(B) for all $0 < k < 1$, $D(\rho)$ is *unimodal*: it is strictly increasing for all $\rho < \rho_0$ and it is strictly decreasing for all $\rho > \rho_0$. The maximum is $D(\rho_0)$.

We set $\varphi(\rho) = \alpha + (1-\alpha)k^{\rho}$ and we call $\varphi^{-1}(\rho) = \frac{1}{\varphi(\rho)}$. Then we can so rewrite $T(\rho)$:

$$T(\rho) = \frac{\alpha\sigma_c}{(1+n)^2}[\sigma_w - \frac{\beta_w\sigma_c}{\beta_c(1-\alpha)}(\alpha k^{-\rho} + 1 - \alpha)](\rho - 1)\varphi^{-1}(\rho) + 1.$$

Thus

$$\frac{\partial T(\rho)}{\partial \rho} = \frac{\alpha\sigma_c}{(1+n)^2}\{(-\frac{\beta_w\sigma_c}{\beta_c(1-\alpha)})(-\alpha k^{-\rho}\ln k)(\rho-1)\varphi^{-1}(\rho) +$$

$$+[\sigma_w - \frac{\beta_w\sigma_c}{\beta_c(1-\alpha)}(\alpha k^{-\rho} + 1 - \alpha)][\varphi^{-1}(\rho) + (\rho-1)(-\varphi^{-2}(\rho)\varphi^{'}(\rho))]\}$$

$$= \frac{\alpha\sigma_c}{(1+n)^2}\varphi^{-1}(\rho)\{(\frac{\alpha\beta_w\sigma_c}{\beta_c(1-\alpha)})(k^{-\rho}\ln k)(\rho-1) +$$

$$+[\sigma_w - \frac{\beta_w\sigma_c}{\beta_c(1-\alpha)}(\alpha k^{-\rho} + 1 - \alpha)][1 + (\rho-1)(-\varphi^{-1}(\rho)\varphi^{'}(\rho))]\}$$

$$= \frac{\alpha\sigma_c}{(1+n)^2}\varphi^{-1}(\rho)\{(\frac{\alpha\beta_w\sigma_c}{\beta_c(1-\alpha)})(k^{-\rho}\ln k)(\rho-1) +$$

$$+[\sigma_w - \frac{\beta_w\sigma_c}{\beta_c(1-\alpha)}(\alpha k^{-\rho} + 1 - \alpha)][1 + (1-\rho)\varphi^{-1}(\rho)\varphi^{'}(\rho)]\}.$$

In order to simplify the expression of $T(\rho)$ we calculate the Taylor expansion of $T$ of one order and with center at $-1$ and we get:

$$T_{Taylor}(\rho) = \frac{A^*k(1-\rho)(\alpha B^*(k-1) + B^* - \sigma_w)}{\alpha(k-1)+1} +$$

$$-\frac{2A^*k(\rho+1)(\alpha^2B^*(k^2-2k+1) + \alpha(2B^*(k-1)+\sigma_w) + B^* - \sigma_w)\ln k}{(\alpha(k-1)+1)^2},$$

where $A^* = \frac{\alpha\sigma_c}{(1+n)^2}$ and $B^* = \frac{\beta_w\sigma_c}{\beta_c(1-\alpha)}$.

Moreover we have that

$\frac{\partial T_{Taylor}(\rho)}{\partial \rho} = -\frac{2A^* k(\alpha^2 B^*(k^2-2k+1)+\alpha(2B^*(k-1)+\sigma_w)+B^*-\sigma_w)\ln k}{(\alpha(k-1)+1)^2} +$

$-\frac{A^* k(\alpha B^*(k-1)+B^*-\sigma_w)}{\alpha(k-1)+1}.$

If we assume $k > 1$ and $B^* > \sigma_w$ we obtain that, for $\rho$ near to $-1$,

$\frac{\partial T_{Taylor}(\rho)}{\partial \rho} < 0.$

We can conclude that, *if* $k < 1$, $T_{Taylor}(\rho)$ *is decreasing for* $\rho$ *near to* $-1$ *(*$\rho \neq 0$*).*

The following figures describe the behaviour of the maps $D(\rho)$, $T(\rho)$ and of the dynamical system $(D(\rho), T(\rho))$ for $n = 0.1$,$\delta = 0.2$,$\alpha = 0.2$,$\beta_c = 0.3$,$\beta_w = 0.2$,$k = 0.4$:

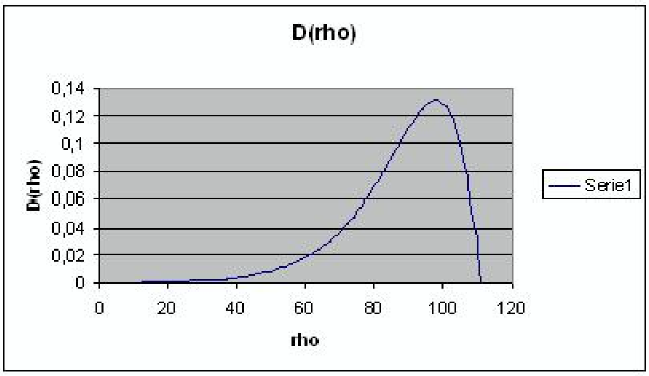


Figure 3.13: *The map* $D(\rho)$

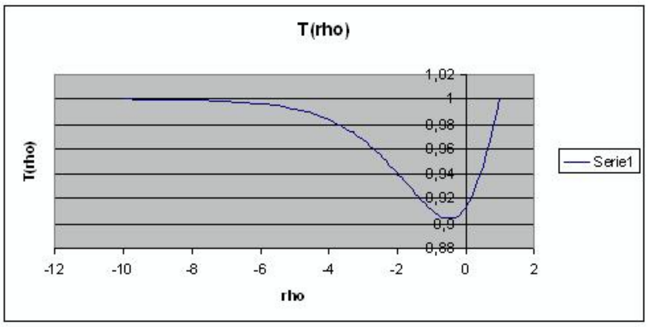


Figure 3.14: *The map* $T(\rho)$

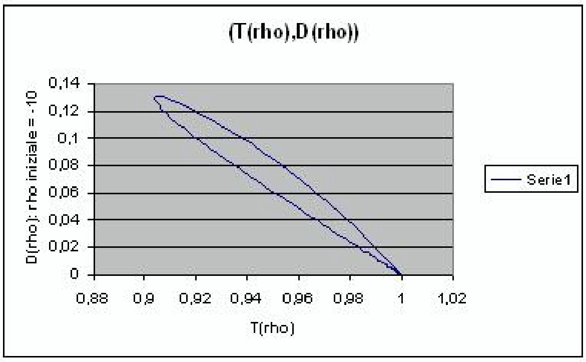


Figure 3.15: *The dynamical system* $(D(\rho), T(\rho))$

## 3.5 Appendix 1

Let $e_f(k) = [\frac{f'(k)k}{f(k)}]$ be the elasticity function of $f(k)$ and let $e_{f'}(k) = \frac{f''(k)k}{f'(k)}$ be the elasticity function of $f'(k)$.

**Proposition** *Consider* $f(k) \geq 0$, $f'(k) > 0$, $f''(k) < 0$ *for all* $k \geq 0$. *Then:*

*(i)* $e_f(k) > 0$ *and* $e_{f'}(k) < 0$*;*

*(ii)* $(e_{f'}(k) > -1) \Leftrightarrow (e_f(k)$ *is monotone increasing).*

**Proof**

(i) Trivial.

(ii) It notices that

$$\frac{de_f(k)}{dk} = \frac{d}{dk}[\frac{f'(k)k}{f(k)}] = \frac{[f''(k)k+f'(k)]f(k)-k[f'(k)]^2}{[f(k)]^2} > 0$$

is equivalent to the following inequalities:

$[f''(k)k + f'(k)]f(k) - k[f'(k)]^2 > 0;$

$\frac{[f''(k)k+f'(k)]f(k)}{k[f'(k)]^2} > 0;$

$f''(k)k > -f'(k);$

$\frac{f''(k)k}{f'(k)} > -1$, that is $e_{f'}(k) > -1$.

**Remark 1** From proof of (ii) it deduces that $\frac{de_f(k)}{dk} \neq e_{f'}(k)$.

**Remark 2** The (ii) is equivalent to $(-1 < e_{f'}(k) < 0)$.

## 3.6 Appendix 2 (Helpful inequalities)

(A) For $n \geq 0, 0 < \beta_c < 1, 0 < \beta_w < 1$ and $0 < \delta < 1$, then:

(1) $\sigma_c = (1+n) - (1-\delta)\beta_c < (1+n)$ and $\sigma_w = (1+n) - (1-\delta)\beta_w < (1+n)$;

(2) $(1+n) \geq 1$; $0 < (1-\delta)\beta_c < 1$ and $0 < (1-\delta)\beta_w < 1$;

(3) $\sigma_c > 0$ and $\sigma_w > 0$;

(4) $(1+n)\beta_c > \beta_c\sigma_c$, $(1+n)\beta_w > \beta_c\sigma_w$, $(1+n)\beta_c > \beta_c\sigma_w$, $(1+n)\beta_w > \beta_w\sigma_c$, and $(1+n)\beta_w > \sigma_c$;

(5) if $(0 < \beta_w < \beta_c < 1)$ then $((1+n)\beta_w - \beta_c\sigma_w < 0)$.

(B) $\frac{1+n}{\sigma_c} > 1$, $\frac{\beta_c}{\beta_w} > 1$, $\frac{1+n}{\sigma_c}\frac{\beta_c}{\beta_w} > 1$, $-\frac{1+n}{\sigma_c}\frac{\beta_c}{\beta_w} < -1$ and $-\frac{1+n}{\sigma_c} < -1$.

(C) $-\frac{1+n}{\sigma_c}\frac{\beta_c}{\beta_w} < -\frac{\beta_c}{\beta_w} < 0$.

(D) if $0 < \beta_w < \beta_c < 1$ then $-\frac{\beta_c}{\beta_w} < -1$.

(E) $F(n, \beta_c, \beta_w, \delta) = -\frac{1+n}{\sigma_c}\frac{\beta_c}{\beta_w} < -1 < 0$.

(F) If $0 < \beta_w < \beta_c < 1$ then $\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} < 1$.

(G) $\sigma_c < (1+n) < (1+n)^2$ and $\frac{\sigma_c}{(1+n)^2} < 1$.

(H) if $(0 < \beta_w < \beta_c < 1)$ then $\sigma_c < \sigma_w$ and $\frac{\sigma_c}{\sigma_w} < 1$.

(I) $\sigma_w - \frac{\beta_w\sigma_c}{(1+n)\beta_c} < \sigma_w$ and $\frac{\sigma_c}{\sigma_w} < \frac{\sigma_c}{\sigma_w - \frac{\beta_w\sigma_c}{(1+n)\beta_c}}$.

(L) $0 < \beta_w\sigma_c < \beta_c\sigma_w < (1+n)\beta_c\sigma_w$, $(1+n)\beta_c\sigma_w$ and

$(1+n)\beta_c\sigma_w - \beta_w\sigma_c > 0$.

(M) $\frac{(1+n)\beta_c\sigma_c}{(1+n)\beta_c\sigma_w - \beta_w\sigma_c} > 0$.

(N) $Det(k^P) = -\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c}e_{f'}(k^P) > 0$ and $Det(k^P) < -e_{f'}(k^P)$.

## 3.7 Appendix 3 (About Flip bifurcations: a condition of existence)

Because $e_{f'}(k) < 0$ for all $k \geq 0$, it notices that

$$e_{f'}(k) = \frac{2}{\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} - (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2}} < 0,$$

if and only if are true the following inequalities:

$$\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} - (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2} < 0,$$

$$\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} > \frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c}\frac{(1+n)^2}{\sigma_c},$$

$$\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} > -\frac{\beta_w(1+n)}{\beta_c},$$

$$\frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} < \frac{\beta_w(1+n)}{\beta_c} - \sigma_w,$$

$$\frac{\sigma_c}{e_f(k)} < (1+n) - \frac{\sigma_w\beta_c}{\sigma_c\beta_w},$$

$$\frac{1}{e_f(k)} < \frac{(1+n)}{\sigma_c} - \frac{\sigma_w\beta_c}{\sigma_c\beta_w},$$

$$\frac{1}{e_f(k)} < \frac{(1+n)\beta_w - \sigma_w\beta_c}{\sigma_c\beta_w},$$

$$e_f(k) > \frac{\sigma_c\beta_w}{(1+n)\beta_w - \sigma_w\beta_c}.$$

(Q) From (5)(A) it has

$\frac{\sigma_c\beta_w}{(1+n)\beta_w - \sigma_w\beta_c} < 0$ and because

$(1+n)\beta_w - \sigma_c\beta_c < (1+n) - \sigma_c < (1+n),$

then

$e_f(k) > \frac{\sigma_c}{(1+n)-\sigma_c} > \frac{\sigma_c}{(1+n)}.$

Obviously $\frac{\sigma_c}{(1+n)-\sigma_c}$ is negative and $\frac{\sigma_c}{(1+n)}$ is positive.

## 3.8 Appendix 4 (About Pitchfork bifurcations: a condition of existence)

The condition

$Det(k^P) - Trace(k^P) + 1 = 0$

is equivalent to formula

$e_{f'}(k) = \frac{2}{\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c}+(\sigma_w-\frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2}}.$

But, because $e_{f'}(k) < 0$, then

$\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c} + (\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2} < 0,$

from which, it obtain the following inequalities:

$(\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)})\frac{\sigma_c}{(1+n)^2} < -\frac{\sigma_c}{1+n}\frac{\beta_w}{\beta_c},$

$\sigma_w - \frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} < -\frac{\beta_w(1+n)}{\beta_c},$

$-\frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} < -\sigma_w - \frac{\beta_w(1+n)}{\beta_c}$,

$\frac{\beta_w}{\beta_c}\frac{\sigma_c}{e_f(k)} > \sigma_w + \frac{\beta_w(1+n)}{\beta_c} = \frac{\beta_c\sigma_w+\beta_w(1+n)}{\beta_c}$,

$e_f(k) < \frac{\sigma_c\beta_w}{\beta_c\sigma_w+\beta_w(1+n)} < \frac{\sigma_c}{\beta_c\sigma_w+\beta_w(1+n)}$.

But

$\beta_c\sigma_w + \beta_w(1+n) < \beta_c\sigma_c + \beta_w(1+n) > \beta_c\sigma_c$,

from which

$e_f(k) < \frac{\sigma_c}{\beta c\sigma_c} = \frac{1}{\beta_c}$.

## 3.9 Appendix 5

If $k^P$ don't depends on $\beta_c$ we have

**Proposition 2** $\frac{\partial D}{\partial\beta_c} < 0$ for all $\beta_c \in ]0,1]$.

**Proof** $\frac{\partial D}{\partial\beta_c} = (-\frac{\beta_w e_{f'}(k^P)}{1+n})\frac{\partial}{\partial\beta_c}(\frac{\sigma_c}{\beta_c})$

$= (-\frac{\beta_w e_{f'}(k^P)}{1+n})(-\frac{1+n}{\beta_c^2}) = \frac{\beta_w e_{f'}(k^P)}{\beta_c^2} < 0$,

because $e_{f'}(k^P)$ is negative while $\beta_w$ and $\beta_c^2$ are positive.

We put $A = e_f(k^P)\sigma_w(\delta-1)$, $B = -\beta_w\sigma_c(\delta-1)$, $C = \beta_w\sigma_c(1+n)$,

$\Delta = B^2 - 4AC$ and recall that $\beta_c \in ]0,1]$.

Then we have

**Proposition 3** The sign of $\frac{\partial T}{\partial \beta_c}$ is

- positive for all $\beta_c$ if $\Delta < 0$;
- positive for all $\beta_c \neq -\frac{B}{2A}$ if $\Delta = 0$;
- negative for all $\beta_c \in ]\frac{-B-\sqrt{\Delta}}{2A}, \frac{-B+\sqrt{\Delta}}{2A}[$ and is positive otherwise if $\Delta > 0$.

**Proof** $\frac{\partial T}{\partial \beta_c} = \frac{e_{f'}(k^P)}{(1+n)^2} \{[\frac{\partial}{\partial \beta_c}(\sigma_w - \frac{\beta_w}{e_f(k^P)} \frac{\sigma_c}{\beta_c})]\sigma_c + (\sigma_w - \frac{\beta_w}{e_f(k^P)} \frac{\sigma_c}{\beta_c}) \frac{\partial}{\partial \beta_c} \sigma_c\}$

$= \frac{e_{f'}(k^P)}{(1+n)^2} \{(-\frac{\beta_w}{e_f(k^P)})(-\frac{1+n}{\beta_c^2})\sigma_c + (\sigma_w - \frac{\beta_w}{e_f(k^P)} \frac{\sigma_c}{\beta_c})(\delta - 1)\}$

$= \frac{e_{f'}(k^P)}{(1+n)^2} [\frac{\beta_w \sigma_c (1+n)}{e_f(k^P)\beta_c^2} + \frac{\sigma_w \beta_c e_f(k^P) - \beta_w \sigma_c}{e_f(k^P)\beta_c}(\delta - 1)]$

$= \frac{e_{f'}(k^P)}{(1+n)^2} [\frac{\beta_w \sigma_c (1+n) + \sigma_w \beta_c^2 e_f(k^P)(\delta-1) - \beta_w \beta_c \sigma_c (\delta-1)}{e_f(k^P)\beta_c^2}]$

$= \frac{e_{f'}(k^P)}{(1+n)^2 e_f(k^P)\beta_c^2} [e_f(k^P)\sigma_w(\delta - 1)\beta_c^2 - \beta_w \sigma_c (\delta - 1)\beta_c + \beta_w \sigma_c (1+n)]$

$= \frac{e_{f'}(k^P)}{(1+n)^2 e_f(k^P)\beta_c^2} (A\beta_c^2 + B\beta_c + C)$.

We conclude observing that

$\frac{e_{f'}(k^P)}{(1+n)^2 e_f(k^P)\beta_c^2} < 0$, $A < 0$ and the sign of $A\beta_c^2 + B\beta_c + C$ depends on $\Delta$.

By the previous propositions we will construct in the TD-plane the phase-diagram of the dynamical system when it depends on $\beta_c$. We get

- Case 1. The dynamical system lies in the first quadrant and it moves from $AA$ to right and down to $T$-axis, it ends at $BB$.

- Case 2. The dynamical system starts from $AA$ (in the first quadrant) going down to T-axis, before to right, after it stops, finally it again moves down and to right. Finally it ends in $BB$.

- Case 3. The dynamical system starts from $AA$ (in the fist quadrant) going down to T-axis and to right, after it stops. Moreover it goes to left, after it stops, and it again goes to right and it ends in $BB$.

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**Conclusions**

In the Thesis we have achieved the following results:

In the first Chapter we presented the various definition of Chaos proposed in the literature evaluating the respective strengths and weaknesses. We noticed that the *Li-Yorke* definition, which is the one usually adopted in economics, has substantial flaws. For example, it cannot be used for the case of non differentiable or two-dimensional maps. In this Chapter we also discussed in detail another method that can be used to detect Chaos in dynamical system, that is, the computation of the *Lyapunov coefficients*. Computational techniques are also useful in detecting and representing graphically other complex phenomena that are generated by dynamical systems. We described in great detail some economic dynamical models that are able to generate the so-called *Arnold Tongues*, which is a complex phenomenon representing a threshold between periodic and aperiodic time evolution. For to dimensional system, an important condition for the presence of Arnold Tongues is the occurrence of a *Neimark-Saker bifurcation*.

In the second Chapter, we reviewed the literature on Chaos in one and two-dimensional economic growth models . In particular, we described in great detail some significant and recent models of economic growth in discrete time. We did not limit ourselves to the description of such models but, in some cases, we developed analytical demonstrations only hinted at by the authors and not fully developed.

In the third Chapter, we presented and developed a new two-dimensional growth model in which two groups of economic agents have optimal but different saving behaviour. This model represents a discrete time version of a model developed in various stages by *Solow, Pasinetti and Samuelson and Modigliani*. A crucial difference from other models presented in the literature is the assumption of optimal saving behaviour of the two different type of agents existing in the literature (the "classes" of "workers" and "capitalists"). For this model we identified the different types of existing equilibria or steady-states and the local stability properties. We verified the emergence of the various types of bifurcations applying the *Hartmann-Grobman theorem*. In particular, a Neimark-Saker bifurcation can occur increasing reducing the elasticity of substitution between the factors of production in the *CES production function*. We also verified how these bifurcations can occur via a diagrammatical tool known as *Triangle of stability* often employed in the economics literature (*Grandmond, Pintus, de Vilder, Cazzavillan, Puu*).