Bell's inequalities in the tomographic representation

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Abstract.
The tomographic approach to quantum mechanics is revisited as a direct tool to investigate violation of Bell-like inequalities. Since quantum tomograms are well defined probability distributions, the tomographic approach is emphasized to be the most natural one to compare the predictions of classical and quantum theory. Examples of inequalities for two qubits an two qutrits are considered in the tomographic probability representation of spin states.

PACS numbers: 03.65.Ud, 03.67.-a

1. Introduction

Bell's inequalities were originally formulated [1] in order to provide a mathematical characterization of classical local hidden variables theories. In their original formulation, Bell's inequalities are propositions concerning expectation values of dichotomic observables (such as spin$-1/2$ polarization), when two spatially separated systems and local measurements are considered, in presence of perfect (anti-) correlations between the two systems relevant observables (such as two spin$-1/2$ in a singlet state). The experimental violation of these inequalities is an evidence against local classical variables models. Later on, other inequalities were proposed that generalize the Bell's idea to the case of non perfectly (anti-) correlated spin$-1/2$ systems [2, 3], to the case of spin of higher value [4] and concerning probability of measurement output instead of measurement expectation value [5].

It is a remarkable fact that not all the states of a (say) bipartite quantum system do violate some Bell-like inequalities: only states that are *entangled* are truly non local and not allowed to be described by means of a *classical* local variables model. With the development of the theory of quantum information and in view of the special role played by entangled states in quantum information protocols, a violation of some Bell-like inequalities has assumed also an operational role as a witness of entanglement. The power of Bell-like inequalities is that they refer only to observables quantities, as expectation value, correlations and probabilities without an explicit link to the underlying theory. If a Bell-like inequality is a proposition that is true for a classical theory, it is nevertheless a well defined proposition (not necessary true) in the framework of quantum theory. Hence the very idea of Bell's inequalities leads to consider a unified description of both classical and quantum mechanics based on fundamental quantities as probability distributions.

The conventional description of pure quantum states is by means of wave functions [6] or state vector in Hilbert space [7]. For mixed states, the density matrix [8, 9] is used to describe quantum states. The problem of measuring the quantum states was considered as the problem of finding the Wigner function [10], by means of which the optical tomograms of the states [11, 12], which are the probability distribution densities of the homodyne photon quadratures, can be determined. In [13] the use of symplectic tomogram as a tool for state reconstruction was extended in order to describe the quantum state by the probability distribution from the very beginning. This approach is called "tomographic probability representation of quantum states". For spin degrees of freedom the probability representation was found in [14, 15] for one qudit and in [16] for two qudits. In the framework of the tomographic representation, the spin state is identified with the probability distribution of spin projection on direction labeled by angular coordinates on the Bloch spheres for arbitrary number of qudits.

The tomographic map from state vectors or density matrices onto fair probability distributions contains complete information on the quantum states. Its mathematical structure was recently found in [17]. The relation of tomographic probability representation with the star-product quantization procedure was established in [18].

The aim of this work is to find new explicit formulas for spin tomograms of two qubits and two qutrits and to analyze, by means of these formulas, some Bell-like inequalities. The paper is organized as follows. In section 2 we review the separability problem using the tomographic probability description of spin states. In section 3 we derive the formulas for spin tomograms of two qubits and study the CHSH inequalities [2]. In section 4 we obtain the probability representation for multiqutrit state. In section 5 we present the conclusions.

2. Tomograms and separability

A tomographic description of quantum system can be formulated for systems with both discrete and continuous variables [17]. Here we are interested in the case of discrete variable systems that we are going to describe in the framework of spin tomography.

For qudit states with spin j the tomographic probability distribution is defined as the diagonal elements of the density operator

$$\rho_U = U^\dagger \rho U \tag{1}$$

in a standard basis $\{|m\rangle\}_{m=-j,\ldots j}$, where U is an operator of the unitary irreducible representation of the SU(2) group. The tomogram of the qudit state reads

$$\omega(m, \vec{n}) = \langle m|\rho_U|m\rangle = \langle m|U^\dagger \rho U|m\rangle \ . \tag{2}$$

Here $\vec{n}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is an unit vector determining a point on the Bloch sphere. The tomogram is, by construction, the probability distribution of the spin projection m onto the direction $\vec{n}$. The probability distribution determines the density matrix ρ. The formula connecting the tomogram $\omega(m,\vec{n})$ with the density matrix ρ was obtained in [15]. For example the tomographic probability of the qubit state

$$\rho=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

reads as follows

$$\omega(1/2,\vec{n}) \quad = \cos^2\theta/2\ , \tag{4}$$

$$\omega(-1/2,\vec{n}) = \sin^2\theta/2\ . \tag{5}$$

We used the matrix U rotating the spinor in the form

$$U=\begin{bmatrix} \cos\theta/2e^{i\frac{\phi+\psi}{2}} & \sin\theta/2e^{i\frac{\phi-\psi}{2}} \\ -\sin\theta/2e^{-i\frac{\phi-\psi}{2}} & \cos\theta/2e^{-i\frac{\phi+\psi}{2}} \end{bmatrix}\ . \tag{6}$$

Here ϕ,θ,ψ are the Euler angles. For two qudits the tomogram is defined as follows:

$$\omega(m_1,m_2;\vec{n_1},\vec{n_2})=\langle m_1m_2|\mathcal{U}^\dagger\rho\mathcal{U}|m_1m_2\rangle\ , \tag{7}$$

where ρ is a density matrix of two qudits, $\mathcal{U}=U_1\otimes U_2$, and the matrices U_1 and U_2 are matrices of irreducible representation of the group SU(2) corresponding to the first and second qudit, respectively. The spin projections m_1 and m_2 onto directions $\vec{n_1}$ and $\vec{n_2}$ are random variables of the tomogram which is joint probability distribution function for the two spin projections. Below we discuss in more details the generic qudits tomograms.

Let us consider an operator $A^{(j)}$ acting on a space of a spin$-j$ irreducible representation of SU(2). Given a standard basis $\{|jm\rangle\}$ with $m=-j,-j+1,...j-1,j$ the matrix elements of the operator

$$A^{(j)}_{m,m'}=\langle m|A^{(j)}|m'\rangle \tag{8}$$

of course completely determine the operator

$$A^{(j)}=\sum A^{(j)}_{m,m'}|m\rangle\langle m'|\ . \tag{9}$$

We consider the diagonal elements in a rotated frame

$$\omega_A(m,\Omega)=\langle m|R^\dagger(\Omega)A^{(j)}R(\Omega)|m\rangle=\mathrm{tr}\left[A^{(j)}R(\Omega)|m\rangle\langle m|R^\dagger(\Omega)\right]\ , \tag{10}$$

where $R(\Omega)$ is a unitary spin$-j$ representation of SU(2) and Ω is a short hand notation for the three Euler angles α, β and γ. The diagonal elements, as functions of the variable m and of the parameters Ω define the spin tomogram of the operator $A^{(j)}$. In the case in which $A^{(j)}$ represents a density operator describing the state of a spin$-j$ system, the tomogram $\omega_A(m,\Omega)$ is interpreted as the probability of finding the system with polarization m along the z axis in a system rotated with Euler angles Ω. The tomogram

(10) is a family of well defined probability distribution on the variable m with parameter $\vec{n}$:

$$\omega_A(m, \vec{n}) \quad \geq 0 \, , \tag{11}$$

$$\sum_m \omega_A(m, \vec{n}) = 1 \, . \tag{12}$$

It is a remarkable result that the knowledge of only diagonal matrix elements in a generic rotated frame is sufficient to reconstruct the operator:

$$A^{(j)} = \sum_{m=-j}^{j} \int d\Omega K(m, \Omega) \omega_A(m, \Omega) \, , \tag{13}$$

where

$$\int d\Omega = \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\gamma \, . \tag{14}$$

The explicit expression for the *quantizer* operator $K(m, \Omega)$ was found in [18].

Notice that as long as the polarization along the z axis is considered, the spin tomogram (10) depends only on two Euler angles: in the following we write

$$\Pi^{(j)}(m, \vec{n}) = R(\Omega)|m\rangle\langle m|R^\dagger(\Omega) \, , \tag{15}$$

where $\vec{n} = (\cos\alpha \sin\beta, \sin\alpha \sin\beta, \cos\beta)$ is the rotated axis of polarization. Hence, in the tomographic approach, the state of a quantum system is described by means of a well defined probability distribution $\omega(m, \vec{n})$ related to a Stern Gerlach-like measurement along the direction $\vec{n}$. Notice that a Bloch sphere description is obtained for the quantum state even for $j > 1/2$.

One of the open problems in quantum mechanics and quantum information theory is to give a complete characterization of entangled states. Given a bipartite system, a quantum state of the system is said to be separable if it can be written as a convex sum of factorized states:

$$\rho = \sum_k p_k \rho_k^{(A)} \otimes \rho_k^{(B)} \, , \quad \sum_k p_k = 1 \, . \tag{16}$$

Otherwise the state is said to be entangled. Let us also recall that a factorized state $\rho = \rho^{(A)} \otimes \rho^{(B)}$ is called a simply separable state. These definitions can be generalized, with some care, to the case of multi-partite systems [19, 20].

The relation between local realism and separability of quantum states was widely studied. It is clear from the definition (16), that every separable states can be described by means of a local hidden variables model (where the hidden variable can be identified with the index k). In [21] it was first shown with an example that the converse is not true, *i.e.* there exist quantum states that can be described by a hidden local variables model but are nevertheless entangled. This means that the violation of a Bell's inequalities by a given quantum states is a sufficient (though not necessary) condition for the state to be entangled. Although a systematic approach to generate all Bell's inequalities exists [22], how to find the inequality that presents a maximal violation for a given entangled state is still an open problem.

From the point of view of entanglement detection and characterization, it is interesting to consider the tomographic description of state of multipartite quantum systems. To fix the ideas, let us consider a bipartite system composed of one spin$-j_1$ and one spin$-j_2$: in this case the spin tomogram of a state of the compound system described by density matrix ρ is written as follows:

$$\omega_\rho(m_1, m_2; \vec{n_1}, \vec{n_2}) = tr\left(\rho \Pi^{(j_1)}(m_1, \vec{n_1}) \otimes \Pi^{(j_2)}(m_2, \vec{n_2})\right) . \tag{17}$$

This definition is simply generalized to the case of multipartite spin systems and refers to local Stern Gerlach-like measurement.

For example the tomographic probability distribution function for the two qubit state

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

reads

$$\omega(1/2, 1/2; \vec{n_1}, \vec{n_2}) = \cos^2\theta_1/2 \cos^2\theta_2/2 , \tag{19}$$

$$\omega(1/2, -1/2; \vec{n_1}, \vec{n_2}) = \cos^2\theta_1/2 \sin^2\theta_2/2 , \tag{20}$$

$$\omega(-1/2, 1/2; \vec{n_1}, \vec{n_2}) = \sin^2\theta_1/2 \cos^2\theta_2/2 , \tag{21}$$

$$\omega(-1/2, -1/2; \vec{n_1}, \vec{n_2}) = \sin^2\theta_1/2 \sin^2\theta_2/2 . \tag{22}$$

The state is simply separable and the tomographic probability has the form of factorized joint probability distribution

$$\omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = \omega_1(m_1, \vec{n_1})\omega_2(m_2, \vec{n_2}) , \tag{23}$$

where the probability distributions ω_1 and ω_2 describe the states of the first and second spin respectively. The joint tomographic probability determines the density matrix by means of inversion formula obtained in [16]. Due to linearity of the tomographic map of density matrices onto joint probability distributions of spin projections, the tomogram of a separable state is the convex sum of factorized joint probability distributions of the simply separable states:

$$\omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = \sum_k p_k \omega_1^{(k)}(m_1, \vec{n_1})\omega_2^{(k)}(m_2, \vec{n_2}) . \tag{24}$$

3. Qubits tomograms

In this section we discuss the tomographic representation for spin$-1/2$ (qubit) systems in its link with standard density matrix description. Let us first consider a one-qubit system. It is well known that a qubit density state can be written in terms of Pauli matrices:

$$\rho_1 = \frac{1}{2}\left(\sigma_0 + x^i\sigma_i\right) , \tag{25}$$

where (the sum over repeated indices is intended)

$$x^i = \delta^{ij}\,\mathrm{tr}(\rho\sigma_j) = \delta^{ij}x_j \tag{26}$$

since

$$\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}\ . \tag{27}$$

In the following we take $m = -1, 1$. With this convention, from the definition (10) it follows that in the tomographic representation:

$$\omega(m,\vec{n}) = \mathrm{tr}\left(\rho_1 \Pi(m,\vec{n})\right)\ , \tag{28}$$

where

$$\Pi(m,\vec{n}) = \frac{1}{2}\left(\sigma_0 + mn^i\sigma_i\right) \tag{29}$$

is the projector on the eigenstate with polarization m along the direction $\vec{n}= (n^1, n^2, n^3)$, where for convenience we have chosen $m = \pm 1$. The operator $\Pi(m,\vec{n})$ plays the role of the de-quantizer operator used in star-product quantization scheme [23].

From (28) and (26) it follows that the explicit expression for a generic qubit tomogram is

$$\omega_1(m,\vec{n}) = \frac{1}{2}\left(1 + m\ \vec{n}\cdot\vec{x}\right)\ , \tag{30}$$

where $\vec{x}= (x_1, x_2, x_3)$ and $\vec{n}\cdot\vec{x}= n^i x_i$. The expression (28) can be immediately generalized to the case of multi-qubit system. In the case of a system of N qubits in a global state ρ_N, the (global) tomogram is given by the following relation:

$$\omega_N(m_1, m_2, \ldots m_N; \vec{n_1}, \vec{n_2}, \ldots \vec{n_N}) = \mathrm{tr}\left[\rho_N \bigotimes_{i=1\ldots N} \Pi(m_i, \vec{n_i})\right]\ . \tag{31}$$

In the case of a system of two qubits (31) simplifies to

$$\omega_2(m_1, m_2; \vec{n_1}, \vec{n_2}) = \mathrm{tr}\left[\rho_2 \frac{1}{4}(\sigma_0 + m_1 n_1^i \sigma_i) \otimes (\tau_0 + m_2 n_2^i \tau_i)\right]\ , \tag{32}$$

where σ_μ and τ_μ are the Pauli matrices respectively related to the first and second qubit.

Defining $x_i = \mathrm{tr}(\rho_2\sigma_i)$, $y_i = \mathrm{tr}(\rho_2\tau_i)$ and $z_{ij} = \mathrm{tr}(\rho_2\sigma_i\otimes\tau_j)$, where σ_i and τ_i are short-hand notation for $\sigma_i\otimes\tau_0$ and $\sigma_0\otimes\tau_i$ respectively, the tomogram (32) reads:

$$\omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = \frac{1}{4}\left(1 + m_1 n_1^i x_i + m_2 n_2^i y_i + m_1 m_2 n_1^i z_{ij} n_2^j\right)\ . \tag{33}$$

Notice that for simply separable states $\mathrm{tr}(\rho_2\sigma_i\otimes\tau_j) = \mathrm{tr}(\rho_2\sigma_i)\,\mathrm{tr}(\rho_2\tau_j)$, *i.e.* $z_{ij} = x_i y_j$ and the tomogram assumes a factorized form:

$$\omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = \frac{1}{4}\left(1 + m_1\ \vec{n_1}\cdot\vec{x}\right)\left(1 + m_2\ \vec{n_2}\cdot\vec{y}\right)\ . \tag{34}$$

3.1. Two spin−1/2 Bell-Wigner inequalities

Let us consider the inequality proposed in [5]. It is related to the case of two spin−1/2 particles with perfect anti-correlation. For each particle the polarization is independently measured along three arbitrary directions. The joint probability of finding the first and the second particles polarized respectively in the $\vec{n_1}$ and $\vec{n_2}$ direction is indicated with $P(\vec{n_1},\vec{n_2})$. The hypothesis of perfect anti correlation implies that the probability of measure parallel polarization along a fixed direction vanishes:

$$P(\vec{n},\vec{n}) = 0 \ . \tag{35}$$

Given three arbitrary directions $\vec{n_a}$, $\vec{n_b}$ and $\vec{n_c}$ the following inequality holds for a classically correlated state [5]:

$$P(\vec{n_a},\vec{n_b}) + P(\vec{n_b},\vec{n_c}) - P(\vec{n_a},\vec{n_c}) \geq 0 \ . \tag{36}$$

Notice that these probability distributions are directly given in the tomographic representation, since

$$P(\vec{n_1},\vec{n_2}) = \omega(1,1;\vec{n_1},\vec{n_2}) \ . \tag{37}$$

Inequality (36) is obtained for perfectly classically anti-correlated states. It is easy to see that a quantum simply separable state cannot exhibit perfect (anti-) correlations, hence we consider non-perfect anti-correlation in a simply separable state of the following form:

$$\omega(m_1,m_2;\vec{n_1},\vec{n_2}) = \frac{1}{4}\left[1 + m_1(\vec{n_1}\cdot\vec{x})\right]\left[1 - m_2(\vec{n_2}\cdot\vec{x})\right] \ . \tag{38}$$

For such a state (36) are always fulfilled and are simply written as follows:

$$\omega(1,1;\vec{n_a},\vec{n_b}) + \omega(1,1;\vec{n_b},\vec{n_c}) - \omega(1,1;\vec{n_a},\vec{n_c}) = \tag{39}$$

$$\frac{1}{4}\left[1 - (\vec{n_a}\cdot\vec{x})(\vec{n_b}\cdot\vec{x}) - (\vec{n_b}\cdot\vec{x})(\vec{n_c}\cdot\vec{x}) + (\vec{n_a}\cdot\vec{x})(\vec{n_c}\cdot\vec{x})\right] \geq 0 \ , \tag{40}$$

that is

$$(\vec{n_a}\cdot\vec{x})(\vec{n_b}\cdot\vec{x}) + (\vec{n_b}\cdot\vec{x})(\vec{n_c}\cdot\vec{x}) - (\vec{n_a}\cdot\vec{x})(\vec{n_c}\cdot\vec{x}) \leq 1 \ . \tag{41}$$

Since the inequalities are fulfilled by non-perfectly anti-correlated particles in a factorized state it follows that the same is true for a generic anti-correlated separable state.

As a simple example, we consider the case of a two-qudit system in the Werner state, defined for $\phi \in [-1,1]$ as follows:

$$\rho_d(\phi) = \frac{1}{d^3 - d^2}\left[(d-\phi)Id_{d^2} + (d\phi - 1)V\right] \ , \tag{42}$$

where Id_{d^2} is the identity operator in the compound system space and V is the swap operator ($V\psi\otimes\phi = \phi\otimes\psi$). These states are symmetric under local unitary operations of the kind $U\otimes U$: hence we expect a particular simple tomographic expression for these states. The state (42) is known to be entangled for $\phi < 0$ and separable otherwise. Notice that a spin−j system can be viewed as a qudit with $d = 2j+1$.

In the case of two qubits ($d = 2$) the tomogram of (42) reads as follows:

$$\omega_W = \frac{1}{4}\left[1 + \frac{2\phi - 1}{3} m_1 m_2 (\vec{n_1} \cdot \vec{n_2})\right] . \tag{43}$$

In terms of tomogram, the inequality (36) is immediately written as

$$\frac{2\phi - 1}{3}\left[(\vec{n_a} \cdot \vec{n_c}) - (\vec{n_a} \cdot \vec{n_b}) - (\vec{n_b} \cdot \vec{n_c})\right] \leq 1 . \tag{44}$$

It follows that the inequality (36) is violated for any $\phi < -1/2$.

3.2. Two spin−1/2 CHSH inequalities

As we have recalled above, both the Bell's inequalities [1] and Bell-Wigner inequalities [5] assume perfect (anti-) correlations between the two system qubits. The inequalities known as CHSH inequalities were introduced [2] in order to relax the hypothesis of perfect correlation between the two systems. Also in this case we deal with dichotomic observables. In the following we consider the case of a composite system of two spin$-1/2$ and the relevant observables are local magnetizations along a couple of directions. As in the original Bell argument, but in contrast with the Wigner approach, these inequalities are expressed in terms of expectation values and correlations of local observables. Some aspects of CHSH inequalities and their relation to tomographic probabilities were discussed in [24].

Given two arbitrary directions $\vec{n_1}$ and $\vec{n_2}$, let us consider the function

$$M(\vec{n_1}, \vec{n_2}) = \mathrm{tr}(\rho_2 n_1^i \sigma_i \otimes n_2^j \tau_j) , \tag{45}$$

that represents the correlation between the polarizations along the $\vec{n_1}$ and $\vec{n_2}$ direction, respectively for the first and second qubit, over the two qubits density state ρ_2. Notice that, in terms of tomograms, the correlation function (45) can be easily written as

$$M(\vec{n_1}, \vec{n_2}) = \sum_{m_1, m_2} m_1 m_2 \omega(m_1, m_2; \vec{n_1}, \vec{n_2}) . \tag{46}$$

Given four arbitrary directions $\vec{n_a}$, $\vec{n_b}$, $\vec{n_c}$ and $\vec{n_{b'}}$, the CHSH inequalities read as follows:

$$|M(\vec{n_a}, \vec{n_b}) - M(\vec{n_a}, \vec{n_c})| + M(\vec{n_{b'}}, \vec{n_b}) + M(\vec{n_{b'}}, \vec{n_c}) - 2 \leq 0 . \tag{47}$$

For two qubits Werner state, using (43), the average magnetization is easily written as

$$M(\vec{n_1}, \vec{n_2}) = \frac{2\phi - 1}{3}\left(\vec{n_1} \cdot \vec{n_2}\right) . \tag{48}$$

The inequality (47) reads

$$\frac{|2\phi - 1|}{3}\left[| \vec{n_a} \cdot (\vec{n_b} - \vec{n_c})| - \vec{n_{b'}} \cdot (\vec{n_b} + \vec{n_c})\right] \leq 2 . \tag{49}$$

Notice that the maximum of the function

$$Y(\vec{n_a}, \vec{n_b}, \vec{n_{b'}}, \vec{n_c}) = | \vec{n_a} \cdot (\vec{n_b} - \vec{n_c})| - \vec{n_{b'}} \cdot (\vec{n_b} + \vec{n_c}) \tag{50}$$

is reached when

$$\vec{n_a} = \pm \frac{\vec{n_b} - \vec{n_c}}{| \vec{n_b} - \vec{n_c} |} \tag{51}$$

$$\vec{n_{b'}} = - \frac{\vec{n_b} + \vec{n_c}}{| \vec{n_b} + \vec{n_c} |} \tag{52}$$

and $\vec{n_b} \cdot \vec{n_c} = 0$, and it is equal to $2\sqrt{2}$. The inequality is violated for any $\phi < -\frac{3\sqrt{2}-2}{4}$. Hence, the violation of the inequality does not detect entanglement when $-1 \leq \phi \leq -\frac{3\sqrt{2}-2}{4}$.

4. Qutrits tomography

In the previous sections we were dealing with qubit systems. Let us now consider the case of qutrits. In order to write the spin tomogram for a generic qutrit state, one has to consider the $s = 1$ irreducible representations of the group SU(2). Let us consider a realization of the angular momentum as qutrits operators J_1, J_2, J_3, such that

$$[J_i, J_j] = i\epsilon_{ij}^k J_k \, . \tag{53}$$

In terms of this given representation, the spin tomogram of a qutrit state is related to the standard density matrix description via the following relation:

$$\omega(m, \vec{n}) = \mathrm{tr}(\rho_1 \Pi(m, \vec{n})) \, , \tag{54}$$

where $m = -1, 0, 1$, and the qutrit *de-quantizer* operator is now given by

$$\Pi(m, \vec{n}) = \left(1 - m^2\right) Id_3 + \frac{m}{2} n^i J_i + \left(\frac{3}{2} m^2 - 1\right) (n^i J_i)^2 \, , \tag{55}$$

where Id_3 is the qutrit identity operator, and $\Pi(m, \vec{n})$ is the projector on the eigenvector of polarization m along the $\vec{n}$ direction. The relation (54) is easily generalized in the case of a system of N qutrits as follows:

$$\omega(m_1, m_2, \ldots m_N; \vec{n_1}, \vec{n_2}, \ldots \vec{n_N}) = \mathrm{tr}\left[\rho_N \bigotimes_{i=1\ldots N} \Pi(m_i, \vec{n_i})\right] \, . \tag{56}$$

As an example let us consider the two-qutrits Werner state obtained from (42) with $d = 3$:

$$\rho_W = \frac{3 - \phi}{24} Id_9 + \frac{3\phi - 1}{24} V \, . \tag{57}$$

The tomographic representation is explicitly given by

$$\omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = \mathrm{tr}\left[\rho_W \Pi(m_1, \vec{n_1}) \otimes \Pi(m_2, \vec{n_2})\right] \, , \tag{58}$$

that yields to

$$\begin{aligned} \omega(m_1, m_2; \vec{n_1}, \vec{n_2}) = {} & \frac{3 - \phi}{24} + \frac{3\phi - 1}{24} \left[3 \left(1 - m_1^2\right) \left(1 - m_2^2\right)\right. \\ & + \left(1 - m_1^2\right) \left(3m_2^2 - 2\right) + \left(1 - m_2^2\right) \left(3m_1^2 - 2\right) \\ & + \frac{m_1 m_2}{2} (\vec{n_1} \cdot \vec{n_2}) \\ & + \left.\left(\frac{3}{2} m_1^2 - 1\right) \left(\frac{3}{2} m_2^2 - 1\right) \left(1 + (\vec{n_1} \cdot \vec{n_2})^2\right)\right] \, . \end{aligned} \tag{59}$$

As another example, we discuss the non-linear Bell-like inequality proposed in [25]:

$$\langle AB' + A'B\rangle^2 + \langle AB - A'B'\rangle^2 \leq 1\,, \tag{60}$$

where A, A' and B, B' are local observables for a system composed of two spins, with the property of orthogonality $\mathrm{tr}(AA') = 0$, $\mathrm{tr}(BB') = 0$. Although this inequality has been formulated for a system of two qubits, it can be considered for a system of two qutrits as well. If $A = n^i_A J_i$, $A' = n^i_{A'} J_i$, $B = n^i_B J_i$, $B' = n^i_{B'} J_i$, from (59) we obtain that

$$\langle AB'\rangle = \frac{3\phi - 1}{12}\,\vec{n}_A \cdot \vec{n}_{B'}\,, \tag{61}$$

and the inequality reads as follows:

$$\left[\frac{3\phi-1}{24}\right]^2 \left[\left(\vec{n}_A \cdot \vec{n}_{B'} + \vec{n}_{A'} \cdot \vec{n}_B\right)^2 + \left(\vec{n}_A \cdot \vec{n}_B - \vec{n}_{A'} \cdot \vec{n}_{B'}\right)^2\right] \leq 1\,. \tag{62}$$

Notice that $\left[\left(\vec{n}_A \cdot \vec{n}_{B'} + \vec{n}_{A'} \cdot \vec{n}_B\right)^2 + \left(\vec{n}_A \cdot \vec{n}_B - \vec{n}_{A'} \cdot \vec{n}_{B'}\right)^2\right] < 8$, therefore the inequality is never violated.

5. Conclusions

To conclude we point out the main results of the paper. We have developed a formulation of Bell's inequalities by means of tomographic probability distribution of spin projections describing the quantum states completely. New formulas convenient for further analysis for tomogram of one qubit, two qubits and tomograms of two qutrits Werner state were obtained. The dequantizer operator for qutrit is also a new result presented in the paper. We demonstrated that both Wigner inequalities and CHSH inequalities as well their violations can be easily explained using joint probability distribution (tomograms) for spin projections. There are bounds for the violation of CHSH inequalities discussed in [26, 27, 28, 29]. The CHSH inequalities (47) are expressed exactly in terms of the function (33), it follows that the bound can be found as the maximum of the left hand side of (47). We will develop the analysis of Bell's inequalities based on tomographic star-product approach in future publications.

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