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CHRISTIAN-ALBRECHTS-UNIVERSITÄT · KIEL

Nr. 20a

Inverse Analysis of the Trimoored Internal Wave Experiment (IWEX)

Part I

by

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DOI 10.3289/IFM_BER_20a

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1977

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INTRODUCTION

This report describes the inverse analysis of the IWEX data. The analysis is entirely confined to the spectral data in the internal wave frequency band averaged over the entire duration of the experiment.

The IWEX experiment has been described by Briscoe (1975). A brief summary and some supplements can be found in part I of this report. The algebra and methods used for the analysis are given in part II and III, the results are described in part IV and V. The individual parts can be read separately. Volume II contains the tables and figures.

A summary of this report will be published in the Jounal of Geophysical Research ('The IWEX-Spectrum' by P. Müller, D.J. Olbers and J. Willebrand).

I. DATA SET

The Internal Wave Experiment (IWEX) was conceived to obtain spectral data of sufficient extent and density to determine the kinematical structure of the motions in the internal wave frequency band in the main thermocline of the deep ocean. Based on the space-time scales of the model of Garrett and Munk (1972) a 3-dimensional array of current meters and temperature sensors was designed to obtain adequate sampling density of the motion field in space and time. Though mainly conceived for kinematical purposes the array was operating long enough to study the time variability of the kinematic structure and to attempt the detection of dynamical processes.

We give a brief description of the experiment, data handling and data which partly reviews and partly supplements the article of Briscoe (1975).

I.1. The IWEX experiment

The IWEX field experiment was performed for 42 days in November - December 1973 at a site ($27^{o}44'N$, $69^{o}51'W$) in the Sargasso Sea shown in Figure I.1 (from Briscoe 1975). This site in the region of the Mid-Ocean-Dynamics-Experiment (MODE) was chosen to benefit by the results of MODE for the design and interpretation of IWEX.

The current meter array was supported by a three-legged mooring with the form of a tetrahedron of about 5 km height (Fig. I.2). The apex was at the top of the main thermocline The array was located between 604 m and 2050 m depths. Current meters and temperature sensors were attached to the three legs on 8 horizontal levels with vertical separations which are nearly equispaced on a logarithmic axis. Instruments are labelled by a letter A, B, or C indicating the associated leg, and by a number indicating the associated level. Table I.1 lists the current meter distribution in the trimooring.

Except for the lowest level the array was instrumented with vector averaging current meters (VACM) which were modified to include two temperature sensors measuring temperature and temperature difference over 1.74 m vertical separation. The temperature measurements were obtained with the intention of estimating the vertical displacement. The lowest level was instrumented with three Geodyne 850 current meters with attached temperature sensors. Here displacement has to be estimated in the traditional way using the temperature gradient obtained from CTD-profiles. The sampling interval was 225 sec for the VACM-DT and 900 sec for the 850-T.

Overall return of data from the temperature and differential temperature

sensors was 100%, from current meters about 95%. Rotor failures caused the loss of current data at C1, C2, A6, B6, A8 and A10 for some days and at B2 for almost the entire experiment. For the first 20 days the high frequency currents at the lowest level (Geodyne 850) are unreliable because the rotors were below the threshold point. Nine current meters (indicated by * in Table I.1) gave high quality data for the entire period of 42 days. The subarray of these current meters contains all essential scales of the total array with the exception of the largest and smallest horizontal and vertical separations.

In addition to current and temperature sensors the mooring included nine pressure-temperature recorders (at intermediate levels) to monitor the mooring motion. The maximum (r.m.s.) values of vertical excursions were 1.4 m (0.2 m) at the apex, 3.5 m (1.0 m) at 1000 m, 5.2 m (3 m) at 1500 m, and 12 m (6 m) at 3000 m depth (Panicker and Schmidt 1975). Almost all energy of the mooring motion was found at semidiurnal period.

Some additional measurements were made: wind speed and direction at the mooring site were monitored during the entire experiment; numerous CTD-profiles were collected at and near the site (Millard 1974); a series of yoyo-CTD-profiles in the main thermocline was made by Hayes (1975); during the deployment of the mooring a simple spar buoy experiment monitored the near surface internal wave field (Briscoe 1974, Zenk 1974).

I.2. Low frequency currents and temperatures

The low frequency data of IWEX show the typical oceanic variability which has been observed during the MODE experiment.

Time series of low frequency currents and temperatures were obtained (Frankignoul 1974) by applying a Gaussian filter of 75 hours width to the time series. Figure I.3a (courtesy of C. Frankignoul) shows the low passed temperatures at different depths. At almost all levels the temperature increases monotonically after the 7th of November with a more abrupt increase in the middle of the experiment. The 2000 m level shows a more irregular pattern than the upper levels. The displacements of the isotherms associated with the increase is about 7 m at the apex, 30 m at 1000 m, 60 m at 2000 m, and 30 m at 3000m depth.

Low passed current vectors are shown in Figure I.3b (courtesy of C. Frankignoul). The currents at the two higher levels show some relation to the low passed temperatures: clockwise rotation before and counterclockwise rotation after the abrupt temperature increase associated with a slight weakening.

The low frequency currents and temperatures and the nearby CTD-profiles can be interpreted as due to a westward advecting anticyclonic mesoscale eddy

reaching IWEX in the middle of November. Bryden (1976) has pointed out that the observed local time change of the low passed temperature is predominantly balanced by horizontal advection.

I.3. Yoyo-CTD-profiles

A 12 hour yoyo experiment was performed at the IWEX site (Hayes 1975). Time series of CTD were collected between 600 m and 900 m, successive lowerings were separated by about 12 minutes.

Hayes computed the spectrum of vertical displacement in the vertical wavelength range 2 m to 300 m (Figure I.4 from Hayes 1975). It shows a change in slope at about 10 m from -2 at larger wavelengths (50 - 300 m) to -2.5 at smaller wavelengths (< 10 m). Investigations of time-lagged coherence indicate that vertical wavelengths of less than 10 m lose their coherence within time lags of less than 12 min while longer wavelengths show significant coherence for almost all time lags considered (0 - 10 h).

I.4. Estimation of vertical displacement

The temperature time series were transformed to

$$\zeta(t) = T(t) \left(\overline{\frac{dT}{dz}} \right)^{-1} \qquad \text{(I.1)}$$

The bar denotes the time average over the entire period of the experiment. For the VACM-DT's the temperature gradient dT/dz is estimated by $\Delta T / \Delta z$ where ΔT is the temperature difference over $\Delta z = 1.74$ m vertical separation. At level 14 the mean temperature gradient is estimated from CTD-profiles.

The quantity $\zeta(t)$ is the traditional estimate of vertical displacement. The following analysis will show that there exist significant deviations between ζ and the true vertical displacement. Therefore ζ will be called "up" following Briscoes's (1975) terminology, though from the definition (I.1) it is actually "down", i.e. positive ζ means downward displacement. This situation is somewhat confusing but we decided to leave the original data set and the name of ζ unchanged. In this report the true sign of ζ will only be important when relating phases of cross-spectra between ζ and the horizontal current to real physical quantities.

I.5. Data handling

To estimate cross-spectra in the internal wave frequency band a record length of 40 days 15 hours of the time series was segmented in 13 nonoverlapping

pieces with a basic length of 75 hours, overlapped 50% by another 12 pieces. The piece length is between 3 inertial periods (77.36 h) and 6 periods (74.52 h) of the semidiurnal tide. Cross-spectra were computed for these 25 pieces by fast Fourier transform routines using a Hanning data window. To increase statistical stability and reduce the amount of data the original cross spectral estimates at 600 equispaced frequency points were averaged (top-hat averaging) to yield estimates at 40 frequencies (between 0.0133 cph and 7.5 cph) which are nearly equispaced on a logarithmic axis. Details of the data handling can be found in Briscoe (1975).

The basic data set of IWEX consists of 25 pieces of 1770 cross- and 60 autospectra at 40 frequency points.

I.6. Notation of cross-spectra

We will use a right handed Cartesian coordinate system $\underset{\sim}{x} = (x_1, x_2, x_3) = (x, y, z)$ with the x_1-axis pointing to the east, the x_2-axis pointing to the north and the x_3-axis pointing upwards. The Cartesian components of the current vector will be denoted by

$$\underset{\sim}{u} = (u_1, u_2, u_3) \tag{I.2}$$

For the cross-spectrum between the component u_m at the position $\underset{\sim}{x}_i$ and the component u_n at the position $\underset{\sim}{x}_j$ we will use the notation

$$A_{mn}^{ij}(\omega) = P_{mn}^{ij}(\omega) - i Q_{mn}^{ij}(\omega) \qquad m,n = 1,2,3 \tag{I.3}$$

Here P_{mn}^{ij} denotes the cospectrum and Q_{mn}^{ij} the quadspectrum. For $i = j$ we will also use

$$A_{mn}^{i}(\omega) = A_{mn}(\omega; \underset{\sim}{x}_i) \tag{I.4}$$

or even A_{mn} if the position is not specifically considered. Definition of the spectra is one-sided so that the integral over positive frequencies yields the total variance.

Coherence and phase are defined by

$$\gamma_{mn}^{ij} = |A_{mn}^{ij}| (A_{mn}^{i} A_{mn}^{j})^{-1/2} \tag{I.5}$$

and

$$\phi_{mn}^{ij} = \arctan(Q_{mn}^{ij}, P_{mn}^{ij}) \tag{I.6}$$

The current vector $\underset{\sim}{u}$ may also be described by its rotary components

$$u_{\pm} = \frac{1}{\sqrt{2}} (u_1 \pm i u_2) \qquad (I.7)$$

$$u_0 = u_3$$

The cross-spectral matrix of these components will be denoted by

$$H_{\mu\nu}^{ij}(\omega) = P_{\mu\nu}^{ij}(\omega) - i Q_{\mu\nu}^{ij}(\omega) \qquad \mu,\nu = +,-,0 \qquad (I.8)$$

Frequently we will use instead of u_3 the vertical displacement ζ (since this is actually estimated from the data). We then refer to the horizontal current component as u and v and denote cross-spectra explicitly by P_{uu}, P_{uv}, $P_{\zeta\zeta}$ etc. Cross-spectra between rotary components and ζ will be denoted by $H_{+\zeta}$, $H_{-\zeta}$ etc. Notice that the transformation from u_3 to ζ leaves the coherence unchanged but changes the phase. We have e.g.

$$\phi_{+0} = \phi_{+\zeta} - \frac{\pi}{2} \qquad (I.9)$$

since $u_0 = -\dot{\zeta} = -i\omega\zeta$.

I.7. Stationarity and Gaussianity of the wave field

The time variability of the energy in the internal wave frequency band has been investigated by Frankignoul and Joyce (1977). From the 25 pieces data set they computed energy time series for different frequency bands. Figure I.5 (from Frankignoul and Joyce 1977) displays some of these energy time series.

The inertial energy shows considerable time variability at all depths. The most striking feature is a large burst of energy of 10 days duration in the middle of the experiment.

Energy in the semidiurnal tide shows even larger variability. By and large, there is a slight increase at higher levels and a decrease at lower levels.

Within the internal wave continuum (10 - 0.4 h) variability of energy on a broad range of scales is observed at all levels, both for horizontal kinetic energy and potential energy. On small time scales the time series of horizontal kinetic energy are uncorrelated between different levels while potential energy shows some significant correlations. On larger time scales almost all time series reflect an increase of energy. By fitting the energy time series to a random process with zero mean and linear increasing variance the increase in the total

energy was found to be about 50%, almost independent of frequency and depth. Temperature alone shows this behavior at high frequencies (4 - 0.4 h), but no significant increase at lower frequencies (10 - 4 h).

Gaussianity and its evolution in time have been investigated by Iida and Briscoe (1975) and Briscoe (1977). Time series of selected instruments have been segmented into nonoverlapping pieces. For each piece the Chi-square and the two-tailed Kolmogorov-Smirnow goodness-of-fit tests were constructed and compared to confidence limits obtained from an artificially generated random process with the correlation structure of the observed internal wave spectrum. It was found that the majority of the current records contain non-Gaussian pieces (even at the 99% confidence limit) whereas the displacement records are Gaussian throughout. In most cases non-Gaussianity is accompanied by high energy events.

I.8. The mean data set

The inverse analysis will only be applied to the mean data set which is defined as the unweighted time average (i.e. average over the 25 pieces) of the basic data set. This data set will be parameterized by nonlinear models so we have to emphasize that due to the small amount of instationarity in the data the parameters of the mean data set differ from the average of the parameter sets determined for each piece separately.

I.8.1 Separations and frequencies

We included only sensors which gave high quality data for the entire period of the experiment. The use of sensors with data gaps poses the problem to construct the average cross-spectrum between sensors with different record length. This problem is solvable but needs cumbersome data management. Moreover, the high quality data cover almost all scales of the total array.

The mean data set contains all temperature data but only current data from the 9 instruments indicated by an asterix in Table I.1, leaving us with 38 autospectra and 1406 co- and quadspectra, i.e. 1444 data points at each of 40 frequencies. The frequencies are logarithmically equispaced between .0133 cph and 7.5 cph (Table I.2) with about 30 frequency points (depending on the level) in the internal wave frequency range between the inertial frequency f = 0.03878 cph and the local Brunt-Väisälä frequency. The third frequency, ω_3 , is slightly larger than f, the sixth frequency, ω_6 , is slightly less than the frequency M_2 of the semidiurnal tide.

Histograms of horizontal and vertical separations between the sensors contributing to this data set are shown in Figure I.6. Table I.3 lists the

total number of spectra, the number of autospectra and the number of cross spectra with specified separation. These numbers vary with frequency. The level 14 is excluded for ω_{22} to ω_{30} and the levels 14, 10 and 8 are excluded for ω_{30} since the frequencies exceed the local Brunt-Väisälä frequency.

I.8.2 Autospectra

Figure I.7a and b (court. of Briscoe) show the autospectra P_{uu} and P_{vv} of east and north components of the current at C6. The spectra are very similar to those obtained in earlier experiments: the energy at the inertial frequency (3rd frequency point) dominates, a smaller tidal peak (6th frequency point) is visible, followed by a steep decrease to higher frequencies. There is a noticeable change in the spectral slope from about -2 in the internal wave continuum ($M_2 < \omega < N$) to -4 in the beyond buoyancy range. This feature is not visible in other deep-sea current data, presumably because of mooring motion.

The autospectrum P_{zz} of displacement at C6 is shown in Figure I.7c (court. of Briscoe). The spectrum has a much smaller inertial peak and a marked tidal peak (the semidiurnal tide contributes about 10% to the total variance). The decrease of the spectrum in the internal wave continuum with a slope close to -2 is interrupted near the local buoyancy frequency by a small hump followed by a strong decrease as steep as the spectral window can estimate. The hump near the local buoyancy frequency has also been reported by Gould (1971) and Cairns (1975). For a possible explanation see Desaubies (1975).

More detailed information about autospectra of IWEX can be found in Tarbell, Briscoe and Chausse (1976).

WKB-scaling of the autospectra (cf. part II) has been demonstrated by Briscoe (1975). Since it works well in the internal wave continuum ($\omega > M_2$) we may construct average autospectra by scaling the available spectra to some level and taking the average. The average autospectra P_{++}, P_{--} and P_{oo} of the rotary components are displayed in Figure I.8. Notice that the clockwise spectrum P_{--} contains the inertial energy while the anticlockwise spectrum P_{++} nearly vanishes at $\omega = f$.

I.8.3 Cross-spectra

The cross-spectra constitute the main part of the data set. They contain all information on the spatial scales of the observed fluctuations.

Figures I.9a-d (courtesy of M. Briscoe) show some selected same component coherences and phases as function of frequency for some slant (same leg) and horizontal distances. In the frequency domain the coherences first increase

up to the local inertial frequency (current) or up to the semidiurnal tidal frequency (displacement) and then decrease almost monotonically, occasionally interrupted by some peaks in the internal wave band (cf. section I.8.4). Coherences of displacement frequently increase towards the buoyancy frequency (for a possible explanation see Desaubies 1975). In the internal wave frequency band the phases are essentially zero.

Figures I.10-14 show coherences and phases for some selected frequencies as function of slant and horizontal distance. Included in the figures are the 95% confidence level for zero true coherence and standard deviations at some points. Presented are coherence and phase for the components u_- and ζ. The same component coherences are found well above the 95% confidence level at almost all separations, the phases being zero in this case. The coherence between u_- and ζ is below the 95% confidence level (the phase is not shown in this case) for almost all frequencies except M_2, very high frequencies and marginally f. By and large the coherences decrease with increasing separation.

I.8.4 Variability in the frequency domain

The mean data set is obtained by frequency and piece averaging and thus represents a highly smoothed picture of the observed fluctuations. The spectra are estimated with about 50 edof's at the lowest 10 frequencies up to 300 edof's at the local buoyancy frequency. Despite this high amount of smoothing some characteristic features in the internal wave continuum survived in almost all auto- and cross-spectra, which cannot obviously be related to a smooth internal wave picture. This is visible by higher energy and/or extreme high or low values for coherence at some frequency points relative to the neighboring frequencies. Some examples are demonstrated in Fig. IV.11 and I.15.

Fig. IV.11 shows the average of all up autospectra. Each spectrum is WKB scaled and normalized by Garrett and Munk's (1972 , 1975) frequency dependence (cf. part V.2). Notice that the averaging did not smooth the humps at the frequencies 10, 14 and 18.

Fig. I.15 shows the up coherences on leg A. Notice the extreme low coherence at frequency 8 and the high coherence at 14 and 18.

Attempts to classify these frequencies as simple harmonics of the inertial or semidiurnal frequency fail at least for the 8th and the 10th frequency.

I.9. Accuracy of the mean data set

Equivalent degrees of freedom (edof) ν for the autospectral estimates have been calculated from Nuttal (1971) and Perrson (1974) for the combined

case of a Hanning data window, 50% overlapping data segments, and top-hat frequency and piece averaging. The effect of non-white internal wave spectra and a small amount of non-stationarity (cf. section I.7) can be taken into account but reduces the edof only slightly (less than 10%) with respect to the case of a white spectrum and stationarity. Values for the edofs can be found in Table I.2.

Confidence intervals for some values of true coherence have been calculated following Carter, Knapp and Nuttal (1973). On figures we will show 95% confidence levels for zero true coherence or the standard deviation for the case that the estimated value is the true coherence.

The covariances between co- and quadspectra at the same frequency are estimated by (Jenkins and Watts 1968)

$$\begin{aligned}
COV[P_{mn}, P_{k\ell}] &= \frac{1}{\nu}\left(P_{m\ell}P_{nk} + Q_{m\ell}Q_{nk} + P_{mk}P_{n\ell} + Q_{mk}Q_{n\ell}\right) \\
COV[P_{mn}, Q_{k\ell}] &= \frac{1}{\nu}\left(P_{nk}Q_{m\ell} - P_{m\ell}Q_{nk} - P_{n\ell}Q_{mk} + P_{mk}Q_{n\ell}\right) \qquad \text{(I.10)} \\
COV[Q_{mn}, Q_{k\ell}] &= \frac{1}{\nu}\left(P_{mk}P_{n\ell} - P_{m\ell}P_{nk} + Q_{mk}Q_{n\ell} - Q_{m\ell}Q_{nk}\right)
\end{aligned}$$

II. ALGEBRAIC STRUCTURE OF A RANDOM INTERNAL WAVE FIELD

II.1. Basic formulas of time series analysis

II.1.1 Cartesian representation

The moored current meters and temperature sensors of the IWEX experiment measure time series of the three velocity components at a fixed point in space. The vertical velocity is inferred from the time series of temperature and temperature gradient. These time series will be denoted by $u_m(\underset{\sim}{x}_i,t)$ where m = 1,2,3 denotes the three velocity components (u_1 eastward, u_2 northward, u_3 upward) and $\underset{\sim}{x}_i$ (i = 1,...,20) the position of the instruments.

Assuming that the measured time series represent realizations of a statistically stationary process the covariance function

$$R_{mn}^{ij}(\tau) = \langle u_m(\underset{\sim}{x}_i,t)\, u_n(\underset{\sim}{x}_j,t+\tau)\rangle \tag{II.1}$$

depends on the time lag τ only. Here cornered brackets denote the ensemble average . The cross-spectral matrix is defined as the Fourier transform of $R_{mn}^{ij}(\tau)$

$$A_{mn}^{ij}(\omega) = P_{mn}^{ij} - i\,Q_{mn}^{ij} = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\tau\, R_{mn}^{ij}(\tau)\, e^{-i\omega\tau} \tag{II.2}$$

Here P_{mn}^{ij} denotes the co-spectrum and Q_{mn}^{ij} the quadrature spectrum. The definition of the spectra is one-sided, i.e. the total variance of a variable is obtained by integrating its auto spectrum over positive frequencies. Inversion of (II.2) yields

$$R_{mn}^{ij}(\tau) = \frac{1}{2}\int_{-\infty}^{+\infty} d\omega\, A_{mn}^{ij}(\omega)\, e^{i\omega\tau} \tag{II.3}$$

The cross spectral matrix may also be defined by

$$2\,A_{mn}^{ij}(\omega)\,\delta(\omega-\omega') = \langle [u_m(\underset{\sim}{x}_i,\omega')]^*\, u_n(\underset{\sim}{x}_j,\omega)\rangle \tag{II.4}$$

where

$$u_m(\underset{\sim}{x},\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dt\, u_m(\underset{\sim}{x},t)\, e^{-i\omega t} \tag{II.5}$$

is the Fourier transform of $u_m(\underset{\sim}{x},t)$.

The cross spectral matrix for negative frequencies is determined by

$$A_{mn}^{ij}(-\omega) = \left[A_{mn}^{ij}(\omega) \right]^{*} \tag{II.6}$$

which follows from the reality condition $u_m = u_m^*$. The statistical stationarity of the time series implies

$$A_{nm}^{ji}(\omega) = A_{mn}^{ij}(-\omega) \tag{II.7}$$

which determines the cross spectral matrix if the order of the time series is interchanged. Because of the relations (II.6) and (II.7) we can restrict ourselves to positive frequencies and $i \leq j$. The cross spectral matrix A_{mn}^{ij} then contains 18 different real functions of $\omega \geq 0$. If $i = j$ this number reduces to 9 because of the additional relations

$$A_{mn}(\omega) = \left[A_{mn}(\omega) \right]^{*} \tag{II.8}$$

The coherence and phase is defined by

$$\gamma_{mn}^{ij} = |A_{mn}^{ij}| \left(P_{mm}^{i} P_{nn}^{j} \right)^{-1/2} \tag{II.9}$$

$$\phi_{mn}^{ij} = \arctan \left(Q_{mn}^{ij}, P_{mn}^{ij} \right) \tag{II.10}$$

Positive phase means that the series $u_m(\underset{\sim}{x}_i, t)$ leads the series $u_n(\underset{\sim}{x}_j, t)$, i.e. the phase propagates from $\underset{\sim}{x}_i$ to $\underset{\sim}{x}_j$.

II.1.2 Rotary representation

For our purposes the velocity field is more conveniently described by its rotary components

$$u_{\pm} = \frac{1}{\sqrt{2}} \left(u_1 \pm i u_2 \right)$$

$$u_0 = u_3 \tag{II.11}$$

with the inverse relations

$$u_1 = \frac{1}{\sqrt{2}} \left(u_+ + u_- \right), \quad u_2 = \frac{i}{\sqrt{2}} \left(u_- - u_+ \right), \quad u_3 = u_0 \tag{II.12}$$

The covariance function between the rotary components is defined by

$$R^{ij}_{\nu\mu}(\tau) = \langle u^*_\nu(\underset{\sim}{x}_i, t)\, u_\mu(\underset{\sim}{x}_j, t+\tau) \rangle \qquad \text{(II.13)}$$

and the cross spectral matrix by

$$A^{ij}_{\nu\mu}(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tau\, R^{ij}_{\nu\mu}(\tau)\, e^{-i\omega\tau} \qquad \text{(II.14)}$$

Note that this definition is not equivalent to the one given in Müller and Siedler (1976). Their cross spectrum $\Gamma^{ij}_{\nu\mu}(\omega)$ is related to $A^{ij}_{\nu\mu}(\omega)$ by

$$\Gamma^{ij}_{\nu\mu}(\omega) = A^{ij}_{-\nu\mu}(\omega) \qquad \text{(II.15)}$$

The "reality condition", $u_\nu(\underset{\sim}{x},t) = u^*_{-\nu}(\underset{\sim}{x},t)$, implies

$$A^{ij}_{\nu\mu}(-\omega) = \left[A^{ij}_{-\nu-\mu}(\omega) \right]^* \qquad \text{(II.16)}$$

The stationarity condition implies

$$A^{ji}_{\mu\nu}(\omega) = A^{ij}_{-\nu-\mu}(\omega) \qquad \text{(II.17)}$$

For $i \neq j$ we have the additional relations

$$A_{\nu\mu}(\omega) = \left[A_{\mu\nu}(\omega) \right]^* \qquad \text{(II.18)}$$

Again we can restrict ourselves to positive frequencies and $i \leq j$.

In analogy to (II.4) the cross spectral matrix may also be defined by

$$2\, A^{ij}_{\nu\mu}(\omega)\, \delta(\omega-\omega') = \langle u^*_\nu(\underset{\sim}{x}_i, \omega')\, u_\mu(\underset{\sim}{x}_j, \omega) \rangle \qquad \text{(II.19)}$$

where

$$u_\nu(\underset{\sim}{x}, \omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dt\, u_\nu(\underset{\sim}{x}, t)\, e^{-i\omega t} \qquad \text{(II.20)}$$

is the Fourier transform of $u_\nu(\underset{\sim}{x},t)$. If we restrict ourselves to positive frequencies, $u_+(\omega)$ describes the counterclockwise rotating part of the motion and $u^*_-(\omega) = u_+(-\omega)$ the clockwise rotating part.

It would have been sufficient to introduce $u_+ = \frac{1}{\sqrt{2}}(u_1 + i u_2)$ only (Mooers 1973) and to retain negative frequencies. In this case inner and outer cross

spectra have to be defined. The inner cross spectrum

$$2 S^{ij}_{++}(\omega)\,\delta(\omega-\omega') = \langle u^{*}_{+}(\underset{\sim}{x}_i,\omega')\,u_{+}(\underset{\sim}{x}_j,\omega)\rangle \qquad \text{(II.21)}$$

is related to our cross spectra $A^{ij}_{\nu\mu}(\omega)$, $\omega \gtrless 0$ by

$$S_{++}(\omega) = \begin{cases} A^{ij}_{++}(\omega) & , \ \omega \gtrless 0 \\ \left[A^{ij}_{--}(-\omega)\right]^{*} & , \ \omega \lessgtr 0 \end{cases} \qquad \text{(II.22)}$$

The outer cross spectrum

$$2 Y^{ij}_{++}(\omega)\,\delta(\omega-\omega') = \langle u_{+}(\underset{\sim}{x}_i,\omega')\,u_{+}(\underset{\sim}{x}_j,\omega)\rangle \qquad \text{(II.23)}$$

is given by

$$Y^{ij}_{++}(\omega) = \begin{cases} A^{ij}_{-+}(\omega) & , \ \omega \gtrless 0 \\ \left[A^{ij}_{+-}(-\omega)\right]^{*} & , \ \omega \lessgtr 0 \end{cases} \qquad \text{(II.24)}$$

The coherence and phase between the rotary components are defined by

$$\gamma^{ij}_{\nu\mu} = |A^{ij}_{\nu\mu}|\,(P^{i}_{\nu\nu}\,P^{j}_{\mu\mu})^{-1/2} \qquad \text{(II.25)}$$

$$\phi^{ij}_{\nu\mu} = \arctan\left(Q^{ij}_{\nu\mu}, P^{ij}_{\nu\mu}\right) \qquad \text{(II.26)}$$

II.1.3 Transformation formulas between the rotary and Cartesian representations

The transformation formulas between the rotary and Cartesian representation are explicitly given by

$$\begin{aligned}
A^{ij}_{++} &= \tfrac{1}{2}\left\{A^{ij}_{11} + A^{ij}_{22} + i\left(A^{ij}_{12} - A^{ij}_{21}\right)\right\} \\
A^{ij}_{+-} &= \tfrac{1}{2}\left\{A^{ij}_{11} - A^{ij}_{22} - i\left(A^{ij}_{12} + A^{ij}_{21}\right)\right\} \\
A^{ij}_{+o} &= \tfrac{1}{\sqrt{2}}\left\{A^{ij}_{13} - i\,A^{ij}_{23}\right\} \\
A^{ij}_{-+} &= \tfrac{1}{2}\left\{A^{ij}_{11} - A^{ij}_{22} + i\left(A^{ij}_{12} + A^{ij}_{21}\right)\right\} \\
A^{ij}_{--} &= \tfrac{1}{2}\left\{A^{ij}_{11} + A^{ij}_{22} - i\left(A^{ij}_{12} - A^{ij}_{21}\right)\right\} \\
A^{ij}_{-o} &= \tfrac{1}{\sqrt{2}}\left\{A^{ij}_{13} + i\,A^{ij}_{23}\right\}
\end{aligned} \qquad \text{(II.27)}$$

$$
\begin{aligned}
A_{0+}^{ij} &= \frac{1}{\sqrt{2}} \{ A_{31}^{ij} + i A_{32}^{ij} \} \\
A_{0-}^{ij} &= \frac{1}{\sqrt{2}} \{ A_{31}^{ij} - i A_{32}^{ij} \} \\
A_{00}^{ij} &= A_{33}^{ij}
\end{aligned}
$$

$$
\begin{aligned}
A_{11}^{ij} &= \frac{1}{2} \{ A_{++}^{ij} + A_{--}^{ij} + A_{+-}^{ij} + A_{-+}^{ij} \} \\
A_{12}^{ij} &= \frac{1}{2i} \{ A_{++}^{ij} - A_{--}^{ij} - A_{+-}^{ij} + A_{-+}^{ij} \} \\
A_{13}^{ij} &= \frac{1}{\sqrt{2}} \{ A_{+0}^{ij} + A_{-0}^{ij} \} \\
A_{21}^{ij} &= \frac{1}{2i} \{ -A_{++}^{ij} + A_{--}^{ij} - A_{+-}^{ij} + A_{-+}^{ij} \} \\
A_{22}^{ij} &= \frac{1}{2} \{ A_{++}^{ij} + A_{--}^{ij} - A_{+-}^{ij} - A_{-+}^{ij} \} \\
A_{23}^{ij} &= \frac{1}{\sqrt{2}i} \{ -A_{+0}^{ij} + A_{-0}^{ij} \} \\
A_{31}^{ij} &= \frac{1}{\sqrt{2}} \{ A_{0+}^{ij} + A_{0-}^{ij} \} \\
A_{32}^{ij} &= \frac{1}{\sqrt{2}i} \{ A_{0+}^{ij} - A_{0-}^{ij} \} \\
A_{33}^{ij} &= A_{00}^{ij}
\end{aligned}
\qquad \text{(II.28)}
$$

For i = j the coherences squared between the rotary components take the form

$$
\gamma_{+-}^2 = \frac{A_{+-} A_{+-}^*}{A_{++} A_{--}} = \frac{(P_{11} - P_{22})^2 + 4 P_{12}^2}{(P_{11} + P_{22})^2 - 4 Q_{12}^2}
$$

$$
\gamma_{+0}^2 = \frac{A_{+0} A_{+0}^*}{A_{++} A_{00}} = \frac{(P_{13} - Q_{23})^2 + (P_{23} + Q_{13})^2}{(P_{11} + P_{22} + 2 Q_{12}) P_{33}} \qquad \text{(II.29)}
$$

$$
\gamma_{-0}^2 = \frac{A_{-0} A_{-0}^*}{A_{--} A_{00}} = \frac{(P_{13} + Q_{23})^2 + (P_{23} - Q_{13})^2}{(P_{11} + P_{22} - 2 Q_{12}) P_{33}}
$$

II.1.4 Relationship to quantities defined in the literature

Fofonoff, 1969:

colinear "coherence": $$C_i = \frac{(A_{+-} A_{+-}^*)^{1/2}}{P_{++} + P_{--}} = \frac{[(P_{11} - P_{22})^2 + 4 P_{12}^2]^{1/2}}{P_{11} + P_{22}} \qquad \text{(II.30)}$$

rotary "coherence": $$C_o = \frac{P_{++} - P_{--}}{P_{++} + P_{--}} = \frac{2 Q_{12}}{P_{11} + P_{22}} \qquad \text{(II.31)}$$

Note that both quantities are neither coherences between the Cartesian components nor coherences between the rotary components.

Gonella, 1972:

ellipse stability: $E = \gamma_{+-}^2$ (II.32)

rotary coefficient: $C_R = C_0$ (II.33)

II.2 Propagating internal waves

II.2.1 Representation of the wave field

A superposition of propagating internal waves can be represented by

$$u_\nu(\underset{\sim}{x},t) = \sum_{\sigma} \int_{f}^{N} d\omega \int d^2\alpha \left[a(\underset{\sim}{q})\, U_\nu(\underset{\sim}{q})\, \eta_\nu(\underset{\sim}{q},x_3) \exp\{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{x} - \omega t)\} + a^*(\underset{\sim}{q})\, U^*_{-\nu}(\underset{\sim}{q})\, \eta^*_{-\nu}(\underset{\sim}{q},x_3) \exp\{i(\underset{\sim}{\alpha}\cdot\underset{\sim}{x} - \omega t)\}\right] \tag{II.34}$$

with

$a(\underset{\sim}{q})$ = amplitude

$U_\nu(\underset{\sim}{q})$ = amplitude factors

$\eta_\nu(\underset{\sim}{q},x_3)$ = vertical eigenfunctions

$\underset{\sim}{\alpha} = (\alpha_1,\alpha_2,0)$ = horizontal wavenumber vector

$\alpha = (\alpha_i\alpha_i)^{1/2}$ = magnitude of the horizontal wavenumber vector

ω = frequency

N = Brunt Väisälä frequency

f = inertial frequency

σ = sign of the vertical wavenumber

In the representation II.34 the internal waves are specified by

$$\underset{\sim}{q} = \{\omega, \underset{\sim}{\alpha}, \sigma\} \tag{II.35}$$

The vertical eigenfunctions $\eta_+ = \eta_- = \frac{1}{\alpha}\partial_3\eta_0$ are explicitly given by

$$\mathcal{Y}_\nu(\underset{\sim}{q}, x_3) = \mathcal{Y}_\nu^\circ \, e^{-i\sigma\Theta(x_3)} \tag{II.36}$$

with

$$\left.\begin{matrix}\mathcal{Y}_+^\circ \\ \mathcal{Y}_-^\circ \\ \mathcal{Y}_0^\circ\end{matrix}\right\} = C(\omega)\,\Omega^{-1/2} \left\{\begin{matrix}-i\sigma\Omega \\ -i\sigma\Omega \\ 1\end{matrix}\right. \tag{II.37}$$

and

$$\Theta(x_3) = \int_{x_3^\circ}^{x_3} dx_3' \, \beta(x_3') \qquad = \text{vertical phase}$$

$$x_3^\circ \qquad = \text{turning depth} \quad (N(x_3^\circ) = \omega)$$

$$\beta(x_3) = \alpha \left(\frac{N^2(x_3) - \omega^2}{\omega^2 - f^2}\right)^{1/2} \qquad = \text{local vertical wavenumber}$$

$$\Omega(x_3) = \frac{\beta(x_3)}{\alpha} \qquad = \text{aspect ratio of wavenumber vector}$$

The normalization constant $C(\omega)$ will be chosen as

$$C(\omega) = \frac{1}{\omega}\,(\omega^2 - f^2)^{1/4} \left[\int dx_3 \frac{N^2 - f^2}{(N^2-\omega^2)^{1/2}}\right]^{-1/2} \tag{II.38}$$

The integration has to be carried out between the turning points.

The amplitude factors are given by

$$\left.\begin{matrix}U_+(\underset{\sim}{q}) \\ U_-(\underset{\sim}{q}) \\ U_0(\underset{\sim}{q})\end{matrix}\right\} = \left\{\begin{matrix}-i\frac{1}{\sqrt{2}}(\omega - f) \\ -i\frac{1}{\sqrt{2}}(\omega + f) \\ \omega\end{matrix}\right\} \left\{\begin{matrix}e^{i\varphi} \\ e^{-i\varphi} \\ 1\end{matrix}\right\} \tag{II.39}$$

They may also be written

$$U_\nu(\underset{\sim}{q}) = f_\nu(\omega)\, g_\nu(\varphi) \quad \text{(no summation)} \tag{II.40}$$

where $f_\nu(\omega)$ denotes the first and $g_\nu(\varphi)$ the second column vector on the right hand side of (II.39). Here φ is the direction of the horizontal wavenumber counted counterclockwise from the east.

In the Cartesian representation the internal wave field is represented by (II.34) if the indices ν and $-\nu$ are replaced by the index m and if the

Cartesian amplitude factors

$$\left.\begin{matrix} U_1(\underset{\sim}{q}) \\ U_2(\underset{\sim}{q}) \\ U_3(\underset{\sim}{q}) \end{matrix}\right\} = \begin{cases} -f \sin\varphi - i\omega\cos\varphi \\ f\cos\varphi - i\omega\sin\varphi \\ \omega \end{cases} \qquad \text{(II.41)}$$

and the Cartesian eigenfunctions

$$\mathcal{Y}_1 = \mathcal{Y}_+ \,,\quad \mathcal{Y}_2 = \mathcal{Y}_+ \,,\quad \mathcal{Y}_3 = \mathcal{Y}_0 \qquad \text{(II.42)}$$

are substituted.

II.2.2 Energy density spectrum

If the wave field is statistically stationary and horizontally homogeneous and if up- and downward propagating waves are uncorrelated the wave amplitudes satisfy the orthogonality relations

$$\begin{aligned} \langle a(\underset{\sim}{q})\, a(\underset{\sim}{q}') \rangle &= 0 \\ \langle a(\underset{\sim}{q})\, a^*(\underset{\sim}{q}') \rangle &= \tfrac{1}{2} E(\underset{\sim}{q})\, \delta(\underset{\sim}{q}-\underset{\sim}{q}') \end{aligned} \qquad \text{(II.43)}$$

Here $E(\underset{\sim}{q})$ denotes the energy density spectrum since the normlaization constant $C(\omega)$ is chosen so that

$$E_0 = \sum_{\sigma} \int_f^N d\omega \int d^2\alpha\, E(\underset{\sim}{q}) \qquad \text{(II.44)}$$

represents the total energy density per unit surface area.

II.2.3 Cross spectra

Using the field representation (II.34) and the orthogonality conditions (II.43) the cross spectral matrix becomes

$$\begin{aligned} H_{\nu\mu}^{ij}(\omega) = \sum_{\sigma} \int d^2\alpha\, E(\underset{\sim}{q})\, U_\nu^*(\underset{\sim}{q})\, U_\mu(\underset{\sim}{q})\, [\mathcal{Y}_\nu^\circ(\underset{\sim}{q}, x_3^i)]^*\, \mathcal{Y}_\mu^\circ(\underset{\sim}{q}, x_3^j) \cdot \\ \cdot \exp\{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij} + \sigma\,\theta_{ij})\} \end{aligned} \qquad \text{(II.45)}$$

where

$$\Theta_{ij} = \Theta(x_3^j) - \Theta(x_3^i) \tag{II.46}$$

is the vertical phase difference and

$$\underset{\sim}{r}_{ij} = (x_1^j - x_1^i , x_2^j - x_2^i , 0) \tag{II.47}$$

the horizontal separation of the instruments. The cross spectra represent weighted projections of the energy density spectrum onto the frequency axis. Explicitly (II.45) takes the form

$$H_{\nu\mu}^{ij}(\omega) = B_{\nu\mu}(\omega) \sum_{\sigma} \int d^2\alpha \; E(\underset{\sim}{q}) \; C_{\nu\mu}(\varphi) \; D_{\nu\mu}(\omega, \sigma; x_3^i, x_3^j) \cdot \tag{II.48}$$

$$\exp\{ -i (\underset{\sim}{\alpha} \cdot \underset{\sim}{r}_{ij} + \sigma \Theta_{ij}) \} \text{ (no summation)}$$

with

$$\{ B_{\nu\mu} \} = \{ f_\nu^* f_\mu \} = \begin{pmatrix} \frac{1}{2}(\omega-f)^2 & \frac{1}{2}(\omega-f)(\omega+f) & \frac{i}{\sqrt{2}}\omega(\omega-f) \\ \frac{1}{2}(\omega-f)(\omega+f) & \frac{1}{2}(\omega+f)^2 & \frac{i}{\sqrt{2}}\omega(\omega+f) \\ -\frac{i}{\sqrt{2}}\omega(\omega-f) & -\frac{i}{\sqrt{2}}\omega(\omega+f) & \omega^2 \end{pmatrix} \tag{II.49}$$

$$\{ C_{\nu\mu} \} = \{ g_\nu^* g_\mu \} = \begin{pmatrix} 1 & e^{-2i\varphi} & e^{-i\varphi} \\ e^{2i\varphi} & 1 & e^{i\varphi} \\ e^{i\varphi} & e^{-i\varphi} & 1 \end{pmatrix} \tag{II.50}$$

$$\{ D_{\nu\mu} \} = \{ [\psi_\nu^\circ]^* \psi_\mu^\circ \} = C^2(\omega) \begin{pmatrix} (\Omega\Omega')^{1/2} & (\Omega\Omega')^{1/2} & i\sigma\left(\frac{\Omega}{\Omega'}\right)^{1/2} \\ (\Omega\Omega')^{1/2} & (\Omega\Omega')^{1/2} & i\sigma\left(\frac{\Omega}{\Omega'}\right)^{1/2} \\ -i\sigma\left(\frac{\Omega'}{\Omega}\right)^{1/2} & -i\sigma\left(\frac{\Omega'}{\Omega}\right)^{1/2} & (\Omega\Omega')^{-1/2} \end{pmatrix} \tag{II.51}$$

where

$$\Omega = \Omega(x_3^i) \quad , \quad \Omega' = \Omega(x_3^j) \tag{II.52}$$

The cross spectrum A_{mn}^{ij} in the Cartesian representation is given by (II.45) if ν and μ are replaced by m and n. Explicitly it takes the form

$$A_{mn}^{ij}(\omega) = \sum_{\sigma} \int d^2\alpha \, E(\underset{\sim}{q}) \, T_{mn}(\omega,\varphi) \, D_{mn}^{ij}(\omega,\sigma,x_3^i,x_3^j) \, e^{-i(\alpha r_{ij} + \sigma\theta_{ij})} \tag{II.53}$$

with

$$\{ D_{mn}^{ij} \} = \{ D_{\nu\mu}^{ij} \} \tag{II.54}$$

and

$$\{T_{mn}\} = \begin{pmatrix} \omega^2\cos^2\varphi + f^2\sin^2\varphi & (\omega^2-f^2)\cos\varphi\sin\varphi + i\omega f & -\omega f\sin\varphi + i\omega^2\cos\varphi \\ (\omega^2-f^2)\cos\varphi\sin\varphi - i\omega f & \omega^2\sin^2\varphi + f^2\cos^2\varphi & \omega f\cos\varphi + i\omega^2\sin\varphi \\ -\omega f\sin\varphi - i\omega^2\cos\varphi & \omega f\cos\varphi - i\omega^2\sin\varphi & \omega^2 \end{pmatrix} \tag{II.55}$$

II.2.4 Scaling

If we consider the x_3-dependence of the cross spectral matrix for i = j we find

$$\{A_{\nu\mu}(x_3)\} \propto \begin{pmatrix} \Omega(x_3) & \Omega(x_3) & 1 \\ \Omega(x_3) & \Omega(x_3) & 1 \\ 1 & 1 & \Omega^{-1}(x_3) \end{pmatrix} \tag{II.56}$$

Hence we find the scaling relations

$$\begin{aligned} &A_{++}(x_3), A_{+-}(x_3), A_{--}(x_3) \propto \left(\frac{N^2(x_3)-\omega^2}{\omega^2-f^2}\right)^{1/2} \propto N(x_3) \\ &A_{+o}(x_3), A_{-o}(x_3) \propto \text{const.} \\ &A_{oo}(x_3) \propto \left(\frac{N^2(x_3)-\omega^2}{\omega^2-f^2}\right)^{-1/2} \propto N^{-1}(x_3) \end{aligned} \tag{II.57}$$

The latter proportionalities hold if $\omega^2 \ll N^2$. The total energy

$$E(\omega, x_3) = \left\{ P_{++} + P_{--} + P_{00} + \frac{N^2}{\omega^2} P_{00} \right\} = \left\{ P_{11} + P_{22} + P_{33} + \frac{N^2}{\omega^2} P_{33} \right\} \qquad \text{(II.58)}$$

scales according to

$$E(\omega, x_3) \propto \frac{N^2(x_3) - f^2}{(N^2(x_3) - \omega^2)^{1/2}} \propto N(x_3) \qquad \text{(II.59)}$$

The latter proportionalitie again holds if $\omega^2 \ll N^2$.

II.2.5 Consistency relations

Not all of the cross spectral components are independent. There exist the following complete set of linearly independent relationships among the cross spectral components which are satisfied for arbitrary energy density spectra $E(\underset{\sim}{q})$ (Müller and Siedler 1976)

$$D_1^{ij} = A_{++}^{ij} + A_{--}^{ij} - \frac{\omega^2 + f^2}{\omega^2} \Omega \Omega' A_{00}^{ij} = 0 \qquad \text{(II.60)}$$

$$D_2^{ij} = (\omega + f)^2 A_{++}^{ij} - (\omega - f)^2 A_{--}^{ij} = 0 \qquad \text{(II.61)}$$

$$D_3^{ij} = \Omega'(\omega + f) A_{+0}^{ij} - \Omega(\omega - f) A_{0-}^{ij} = 0 \qquad \text{(II.62)}$$

$$D_4^{ij} = \Omega'(\omega - f) A_{-0}^{ij} - \Omega(\omega + f) A_{0+}^{ij} = 0 \qquad \text{(II.63)}$$

These relations define 8 consistency relations (counting real and imaginary part separately) which can be utilized to test whether or not the observed fluctuations represent a field of propagating internal waves. We have formulated the consistency relations as linear relations among the cross spectral components since this is convenient for algebraic manipulations. For $i \neq j$ the consistency relations take a simpler form if expressed in terms of coherences and phases

$$\gamma_{--}^{ij} = \gamma_{00}^{ij} \quad , \quad \phi_{--}^{ij} = \phi_{00}^{ij}$$

$$\gamma_{++}^{ij} = \gamma_{00}^{ij} \quad , \quad \phi_{++}^{ij} = \phi_{00}^{ij}$$

$$\gamma_{+0}^{ij} = \gamma_{0-}^{ij} \quad , \quad \phi_{+0}^{ij} = \phi_{0-}^{ij} \qquad \text{(II.64)}$$

$$\gamma_{-0}^{ij} = \gamma_{0+}^{ij} \quad , \quad \phi_{-0}^{ij} = \phi_{0+}^{ij}$$

In deriving these formulas we have used the scaling laws (II.56).

If i = j the number of consistency relations reduces to 4. In this case the relation $D_1 = 0$ predicts the ratio of the horizontal and vertical kinetic energy (Fofonoff 1969)

$$\frac{P_{++} + P_{--}}{P_{00}} = \frac{\omega^2 + f^2}{\omega^2} \frac{N^2 - \omega^2}{\omega^2 - f^2} \qquad \text{(II.65)}$$

This ratio is zero for $\omega = N$ and infinite for $\omega = f$. The relation $D_2 = 0$ predicts the ratio between the anticlockwise and clockwise rotating part of the motion

$$\frac{P_{++}}{P_{--}} = \frac{(\omega - f)^2}{(\omega + f)^2} \qquad \text{(II.66)}$$

which is zero for inertial oscillations. The relation $D_3 = 0$ may also be written

$$\gamma_{+0} = \gamma_{-0} \quad , \quad \phi_{+0} = -\phi_{-0} \qquad \text{(II.67)}$$

In the Cartesian representation the consistency relations (II.60)-(II.63) take the form

$$D_1^{ij} = A_{11}^{ij} + A_{22}^{ij} - \frac{\omega^2 + f^2}{\omega^2} \Omega \Omega' A_{33}^{ij} = 0$$

$$D_2^{ij} = \frac{2\omega f}{\omega^2 + f^2} \left(A_{11}^{ij} + A_{22}^{ij} \right) + i \left(A_{12}^{ij} - A_{21}^{ij} \right) = 0$$

$$D_3^{ij} = \frac{\omega}{f} \left(\Omega A_{31}^{ij} - \Omega' A_{13}^{ij} \right) + i \left(\Omega A_{32}^{ij} + \Omega' A_{23}^{ij} \right) = 0 \qquad \text{(II.68)}$$

$$D_4^{ij} = \frac{\omega}{f} \left(\Omega A_{32}^{ij} - \Omega' A_{23}^{ij} \right) - i \left(\Omega A_{31}^{ij} + \Omega' A_{13}^{ij} \right) = 0$$

II.2.6 Independent moments

If the observed fluctuations represent propagating waves only 10 linearly independent moments are measured, namely (in complex notation)

$$M_m^{ij}(\omega) = \frac{1}{2\pi} \sum_{\sigma} \int d^2\alpha \; E(\underset{\sim}{q})\, \sigma^{-m} e^{-im\varphi} \; e^{-i(\underset{\sim}{q}\cdot\underset{\sim}{r}_{ij} + \sigma\Theta_{ij})} \quad , \; m = 0, \pm 1, \pm 2 \qquad \text{(II.69)}$$

If i = j the number of independent moments reduces to 5 since

$$M_m = M_{-m}^{*} \qquad \text{(II.70)}$$

The cross spectra $A_{\nu\mu}^{ij}$ can be expressed in terms of these normalized moments by

$$\left.\begin{matrix} A_{++}^{ij} \\ A_{+-}^{ij} \\ A_{+o}^{ij} \\ A_{-+}^{ij} \\ A_{--}^{ij} \\ A_{-o}^{ij} \\ A_{o+}^{ij} \\ A_{o-}^{ij} \\ A_{oo}^{ij} \end{matrix}\right\} = 2\pi C^2 (\Omega\Omega')^{-1/2} \left\{\begin{matrix} \frac{1}{2}\Omega\Omega'(\omega-f)^2 M_0^{ij} \\ \frac{1}{2}\Omega\Omega'(\omega+f)(\omega-f) M_2^{ij} \\ -\frac{1}{\sqrt{2}}\Omega(\omega-f)\omega M_1^{ij} \\ \frac{1}{2}\Omega\Omega'(\omega+f)(\omega-f) M_{-2}^{ij} \\ \frac{1}{2}\Omega\Omega'(\omega+f)^2 M_0^{ij} \\ -\frac{1}{\sqrt{2}}\Omega(\omega+f)\omega M_{-1}^{ij} \\ -\frac{1}{\sqrt{2}}\Omega'(\omega-f)\omega M_{-1}^{ij} \\ -\frac{1}{\sqrt{2}}\Omega'(\omega+f)\omega M_1^{ij} \\ \omega^2 M_0^{ij} \end{matrix}\right. \qquad \text{(II.71)}$$

where the factors depend on frequency and positions only.

The significance of these moments can easily be interpreted for i ≠ j. Using polar coordinates for the horizontal wavenumber vector the normalized moments reduce to

$$M_m(\omega) = \frac{1}{2\pi} \sum_{\sigma} \int d\alpha\, d\varphi \; E^{\sigma}(\omega, \alpha, \varphi)\, \sigma^{-m} e^{-im\varphi} \qquad \text{(II.72)}$$

Decomposing the spectrum into its even and odd component

$$E^{e,o}(\omega, \alpha, \varphi) = E^{+}(\omega, \alpha, \varphi) \pm E^{-}(\omega, \alpha, \varphi) \qquad \text{(II.73)}$$

we find

$$M_m = \frac{1}{2\pi} \int d\alpha\, d\varphi\, e^{-im\varphi}\, E^e(\omega,\alpha,\varphi) \quad , m = 0,2$$
$$M_m = \frac{1}{2\pi} \int d\alpha\, d\varphi\, e^{-im\varphi}\, E^o(\omega,\alpha,\varphi) \quad , m = 1 \tag{II.74}$$

or

$$\begin{aligned} M_0 &= \int d\alpha\, c_0^e(\omega,\alpha) \\ M_1 &= \int d\alpha\, c_1^o(\omega,\alpha) \\ M_2 &= \int d\alpha\, c_2^e(\omega,\alpha) \end{aligned} \tag{II.75}$$

where

$$c_m^{e,o}(\omega,\alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, E^{e,o}(\omega,\alpha,\varphi)\, e^{-im\varphi} \tag{II.76}$$

are the Fourier coefficients of the expansion

$$E^{e,o}(\omega,\alpha,\varphi) = \sum_{m=-\infty}^{\infty} c_m^{e,o}(\omega,\alpha)\, e^{im\varphi} \tag{II.77}$$

Hence cross spectra obtained from a single instrument only provide information about the Fourier coefficients c_0^e, c_1^o, c_2^e.

II.2.7 Isotropy and symmetry relations

In case of isotropy and (or) symmetry of the energy density spectrum further relations hold among the independent moments.

For i = j these relations can easily be inferred from (II.75). If the energy density spectrum $E(\underset{\sim}{q})$ is vertically symmetric, i.e. independent of the sign of the vertical wavenumber, the odd component of the spectrum is zero and we find

$$M_1 = 0 \tag{II.78}$$

If the energy density spectrum is horizontally isotropic, i.e. independent of the direction φ of the horizontal wavenumber, the Fourier coefficients $c_m\ (m \neq 0)$ are zero and we find

$$M_1 = 0 \quad , \quad M_2 = 0 \tag{II.79}$$

In terms of the cross spectra the symmetry relation is given by

$$R_{+o} = R_{-o} = 0 \tag{II.80}$$

and the isotropy relation by

$$R_{+o} = R_{-o} = 0 \quad , \quad R_{+-} = 0 \tag{II.81}$$

In terms of coherences we find

$$\gamma_{+o} = \gamma_{-o} = \frac{|M_1|}{M_o} = 0 \tag{II.82}$$

and

$$\gamma_{+-} = \frac{|M_2|}{M_o} = 0 \tag{II.83}$$

The corresponding relation for slanted separation ($\underset{\sim}{r}_{ij} \neq 0$, $x_3^j - x_3^i \neq 0$) of the instruments may be derived as follows.

If $E(\underset{\sim}{q})$ is vertically symmetric the normalized moments simplify to

$$M_m^{ij} = \frac{1}{2\pi} \int d\alpha\, d\varphi\, E^{\sigma}(\omega, \alpha, \varphi)\, e^{-im\varphi}\, e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij}} \sum_{\sigma} \sigma^{-m}\, e^{-i\sigma\theta_{ij}} \tag{II.84}$$

or

$$M_m^{ij} = \frac{1}{2\pi} \int d\alpha\, d\varphi\, E^{\sigma}(\omega, \alpha, \varphi)\, e^{-im\varphi}\, e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij}} \begin{cases} 2\cos\theta_{ij} & , \ m = 0, \pm 2 \\ -2i\sin\theta_{ij} & , \ m = \pm 1 \end{cases} \tag{II.85}$$

No linear relations hold among the independent moments in this case.

If $E(\underset{\sim}{q})$ is horizontally isotropic the normalized moments reduce to

$$\begin{aligned} M_m^{ij} &= \frac{1}{2\pi} \sum_{\sigma} \int d\alpha\, E^{\sigma}(\omega, \alpha, \varphi)\, \sigma^{-m}\, e^{-i\sigma\theta_{ij}} \int_0^{2\pi} d\varphi\, e^{-im\varphi}\, e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij}} \\ &= \frac{1}{2\pi} \sum_{\sigma} \int d\alpha\, E^{\sigma}(\omega, \alpha, \varphi)\, \sigma^{-m}\, e^{-i\sigma\theta_{ij}} \int_0^{2\pi} d\varphi\, e^{-im\varphi}\, e^{-i\alpha r_{ij}\cos(\varphi - \psi)} \end{aligned} \tag{II.86}$$

where ψ is the direction of the vector $\underset{\sim}{r}_{ij}$. The φ-integration can be carried out analytically yielding

$$M_m^{ij} = \sum_{\sigma} \int d\alpha\, E^{\sigma}(\omega, \alpha, \varphi)\, \sigma^{-m}\, e^{-i\sigma\theta_{ij}}\, i^{-m}\, J_{-m}(-\alpha r_{ij})\, e^{-im\psi} \tag{II.87}$$

where J_m is the Bessel function of order m. Hence the following 4

relationships are satisfied for an arbitrary but isotropic internal wave field

$$\begin{aligned} e^{-i\psi} M_{-1}^{ij} - e^{i\psi} M_{1}^{ij} &= 0 \\ e^{-2i\psi} M_{-2}^{ij} - e^{2i\psi} M_{2}^{ij} &= 0 \end{aligned} \qquad \text{(II.88)}$$

If the wave field is both isotropic and symmetric, the normalized moments reduce to

$$M_m^{ij} = \int d\alpha\, E^{\sigma}(\omega,\alpha,\varphi)\, i^{-m}\, J_{-m}(\alpha r_{ij})\, e^{-im\psi} \begin{cases} 2\cos\Theta_{ij} & , m = 0, \pm 2 \\ -2i\sin\Theta_{ij} & , m = \pm 1 \end{cases} \qquad \text{(II.89)}$$

yielding the 7 relations

$$\begin{aligned} Im\{M_0^{ij}\} &= 0 \\ M_1^{ij} - [M_{-1}^{ij}]^* &= 0 \\ Im\{e^{-i\psi} M_{-1}^{ij}\} &= 0 \\ M_2^{ij} - [M_{-2}^{ij}]^* &= 0 \\ Im\{e^{-2i\psi} M_{-2}^{ij}\} &= 0 \end{aligned} \qquad \text{(II.90)}$$

In this case only the 3 moments

$$\begin{aligned} Re\{M_0^{ij}\} &= \int d\alpha\, E^{\sigma}(\omega,\alpha,\varphi)\, J_0(\alpha r_{ij}) \cos\Theta_{ij} \\ Re\{e^{-i\psi} M_{-1}^{ij}\} &= -\int d\alpha\, E^{\sigma}(\omega,\alpha,\varphi)\, J_1(\alpha r_{ij}) \sin\Theta_{ij} \\ Re\{e^{-2i\psi} M_{-2}^{ij}\} &= -\int d\alpha\, E^{\sigma}(\omega,\alpha,\varphi)\, J_2(\alpha r_{ij}) \cos\Theta_{ij} \end{aligned} \qquad \text{(II.91)}$$

do not vanish and provide a tool to determine the (ω,α) dependence of $E(\omega,\alpha)$.

The above isotropy and symmetry relations together with the corresponding relations for purely horizontal and purely vertical separation are listed in Table II.1. When comparing these relations with those of Müller and Siedler (1976) it must be considered that their definition $\tilde{M}_m^{ij}$ of the independent

moments is related to our definition by

$$\tilde{M}_m^{ij} \propto M_{-m}^{ij} \qquad \text{(II.92)}$$

The relations are expressed in terms of the independent moments M_m^{ij} . The relation in terms of cross spectra or coherences and phases can be obtained by using the transformation formulas (II.71). The number of isotropy and symmetry relations is listed in Table II.2.

II.3. Standing internal wave modes

II.3.1 Representation of the wave field

In the modal representation the internal wave field is given by

$$u_\nu(\underset{\sim}{x},t) = \int_f^N d\omega \int d^2\alpha \left[a(\omega,\underset{\sim}{\alpha})\, \tilde{U}_\nu(\omega,\underset{\sim}{\alpha})\, \tilde{\eta}_\nu(\omega,\alpha,x_3)\, e^{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{x}-\omega t)} + a^*(\omega,\underset{\sim}{\alpha})\, \tilde{U}^*_{-\nu}(\omega,\underset{\sim}{\alpha})\, \tilde{\eta}_{-\nu}(\omega,\alpha,x_3)\, e^{i(\underset{\sim}{\alpha}\cdot\underset{\sim}{x}-\omega t)} \right] \qquad \text{(II.93)}$$

Here the integration over α reduces to a one-dimensional integration and a sum over discrete eigenvalues.

The vertical eigenfunctions ($\tilde{\eta}_+ = \tilde{\eta}_- = \frac{1}{\alpha}\partial_3\tilde{\eta}_0$) have to be determined from the eigenvalue problem

$$\partial_3\partial_3\tilde{\eta}_0 + \alpha^2 \frac{N^2(x_3)-\omega^2}{\omega^2-f^2}\tilde{\eta}_0 = 0 \qquad \text{(II.94)}$$

with the boundary conditions

$$\partial_3\tilde{\eta}_0 - g\frac{\alpha^2}{\omega^2-f^2}\tilde{\eta}_0 = 0 \quad \text{at} \quad x_3 = 0 \qquad \text{(II.95)}$$

$$\tilde{\eta}_0 = 0 \quad \text{at} \quad x_3 = -h_0 \qquad \text{(II.96)}$$

The eigenfunctions are conveniently normalized according to

$$\int_{-h_0}^{0} dx_3 \frac{\omega^2(N^2-f^2)}{\omega^2-f^2}\tilde{\eta}_0(x_3)\tilde{\eta}_0(x_3) + g\frac{\omega^2}{\omega^2-f^2}\tilde{\eta}_0(x_3=0)\tilde{\eta}_0(x_3=0) = 1 \qquad \text{(II.97)}$$

The amplitude factors are given by

$$\left.\begin{matrix}\tilde{U}_+ \\ \tilde{U}_- \\ \tilde{U}_o\end{matrix}\right\} = \left\{\begin{matrix}-\frac{i}{\sqrt{2}}(\omega - f) \\ -\frac{i}{\sqrt{2}}(\omega + f) \\ \omega\end{matrix}\right\} \left\{\begin{matrix}e^{i\varphi} \\ e^{-i\varphi} \\ 1\end{matrix}\right\} \qquad \text{(II.98)}$$

Note that the amplitude factors $\tilde{U}_\nu$ for standing modes are identical to those for propagating waves.

II.3.2 Energy density spectrum

Assuming the field to be statistically stationary and horizontally homogeneous the amplitudes satisfy

$$\begin{aligned} &\langle a(\omega, \underset{\sim}{\alpha})\, a(\omega', \underset{\sim}{\alpha}')\rangle = 0 \\ &\langle a(\omega, \underset{\sim}{\alpha})\, a^*(\omega', \underset{\sim}{\alpha}')\rangle = \tfrac{1}{2}\tilde{E}(\omega, \underset{\sim}{\alpha})\,\delta(\omega - \omega')\,\delta(\underset{\sim}{\alpha} - \underset{\sim}{\alpha}') \end{aligned} \qquad \text{(II.99)}$$

Here $\tilde{E}(\omega, \underset{\sim}{\alpha})$ represents the energy density spectrum since the normalization (II.97) is chosen so that

$$\tilde{E}_o = \int_f^N d\omega \int d^2\alpha\ \tilde{E}(\omega, \underset{\sim}{\alpha}) \qquad \text{(II.100)}$$

represents the total energy density per unit surface area.

II.3.3 Cross spectra

The cross spectral matrix for $\omega \gtrless 0$ is given by

$$\tilde{H}^{ij}_{\nu\mu}(\omega) = \int d^2\alpha\ \tilde{E}(\omega, \underset{\sim}{\alpha})\, \tilde{U}^*_\nu(\omega, \underset{\sim}{\alpha})\, \tilde{U}_\mu(\omega, \underset{\sim}{\alpha})\, \tilde{q}_\nu(\omega, \underset{\sim}{\alpha}, x_3^i)\, \tilde{q}_\mu(\omega, \underset{\sim}{\alpha}, x_3^j) \cdot \exp\{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij}\} \qquad \text{(II.101)}$$

or explicitly by

$$\tilde{H}^{ij}_{\nu\mu}(\omega) = B_{\nu\mu}(\omega) \int d^2\alpha\, \tilde{E}(\omega, \underset{\sim}{\alpha})\, C_{\nu\mu}(\varphi)\, \tilde{D}^{ij}_{\nu\mu}(\omega, \underset{\sim}{\alpha}, x_3^i, x_3^j)\, e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij}} \qquad \text{(II.102)}$$

where $B_{\nu\mu}(\omega)$ is given by (II.49), $C_{\nu\mu}(\varphi)$ by (II.50) and

$$\tilde{D}^{ij}_{\nu\mu} = \tilde{q}_\nu(\omega, \underset{\sim}{\alpha}, x_3^i)\, \tilde{q}_\mu(\omega, \underset{\sim}{\alpha}, x_3^j) \qquad \text{(II.103)}$$

II.3.4 Consistency relations

As for progagating waves there exist linear relationships among the different components of the cross spectral matrix which are satisfied for arbitrary energy density spectra. These consistency relations for standing modes are listed in Table II.3. They differ from the corresponding relations for propagating waves. Our definition of propagating waves implies

(i) that the solutions of (II.94) can be approximated by the WKBJ solutions

$$\psi_0 = C(\omega)\,\Omega^{-1/2}\,e^{-i\sigma\Theta(x_3)}$$
$$\psi_\pm = -C(\omega)\,\Omega^{1/2}\,i\sigma\,e^{-i\sigma\Theta(x_3)} \qquad \text{(II.104)}$$

(ii) that up- ($\sigma = -1$) and downward ($\sigma = +1$) propagating waves are uncorrelated and

(iii) that there exists a continuum of solutions, i.e. to each value of ω and $\underset{\sim}{\alpha}$ there exist two eigensolutions with $\beta = \alpha\,(N^2-\omega^2)^{1/2}\,(\omega^2-f^2)^{-1/2}$.

Our definition of standing modes, on the contrary, implies

(i) that we have to use the exact solution of (II.94)

(ii) that there exist a fixed phase and amplitude relation between up- and downward propagating solutions (prescribed by the boundary condition (II.95)) so that a standing mode can be formed and

(iii) that there only exists a discrete set of solutions, i.e. for a given value of ω, φ there exist eigensolutions only for discrete values $\alpha_n (n = 0,1,2,\ldots)$. The values of α_n are determined by the boundary condition (II.96).

Whether the energy at a given frequency is confined to discrete values of or spread over a continuum is a property of the energy density spectrum E and does not affect the consistency relations which are to hold for arbitrary energy density spectra. Also, whether the WKBJ solutions provide a sufficient description or whether we have to consider the correct solutions of (II.94) also does not affect the consistency relations since the difference between the WKBJ solutions and the exact solutions may be arbitrarily small depending on the Brunt Väisälä profile and the values of ω and $\underset{\sim}{\alpha}$. The difference which affects the structure of the consistency relations is the fact that the standing mode solution consists of a single standing (i.e. real) mode whereas the propagating wave solution consists of two propagating i.e. (complex) modes which are statistically uncorrelated. This difference is reflected by the different structure of the matrices $D_{\nu\mu}$ and $\tilde{D}_{\nu\mu}$.

Specifically we find for slanted separation that the relation

$$D_2^{ij} = (\omega+f)^2\,A_{++}^{ij} - (\omega-f)^2\,A_{--}^{ij} = 0 \qquad \text{(II.105)}$$

must be satisfied for both standing and propagating waves. This relation only involves the horizontal velocity components. The relations $D_1^{ij} = 0$, $D_3^{ij} = 0$ and $D_4^{ij} = 0$ are not satisfied by standing modes. However, evaluating D_1 for standing modes we find

$$D_1^{ij} = \tilde{A}_{++}^{ij} + \tilde{A}_{--}^{ij} - \frac{\omega^2+f^2}{\omega^2}\,\Omega\Omega'\,\tilde{A}_{00}^{ij}$$

$$= (\omega^2+f^2)\int d^2\alpha\,\tilde{E}(\omega,\underset{\sim}{\alpha})\,e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{x}_{ij}}\left\{\tilde{\psi}_+(x_3^i)\,\tilde{\psi}_+(x_3^j) - \Omega\Omega'\,\tilde{\psi}_0(x_3^i)\,\tilde{\psi}_0(x_3^j)\right\} \qquad \text{(II.106)}$$

which may be arbitrarily small depending on the energy density spectrum E and the eigenfunction $\tilde{\psi}_\nu$. The same is true for the relations $D_3^{ij} = 0$ and $D_4^{ij} = 0$. Hence the acceptance of the consistency relations $D_l^{ij} = 0$ $(l = 1,\ldots 4)$ for propagating waves does not imply the rejection of standing modes.

The situation is different for horizontal separations of the instruments (including no separation). Here we find instead of the consistency relations $D_3^{ij} = 0$ and $D_4^{ij} = 0$ for propagating waves the relations

$$\tilde{D}_3^{ij} = (\omega - f)\,\tilde{A}_{-0}^{ij} + (\omega+f)\,\tilde{A}_{0+}^{ij} = 0 \qquad \text{(II.107)}$$

$$\tilde{D}_4^{ij} = (\omega + f)\,\tilde{A}_{+0}^{ij} + (\omega - f)\,\tilde{A}_{0-}^{ij} = 0 \qquad \text{(II.108)}$$

which allow to discriminate between standing and propagating modes. However, if the consistency relations $D_i^{ij} = 0 (i = 1,\ldots 4)$ and the symmetry relations $M_{1,2}^{ij} = 0$ for propagating waves are satisfied then the relations $\tilde{D}_{3,4}^{ij} = 0$ for standing modes are also satisfied. Hence if we find the measurements consistent with a symmetric field of propagating waves we also find them consistent with a field of standing modes. If the consistency relations for standing modes are satisfied nothing can be said about propagating waves. The same arguments hold for vertically separated instruments.

II.3.5 *Independent moments, isotropy relations and nonvanishing moments*

Also listed in Table II.3 are the independent moments

$$M_m^{\nu\mu}(\omega) = \frac{1}{2\pi}\int d^2\alpha\,\tilde{E}(\omega,\underset{\sim}{\alpha})\,e^{-im\varphi}\,\tilde{\psi}_\nu(x_3^i)\,\tilde{\psi}_\mu(x_3^j)\,e^{-i\underset{\sim}{\alpha}\cdot\underset{\sim}{x}_{ij}} \qquad \text{(II.109)}$$

the isotropy relations and the nonvanishing moments in case of isotropy. Table II.4 summarizes the number of moments and relations. The results can be

converted to relations involving the cross spectra $\tilde{A}^{ij}_{\nu\mu}$ by means of the transformation formulas

$$\left.\begin{matrix} A^{ij}_{++} \\ A^{ij}_{+-} \\ A^{ij}_{+o} \\ A^{ij}_{-+} \\ A^{ij}_{--} \\ A^{ij}_{-o} \\ A^{ij}_{o+} \\ A^{ij}_{o-} \\ A^{ij}_{oo} \end{matrix}\right\} = 2\pi \left[\begin{matrix} \frac{1}{2}(\omega-f)^2 M_o^{++} \\ \frac{1}{2}(\omega-f)(\omega+f) M_2^{++} \\ \frac{i}{\sqrt{2}}(\omega-f)\omega M_1^{+o} \\ \frac{1}{2}(\omega-f)(\omega+f) M_{-2}^{++} \\ \frac{1}{2}(\omega+f)^2 M_o^{++} \\ \frac{i}{\sqrt{2}}(\omega+f)\omega M_{-1}^{+o} \\ -\frac{i}{\sqrt{2}}(\omega-f)\omega M_{-1}^{o+} \\ -\frac{i}{\sqrt{2}}(\omega+f)\omega M_1^{o+} \\ \omega^2 M_o^{oo} \end{matrix}\right] \qquad \text{(II.110)}$$

II.4 Isotropy and symmetry relations independent of the kinematic structure

Here we will derive those relations among the different components of the cross spectral matrix which express the isotropy and symmetry of the field without specifying the kinematics. The method of establishing these relations has been described by Robertson (1940) and Batchelor (1953). We will restrict ourselves to the cross spectral matrix obtained from one instrument since the field is vertically inhomogeneous.

Consider the cross spectral matrix

$$A_{mn}(\omega) \propto \langle u_m^*(\omega)\, u_n(\omega) \rangle \qquad \text{(II.111)}$$

and its projection onto 2 arbitrary unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$Q(\underset{\sim}{a},\underset{\sim}{b}) = a_m b_n A_{mn} \qquad \text{(II.112)}$$

If we require the velocity field to be invariant under a certain transformation group the scalar Q has to be invariant under the same group. It is a rigorous result of group theory that an invariant scalar can be expressed in terms of

the fundamental invariants of the transformation.

As an example let us consider a vertically symmetric field. If $\underset{\sim}{\nu} = (1,0,0)$ and $\underset{\sim}{\mu} = (0,1,0)$ denote the two unit vectors which define the horizontal plane, Q can be expressed in terms of the fundamental invariants $\underset{\sim}{a}\cdot\underset{\sim}{b}$, $\underset{\sim}{a}\cdot\underset{\sim}{\nu}$, $\underset{\sim}{b}\cdot\underset{\sim}{\nu}$, $\underset{\sim}{a}\cdot\underset{\sim}{\mu}$ and $\underset{\sim}{b}\cdot\underset{\sim}{\mu}$. According to (II.112) Q is linear in the components of each of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and is a homogeneous function of these vectors. Hence

$$Q(\underset{\sim}{a}\cdot\underset{\sim}{b}, \underset{\sim}{a}\cdot\underset{\sim}{\nu}, \underset{\sim}{b}\cdot\underset{\sim}{\nu}, \underset{\sim}{a}\cdot\underset{\sim}{\mu}, \underset{\sim}{b}\cdot\underset{\sim}{\mu}) = A a_m b_m + B a_m \nu_m b_n \nu_n + C a_m \nu_m b_n \mu_n + D a_m \mu_m b_n \nu_n + E a_m \mu_m b_n \mu_n \tag{II.113}$$

where A,B,C,D and E represent arbitrary constants. Hence A_{mn} must have the form

$$A_{mn} = A\delta_{mn} + B\nu_m\nu_n + C\nu_m\mu_n + D\mu_m\nu_n + E\mu_m\mu_n \tag{II.114}$$

or

$$\{A_{mn}\} = \begin{pmatrix} A+B & C & 0 \\ D & A+E & 0 \\ 0 & 0 & A \end{pmatrix} \tag{II.115}$$

A symmetric field hence satisfies the invariance relations

$$A_{13} = 0 \quad , \quad A_{23} = 0 \tag{II.116}$$

Table II.5 lists the fundamental invariants, the structure of A_{mn} and the invariance relations for various transformation classes.

III. BASIC METHODS OF ANALYSIS

III.1. Introduction

The basic goals of the IWEX-experiment are to establish

a) the existence of internal waves and

b) the distribution of wave energy among the different wavenumbers and frequencies.

Because of the inhomogeneous spatial resolution of the IWEX measurements both these goals cannot be accomplished by the standard methods of time series analysis. More general techniques have to be used for optimizing the conclusion. Here we describe the two techniques which are applied to the IWEX data set. The consistency tests provide a technique to test general hypotheses concerning the kinematical structure of the observed fluctuations. The inverse technique provides a technique to estimate parameters of the energy distribution.

To be more specific consider the following example. Let us assume that the observed fluctuations within the internal wave frequency band represent a statistically stationary and homogeneous ensemble of linear internal waves. We will refer to such an assumption as an assumption about the model class. The cross spectrum A_{mn}^{ij} between the velocity component u_m at position $\underset{\sim}{x}^i$ and the velocity component u_n at position $\underset{\sim}{x}^j$ can then be expressed as a weighted projection of the energy density spectrum $E(\underset{\sim}{q})$ onto the frequency axis (cf. part II).

$$\hat{A}_{mn}^{ij}(\omega) = \sum_{\sigma} \int d^2\alpha \, E(\underset{\sim}{q}) \, U_{mn}^{ij}(\underset{\sim}{q}) \, e^{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij} + \sigma\theta_{ij})} \qquad \text{(III.1)}$$

The kernel U_{mn}^{ij} is determined by the model class, i.e. by the structure of the internal wave eigensolutions. The consistency tests then provide a technique to test whether or not a random superposition of linear internal waves represents a consistent description of the observed fluctuations. The inverse technique then provides a technique to invert the relation (III.1), i.e. to determine the energy spectrum $E(\underset{\sim}{q})$ from the observed cross spectra $A_{mn}^{ij}(\omega)$. For the inversion the spectrum $E(\underset{\sim}{q})$ will be represented by an analytical function $E(\underset{\sim}{q},\underset{\sim}{x})$ containing a set of free parameters $\underset{\sim}{x} = (x_\alpha)$, $\alpha = 1,\ldots,P$. The inverse technique then reduces to the determination of these parameters.

A summary of this part can be found in Olbers, Müller and Willebrand (1976).

III.2. Basic geometry

III.2.1 The data point

For the general description of the inverse method and the consistency tests the following definitions will be convenient. For a single frequency all measured cross spectra $A_{mn}^{ij}(\omega)$ will be represented by the column vector

$$\underset{\sim}{y} = (y_\ell) = \{ A_{mn}^{ij}(\omega) \} \qquad \ell = 1, \ldots, L \qquad \text{(III.2)}$$

We will refer to $\underset{\sim}{y}$ as the data point and represent it by a point in the L-dimensional data space.

Time series of finite length only provide estimates of the cross spectral components. These estimates, and hence the data point $\underset{\sim}{y}$, have to be regarded as realizations of random variables. For the following we assume that the covariance matrix of the data is known and given by

$$COV[y_\ell, y_{\ell'}] = S_{\ell\ell'} \qquad \ell, \ell' = 1, \ldots, L \qquad \text{(III.3)}$$

where $\underset{\sim}{S}$ is the usual estimator of the covariance matrix as given by (I.10). Furthermore, we assume that $\underset{\sim}{y}$ is normally distributed. This assumption is justified since both the Chi-squared distribution for autospectra and the complex Wishart distribution for cross spectra can be approximated by a normal distribution if the equivalent number of degrees of freedom ν is sufficiently large. For the IWEX data set ν is typically O (50-300) (cf. Table I.2). Knowing the distribution function we can construct confidence levels for the data point in the usual manner.

In the following we will not try to represent the actual data point $\underset{\sim}{y}$ by a theoretical model but its expectation value $\langle \underset{\sim}{y} \rangle$.

III.2.2 Characterization of model classes

A model class is characterized by all relations among the cross spectra which are satisfied for arbitrary energy density spectra $E(\underset{\sim}{q})$. We will restrict ourselves mainly to linear relations of the form

$$L_{r\ell} \hat{y}_\ell = 0 \qquad r = 1, \ldots, R \;;\; \ell = 1, \ldots, L \qquad \text{(III.4)}$$

The index r counts the number of linearly independent relations. These relations will be referred to as consistency relations. For the model class "linear internal waves" the complete set of linearly independent consistency relations

is listed in part II (cf. Müller and Siedler 1976). Also listed are the consistency relations for various other model classes. All model classes are constructed so that they automatically satisfy those constraints which follow from the fact that the data represent cross spectra (e.g. reality condition, coherence less or equal one).

Within the data space the consistency relations $\underset{\sim}{L}\hat{\underset{\sim}{y}} = 0$ define a (L-R)-dimensional hyperplane in which the data point must lie if the data satisfy the consistency relations.

III.2.3 The model point

If we specify the model class, the analytical form of the energy density spectrum, and the parameters relations like (III.1) provide us with a model point

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = (y_\ell(\underset{\sim}{x})) = \{\hat{A}^{ij}_{mn}(\omega)\} \qquad \ell = 1,\ldots,L \qquad \text{(III.5)}$$

which is a specific point on the hyperplane $\underset{\sim}{L}\hat{\underset{\sim}{y}} = 0$.

The geometric relations between the data point $\underset{\sim}{y}$ with its 95% confidence ellipsiod, the hyperplane $\underset{\sim}{L}\hat{\underset{\sim}{y}} = 0$ and the model point $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ are displayed in Figure III.1.

Generally the data point does not lie in the hyperplane, nor does the data point coincide with the model point. When will we tolerate these differences? Intuitively we would accept a model class if its hyperplane interseots the, say, 95% confidence ellipsoid of the data point. Similarly we would accept a model point if it lies within the 95% confidence ellipsoid of the data point. The model and the data point are statistically indistinguishable in this case. The precise formulation of these criteria is given by the likelihood ratio test.

III.3. Likelihood ratio test

Let us first consider the question whether or not a given model point $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ provides a consistent representation of the data point $\underset{\sim}{y}$. In this case we have to test the hypothesis whether the observed data point $\underset{\sim}{y}$ can be regarded as a realization of a Gaussian variable with known covariance matrix $\underset{\sim}{S}$ and expectation value

$$\langle \underset{\sim}{y} \rangle = \hat{\underset{\sim}{y}}(\underset{\sim}{x}) \qquad \text{(III.6)}$$

Here the cornered bracket denotes the ensemble average. The likelihood ratio

test yields as the appropriate statistic

$$\varepsilon^2 = (\underset{\sim}{y}^T - \hat{\underset{\sim}{y}}^T(\underset{\sim}{x}))\, \underset{\sim}{W}\, (\underset{\sim}{y} - \hat{\underset{\sim}{y}}(\underset{\sim}{x})) \qquad \text{(III.7)}$$

where

$$\underset{\sim}{W} = \underset{\sim}{S}^{-1} \qquad \text{(III.8)}$$

is the inverse of the covariance matrix. If the hypothesis $\langle \underset{\sim}{y} \rangle = \hat{\underset{\sim}{y}}(\underset{\sim}{x})$ is true, the statistic ε^2 follows a Chi-squared distribution with L degrees of freedom. We hence reject the hypothesis at the 95% significance level if

$$\varepsilon^2 \geq \chi^2_{L,0.05} \qquad \text{(III.9)}$$

since there is only a small probability 0.05 that a realization satisfies (III.9) when the hypothesis is true. Here $\chi^2_{L,0.05}$ denotes the 0.05 percentile of the χ^2_L -distribution. If $\varepsilon^2 < \chi^2_{L,0.05}$ we accept the hypothesis, but being aware that alternative hypotheses are not precluded.

In formula (III.7) the matrix $\underset{\sim}{W}$ serves as a metric in the data space so that the condition (III.9) may also be formulated that we reject the hypothesis $\langle \underset{\sim}{y} \rangle = \hat{\underset{\sim}{y}}(\underset{\sim}{x})$ when the distance squared between $\underset{\sim}{y}$ and $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ exceeds the 0.05 percentile of the χ^2_L -distribution.

As regards the problem whether or not a given model class, characterized by the hyperplane $\underset{\sim}{L}\hat{\underset{\sim}{y}} = 0$, represents a consistent description of the data point $\underset{\sim}{y}$ we consider all distances between $\underset{\sim}{y}$ and the points $\hat{\underset{\sim}{y}}$ on the hyperplane and ask if the minimal distance is sufficiently small. More precisely we consider the point $\hat{\underset{\sim}{y}}'$ on the hyperplane which is closest to the data point $\underset{\sim}{y}$ (see Fig. III.1). By straightforward algebra we find

$$\hat{\underset{\sim}{y}}' = \underset{\sim}{y} - \underset{\sim}{W}^{-1}\, \underset{\sim}{L}^T\, \underset{\sim}{T}^{-1}\, \underset{\sim}{L}\, \underset{\sim}{y} \qquad \text{(III.10)}$$

where

$$\underset{\sim}{T} = \underset{\sim}{L}\, \underset{\sim}{W}^{-1}\, \underset{\sim}{L}^+ \qquad \text{(III.11)}$$

We now test the hypothesis

$$\langle \underset{\sim}{y} \rangle = \langle \hat{\underset{\sim}{y}}' \rangle \qquad \text{(III.12)}$$

or equivalently

$$\underset{\sim}{L} \langle \underset{\sim}{y} \rangle = 0 \tag{III.13}$$

The distance squared between $\underset{\sim}{\hat{y}}'$ and $\underset{\sim}{y}$ becomes

$$\varepsilon^2_{min} = \underset{\sim}{y}^T \underset{\sim}{L}^T \underset{\sim}{T}^{-1} \underset{\sim}{L} \underset{\sim}{y} \tag{III.14}$$

and follows a χ^2_R -distribution if the hypothesis (III.12) is true. We hence reject the hypothesis that the model class provides a consistent description of the data if

$$\varepsilon^2_{min} \geq \chi^2_{R,0.05} \tag{III.15}$$

Note that the number of degrees of freedom is given by the number R of independent consistency relations, not by the total number of data. This is due to the fact that the consistency relations only ascribe specific values to R linear combinations of the data whereas the other (L-P) linear combinations are adjusted so that they coincide with the data.

When applying the formulas (III.9) and (III.15) we meet with the problem that the inverse of the matrices $\underset{\sim}{S}$ (a LxL matrix) and $\underset{\sim}{T}$ (a RxR matrix) does not always exist or that the inversion of the matrices is very elaborate. In order to limit time and storage requirements modified versions of the likelihood ratio test have been used. The modifications are slightly different for the consistency tests and for the inverse technique.

III.4. Modified likelihood ratio test for the consistency relations

Let us assume that there only exists one consistency test

$$L_{1\ell} \hat{y}_\ell = 0 \tag{III.16}$$

In this case

$$\underset{\sim}{T}^{-1} = (L S L^T)^{-1} = \frac{1}{\sum_{\ell,\ell'} L_{1\ell} S_{\ell\ell'} L_{1\ell'}} = \frac{1}{VAR[\underset{\sim}{L} \underset{\sim}{y}]} \tag{III.17}$$

and (III.14) reduces to

$$\varepsilon^2_{min} = \Delta_1^2 = \frac{\underset{\sim}{y}^T \underset{\sim}{L}^T \underset{\sim}{L} \underset{\sim}{y}}{VAR[\underset{\sim}{L} \underset{\sim}{y}]} \tag{III.18}$$

Hence Δ_1 is a Gaussian variable with zero mean and variance unity. We reject

the hypothesis $L_{il}\hat{y}_l = 0$ if

$$\Delta_i^2 \gtrsim \chi^2_{1,0.05} \tag{III.19}$$

If there is more than one consistency test we consider the statistics

$$\Delta^2 = \sum_r \Delta_r^2 = \sum_r \frac{\left(\sum_l L_{rl} y_l\right)^2}{\sum_{l,l'} L_{rl} S_{ll'} L_{rl'}} = \sum_r \frac{\left(\sum_l L_{rl} y_l\right)^2}{T_{rr}} \tag{III.20}$$

i.e. we replace $\underset{\sim}{T}^{-1}$ in (III.14) by

$$T^{-1}_{rr'} \rightarrow \delta_{rr'} \frac{1}{T_{rr}} \tag{III.21}$$

The expectation value of Δ^2 is given by

$$\langle \Delta^2 \rangle = R \tag{III.22}$$

its variance by

$$VAR[\Delta^2] = 2 \sum_{r,r'} \frac{T_{rr'} T_{rr'}}{T_{rr} T_{r'r'}} \tag{III.23}$$

If all the single contributions Δ_r (r = 1,...,R) are uncorrelated the statistic Δ^2 would follow a χ^2_R -distribution with R degrees of freedom. However in our case the Δ_r (r = 1,...,R) are generally correlated. In this case the distribution function of Δ^2 can be approximated by a nonnormalized $\chi^2_{R_{eff}}$ -distribution with

$$R_{eff} = 2 \frac{\langle \Delta^2 \rangle^2}{VAR[\Delta^2]} \tag{III.24}$$

degrees of freedom where always $R_{eff} \leq R$. We hence reject the hypothesis that a given model class provides a consistent description of the fluctuations if the normalized statistic

$$\tilde{\Delta}^2 = \frac{1}{R} \Delta^2 \tag{III.25}$$

satisfies

$$\tilde{\Delta}^2 \gtrsim \tilde{\chi}^2_{R_{eff},0.05} \tag{III.26}$$

where

$$\tilde{\chi}^2_{R_{eff},0.05} = \frac{1}{R_{eff}} \chi^2_{R_{eff},0.05} \qquad \text{(III.27)}$$

is the normalized 0.05 percentile of the $\chi^2_{R_{eff}}$-distribution.

The reduction factor $\gamma_c = R_{eff}/R$ will not explicitly be calculated but can be assumed to be of the same order as the reduction factor $\gamma = L_{eff}/L$ which will be discussed in section III.5.

So given the model class, i.e. its consistency relations $\underset{\sim}{L}\hat{\underset{\sim}{y}} = 0$, it is straightforward to compute $\tilde{\Delta}^2$ and to decide whether or not this model class must be rejected.

III.5. Modified likelihood ratio test for the inverse method

Here we consider the modification of the likelihood ratio test which is applied to the inverse method. For the consistency tests we have approximated the inverse of T_{rr}, by δ_{rr}/T_{rr}. Here we will use a slightly better approximation for the inverse of $\underset{\sim}{S}$ whose rationalization will be given in this section.

III.5.1 Metric of the likelihood ratio test

The matrix $\underset{\sim}{W}$ in formula (III.7) defines a metric in the data space. The likelihood ratio test suggests the metric $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$. Generally, a metric $\underset{\sim}{W}$ corresponds to a weighting of the data by $\underset{\sim}{W}^{1/2}$. The specific weighting provided by the likelihood ratio test may be rationalized as follows: If the random variables $\underset{\sim}{y} = (y_1,\ldots,y_L)$ are standardized, i.e. are transformed to a set of uncorrelated variables $\tilde{\underset{\sim}{y}} = (\tilde{y}_1,\ldots,\tilde{y}_L)$ which have zero mean and variance unity, the statistic reduces to

$$\varepsilon^2 = \tilde{\underset{\sim}{y}}^T \tilde{\underset{\sim}{y}} \qquad \text{(III.28)}$$

Hence the likelihood ratio test postulates equal weights for the standardized variables. This is sensible if the data represent the same physical quantity (e.g. independent measurements of the same quantity). In our case the data represent different physical quantities and there exists no generally accepted weighting function. The statistician has to decide what he regards an optimal weighting function. His choice will be based on a priori information about the data and the goals of his analysis.

Besides the uniform weighting of the standardized variables the maximum

likelihood metric $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$ has the following properties. First of all it makes the variables nondimensional and hence comparable. Secondly, the effective number of data

$$L_{eff} = \frac{2\langle\varepsilon^2\rangle^2}{VAR[\varepsilon^2]} = \frac{tr\{\underset{\sim}{S}\underset{\sim}{W}\}^2}{tr\{\underset{\sim}{S}\underset{\sim}{W}\underset{\sim}{S}\underset{\sim}{W}\}} \qquad \text{(III.29)}$$

takes its maximal value $L_{eff} = L$ for the maximum likelihood metric. Although these are desirable properties we cannot apply $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$ to the analysis of the IWEX data since $\underset{\sim}{S}^{-1}$ may not always exist and, more important, since the inversion of the covariance matrix $\underset{\sim}{S}$ is too elaborate. Furthermore, in order to limit the time and storage requirements we have to restrict ourselves to diagonal metrics of the form

$$W_{ll'} = \delta_{ll'} W_l \qquad \text{(III.30)}$$

III.5.2 Diagonal metrics

The trivial metric $w_l = 1$ $(l = 1,\ldots,L)$ cannot be applied since it does not consider the different dimensions of the components of the data vector $\underset{\sim}{y}$. The basic variables of the IWEX data set are the horizontal velocity components and the vertical displacement. A sensible way to avoid the problem of different dimensions is to normalize each component of the data vector by its standard deviation, i.e. to apply the metric

$$W_l^{(a)} = \frac{1}{S_{ll}} \qquad \text{(III.31)}$$

The effective number of data is then given by

$$L_{eff}^{(a)} = \frac{L^2}{\sum_{l,l'} C_{ll'}} \qquad \text{(III.32)}$$

where

$$C_{ll'} = \frac{S_{ll'} S_{ll'}}{S_{ll} S_{l'l'}} \qquad \text{(III.33)}$$

As can be seen from Figure III.2 the effective number of data $L_{eff}^{(a)}$ is approximately 5% of the total number of data. Hence when applying the metric $\underset{\sim}{W}^{(a)}$ we use the information of the data set rather ineffectively.

In order to increase the effective number of data that diagonal metric can be chosen which maximizes this number. Its diagonal elements are given by

$$W_\ell^{(b)} = \frac{1}{S_{\ell\ell}} \sum_{\ell'} C^{-1}_{\ell\ell'} \qquad \text{(III.34)}$$

and the effective number of data by

$$L_{eff}^{(b)} = \sum_{\ell,\ell'} C^{-1}_{\ell\ell'} \qquad \text{(III.35)}$$

Again we meet with the problem that the inversion of the matrix $\underset{\sim}{C}$ requires too much effort.

II.5.3 Metric applied to the IWEX data set

As a less optimal but manageable metric we have chosen that diagonal metric which minimizes

$$\sum_{\ell,\ell'} \left(S_{\ell\ell'} W_{\ell'} - \delta_{\ell\ell'} \right)^2 \frac{S_{\ell\ell'}}{S_{\ell\ell}} = \min \qquad \text{(III.36)}$$

Its diagonal elements are given by

$$W_\ell^{IWEX} = \frac{1}{S_{\ell\ell}} \cdot \frac{1}{\sum_{\ell'} C_{\ell\ell'}} \qquad \text{(III.37)}$$

and its effective number of data by

$$L_{eff}^{IWEX} = \frac{\sum_{\ell,\ell'} K_\ell K_{\ell'}}{\sum_{\ell,\ell'} C_{\ell\ell'} K_\ell K_{\ell'}} \qquad \text{(III.38)}$$

where

$$K_\ell = \frac{1}{\sum_{\ell'} C_{\ell\ell'}} \qquad \text{(III.39)}$$

This choice of the metric improves the effective number of data (see Figure III.2) but still leaves us with a reduction factor

$$\gamma^{IWEX} = \frac{L_{eff}^{IWEX}}{L} \qquad \text{(III.40)}$$

which is of the order of 0.1.

For the various metrices discussed the weighting function, the expectation value and variance of ε^2, and the effective number of data are listed in Table III.1.

The distance square calculated from our metric W^{IWEX} is explicitly given by

$$\varepsilon^2 = \delta\underset{\sim}{y}^T \underset{\sim}{W}^{IWEX} \delta\underset{\sim}{y} = \sum_{\ell} \frac{\delta y_\ell}{(S_{\ell\ell})^{1/2}} \cdot \frac{\delta y_\ell}{(S_{\ell\ell})^{1/2}} \cdot \frac{1}{\sum_{\ell'} C_{\ell\ell'}} \qquad \text{(III.41)}$$

where

$$\delta\underset{\sim}{y} = \underset{\sim}{y} - \hat{\underset{\sim}{y}}(\underset{\sim}{x}) \qquad \text{(III.42)}$$

It implies that the components of the data vector are normalized by their standard deviations and then weighted according to their correlation with other components. Components which are strongly correlated with other components are less weighted than those components which are less correlated.

With the choice of a different metric we have to reformulate our criterion (III.9) for the rejection of the hypothesis $\langle \underset{\sim}{y} \rangle = \hat{\underset{\sim}{y}}(\underset{\sim}{x})$. We now have to reject the hypothesis if

$$\tilde{\varepsilon}^2 = \frac{\varepsilon^2}{\langle \varepsilon^2 \rangle} \geq \chi^2_{L_{eff},\,0.05} \qquad \text{(III.43)}$$

Note that this condition has further to be changed since $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ is a random variable.

III.6. The inverse method

As regards the specification of the analytical form and the parameters of the energy density spectrum we have so far only established the criterion when we regard a model point $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ as a consistent representation of the data point. The algorithm to construct such a model point will be discussed in this chapter.

III.6.1 Least square fit

Suppose a consistent model class has been found. For simplicity we further assume that the analytical form of the spectrum and the number of parameters is specified. Since the number R of data is typically $O(10^3)$ whereas the number P of parameters is typically O(20) the system is overconstrained, meaning that the equation $\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x})$, or equivalently $\varepsilon^2(\underset{\sim}{x}) = 0$, has no solution for $\underset{\sim}{x}$.

However, since we reject a model point only if its distance squared from the data point exceeds a certain critical value it is sensible to determine $\underset{\sim}{x}$ by minimizing

$$\varepsilon^2(\underset{\sim}{x}) = \delta\underset{\sim}{y}^T \underset{\sim}{W}\, \delta\underset{\sim}{y} = \min \qquad \text{(III.44)}$$

This is a generalized least square condition. Our solution will mainly follow Jackson (1972):

We first determine a zero order estimate $\underset{\sim}{x}_o$ of the parameter partly by using a priori information and partly by a rough determination of the minimum of ϵ^2 following a strategy proposed by Powell (1964).

We linearize the model $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$, which is generally highly nonlinear, at $\underset{\sim}{x} = \underset{\sim}{x}_o$

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \hat{\underset{\sim}{y}}(\underset{\sim}{x}_o) + \underset{\sim}{A}\,\Delta\underset{\sim}{x} \tag{III.45}$$

where

$$A_{l\alpha} = \left.\frac{\partial y_l}{\partial x_\alpha}\right|_{\underset{\sim}{x}=\underset{\sim}{x}_o} \qquad l = 1,\ldots,L\,;\ \alpha = 1,\ldots,P \tag{III.46}$$

and

$$\Delta\underset{\sim}{x} = \underset{\sim}{x} - \underset{\sim}{x}_o \tag{III.47}$$

Substituting (III.45) into the expression (III.44) for the distance squared we find

$$\epsilon^2(\underset{\sim}{x}) = (\Delta\underset{\sim}{y}^T - \Delta\underset{\sim}{x}^T\underset{\sim}{A}^T)\,\underset{\sim}{W}\,(\Delta\underset{\sim}{y} - \underset{\sim}{A}\,\Delta\underset{\sim}{x}) \tag{III.48}$$

with

$$\Delta\underset{\sim}{y} = \underset{\sim}{y} - \hat{\underset{\sim}{y}}(\underset{\sim}{x}_o) \tag{III.49}$$

The minimum of $\epsilon^2(\underset{\sim}{x})$ is obtained by setting the derivatives with respect to the parameters equal to zero ($\partial\epsilon^2/\partial x_\alpha = 0$, $\alpha = 1,\ldots,P$). This yields the normal equation

$$\underset{\sim}{A}^T\underset{\sim}{W}\underset{\sim}{A}\,\Delta\underset{\sim}{x} = \underset{\sim}{A}^T\underset{\sim}{W}\Delta\underset{\sim}{y} \tag{III.50}$$

or, equivalently,

$$\underset{\sim}{M}\Delta\underset{\sim}{x} = \Delta\underset{\sim}{z} \tag{III.51}$$

if we define

$$\underset{\sim}{M} = \underset{\sim}{A}^T\underset{\sim}{W}\underset{\sim}{A} \tag{III.52}$$

and

$$\Delta\underset{\sim}{z} = \underset{\sim}{A}^T\underset{\sim}{W}\Delta\underset{\sim}{y} \tag{III.53}$$

The normal equation is a set of P linear equations for P unknowns. Its solution may formally be written

$$\Delta \underset{\sim}{x} = \underset{\sim}{M}^{(-1)} \Delta \underset{\sim}{z} \qquad \text{(III.54)}$$

If the matrix $\underset{\sim}{M}$ is nonsingular and well conditioned we have

$$\underset{\sim}{M}^{(-1)} = \underset{\sim}{M}^{-1} \qquad \text{(III.55)}$$

Note that only the P linear combinations Δz_α ($\alpha = 1,\ldots,P$) of the data determine the parameter increments Δx_α .

If the matrix $\underset{\sim}{M}$ is singular (with rank $P' < P$) the system is underdetermined and the above standard least square procedure fails. In this case the least square condition (III.44) has an infinity of solutions. We construct a unique solution by specifying $\underset{\sim}{M}^{(-1)}$ as follows. First we diagonalize $\underset{\sim}{M}$ by an orthogonal transformation $\underset{\sim}{T}$, i.e. construct

$$M_d = \underset{\sim}{T}^T \underset{\sim}{M} \underset{\sim}{T} \qquad \text{(III.56)}$$

where

$$M^d_{\alpha\beta} = m_\alpha \delta_{\alpha\beta} \qquad \text{(III.57)}$$

The normal equation (III.51) then transforms to

$$\underset{\sim}{M}_d \, \Delta \underset{\sim}{x}' = \Delta \underset{\sim}{z}' \qquad \text{(III.58)}$$

with

$$\Delta \underset{\sim}{x}' = \underset{\sim}{T}^T \Delta \underset{\sim}{x} \qquad \text{(III.59)}$$

and

$$\Delta \underset{\sim}{z}' = \underset{\sim}{T}^T \Delta \underset{\sim}{z} \qquad \text{(III.60)}$$

We now determine the parameter increments by

$$\Delta \underset{\sim}{x}' = \underset{\sim}{M}_d^{(-1)} \Delta \underset{\sim}{z}' \qquad \text{(III.61)}$$

where $\underset{\sim}{M}_d^{(-1)}$ is another diagonal matrix whose elements are m_α^{-1} if $m_\alpha \neq 0$ and zero if $m_\alpha = 0$. Transforming back to the unprimed variables we find

$$\Delta \underset{\sim}{x} = \underset{\sim}{M}^{(-1)} \Delta \underset{\sim}{z} \tag{III.62}$$

with

$$\underset{\sim}{M}^{(-1)} = \underset{\sim}{T}\, \underset{\sim}{M}_d^{(-1)}\, \underset{\sim}{T}^T \tag{III.63}$$

The parameter increments (III.62) satisfy the least square condition (III.44). Out of the infinity of solutions, the solution (III.62) is that solution which minimizes

$$\Delta \underset{\sim}{x}^T \Delta \underset{\sim}{x} = \min \tag{III.64}$$

Note that in the underdetermined case only $P' < P$ linear combinations of the parameter increments are determined by the data. Those linear combinations of the parameter increments which are not uniquely determined by the least square condition have been set equal to zero.

When expressed in terms of the original data $\underset{\sim}{y}$ the solutions take the form

$$\Delta \underset{\sim}{x} = \underset{\sim}{H} \Delta \underset{\sim}{y} = \begin{cases} \underset{\sim}{M}^{-1} \underset{\sim}{A}^T \underset{\sim}{W} \Delta \underset{\sim}{y} & \text{(well-posed case)} \\ \underset{\sim}{M}^{(-1)} \underset{\sim}{A}^T \underset{\sim}{W} \Delta \underset{\sim}{y} & \text{(underdetermined case)} \end{cases} \tag{III.65}$$

The matrix $\underset{\sim}{H}$ is known as the generalized (Lanczos) inverse of the matrix $\underset{\sim}{A}$.

Since we have linearized the functional dependence of the model point $\hat{\underset{\sim}{y}}$ on the parameters, the above results are valid only for small parameter increments. If $\Delta \underset{\sim}{x}$ must be large to satisfy the observation then the results must be checked by expanding about the new parameters.

III.6.2 Covariance matrix of the parameters

Since the estimated parameter increments depend on the data point $\underset{\sim}{y}$ they have to be regarded as realizations of a random variable. Their covariance matrix $\underset{\sim}{R}$ is estimated from the covariance matrix $\underset{\sim}{S}$ of the data by

$$\underset{\sim}{R} = \langle \delta \underset{\sim}{x}\, \delta \underset{\sim}{x}^T \rangle = \underset{\sim}{H}\, \underset{\sim}{S}\, \underset{\sim}{H}^T \tag{III.66}$$

The parameter increments are generally correlated. Statistically orthogonal parameter increments can be constructed by diagonalizing the parameter covariance matrix $\underset{\sim}{R}$, but the new parameters usually do not have any clear physical meanings.

III.6.3 Reformulation of the likelihood ratio test

Since $\Delta \underset{\sim}{x}$ is a random variable our model point $\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \hat{\underset{\sim}{y}}(\underset{\sim}{x}_o) + \underset{\sim}{A}\,\Delta \underset{\sim}{x}$ is a random variable as well. Hence we have to reformulate the likelihood ratio test. We now test the hypothesis whether or not the data point $\underset{\sim}{y}$ may be regarded as a realization of a Gaussian random variable with known covariance matrix $\underset{\sim}{S}$ and mean

$$\langle \underset{\sim}{y} \rangle = \hat{\underset{\sim}{y}}(\underset{\sim}{x}_o) + \underset{\sim}{A} \langle \Delta \underset{\sim}{x} \rangle \qquad \text{(III.67)}$$

The statistic

$$\varepsilon^2 = (\underset{\sim}{y}^T - \hat{\underset{\sim}{y}}^T(\underset{\sim}{x}))\, \underset{\sim}{W}\, (\underset{\sim}{y} - \hat{\underset{\sim}{y}}(\underset{\sim}{x})) \qquad \text{(III.68)}$$

then reduces to

$$\varepsilon^2 = (\delta \underset{\sim}{y}^T - \delta \underset{\sim}{x}^T \underset{\sim}{A}^T)\, \underset{\sim}{W}\, (\delta \underset{\sim}{y} - \underset{\sim}{A}\, \delta \underset{\sim}{x}) \qquad \text{(III.69)}$$

Its expectation value is given by

$$\langle \varepsilon^2 \rangle = tr\{ \underset{\sim}{W}\underset{\sim}{S} \} - tr\{ \underset{\sim}{M}\underset{\sim}{R} \} \qquad \text{(III.70)}$$

For $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$ and $\underset{\sim}{H} = \underset{\sim}{M}^{-1}\, \underset{\sim}{A}^T \underset{\sim}{W}$ the distribution of ε^2 is a χ^2_{L-P}-distribution with (L-P) degrees of freedom. For our choice of the weighting function $\underset{\sim}{W} = \underset{\sim}{W}^{IWEX}$ we assume that ε^2 can be approximated by a $\chi^2_{L_{eff}-P}$-distribution with $(L_{eff} - P)$ degrees of freedom and reject the hpyothesis (III.67) at the 95% significance level, if

$$\tilde{\varepsilon}^2 = \frac{\varepsilon^2}{\langle \varepsilon^2 \rangle} \geq \chi^2_{L_{eff}-P,\,0.05} \qquad \text{(III.71)}$$

III.6.4 General strategy

In order to determine the number of parameters which are well determined by the data we use an analytical form of the energy density spectrum with more parameters than are likely to be uniquely determined by the data. Since the data only have a certain numerical accuracy we define the diagonal elements of the matrix $\underset{\sim}{M}_d^{(-1)}$ by m_α^{-1} if $m_\alpha \geq \hat{m}$ and 0 if $m_\alpha < \hat{m}$. Here the threshold $\hat{m}$ is given by the numerical accuracy of the data. This procedure then yields the number of parameters which are determined by the data. However, the variance of the parameter

$$VAR[\Delta x'_\alpha] = \frac{1}{m_\alpha^2} VAR[\Delta z'_\alpha] \qquad \text{(III.72)}$$

although finite, may be unacceptably large. A sensible way to control the variance is to consider an eigenvalue m_α also to be zero if $\hat{m} \leq m_\alpha < m_\alpha^0$ where m_α^0 is some other threshold and then construct the Lanczos inverse. This reduces the resolution, i.e. the number of parameters which are determined by the data, but improves the variance of the remaining parameters. This tradeoff must be carefully chosen in order not to try to resolve poorly determined features of the data.

The threshold m_α^0 may be set according to the following rationalization (Curl 1970). One takes an a priori guess of the original physical parameters P_α^0 ($\alpha = 1,\ldots,P$) and constructs to each parameter an interval $P_\alpha^L < P_\alpha^0 < P_\alpha^u$ in which the parameters fall according to information from other experiments in similar situations. One then considers the variables

$$\Delta x_\alpha = \frac{P_\alpha - P_\alpha^0}{P_\alpha^u - P_\alpha^L} \qquad \text{(III.73)}$$

and calculates the variance (III.72). If the variance of $\Delta x'_\alpha$ is smaller than 1, i.e.

$$VAR[\Delta x'_\alpha] = \frac{1}{m_\alpha^2} VAR[\Delta z'_\alpha] \leq 1 \qquad \text{(III.74)}$$

the experiment under consideration provides a more precise estimate of the parameter than what one knows from other experiments. If, on the contrary, the variance of $\Delta x'_\alpha$ is larger than 1 the estimates are less precise than what one knows from other experiments. In this case the parameters should not be determined from the experiment under consideration. The threshold should hence be set

$$m_\alpha^* = \mathrm{VAR}^{1/2}[\Delta z_\alpha'] \qquad \text{(III.75)}$$

The choice of the threshold m_α^* for the IWEX data essentially followed this rationalization, though no formal algorithm was constructed.

III.7. Computational problems

The inverse technique mainly requires three algorithms:

1) computing the model cross-spectra for a given parameter set
2) minimizing ε^2
3) computing parameter covariance

The realization of these algorithms meets with a number of problems concerning computer time, storage and accuracy.

1. The computation of model cross-spectra requires the numerical evaluation of the integral (III.1). Analytical solutions could only be found for a few special cases (cf. Desaubies 1977). The integrand is a product of a slowly converging (energy spectrum) and a more or less rapidly oscillating (exponential or Bessel) function. This type of integral requires special methods. Conventional algorithms like Simpson or Rhomberg fail completely. Also, to limit the computer time, we had to use the fact that for each instrument pair (corresponding to a group of 18 data) the integral is essentially the same. This is one of the reasons that the inverse program is rather specialized to internal wave problems.

2. Because of the high nonlinearity of the model the iteration procedure described in section III.6.1 needed initial parameter values rather close to the final solution. Besides estimating initial values from some obvious data features (see part V) we needed a very efficient minimum strategy. Also, the success of the iteration was very sensitive to the accuracy of the numerical differentiation $\partial y_\iota / \partial x_\alpha$ necessary for the calculation of the matrix $\underset{\sim}{A}$.

3. For large data sets, the exact computation of the parameter covariance is the most time consuming step in the analysis, since the amount of operations is proportional to the square of L (total data number), i.e. to the fourth power of the number of sensors, whereas the amount for the other calculations of the inverse program is proportional to L.

A rough estimate of $\underset{\sim}{R}$ can be obtained from (III.66). If $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$ one finds

$$\underset{\sim}{R} = \frac{\langle \varepsilon^2 \rangle}{L - P} \; \underset{\sim}{M}^{-1} \qquad \text{(III.76)}$$

This formula frequently occurs in the literature (e.g. Zurmühl, 1965). For diagonal weights, it is exact only if the data are completely uncorrelated. As shown in section III.5. the data correlation reduces the effective number of data considerably. We found, when replacing L by L_{eff} in (III.76), that the parameter covariance $\underset{\sim}{R}$ usually coincides within 20% with its exact value given by (III.66). Despite this convenient approximation we have calculated the covariance matrix rigorously.

In contrast to the inverse technique, the evaluation of the consistency tests posed almost no severe problems. The computer time needed for the consistency tests was only about 1‰ of the total time spent for the analysis of the IWEX data.

IV. RESULTS OF CONSISTENCY TESTS

IV.1. Consistency relations

When representing the observed cross spectra by a theoretical model both the model class and the energy distribution have to be specified. A model class determines those general properties of the fluctuations which are independent of the specific distribution of energy among the different wavenumber and frequencies. Typical specifications of the model class are: the observed fluctuations represent a field of random propagating internal waves; the fluctuations represent a field of standing internal waves; the fluctuations represent a field of internal waves contaminated by density and (or) velocity finestructure.

If the observed fluctuations represent a random field of propagating internal waves the observed cross spectra are weighted projections of the energy density spectrum $E^{\sigma}(\omega, \underset{\sim}{\alpha})$ (ω = frequency, $\underset{\sim}{\alpha}$ = horizontal wavenumber vector, σ = sign of the vertical wavenumber) onto the frequency axis (cf. II.46)

$$\hat{A}^{ij}_{mn}(\omega) = \sum_{\sigma} \int d^2\alpha \, E^{\sigma}(\omega, \underset{\sim}{\alpha}) \, U^{ij}_{mn} \, e^{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij} + \sigma\Theta_{ij})} \qquad \text{(IV.1)}$$

where $\underset{\sim}{r}_{ij}$ is the separation vector of the instruments and Θ_{ij} the vertical phase difference. The kernel U^{ij}_{mn} is determined by the structure of the internal wave eigensolutions, i.e. by the model class.

A model class may be characterized by all relations among the cross spectra which are satisfied for arbitrary energy density spectra. We will restrict ourselves to linear relations of the form

$$\sum_{m,n,i,j} L_{\tau,mnij}(\omega) \, \hat{A}^{ij}_{mn}(\omega) = 0 \quad , \tau = 1, \ldots, R \qquad \text{(IV.2)}$$

where R denotes the number of linearly independent relations. Such relations will be referred to as consistency relations. They may also be written in vector notation

$$\sum_{\ell} L_{\tau\ell} \, \hat{y}_{\ell} = 0 \qquad \tau = 1, \ldots, R \; ; \; \ell = 1, \ldots, L \qquad \text{(IV.3)}$$

where $\underset{\sim}{y}$ denotes the vector

$$\underset{\sim}{\hat{y}} = (\hat{y}_{\ell}) = \{ \hat{A}^{ij}_{mn}(\omega) \} \qquad \text{(IV.4)}$$

constructed from all the theoretical cross spectra $\hat{A}^{ij}_{mn}(\omega)$ The total number of cross spectra is denoted by L.

The consistency relations for the model classes: "propagating and standing internal waves" have been given by Müller and Siedler (1976). These consistency relations together with the consistency relations for various other model classes are listed and discussed in part II.

IV.2. General formulation of consistency tests

In part III we discussed the statistical framework for testing whether or not the observed cross spectra satisfy a given set of consistency relations. As a convenient statistic we found

$$\Delta^2 = \sum_r \Delta_r^2 \tag{IV.5}$$

where

$$\Delta_r^2 = \frac{\left(\sum_\ell L_{r\ell}\, y_\ell\right)^2}{VAR\left[\sum_\ell L_{r\ell}\, y_\ell\right]} \tag{IV.6}$$

Here $\underset{\sim}{y}$ denotes the vector

$$\underset{\sim}{y} = (y_\ell) = \left\{ A^{ij}_{mn}(\omega) \right\} \tag{IV.7}$$

constructed from all the observed cross spectra $A^{ij}_{mn}(\omega)$

Assuming the observed cross spectra to be normally distributed, Δ_r is normally distributed with variance 1 and zero mean. If all the single contributions Δ_r $(r = 1,\ldots,R)$ are uncorrelated the statistic Δ^2 follows a χ^2_R -distribution with R degrees of freedom. Generally the single contributions Δ_r are correlated. In this case the distribution of Δ^2 can still be approximated by a χ^2 -distribution but with

$$R_{eff} = 2\frac{\langle \Delta^2 \rangle^2}{VAR[\Delta^2]} = 2\frac{R^2}{VAR[\Delta^2]} \tag{IV.8}$$

degrees of freedom. We hence reject the hypothesis that the observed fluctuations satisfy a certain set of consistency relations at the 95% significance level if

$$\tilde{\Delta}^2 = \frac{1}{R}\Delta^2 \geq \frac{1}{R_{eff}}\chi^2_{R_{eff},\,0.05} = \tilde{\chi}^2_{R_{eff},\,0.05} \tag{IV.9}$$

Here $\chi^2_{R_{eff},\,0.05}$ denotes the 0.05 percentile of the $\chi^2_{R_{eff}}$ -distribution.

Given the consistency relation for a certain model class, i.e. the consistency matrix $\underset{\sim}{L}$, the normalized statistic $\tilde{\Delta}^2$ can easily be computed if

we approximate

$$VAR\left[\sum_{\ell} L_{r\ell}\, y_\ell\right] = \sum_{\ell,\ell'} L_{r\ell}\, L_{r\ell'}\, COV\left[y_\ell, y_{\ell'}\right] = \sum_{\ell,\ell'} L_{r\ell}\, L_{r\ell'}\, S_{\ell\ell'} \qquad \text{(IV.10)}$$

where $\underset{\sim}{S}$ is the usual estimator of the covariance matrix given by (I.10).

The calculation of the effective number of degrees of freedom requires more computational effort since it is a fourth order quantity in terms of the cross spectra. We will not calculate R_{eff} explicitly but will assume

$$R_{eff} = \gamma\, R \qquad \text{(IV.11)}$$

where γ is the reduction factor shown in Figure III.2 and discussed in part III. Generally $\gamma = 0$ (0.1). Our results are not very sensitive to the proper choice of γ since $\chi^2_{R_{eff},0.05}$ is only slightly dependent on R_{eff}. For example $\chi^2_{R_{eff},0.05}$ = 3.84, 2.21, 1.83, 1.35, 1.24, 1.11 for R_{eff} = 1, 5, 10, 50, 100, 500 respectively.

In order to test whether or not a certain model class provides a consistent description of the observed cross spectra we calculate the normalized statistic $\tilde{\Delta}^2$. This value has to be compared with the normalized 0.05 percentile $\tilde{\chi}^2_{R_{eff},0.05}$ of the $\chi^2_{R_{eff}}$-distribution. Since we do not know R_{eff} exactly two criteria for the rejection will be applied.

For a single frequency we reject a model class if $\tilde{\Delta}^2$ is much larger than $\tilde{\chi}^2_{R_{eff},0.05}$ with R_{eff} = 0.1 R. If $\tilde{\Delta}^2$ is smaller or slightly larger than $\tilde{\chi}^2_{R_{eff},0.05}$ we accept the model class but do not preclude any alternative hypothesis.

For a frequency band we expect the values of $\tilde{\Delta}^2$ to scatter around its mean value

$$\langle \tilde{\Delta}^2 \rangle = 1 \qquad \text{(IV.12)}$$

if the consistency relations are valid. If we find that the values of $\tilde{\Delta}^2$ are systematically larger than 1 we have to reject the model class for this frequency band as well. Formally this criterion may be derived as follows. Consider the mean value of $\tilde{\Delta}^2$ over a certain frequency band

$$\overline{\tilde{\Delta}^2} = \frac{1}{N} \sum_{i=1}^{N} \tilde{\Delta}^2(\omega_i) \qquad \text{(IV.13)}$$

Since the single contributions $\tilde{\Delta}^2(\omega_i)$ follow a χ^2-distribution with R_{eff} degrees of freedom, the mean value $\overline{\tilde{\Delta}^2}$ follows a χ^2-distribution with $N \cdot R_{eff}$ degrees of freedom. The different frequencies are assumed to be statistically independent.

We hence have to reject the model if

$$\overline{\tilde{\Delta}^2} \geq \tilde{\chi}^2_{NR_{eff},0.05} \qquad \text{(IV.14)}$$

Here $\tilde{\chi}^2_{NR_{eff},0.05}$ is typically O (1.1) if we take R = 500, R_{eff} = 50 and N = 10.

The results of the consistency tests are presented by graphically displaying $\tilde{\Delta}^2$ as a function of frequency. The number R of consistency relations or tests which go into the calculation of $\tilde{\Delta}^2$ is given in the inlet of the figures. This number varies with frequency since for higher frequencies those levels of the array are excluded where ω exceeds the local Brunt Väisälä frequency. Only the value R is given which is valid for the 3rd to the 21st frequency point. Values of R for the other frequency intervals can be found in Table IV.1. Also shown in the figures is $\tilde{\chi}^2_{R_{eff},0.05}$ for values of R_{eff} = O (0.1 R).

IV.3. Simple models

Before we systematically investigate physically reasonable model classes we test some simple models which will be used later in hybrid models.

IV.3.1 Zero model

As the simplest hypothesis we assume that the expectation value of all cross spectra vanishes

$$\langle A^{ij}_{mn} \rangle = 0 \qquad (\text{all } m,n,i,j) \qquad \text{(IV.15)}$$

or

$$\langle y_\ell \rangle = 0 \qquad \ell = 1, \ldots, L \qquad \text{(IV.16)}$$

The consistency matrix $\underset{\sim}{L}$ for this model class is given by

$$L_{r\ell} = \delta_{r\ell} \qquad r,\ell = 1, \ldots, L \qquad \text{(IV.17)}$$

The statistic $\tilde{\Delta}^2$ reduces to

$$\tilde{\Delta}^2_0 = \frac{1}{L} \sum_{r=1}^{L} \frac{\left(\sum_\ell L_{r\ell} y_\ell\right)^2}{\sum_{\ell,\ell'} L_{r\ell} L_{r\ell'} S_{\ell\ell'}} = \frac{1}{L} \sum_{r=1}^{L} \frac{y_r^2}{S_{rr}} \qquad \text{(IV.18)}$$

Figure IV.1 shows $\tilde{\Delta}^2_0$ as a function of frequency and demonstrates that the hypothesis $\langle \underset{\sim}{y} \rangle = 0$ must be rejected at all frequencies.

The value of $\tilde{\Delta}_0^2$ is mainly determined by the equivalent number of degrees of freedom ν of the spectral estimates. Since $S_{rr} \propto \nu^{-1}$ (see I.10) we find for the statistic

$$\tilde{\Delta}_0^2 \propto \nu \qquad \text{(IV.19)}$$

Hence the large values of $\tilde{\Delta}_0^2$ at high frequencies reflect the large edof at these frequencies. In order to illustrate this dependence, ν is also shown in Figure IV.1.

IV.3.2 White noise

White noise will be defined in our context as a field for which all cross spectra are zero whereas the autospectra may take arbitrary values. White noise must hence satisfy

$$\langle R_{mn}^{ij} \rangle = 0 \quad \text{if } m \neq n \text{ or } i \neq j \qquad \text{(IV.20)}$$

There are L_c such relations if L_c denotes the number of cross spectra. The statistic $\tilde{\Delta}^2$ is given in this case by

$$\tilde{\Delta}_{wn}^2 = \frac{1}{L_c} \sum_r \frac{y_r}{S_{rr}} \qquad \text{(IV.21)}$$

where the summation is over all cross spectra. The values of $\tilde{\Delta}_{wn}^2$ are also shown in Figure IV.1. They behave similar to those of $\tilde{\Delta}_0^2$. The assumption that the observed fluctuations represent white noise must be rejected as well.

IV.3.3. Finestructure

As a modification of the white noise model we have also included in Figure IV.1 the values $\tilde{\Delta}_{fs}^2$ for a model where all cross spectra from vertically separated instruments vanish whereas the other cross spectra and the autospectra can take arbitrary values. Such a model reflects the principal properties of density and current finestructure in the ocean, which has a small vertical and a large horizontal coherence scale. In our definition the vertical coherence scale is assumed to be smaller than the smallest vertical separation of the array whereas the horizontal coherence scale remains unspecified. This finestructure model must be rejected as well.

Generally, we observe

$$\tilde{\Delta}_0^2 > \tilde{\Delta}_{wn}^2 > \tilde{\Delta}_{fs}^2 \qquad \text{(IV.22)}$$

consistent with physical intuition.

IV.4. Basic assumptions of the Garrett and Munk model

The first attempt to reconstruct the energy density spectrum of the internal wave field in the deep ocean from the observed projections was made by Garrett and Munk (1972). Combining measurements from different locations, depths and instruments they proposed an energy distribution which was believed to reflect principal features of the mean internal wave field in the deep ocean. Here we will test the basic assumptions of the Garrett and Munk model. The energy and wavenumber scales and the form of the spectrum will be discussed in part V. Although Garrett and Munk start from vertically standing modes, they take locally defined vertical averages and smear out the modal structure into a continuum. This corresponds to a WKBJ approximation. Within this WKBJ framework the basic assumptions of the Garrett and Munk spectrum are:

(i) the observed fluctuations within the internal wave field range represent realizations of a statistically stationary and horizontally homogeneous process;

(ii) the fluctuations represent a superposition of free propagating internal waves

(iii) the internal wave field scales in the vertical according to WKBJ theory

(iv) the internal wave field is horizontally isotropic, i.e. independent of the direction of the horizontal wavenumber

(v) the internal wave field is vertically symmetric, i.e. independent of the sign of the vertical wavenumber.

The last assumption is a consequence of the modal approach.

The stationarity has been discussed in part I. Here we will test all linear relations among the cross spectra which follow from the other assumptions. Formally these relations can be divided into two classes. In the first class cross spectral components between two instruments or from one instrument are compared, i.e. the relations take the form

$$\sum_{m,n} L_{mn}(\omega)\, \hat{A}^{ii}_{mn}(\omega) = 0 \qquad \text{(IV.23)}$$

or

$$\sum_{m,n} L_{mn}(\omega)\, \hat{A}_{mn}(\omega) = 0 \qquad \text{(IV.24)}$$

where $L_{mn}(\omega)$ is a known function of frequency. The consistency, isotropy and symmetry conditions fall into this class. The second class compares cross spectra between two instruments with cross spectra from another two instruments,

$$\sum_{m,n,i,j} L_{mnij}(\omega)\,\hat{A}^{ij}_{mn}(\omega) = 0 \qquad \text{(IV.25)}$$

or cross spectra from one instrument with the cross spectra from another instrument

$$\sum_{m,n,i} L_{mni}(\omega)\,\hat{A}^{i}_{mn}(\omega) = 0 \qquad \text{(IV.26)}$$

The homogeneity and WKB-scaling relations take for example the form (IV.26). Here we will only test all relations which take the forms (IV.23), (IV.24) and (IV.26). Relations of the form (IV.25) are possible because of the specific array geometry but will not be considered. Nevertheless, there remain 854 relations among 1444 cross spectral components (cf. Table IV.1) which can be utilized to test the basic assumptions of the Garrett and Munk model.

The results of these tests are shown in Figure IV.2, which demonstrates that the basic assumptions of the Garrett and Munk model cannot be accepted. They must clearly be rejected for high and low frequencies. They must, however, also be rejected for medium frequencies. Although the values of $\tilde{d}^2$ are acceptable for each single frequency they are not randomly scattered around the expectation value $\langle \tilde{d}^2 \rangle = 1$, indicating that there exist slight but systematic inconsistencies.

The violation at the inertial frequency f is not real but must be ascribed to the finite frequency resolution of the estimated cross spectra (see section IV.7.1).

In the following we will investigate in detail the various tests which contribute to the above result in order to determine which assumptions are inconsistent and need to be modified.

IV.5. Homogeneity and WKB-scaling

If the fluctuations represent realizations of a horizontally homogeneous process the cross spectra should satisfy the homogeneity relations

$$A_{mn}(x_3^i) - A_{mn}(x_3^j) = 0 \quad \text{if } x_3^j - x_3^i = 0 \qquad \text{(IV.27)}$$

If the fluctuations represent a field of propagating internal waves the spectra should additionally satisfy the WKB-scaling relations

$$\frac{A_{mn}(x_3^i)}{D_{mn}(x_3^i)} - \frac{A_{mn}(x_3^j)}{D_{mn}(x_3^j)} = 0 \qquad \text{(IV.28)}$$

or

$$\tilde{A}_{mn} = \frac{A_{mn}}{D_{mn}} = \text{const.} \qquad \text{(IV.29)}$$

where

$$\{D_{mn}(x_3)\} = \begin{pmatrix} \Omega(x_3) & \Omega(x_3) & 1 \\ \Omega(x_3) & \Omega(x_3) & 1 \\ 1 & 1 & \Omega^{-1}(x_3) \end{pmatrix}, \quad \Omega(x_3) = \left[\frac{N^2(x_3)+\omega^2}{\omega^2-f^2}\right]^{1/2} \qquad \text{(IV.30)}$$

Specifically we find for $\omega^2 \ll N^2$ the well known scaling relations

$$P_{11}(x_3)\,N^{-1}(x_3),\ P_{22}(x_3)\,N^{-1}(x_3) = \text{const.}$$

$$P_{33}(x_3)\,N(x_3) = \text{const.} \qquad \text{(IV.31)}$$

As already demonstrated by Briscoe (1975) these specific scaling relations are fairly well satisfied.

In order to systematically investigate whether or not there exist significant deviations from the scaling and homogeneity relations we determine the mean scaled spectra

$$\langle \tilde{A}_{mn} \rangle = \frac{1}{N} \sum_{i=1}^{N} \tilde{A}^{i}_{mn} \qquad \text{(IV.32)}$$

where N denotes the number of different instruments which may depend on m and n since we have 20 instruments measuring u_3 and 9 instruments measuring u_1 and u_2. We then test whether or not

$$\langle \tilde{A}_{mn} \rangle - \tilde{A}^{i}_{mn} = 0 \qquad \text{(IV.33)}$$

The appropriate statistic

$$\tilde{\Delta}^2 = \frac{1}{N-1}\frac{1}{9} \sum_{i=1}^{N} \sum_{m,n} \frac{\left(\tilde{A}^{i}_{mn} - \langle \tilde{A}_{mn} \rangle\right)^2}{VAR\left[\tilde{A}^{i}_{mn} - \langle \tilde{A}_{mn} \rangle\right]} \qquad \text{(IV.34)}$$

is shown in Figure IV.3. The assumption that the spectra satisfy the homogeneity and WKB-scaling relations must be rejected for the 5 highest frequency points. For the other frequency points this assumption might be accepted although the distribution of the values indicates slight inconsistencies.

In order to demonstrate the sensitivity of these tests Figure IV.3 also shows $\tilde{\Delta}^2$ for a constant Brunt Väisälä profile or, equivalently, for a vertically

homogeneous field. This assumption must be rejected at all frequencies. The anomalously high values are mainly due to the fact that the autospectra at level 14 differ by nearly a factor of 5 from the spectra at other levels. Since level 14 is excluded from the 22nd frequency point the value of $\tilde{\Delta}^2$ drops drastically.

Figure IV.4 shows the test results separately for the horizontal velocity field and the displacement field, i.e.

$$\tilde{\Delta}^2 = \frac{1}{N-1}\,\frac{1}{4}\sum_{i=1}^{N}\sum_{m,n=1}^{2}\frac{(\tilde{A}^i_{mn} - \langle\tilde{A}_{mn}\rangle)^2}{VAR[\tilde{A}^i_{mn} - \langle\tilde{A}_{mn}\rangle]} \qquad \text{(IV.35)}$$

and

$$\tilde{\Delta}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(\tilde{A}^i_{33} - \langle\tilde{A}_{33}\rangle)^2}{VAR[\tilde{A}^i_{33} - \langle\tilde{A}_{33}\rangle]} \qquad \text{(IV.36)}$$

The violation of the homogeneity and scaling relations at high frequencies is mainly due to the displacement field. The largest contributions arise from the spectra at level 14 (not discernible from the figure).

The inconsistencies at high frequencies are presumably due to turning point effects. As the frequency approaches N the WKBJ solutions become less appropriate and break down at $\omega = N$ where they predict an infinite vertical wavelength. This result is confirmed by the drop of $\tilde{\Delta}^2$ at the 22nd frequency since for this and higher frequencies level 14 is excluded from the analysis.

The violation of the basic assumptions of the Garrett and Munk model cannot be ascribed to a violation of the homogeneity and scaling relations. This result is confirmed if we only test the consistency relations and the isotropy and symmetry relations. For the IWEX array this model is determined by 771 independent relations per frequency. The result is shown in Figure IV.5. Again the violation is most pronounced at tidal frequency and at high frequencies whereas for medium frequencies the violation is small but significant. The estimate at inertial frequency is again not reliable because of finite frequency resolution.

Result. The homogeneity and WKB-scaling relations are only slightly violated for low and medium frequencies. For the highest frequencies there exist larger inconsistencies but these must be ascribed to the proximity of the turning point. The rejection of the basic assumptions of the Garrett and Munk model is not mainly due to the violation of the homogeneity and scaling relations. This result implies that contaminating fields also satisfy the homogeneity and WKB-scaling relations.

IV.6. Symmetry and isotropy

Here we will investigate to what extent the violations are due to the assumptions of symmetry and (or) isotropy. For this we test those relations among the cross spectra which must hold for a symmetric or isotropic field independent of the specific kinematic structure. These relations can easily be inferred from the following arguments.

Vertical symmetry implies that cross spectra do not change when reflected at a horizontal plane

$$R_{u_\alpha u_3} = R_{u_\alpha - u_3} = - R_{u_\alpha u_3} \quad , \alpha = 1,2 \tag{IV.37}$$

from which the symmetry relations

$$R_{\alpha 3} = 0 \quad , \alpha = 1,2 \tag{IV.38}$$

are inferred.

Similarly horizontal isotropy implies that cross spectra do not change when rotated about the vertical axis

$$\begin{aligned} R_{u_\alpha u_3} &= R_{-u_\alpha u_3} = - R_{u_\alpha u_3} \quad , \alpha = 1,2 \\ R_{u_1 u_1} &= R_{u_2 u_2} \quad , \quad R_{u_1 u_2} = R_{u_2 - u_1} = - R_{u_2 u_1} \end{aligned} \tag{IV.39}$$

yielding the isotropy relations

$$\begin{aligned} R_{\alpha 3} &= 0 \quad , \alpha = 1,2 \\ P_{11} - P_{22} &= 0 \quad , \quad P_{12} = 0 \end{aligned} \tag{IV.40}$$

These relations are all symmetry and isotropy relations which hold among cross spectra measured at the same position as can be inferred from the general theory presented in section II.4. Because of the specific array geometry (a perfect tetrahedron) there exist also isotropy relations for cross spectra obtained from horizontally separated instruments. Some of these have been tested by Briscoe (1975) but will not be considered here. It should also be mentioned that there exists a more extensive set of isotropy and symmetry relations (listed in Table II.1 and II.3) if one assumes that the field is an internal wave field.

The estimates $\tilde{\Delta}^2$ for the isotropy relations are shown in Figure IV.6. The assumption of isotropy can be accepted for medium frequencies. At high

frequencies we find a slight anisotropy. At low frequencies, especially at inertial and tidal frequencies, a significant anisotropy is observed.

The test of the symmetry relations, shown in Figure IV.7, shows essentially the same features.

These above results are confirmed when the mean value of the square root of the ellipse stability

$$\langle \gamma_{+-} \rangle = \frac{1}{N} \sum_{i=1}^{N} \gamma_{+-}^{i} \qquad \text{(IV.41)}$$

with

$$\gamma_{+-}^{i} = \frac{|A_{+-}^{i}|}{(A_{++}^{i} A_{--}^{i})^{1/2}} = \left[\frac{(P_{11}^{i} - P_{22}^{i})^2 + 4 P_{12}^{i2}}{(P_{11}^{i} + P_{22}^{i})^2 - 4 Q_{12}^{i2}} \right]^{1/2} \qquad \text{(IV.42)}$$

is considered (Figure IV.8). For an isotropic field γ_{+-} should vanish. However the expectation value of the estimate (IV.41) is not zero since the coherence estimate is biased. The bias is also shown. Figure IV.8 also displays the mean values of the coherences

$$\gamma_{+o}^{i} = \frac{|A_{+o}^{i}|}{(A_{++}^{i} A_{oo}^{i})^{1/2}} = \left[\frac{(P_{13}^{i} - Q_{23}^{i})^2 + (P_{23}^{i} + Q_{13}^{i})^2}{(P_{11}^{i} + P_{22}^{i} + 2 Q_{12}^{i}) P_{33}^{i}} \right]^{1/2} \qquad \text{(IV.43)}$$

$$\gamma_{-o}^{i} = \frac{|A_{-o}^{i}|}{(A_{--}^{i} A_{oo}^{i})^{1/2}} = \left[\frac{(P_{13}^{i} + Q_{23}^{i})^2 + (P_{23}^{i} - Q_{23}^{i})^2}{(P_{11}^{i} + P_{22}^{i} - 2 Q_{12}^{i}) P_{33}^{i}} \right]^{1/2} \qquad \text{(IV.44)}$$

which should vanish for isotropic or symmetric fields.

Generally the acceptance of a certain hypothesis does not preclude any alternative hypothesis. In our case: the coherence γ_{+-} vanishes for an isotropic field. Thus we can conclude from γ_{+-} being unequal zero that the field must be anisotropic. However, if γ_{+-} is found to be zero no exact statement can be made except the vague one that the finding is consistent with an isotropic field. More specifically, if the field is governed by internal wave kinematics the coherence

$$\gamma_{+-}(\omega) = \frac{\int d\alpha \, c_2^{e}(\omega,\alpha)}{\int d\alpha \, c_o^{e}(\omega,\alpha)} \qquad \text{(IV.45)}$$

is determined by the second coefficient in the Fourier expansion

$$E^{e,o}(\omega,\alpha,\varphi) = \sum_{m=-\infty}^{\infty} c_m^{e,o}(\omega,\alpha) \, e^{i m \varphi} \qquad \text{(IV.46)}$$

of the energy density spectrum. Hence $\gamma_{+-}(\omega) = 0$ implies $\int d\alpha \, c_2^{e}(\omega,\alpha) = 0$

which does not imply that the energy density spectrum is isotropic. However, from c_2^e being zero some implications can be drawn on the other Fourier coefficients since E must be a positive function.

Result. The symmetry and isotropy relations are well satisfied for medium frequencies. For low frequencies, especially for inertial and tidal frequency, there exist significant deviations from both isotropy and symmetry which must be considered when modelling the observed spectra. At high frequencies slight asymmetries and anisotropies are found. Again these violations cannot account for the violation of the basic assumptions of the Garrett and Munk model.

IV.7. Internal wave models

IV.7.1 Propagating waves

The preceeding discussions clearly demonstrate that the large violations of the basic assumptions of the Garrett and Munk model are not due to the violation of the homogeneity, scaling, isotropy and symmetry relations but must be due to the invalidity of the assumption that the fluctuations represent propagating internal waves. This statement is confirmed by Figure IV.9 where the result of the consistency relations for propagating waves are shown.

The consistency relations for propagating waves take the form (cf. part II)

$$D_1^{ij} = A_{++}^{ij} + A_{--}^{ij} - \frac{\omega^2+f^2}{\omega^2}\,\Omega^i\Omega^j A_{oo}^{ij} = 0 \qquad \text{(IV.47)}$$

$$D_2^{ij} = (\omega+f)^2 A_{++}^{ij} - (\omega-f)^2 A_{--}^{ij} = 0 \qquad \text{(IV.48)}$$

$$D_3^{ij} = \Omega^j(\omega+f) A_{+o}^{ij} - \Omega^i(\omega-f) A_{o-}^{ij} = 0 \qquad \text{(IV.49)}$$

$$D_4^{ij} = \Omega^j(\omega-f) A_{-o}^{ij} - \Omega^i(\omega+f) A_{o+}^{ij} = 0 \qquad \text{(IV.50)}$$

Let us consider these consistency relations for i = j first.

The consistency relations $D_1 = 0$ and $D_2 = 0$ involve the autospectra P_{++}, P_{--} and P_{zz}. The mean values of the scaled autospectra (scaled to the depth x_3^o of the apex of the array)

$$\langle \tilde{P}_{++} \rangle = \frac{1}{N} \sum_{i=1}^{N} P_{++}^{i} \frac{\Omega(x_3^\circ)}{\Omega(x_3^i)} = \frac{1}{N} \sum_i \tilde{P}_{++}^{i}$$

$$\langle \tilde{P}_{--} \rangle = \frac{1}{N} \sum_i P_{--}^{i} \frac{\Omega(x_3^\circ)}{\Omega(x_3^i)} = \frac{1}{N} \sum_i \tilde{P}_{--}^{i} \qquad \text{(IV.51)}$$

$$\langle \tilde{P}_{33} \rangle = \frac{1}{N} \sum_i P_{33}^{i} \frac{\Omega(x_3^i)}{\Omega(x_3^\circ)} = \frac{1}{N} \sum_i \tilde{P}_{33}^{i}$$

are shown in Figure I.8.

The consistency relation $D_1 = 0$ states that the ratio between the horizontal and vertical kinetic energy is a given function of frequency

$$\frac{P_{++} + P_{--}}{P_{00}} = \frac{\omega^2 + f^2}{\omega^2} \Omega^2 \qquad \text{(IV.52)}$$

This ratio becomes zero for buoyancy oscillations ($\omega = N$) and infinite for inertial oscillations ($\omega = f$). The relation (IV.52) may also be written

$$\frac{\tilde{P}_{++}^{i} + \tilde{P}_{--}^{i}}{\omega^2 + f^2} \frac{1}{\Omega(x_3^\circ)} = \Omega(x_3^\circ) \tilde{P}_{33}^{i} \qquad \text{(IV.53)}$$

The mean values of both terms are shown in Figure IV.10. They are normalized by the factor $C^2(\omega) E^{GM}(\omega)$ where $C(\omega)$ is the normalization constant (II.38) and $E^{GM}(\omega)$ the Garrett and Munk frequency spectrum. For most frequencies the observed horizontal kinetic energy spectrum is significantly larger than the horizontal kinetic energy spectrum calculated from the observed displacement spectrum using internal wave theory. We will refer to this discrepancy as the energy up-current disparity.

The consistency relation $D_2 = 0$ compares the anticlockwise, P_{++}, and the clockwise, P_{--}, rotating part of the motion

$$\frac{P_{++}}{P_{--}} = \frac{(\omega - f)^2}{(\omega + f)^2} \qquad \text{(IV.54)}$$

This ratio should become zero for inertial oscillations. The relations $D_2 = 0$ can also be written

$$\frac{2}{(\omega-f)^2}\frac{1}{\Omega(x_3^\circ)}\tilde{P}^i_{++} = \frac{2}{(\omega+f)^2}\frac{1}{\Omega(x_3^\circ)}\tilde{P}^i_{--} \qquad \text{(IV.55)}$$

The normalized mean values of both terms are shown in Figure IV.11. Except for high frequencies the agreement is rather poor. Figure IV.11 also shows the normalized mean value of $\tilde{P}^i_{33}\,\Omega(x_3^\circ)$ which ought to be equal to the two other curves because of $D_1 = 0$.

The discrepancy between P_{++} and P_{--} at the inertial frequency f is not real but must be ascribed to the finite frequency resolution of the estimated spectra. Our estimate $P_{\nu\nu}(\omega)$ is the average of the true spectrum over a finite frequency interval. If the true clockwise spectrum, P_{--}, behaves smoothly at ω the observed value of the anticlockwise spectrum is approximately

$$P_{++}(\omega) = P_{--}(\omega)\int_{\omega-\Delta}^{\omega+\Delta} d\omega' \left(\frac{\omega'-f}{\omega'+f}\right)^2 \qquad \text{(IV.56)}$$

which yields $P_{++}/P_{--} = \frac{1}{12}\Delta^2/f^2$ at $\omega = f$. At low frequencies the resolution is $\Delta = f/3$ and we find $P_{++}/P_{--} = 10^{-2}$ as observed (Figure I.8).

The consistency relations $D_3 = 0$ and $D_4 = 0$ can be expressed in terms of coherences and phases as

$$\gamma_{+o} = \gamma_{-o} \qquad \text{(IV.57)}$$

$$\phi_{+o} = -\phi_{-o} \qquad \text{(IV.58)}$$

Mean values of γ_{+o} and γ_{-o} were shown in Fig. IV.9. There is no strong violation of the relation (IV.57).

Fir $i \neq j$ the consistency relations take the form

$$\gamma^{ij}_{++} = \gamma^{ij}_{oo}, \qquad \phi^{ij}_{++} = \phi^{ij}_{oo} \qquad \text{(IV.59)}$$

$$\gamma^{ij}_{--} = \gamma^{ij}_{oo}, \qquad \phi^{ij}_{--} = \phi^{ij}_{oo} \qquad \text{(IV.60)}$$

$$\gamma^{ij}_{+o} = \gamma^{ij}_{o-}, \qquad \phi^{ij}_{+o} = \phi^{ij}_{o-} \qquad \text{(IV.61)}$$

$$\gamma^{ij}_{-o} = \gamma^{ij}_{o+}, \qquad \phi^{ij}_{-o} = \phi^{ij}_{o+} \qquad \text{(IV.62)}$$

when expressed in terms of coherences and phases. In deriving (IV.59)...(IV.62)

the WKB scaling relations have been used. The coherences γ_{--}^{ij} and γ_{oo}^{ij} are shown for two instrument separations in Figure V.22. A large discrepancy is observed which will be referred to as the coherence up-current disparity (cf. Briscoe 1975).

All these discrepancies contribute to the violation of the consistency relations shown in Figure IV.9. The discrepancies are most pronounced for low and high frequencies, less pronounced but significnat for medium frequencies.

To what degree the discrepancy at high frequencies can be removed by choosing standing modes instead of propagating waves will be discussed in the next section.

IV.7.2 Standing modes

The consistency relations for standing modes differ from those for propagating waves. This is only due to the fact that up- and downward propagating waves are assumed to be uncorrelated for propagating waves whereas up- and downward propagating waves have a fixed phase and amplitude relation for standing modes. The fact that there exists a continuum of solutions for propagating waves whereas there only exists a discrete set of solutions for standing modes and the fact that the vertical eigenfunctions are appoximated by WKB solutions in case of propagating waves do not affect the structure of the consistency relations.

For slant separations we only find (cf. Table II.3)

$$\tilde{D}_2^{ij} = (\omega+f)^2 A_{++}^{ij} - (\omega-f)^2 A_{--}^{ij} = 0 \qquad \text{(IV.63)}$$

as a consistency relation for standing modes. This relation is identical to the relations $D_2^{ij} = 0$ for propagating waves.

For purely horizontal separation we additionally find

$$\tilde{D}_3^{ij} = (\omega+f) A_{+o}^{ij} + (\omega-f) A_{o-}^{ij} = 0 \qquad \text{(IV.64)}$$

$$\tilde{D}_4^{ij} = (\omega-f) A_{-o}^{ij} + (\omega+f) A_{o+}^{ij} = 0 \qquad \text{(IV.65)}$$

which differ from the corresponding consistency relations $D_3^{ij} = 0$ and $D_4^{ij} = 0$ for propagating waves and provide a tool to discriminate between standing and propagating wave. In terms of coherences and phases the relations (IV.64) and (IV.65) take the form

$$\gamma_{+o}^{ij} = \gamma_{o-}^{ij} \quad , \quad \phi_{+o}^{ij} = \phi_{o-}^{ij} + \pi \qquad \text{(IV.66)}$$

$$\gamma_{-o}^{ij} = \gamma_{o+}^{ij} \quad , \quad \phi_{-o}^{ij} = \phi_{o+}^{ij} + \pi \qquad \text{(IV.67)}$$

The phase relations differ from the phase relations

$$\phi_{+o}^{ij} = \phi_{o-}^{ij} \quad , \quad \phi_{-o}^{ij} = \phi_{o+}^{ij} \qquad \text{(IV.68)}$$

for propagating modes.

In order to discriminate between standing and propagating modes one has to compare the phases ϕ_{+o}^{ij} and ϕ_{-o}^{ij} with the phases ϕ_{o-}^{ij} and ϕ_{o+}^{ij} for horizontal separations. They are equal for propagating waves. The differ by 180° for standing modes. However, if the corresponding coherences are zero the phases are not well-defined and one cannot discriminate between standing and propagating modes. A vertically symmetric field of propagating waves, for example, satisfies $\gamma_{\nu o} = 0$ ($\nu = +,-$) and $\gamma_{o\nu} = 0$ ($\nu = +,-$) for horizontal separations. A symmetric field of propagating waves hence satisfies the consistency relations $\tilde{D}_3^{ij} = 0$ and $\tilde{D}_4^{ij} = 0$ for standing modes. We cannot discriminate between a symmetric field of propagating waves and a field of standing modes.

For i = j the phase relations between u_+, u_- and ζ (= down!) take the form

$$\phi_{+\zeta} + \phi_{-\zeta} = 0 \qquad \text{for standing modes} \qquad \text{(IV.69)}$$

and

$$\phi_{+\zeta} + \phi_{-\zeta} = \pi \qquad \text{for propagating modes} \qquad \text{(IV.70)}$$

Figure IV.12 shows the phases of the mean scaled spectra $\langle \tilde{A}_{+\zeta} \rangle$ and $\langle \tilde{A}_{-\zeta} \rangle$, i.e.

$$\phi_{\nu\zeta} = \arctan \left(\langle \tilde{Q}_{\nu\zeta} \rangle , \langle \tilde{P}_{\nu\zeta} \rangle \right) \quad , \nu = +,- \qquad \text{(IV.71)}$$

At low frequencies where at least $\gamma_{-\zeta}$ can be said to differ significantly from zero (cf. Figure IV.8) we rather accept $\phi_{+\zeta} + \phi_{-\zeta} = \pi$, i.e. propagating waves. At medium frequencies where the phases are not well-defined the result is ambiguous. At high frequencies standing modes, i.e. $\phi_{+\zeta} = -\phi_{-\zeta}$ seems to be preferable.

The same picture emerges when the consistency relations $\tilde{D}_3^{ij}$, $\tilde{D}_4^{ij} = 0$ and D_3^{ij}, $D_4^{ij} = 0$ are tested for zero and horizontal separations. The result is shown in Figure IV.13. At tidal frequency standing modes must be rejected. At the other low and at medium frequencies both models are consistent. At high frequencies propagating modes must be rejected.

If all the consistency relations $\tilde{D}_2^{ij}$, $\tilde{D}_3^{ij}$, $\tilde{D}_4^{ij} = 0$ for standing modes are tested the above result is not significantly changed (Figure IV.14). At low frequencies, i.e. at inertial and tidal frequencies, standing modes must clearly be rejected. At the other frequencies standing modes seem to provide a consistent description of the data. Also shown on Figure IV.14 are the corresponding tests D_2^{ij}, D_3^{ij}, $D_4^{ij} = 0$ for propagating waves which indicate that propagating waves seem to be less appropriate at high frequencies. This presumably points to a genuine physical effect, namely that the waves become phase-locked when approaching the turning point.

The test results for an isotropic field of standing modes are shown on Figure IV.15. This model must also be rejected at high frequencies where, as we already know, the field is slightly but significantly anisotropic.

At high frequencies a standing mode model seems to be more appropriate than a propagating wave model. However, there exist more tests for propagating waves than for standing modes. The consistency relations D_3^{ij}, $D_4^{ij} = 0$ also hold for slant separations. Additionally, there exists the test $D_1^{ij} = 0$, which has no counterpart. Evaluating D_1 for standing modes we find

$$D_1 = \tilde{P}_{++} + \tilde{P}_{--} - \frac{\omega^2+f^2}{\omega^2}\,\frac{N^2-\omega^2}{\omega^2-f^2}\,\tilde{P}_{00} \qquad \text{(IV.72)}$$

$$= \frac{\omega^2+f^2}{\omega^2}\int d^2\alpha\, \tilde{E}(\omega,\underset{\sim}{\alpha})\left\{\frac{1}{\alpha^2}(\partial_3\tilde{\eta}_0)^2 - \frac{N^2-\omega^2}{\omega^2-f^2}\,\tilde{\eta}_0^2\right\}$$

where $\tilde{\eta}_0$ denotes the vertical eigenfunction. The measured spectra $A_{\nu\mu}$ have been obtained by averaging in time. Because of ergodicity this also corresponds to a certain averaging in space. The vertical average of the first term in the parenthesis becomes

$$\frac{1}{\alpha^2}\int_{x_3^l}^{x_3^u} dx_3'\,(\partial_3\tilde{\eta}_0)^2 = -\frac{1}{\alpha^2}\int_{x_3^l}^{x_3^u} dx_3'\,\tilde{\eta}_0\,\partial_3\partial_3\tilde{\eta}_0 + \frac{1}{\alpha^2}\,\tilde{\eta}_0\,\partial_3\tilde{\eta}_0\Big|_{x_3^l}^{x_3^u} \qquad \text{(IV.73)}$$

where we have integrated by parts. The first term becomes equal to the vertical average of $(N^2-\omega^2)(\omega^2-f^2)^{-1}\tilde{\eta}_0^2$ if the eigenequation $\partial_3\partial_3\tilde{\eta}_0 + \alpha^2(N^2-\omega^2)(\omega^2-f^2)^{-1}\tilde{\eta}_0 = 0$ for $\tilde{\eta}_0$ is used (cf. II.94). The second term vanishes if the integration is carried out between the nodes of the eigenfunctions $\tilde{\eta}_0$. If only high mode numbers are excited we hence expect standing modes to satisfy the relation $D_1 = 0$ as well. The same argument holds if many modes are excited since an average over many modes

is equivalent to a vertical average. A standing mode field containing many modes or high modes hence does not remove the observed energy up-current disparity (see Figure IV.10). Similar arguments are true if the test $D_1^{ij} = 0$ is considered for $i \neq j$. A standing mode field consisting of many modes or high modes also does not remove the observed coherence up-current disparity (see Figure V.22). A standing mode field consisting of a few low modes might produce systematic violations of the relation $D_1^{ij} = 0$ but is inconsistent with the observed coherence scale.

Turning to the question to what degree the violation of the consistency relations for propagating waves at high frequencies (Figure IV.9) can be removed by standing modes the above discussion suggests that standing modes though preferable cannot account for all of the observed discrepancy shown in Figure IV.9. For the following we disregard the standing mode model and further investigate the conceptually simpler propagating wave model, but being aware of the fact that part of the high frequency discrepancy can be ascribed to the proximity of the turning point and its phase-locking effect.

IV.7.3 Combination of standing and propagating waves

A more sophisticated model would represent the observed spectra by a combination of standing and propagating waves. This model is described by the consistency relation $D_2^{ij} = 0$ which is valid for both standing and propagating waves. Although acceptable for medium and high frequencies (Figure IV.16) the above discussion implies that this model also cannot account for the observed up-current disparities. The test however implies that random internal waves - propagating, standing or a combination of both - do not represent a consistent description at tidal frequency. Note that the estimate of $\tilde{4}^2$ at inertial frequency is not reliable because of finite frequency resolution. The argument of section IV.7.1 suggests that the observed fluctuations at inertial frequency are consistent with free propagating internal waves.

Result. The consistency relations for propagating waves are violated for all frequencies. The violation is most pronounced at low (especially at tidal frequency) and at high frequencies, less pronounced but significant for medium frequencies. The violation at high frequencies is partly due to the proximity of the turning point and its phase-locking effect. The consistency relations for standing modes are violated for low frequencies, but are satisfied for medium and high frequencies. Discussion of standing mode solutions, however, suggests that standing modes - though preferable at high frequencies - cannot remove the energy and coherence up-current disparity and neither provide a consistent description of the observed cross spectra.

IV.8. Up-current disparity

The preceeding discussion clearly demonstrates that a model of propagating internal waves does not represent a consistent description of the observed fluctuations. Here we ask which tests mainly contribute to this discrepancy.

Figure IV.16 already showed that all of the observed inconsistencies cannot be ascribed to the violation of the relation $D_2^{ij} = 0$. This relation only involves the horizontal current velocity. It is significantly violated only for tidal frequency. The violation at inertial frequency is not real and must be ascribed to the finite frequency resolution. The discrepancies of the propagating wave model at medium and high frequencies can hence not be ascribed to inconsistencies within the horizontal velocity field.

The consistency relations $D_3^{ij}, D_4^{ij} = 0$ only involve the cross spectra $R_{\nu 0}^{ij}$ and $R_{0\nu}^{ij}$ ($\nu = +,-$). The result of the corresponding tests is shown in Figure IV.17. These relations are significantly violated at high frequencies only. This violation is presumably due to the proximity of the turning point as might be inferred from the drop at the 30th frequency point where the levels 8, 10 and 14 are excluded.

The most significant contribution to the violation of the consistency relations for propagating waves arises from the test $D_1^{ij} = 0$ which compares the displacement field with the horizontal current field (Figure IV.18). Significant violations are obvious for all frequency bands, especially for the high frequency band. The violation of the relation $D_1^{ij} = 0$ formally expresses the up-current disparity of the observed fluctuations since both the energy up-current disparity (Figure IV.10) and the coherence up-current disparity (Figure V.22) contribute to the violation of $D_1^{ij} = 0$. Also shown in Figure IV.18 is the violation of the relation $D_1^{ij} = 0$ for zero separation, i.e. the energy up-current disparity. Generally, the main contribution to the violation of the relation $D_1^{ij} = 0$ does not seem to arise from the violation of the energy relation $D_1 = 0$ although some spikes hinder a uniformly valid statement. However, except for the low frequencies, the relation $D_1^{ij} = 0$ is much better satisfied for horizontally separated instruments than for vertically separated instruments (Figure IV.10).

A more quantitative picture emerges when the energies (Figure IV.10) and the coherences (Figure V.22) are considered. Inspection of both these figures reveals that the energy up-current disparity

$$\delta_E = \frac{P_{++} + P_{--} - \frac{\omega^2 + f^2}{\omega^2} \frac{N^2 - \omega^2}{\omega^2 - f^2} P_{00}}{P_{++} + P_{--}} \qquad \text{(IV.74)}$$

is generally of the order of 0.1 but significantly larger for specific frequency points. The coherence up-current disparity

$$\delta_\gamma = (\gamma_{00}^{ij} - \gamma_{--}^{ij})(\gamma_{00}^{ij})^{-1} \qquad \text{(IV.75)}$$

is generally of the order of 0.4 for vertically separated instruments and of the order of 0.1 for horizontally separated instruments. This discrepancy between the energy up-current disparity and the coherence up-current disparity for vertical separations will be referred to as the disparity between the energy and coherence disparity.

Result. The violation of the consistency relations for propagating modes is mostly due to the up-current disparity. Inconsistencies within the horizontal velocity field are only apparent for the tidal frequency.

IV.9. Hybrid model classes

The observed fluctuations within the internal wave band cannot be described by a field consisting of internal waves only. Here we will investigate whether or not internal waves contaminated by non-internal wave fluctuations will represent a consistent model class. Such model classes will be named hybrid model classes. Before we investigate specific hybrid model classes, we briefly describe the general construction of the consistency relations for hybrid model classes.

IV.9.1 Consistency relations for hybrid model classes

Within a hybrid model class the observed fluctuations are represented by a linear combination

$$\hat{\underset{\sim}{y}} = \hat{\underset{\sim}{y}}^{(1)} + \hat{\underset{\sim}{y}}^{(2)} \qquad \text{(IV.76)}$$

where $\hat{\underset{\sim}{y}}^{(1)}$ represents the internal wave field and $\hat{\underset{\sim}{y}}^{(2)}$ the contaminating field. Suppose the consistency relations for the individual model classes $\hat{\underset{\sim}{y}}^{(1)}$ and $\hat{\underset{\sim}{y}}^{(2)}$ are known. The problem then reduces to the construction of the consistency relations for the hybrid model class $\hat{\underset{\sim}{y}}$.

Let the first model class be defined by the relations

$$L^{(1)}_{\alpha\ell}\,\hat{y}^{(1)}_{\ell} = 0 \quad , \alpha = 1, \ldots, R^{(1)} \qquad \text{(IV.77)}$$

The $R^{(1)}$ linear forms $L^{(1)}_{\alpha\ell}$ $(\alpha = 1, \ldots, R^{(1)})$ define a $R^{(1)}$-dimensional subspace

$$W^{(1)} = \{ L^{(1)}_{1\ell}, \ldots, L^{(1)}_{R^{(1)}\ell} \} \qquad \text{(IV.78)}$$

in the L-dimensional data space. All vectors $\hat{\underset{\sim}{y}}^{(1)}$ which satisfy $\underset{\sim}{L}^{(1)}\hat{\underset{\sim}{y}}^{(1)} = 0$ lie in a $(L - R^{(1)})$-dimensional hyperplane

$$W_{(1)}^{\perp} \perp W^{(1)} \tag{IV.79}$$

which is orthogonal to $W^{(1)}$. If the second model class is defined by

$$L_{\alpha\ell}^{(2)} \hat{y}_{\ell}^{(2)} = 0 \quad , \quad \alpha = 1, \ldots, R^{(2)} \tag{IV.80}$$

we can construct in the same manner

$$W^{(2)} = \{ L_{1\ell}^{(2)}, \ldots, L_{R^{(2)}\ell}^{(2)} \} \tag{IV.81}$$

and

$$W_{(2)}^{\perp} \perp W^{(2)} \tag{IV.82}$$

The vectors $\hat{\underset{\sim}{y}} = \hat{\underset{\sim}{y}}^{(1)} + \hat{\underset{\sim}{y}}^{(2)}$ are elements of $W_{(1)}^{\perp} + W_{(2)}^{\perp}$ which is orthogonal to $W^{(1)} \cap W^{(2)}$

$$W_{(1)}^{\perp} + W_{(2)}^{\perp} = (W^{(1)} \cap W^{(2)})^{\perp} \tag{IV.83}$$

The consistency relations for the hybrid model class are hence given by the intersection of the consistency relations $\underset{\sim}{L}^{(1)} \hat{\underset{\sim}{y}}^{(1)} = 0$ and $\underset{\sim}{L}^{(2)} \hat{\underset{\sim}{y}}^{(2)} = 0$ for the individual model classes.

IV.9.2 Propagating waves contaminated by white noise

Let us first consider propagating waves which are contaminated by white noise. In this case $\underset{\sim}{L}^{(1)} \hat{\underset{\sim}{y}}^{(1)} = 0$ is given by the consistency relations $D_1^{ij}, D_2^{ij}, D_3^{ij}, D_4^{ij} = 0$. White noise will be defined by

$$\begin{aligned} i = j : \quad & A_{mn}^{ij} = \begin{cases} \text{arbitrary if } m = n \\ 0 \text{ otherwise} \end{cases} \\ i \neq j : \quad & A_{mn}^{ij} = 0 \end{aligned} \tag{IV.84}$$

or, in terms of coherences, by

$$\begin{aligned} i = j : \quad & \gamma_{mn}^{ij} = 0 \quad \text{if } m \neq n \\ i \neq j : \quad & \gamma_{mn}^{ij} = 0 \end{aligned} \tag{IV.85}$$

Hence we characterize white noise by arbitrary autospectra but vanishing cross spectra. Instrumental noise may for example be modelled in this way.

For the hybrid model class we find the consistency relations

$$\begin{aligned} i = j &: \quad D_3^{ij}\,,\ D_4^{ij} = 0 \\ i \neq j &: \quad D_1^{ij}\,,\ D_2^{ij}\,,\ D_3^{ij}\,,\ D_4^{ij} = 0 \end{aligned} \tag{IV.86}$$

Because of the addition of white noise the consistency relations $D_1^{ij},\ D_2^{ij} = 0$ need not to be satisfied for zero separation. The test results are shown in Figure IV.20. No significant improvement can be observed. This is understandable since white noise cannot remove the disparity between the energy and coherence up-current disparity. If there exist an energy disparity δ_E we find for the coherence disparity.

$$\begin{aligned} \delta_c = \frac{\gamma_{oo}^{ij} - \gamma_{--}^{ij}}{\gamma_{oo}^{ij}} &= 1 - \frac{|A_{--}^{ij}|}{|A_{oo}^{ij}|} \cdot \left(\frac{A_{oo}^{ii}\ A_{oo}^{jj}}{A_{--}^{ii}\ A_{--}^{jj}} \right)^{1/2} \\ &= 1 - (1 - \delta_E) = \delta_E \end{aligned} \tag{IV.87}$$

inconsistent with the data.

IV.9.3 Propagating waves contaminated by finestructure

A similar result is obtained if we add current and temperature finestructure to the internal wave field. In section IV.3.3 we have characterized finestructure as a field for which all cross spectra from vertically separated instruments vanish whereas all other cross spectra and the autospectra can take arbitrary values. Here we use a more restrictive model of finestructure which is defined by the relations

$$\begin{aligned} i = j: &\quad A_{mn}^{ij} = \begin{cases} \text{arbitrary if } m = n \\ 0 \text{ otherwise} \end{cases} \\ i \neq j\,,\ \Delta z = 0: &\quad A_{mn}^{ij} = \begin{cases} \text{arbitrary if } m = n \\ 0 \text{ otherwise} \end{cases} \\ i \neq j\,,\ \Delta z \neq 0: &\quad A_{mn}^{ij} = 0 \end{aligned} \tag{IV.88}$$

or, in terms of coherences, by

$$
\begin{aligned}
i = j &: \quad \gamma_{mn}^{ij} = 0 \quad \text{if } m \neq n \\
i \neq j,\ \Delta z = 0 &: \quad \gamma_{mn}^{ij} = 0 \quad \text{if } m = n \\
i \neq j,\ \Delta z \neq 0 &: \quad \gamma_{mn}^{ij} = 0
\end{aligned}
\tag{IV.89}
$$

Again the vertical correlation length of the finestructure field is assumed to be smaller than the smallest vertical separation of the IWEX array. However, we additionally assume that the different velocity components are uncorrelated. The autospectra P_{11}, P_{22}, P_{33} and the cross spectra $R_{11}^{ij}, R_{22}^{ij}, R_{33}^{ij}$ for horizontal separations can take arbitrary values.

The consistency relations for the hybrid model are given in this case by

$$
\begin{aligned}
i = j &: \quad D_3^{ij}, D_4^{ij} = 0 \\
i \neq j,\ \Delta z = 0 &: \quad D_3^{ij}, D_4^{ij} = 0 \\
i \neq j,\ \Delta z \neq 0 &: \quad D_1^{ij}, D_2^{ij}, D_3^{ij}, D_4^{ij} = 0
\end{aligned}
\tag{IV.90}
$$

Because of the addition of finestructure the consistency relations $D_1^{ij}, D_2^{ij} = 0$ need not to be satisfied for zero and for horizontal separations. The corresponding test is shown in Figure IV.21. Again, no significant improvement occurs, since finestructure does not remove the disparity between the energy disparity and the coherence disparity for vertical separations.

IV.9.4 Propagating waves contaminated by coherent noise

Coherent noise, characterized by $\gamma_{mn}^{ij} \neq 0$, does not fit into our framework of consistency relations since it is characterized by nonlinear relationships among the cross spectra. Nevertheless, an incomplete set of consistency relations may be obtained by physical reasoning.

When interpreting the IWEX time series one main problem is whether or not

$$
u_3(t) = -\dot{\bar{T}}(t) \left(\frac{\overline{\Delta T}}{\Delta z} \right)^{-1} \tag{IV.91}
$$

derived from

$$\partial_t T + u_3 \partial_3 \overline{T} = 0 \qquad \text{(IV.92)}$$

yields a sufficiently correct estimate of the vertical velocity or displacement field. Two effects might contribute to the invalidity of the above estimate: (i) the terms neglected in the equation of motion for the temperature field might not be negligible, (ii) our estimate of the mean temperature gradient (= temperature difference over 1.74 m averaged over 42 days) might not be appropriate to our WKBJ theory of the inernal wave field. The last argument also applies to the Brunt Väisälä frequency N and to the local vertical wavenumber which depends on N. Suppose the true values of the mean temperature gradient and of the local vertical wavenumber are given by

$$\partial_3 \hat{\overline{T}} = (1 + \delta_1)\, \partial_3 \overline{T} \qquad \text{(IV.93)}$$

and by

$$\hat{\beta} = \left(\frac{\hat{N}^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2} \alpha = (1 + \delta_2)\, \beta \qquad \text{(IV.94)}$$

Here $\partial_3 \overline{T}$ and β denote the values used throughout the calculations. The "true" cross spectra are then given by

$$\begin{aligned}
\hat{A}^{ij}_{\nu\mu} &= A^{ij}_{\nu\mu} && , \nu = +,- \\
\hat{A}^{ij}_{\nu 0} &= \frac{1}{1+\delta_1} A^{ij}_{\nu 0} \approx (1-\delta_1) A^{ij}_{\nu 0} && , \nu = +,- \\
\hat{A}^{ij}_{0\nu} &= \frac{1}{1+\delta_1} A^{ij}_{0\nu} \approx (1-\delta_1) A^{ij}_{0\nu} && , \nu = +,- \\
\hat{A}^{ij}_{00} &= \frac{1}{(1+\delta_1)^2} A^{ij}_{00} \approx (1-2\delta_1) A^{ij}_{00}
\end{aligned} \qquad \text{(IV.95)}$$

and the "true" consistency relations $\hat{D}^{ij}_1, \hat{D}^{ij}_2, \hat{D}^{ij}_3, \hat{D}^{ij}_4 = 0$ by

$$\hat{D}_1^{ij} = A_{++}^{ij} + A_{--}^{ij} - \frac{\omega^2+f^2}{\omega^2}\,\frac{\beta^i}{\alpha}\,\frac{\beta^j}{\alpha}\,A_{oo}^{ij}\,(1+2\delta_2-2\delta_1) = 0$$

$$\hat{D}_2^{ij} = (\omega+f)^2 A_{++}^{ij} - (\omega-f)^2 A_{--}^{ij} = 0$$

(IV.96)

$$\hat{D}_3^{ij} = (1+\delta_2)(1-\delta_1)\left\{\frac{\beta^j}{\alpha}(\omega+f)A_{+o}^{ij} - \frac{\beta^i}{\alpha}(\omega-f)A_{o-}^{ij}\right\} = 0$$

$$\hat{D}_4^{ij} = (1+\delta_2)(1-\delta_1)\left\{\frac{\beta^j}{\alpha}(\omega-f)A_{-o}^{ij} - \frac{\beta^i}{\alpha}(\omega+f)A_{o+}^{ij}\right\} = 0$$

The violation of the consistency relations $D_1^{ij} = 0$ might hence be due to our wrong choice of the mean temperature gradient and of the Brunt Väisälä frequency. Taking this possibility into account only the relations D_2^{ij}, D_3^{ij}, $D_4^{ij} = 0$ need to be satisfied. They are clearly much better satisfied (Figure IV.22) than the whole set of consistency relations for propagating waves since the relation $D_1^{ij} = 0$ is violated most of all.

Interpreting the above modification of the consistency relations as an additional temperature signal we indeed find that it represents a coherent contamination, as coherent as the temperature signal itself. The consistency relations D_2^{ij}, D_3^{ij}, $D_4^{ij} = 0$ are incomplete in the sense that additional relations must hold among the cross spectra since only 2 free adjustable parameter δ_1 and δ_2 are introduced in order to satisfy the 81 consistency relations $D_1^{ij} = 0$. This figure improves when the parameters are allowed to depend on the depth.

IV.9.5 The IWEX model class

We have seen that the addition of coherent noise provides a considerable improvement of the model class whereas the addition of finestructure only provides a small but significant improvement. As a sufficiently consistent model class for the IWEX data set we choose the model class where both finestructure and coherent noise are used as contaminating agents. This model class is characterized by the consistency relations

$$i = j : \qquad D_3^{ij},\ D_4^{ij} = 0$$

(IV.97)

$$i \neq j,\ \Delta z = 0 : \quad D_3^{ij},\ D_4^{ij} = 0$$

$$i \neq j,\ \Delta z \neq 0 : \quad D_2^{ij},\ D_3^{ij},\ D_4^{ij} = 0$$

The corresponding tests are shown in the final figure of this part, Figure IV.23. This model class seems to provide a consistent description for most frequencies. Slight inconsistencies may occur for the tidal frequency and for the high frequency band. The drop at the 30th frequency point indicates that the slight inconsistencies at high frequencies can be ascribed to the proximity of the turning point.

Result. The consistency relations for various hybrid model classes were tested. Contaminating the internal wave field by white noise or finestructure only provides a small improvement since both these model classes do not remove the disparity between the energy and coherence up-current disparity. Contaminating the internal wave field by coherent noise provides a considerable improvement. The model class used for the following inverse analysis accounts for both contamination by finestructure and by coherent noise. This model class provides a consistent description for most frequencies. Slight inconsistencies may occur at tidal frequency and at high frequencies.

IV.10. Summary

The results of the consistency relations are summarized in Table IV.2. We distinguish 3 frequency bands:

(i) the low frequency band from the 3rd to about the 9th frequency point dominated by fluctuations with inertial and tidal frequency

(ii) the medium frequency band ranging from about the 10th to the 20th frequency point and

(iii) the high frequency band ranging from about the 21st to the 30th frequency point.

Within each frequency band the tested relations are called

(i) strongly violated when the relations are violated for each frequency point

(ii) slightly violated when the calculated values of the statistic imply acceptance for each frequency point but do not scatter randomly about their expectation value

(iii) satisfied when both the single values of the statistic and its distribution imply acceptance.

The question marks indicate that the results at inertial frequency are not reliable because of finite frequency resolution.

V. PARAMETERS OF THE CONTAMINATED INTERNAL WAVE SPECTRUM

V.1. Internal wave energy spectrum

When interpreting deep-sea current measurements it is generally assumed that motions in the internal wave frequency band are primarily governed by linear internal wave dynamics. For a pure linear random internal wave field the cross-spectral matrix $A_{mn}^{ij}(\omega)$ can be expressed rigorously in terms of the complete energy spectrum of the wave field which describes the distribution of wave energy in wavenumber-frequency space.

The definition of the energy spectrum depends on the representation and the statistical conception about the wave components constituting the wave field. In a modal representation vertically up- and downward propagating wave components (with the same horizontal wavenumber) have a fixed phase and amplitude relation thus forming a vertically standing mode. The WKB representation allows for independent phases between these components so that the wave field is a superposition of wave groups which propagate horizontally as well as vertically. The statistical models appropriate to these contrary conceptions of the wave field naturally differ: in the WKB picture up- and downward propagating waves are statistically independent while in the mode picture they are deterministically related. Which of these descriptions is more appropriate to oceanic conditions must be determined from measurements.

The analysis of this part will be carried out within a WKB representation. In the WKB picture the cross-spectral matrix of a pure linear internal wave field takes the form

$$A_{mn}^{ij}(\omega) = \sum_{\sigma} \int d\alpha\, d\varphi\; E^{\sigma}(\omega,\alpha,\varphi)\; U_{mn}^{ij}(\varphi,\omega,\sigma)\; e^{-i(\underset{\sim}{\alpha}\cdot\underset{\sim}{r}_{ij} + \sigma\Theta_{ij})} \qquad (V.1)$$

where α and φ are polar coordinates of the horizontal wavenumber, $\underset{\sim}{r}_{ij}$ is the horizontal distance vector between position i and j (pointing from i to j), and Θ_{ij} is the vertical phase difference given by

$$\Theta_{ij} = \int_{z_i}^{z_j} dz\; \beta(z) \qquad (V.2)$$

where

$$\beta(z) = \alpha \left[\frac{N^2(z) - \omega^2}{\omega^2 - f^2} \right]^{1/2} \qquad (V.3)$$

is the modulus of the vertical wavenumber. The matrix U^{ij}_{mn} is constructed from the amplitude factors of the linear wave field (cf. Schott and Willebrand 1973, Müller and Siedler 1976) and is given in part II where also the modal counterpart of (V.1) is derived.

The energy spectrum $E^{\sigma}(\omega,\alpha,\varphi)$ of a WKB field consists of two branches, characterized by $\sigma = \pm$ which is the sign of the vertical wavenumber. E^{-} is the spectrum of upward propagating energy, E^{+} of downward propagating energy. Alternatively we will use

$$E^{\uparrow}(\omega,\alpha,\varphi) = E^{-}(\omega,\alpha,\varphi)$$
$$E^{\downarrow}(\omega,\alpha,\varphi) = E^{+}(\omega,\alpha,\varphi) \qquad (V.4)$$

which indicates the direction of energy propagation. If

$$E^{\uparrow}(\omega,\alpha,\varphi) = E^{\downarrow}(\omega,\alpha,\varphi) \qquad (V.5)$$

we will call the wave field (vertically) symmetric.

It is convenient to factorize the spectrum in the form

$$E^{\sigma}(\omega,\alpha,\varphi) = E^{\sigma}(\omega)\, A^{\sigma}(\alpha;\omega)\, S^{\sigma}(\varphi;\alpha,\omega) \qquad (V.6)$$

and discuss successively the frequency distribution $E^{\sigma}(\omega)$ of the energy, the wavenumber distribution $A^{\sigma}(\alpha;\omega)$ at each frequency, and the directional distribution $S^{\sigma}(\varphi;\alpha,\omega)$ at each frequency and wavenumber. These distribution functions are conveniently normalized according to

$$\int_{f}^{N} d\omega\, E^{\sigma}(\omega) = E^{\sigma}_{o}$$
$$\int_{0}^{\infty} d\alpha\, A^{\sigma}(\alpha;\omega) = 1 \qquad (V.7)$$
$$\int_{0}^{2\pi} d\varphi\, S^{\sigma}(\varphi;\alpha,\omega) = 1$$

The total energy per unit surface area is then given by

$$E_o = \frac{1}{2} \int_{-h_o}^{0} dz \, \langle u_j u_j + N^2 \zeta^2 \rangle$$

$$= \sum_{\sigma} \int_{f}^{N} d\omega \int_{0}^{\infty} d\alpha \int_{0}^{2\pi} d\varphi \; E^{\sigma}(\omega, \alpha, \varphi) \qquad (V.8)$$

$$= E_o^{+} + E_o^{-}$$

The vertical distribution of wave energy can be derived from WKB theory. The spectrum of energy per unit volume is given by (cf. part II)

$$E^{\sigma}(\omega, \alpha, \varphi, z) = \frac{E^{\sigma}(\omega, z)}{E^{\sigma}(\omega)} \; E^{\sigma}(\omega, \alpha, \varphi) \qquad (V.9)$$

where

$$E^{\sigma}(\omega, z) = E^{\sigma}(\omega) \frac{N^2(z) - f^2}{(N^2(z) - \omega^2)^{1/2}} \left\{ \int dz' \frac{N^2(z') - f^2}{(N^2(z') - \omega^2)^{1/2}} \right\}^{-1} \qquad (V.10)$$

The integration must be performed between the turning points. For $\omega \ll N(z)$ the energy varies with depth approximately as the buoyancy frequency.

The wave field will be called (horizontally) isotropic if

$$S^{\sigma}(\varphi, \alpha, \omega) = \frac{1}{2\pi} \qquad (V.11)$$

V.2. The GM-model

The most extensive and far-reaching attempt to reconstruct the energy spectrum from ocean current and displacement measurements has been undertaken by Garrett and Munk (1972 , 1975). Based on internal wave spectral data made at different locations and times and by different instruments, they proposed a spectral model with the following features

- horizontal isotropy,
- vertical symmetry (which originates from their basic modal approach),
- horizontal scales of the order of some km's,

- vertical scales down to 100 m,
- a -2 slope in the frequency domain of the horizontal kinetic energy spectrum.

These features and in particular the form of the wavenumber distribution function $A^{\sigma}(\alpha;\omega)$ are still subject of investigations. Also, the problem of contamination, i.e. to what degree the observed fluctuations represent internal waves, is still unsolved.

Horizontal isotropy and vertical symmetry implies

$$E^{\uparrow}(\alpha,\varphi,\omega) = E^{\downarrow}(\alpha,\varphi,\omega) = \frac{1}{4\pi} E(\alpha,\omega) \qquad (V.12)$$

The spectrum $E(\alpha,\omega)$ may be factorized

$$E(\alpha,\omega) = E_0\, B(\omega)\, A(\alpha,\omega) \qquad (V.13)$$

For the frequency distribution function (normalized to one) GM chose

$$B^{GM}(\omega) = \frac{2}{\pi}\frac{f}{\omega}\left(\omega^2 - f^2\right)^{-1/2} \qquad (V.14)$$

which is the well-established ω^{-2} law away from f, modified by an integrable cusp at f to represent the inertial peak.

The wavenumber distribution function of GM is confined to the self-similarity class

$$A^{GM}(\alpha,\omega) = A\left(\frac{\alpha}{\alpha_*(\omega)}\right) / \alpha_*(\omega) \qquad (V.15)$$

which is characterized by a 'bandwidth' $\alpha_*(\omega)$. For $A(\lambda)$ GM took a simple top-hat distribution

$$A(\lambda) = \begin{cases} 1 & \text{for } 0 \le \lambda \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(GM 72)} \qquad (V.16)$$

in their 1972 model, and

$$A(\lambda) = (t-1)(1+\lambda)^{-t} \qquad \text{(GM 75)} \qquad (V.17)$$

in their more sophisticated 1975 model. Cairns and Williams (1976) used a slightly different distribution

$$A(\lambda) = I(t,2)\,(1+\lambda^2)^{-t/2} \quad \text{(Cairns \& Williams)} \qquad \text{(V.18)}$$

The normalization constant $I(t,2)$ is given by (V.33).

For the 'bandwidth' $\alpha_*(\omega)$ GM chose

$$\alpha_*^{GM}(\omega) = j_* \frac{\pi}{bN_0}\,(\omega^2 - f^2)^{1/2} \qquad \text{(V.19)}$$

where bN_0 is the stratification parameter and j_* the mode number 'bandwidth'. This form of the bandwidth is based on the WKB-dispersion relation

$$\alpha^{(j)}(\omega) = \frac{j\pi\,(\omega^2 - f^2)^{1/2}}{\int_{-h}^{0} dz\,(N^2(z) - \omega^2)^{1/2}} \qquad \text{(V.20)}$$

For $\omega^2 \ll N^2$ and for an exponential profile of the buoyancy frequency, $N(z) = N_0 \exp\{z/b\}$ (V.20) reduces to (V.19) since

$$\int_{-h}^{0} dz\,(N^2(z) - \omega^2)^{1/2} \approx bN_0 \qquad \text{(V.21)}$$

if $b \ll h$. If $N_0 b$ is defined by (V.21) the relation (V.19) applies generally to smooth profiles of $N(x_3)$.

The box model GM 72 is described by two scale parameters: the energy level E_0 and the mode number j_*. The GM 75 model involves in addition to E_0 and j_* one shape parameter, the wavenumber slope t. For determining the values of these parameters and verifying the whole model, GM not only used moored data as provided by IWEX but all kinds of available data such as towed and dropped spectra. There are various experimental efforts to get accurate estimates of the parameters (in particular t and j_*). Recent values can be found in Table 1.

The investigations of Garrett and Munk were followed by various efforts which improved their model and pointed out its limitations and shortcomings (see various papers in JGR 80, 1975, Bell 1976; also IWEX was conceived for this purpose). The most obvious shortcoming of GM's approach is the fact that they had to use an inhomogeneous data set and therefore could not treat the contamination problem adequately. Many models of contamination have been developed and partly tested against data (e.g. Garrett and Munk 1971, Siedler 1974, Joyce 1974, McKean 1974). However, progress in the contamination problem can only be expected from a homogeneous data set such as provided by IWEX where horizontal currents, vertical displacements and temperature measurements can be studied simultaneously.

V.3. Parametrization of the internal wave spectrum

In the following sections we discuss the distribution of internal wave energy in wavenumber-frequency space as determined by applying inverse techniques (part III) to the mean data set (part I). It is found that this data set can not be explained by a pure internal wave field but that a considerable amount of contamination is present in the current and displacement records. The final model of the energy spectrum involves internal waves and three kinds of contaminations. This complex model will be presented step by step, always pointing out those features in the data which lead us to introduce corresponding features in the model. First we discuss the internal wave spectrum, then the contamination and finally the hybrid model which will be referred to as the IWEX model.

We want to recall that the parametrization of the data had to be done for each frequency separately so that a second parametrization step is necessary which treats the frequency behavior of the parameters obtained by the inverse technique.

V.3.1 Energy level and vertical symmetry

The IWEX-autospectra of horizontal current and vertical displacement show the features already observed in earlier experiments: peaks at the inertial and the tidal frequencies, followed by an almost -2 slope in the internal wave continuum. WKB-scaling works well (Briscoe 1975 and section IV.5).

We can estimate the frequency spectrum of the total energy per unit surface area

$$E^e(\omega) = E^{\uparrow}(\omega) + E^{\downarrow}(\omega) \tag{V.22}$$

either from the displacement or from current auto-spectra

$$P_{zz}(\omega,z) = \frac{1}{\omega^2}\,\frac{\omega^2-f^2}{N^2-f^2}\; C(\omega,z)\, E^e(\omega) \tag{V.23}$$

$$P_{11}(\omega,z) + P_{22}(\omega,z) = \frac{\omega^2+f^2}{\omega^2}\,\frac{N^2-\omega^2}{N^2-f^2}\; C(\omega,z)\, E^e(\omega) \tag{V.24}$$

Here

$$C(\omega,z) = \frac{N^2-f^2}{(N^2-\omega^2)^{1/2}}\left\{\int dz'\,\frac{N^2(z')-f^2}{(N^2(z')-\omega^2)^{1/2}}\right\}^{-1} \tag{V.25}$$

denotes the ratio between energy per unit volume and energy per unit surface area (cf. V.10). The integral must be performed between the turning points.

Fig. IV.10 displays the two estimates of $E^e(\omega)$ obtained from (V.23) and (V.24) ;scaling the auto-spectra by the corresponding factor on the right hand side and averaging over the entire array. The estimates $E^e(\omega)$ are normalized by GM's frequency distribution function $B^{GM}(\omega)$ (this will be done for all energy distributions in this report). Notice that the estimates differ with statistical significance at almost all frequencies. In the internal wave continuum (with the exception of the frequencies 10, 14 and 18) the energy in the horizontal currents systematically exceeds the one in the ups by about 10%.

Fig. IV.10 also demonstrates that the observed inertial peak (as the sum of the 3rd and 4th frequency) exceeds the one of GM's model significantly. The slope of $E^e(\omega)$ is about -2 and thus steeper than those of the observed auto-spectra which are close to -5/3. The difference comes about by the frequency dependence of $C(\omega, z)$. This quantity feels already at intermediate frequencies $\omega \ll N(z)$ ($\approx \omega_{20}$ in our case) the turning point (through the integral) and expresses the fact that with increasing frequency a lesser part of the water column can be filled with internal wave energy. We should mention that this 'turning point effect' is described excellantly by the WKB approximation though the singularity in the integral looks rather suspicious. The Airy function approximation leads with high accuracy to the same relations between the local autospectra and the energy spectrum $E^e(\omega)$ (Desaubies 1973). Our equations (V.10), (V.23), (V.24) are identical to Desaubies' equations (3.8), (3.9), (3.17).

We will see later that the estimation of $E^e(\omega)$ by means of (V.23) and (V.24) is rather poor since currents as well as ups are highly contaminated.

Auto-spectra involve only the even part of the energy spectrum. The decomposition into the up- and downward propagating part becomes possible by considering additionally the cross-spectra A^{ij}_{mm} and A^{ij}_{12} at nonzero vertical separation and A^{ij}_{m3} , m = 1,2 which depend on the even part and the odd part

$$E^e(\alpha, \varphi, \omega) = E^{\uparrow}(\alpha, \varphi, \omega) + E^{\downarrow}(\alpha, \varphi, \omega)$$
$$E^o(\alpha, \varphi, \omega) = E^{\uparrow}(\alpha, \varphi, \omega) - E^{\downarrow}(\alpha, \varphi, \omega) \qquad (V.26)$$

of the energy spectrum. For a horizontally isotropic wave field (which is a good approximation for $\omega \gg M_2$, cf. section IV.6.) many of these cross-spectra vanish and only Q^{ij}_{mm} and Q^{ij}_{12} remain to determine $E^o(\omega)$.

Figure V.1 shows the even part $E^{\uparrow}(\omega) + E^{\downarrow}(\omega)$ and the odd part $E^{\uparrow}(\omega) - E^{\downarrow}(\omega)$ of the internal wave energy (normalized by the GM distribution $B^{GM}(\omega)$) obtained by the final IWEX model. In the continuum ($\omega > M_2$) the wave

field turns out to be symmetric. Only at low frequencies asymmetries are significant: inertial and tidal waves propagate downward. These results have been obtained using currents and displacements measurements simultaneously. Currents and displacements separately show quite a different behaviour as demonstrated in Fig. V.2.

Since asymmetries are unimportant at high frequencies the internal wave continuum will be modelled by a symmetric spectrum. At M_2 none of the models which we considered gave a consistent representation of the data. An asymmetric wave field gave better results than a symmetric one or a modal field. We chose a model which allows for asymmetry in the energy but is symmetric otherwise because further asymmetries in other parameters did not improve the fit.

V.3.2 Wavenumber structure

There are more direct methods to study the wavenumber structure of a wave field than using moored instruments. For example, dropped or towed temperature sensors yield displacement spectra of vertical wavenumber (Hayes 1975) or horizontal wavenumber (Katz 1975). Measurements with towed (Bell 1976) thermistor chains can be used to estimate two-dimensional (α, β)-spectra of the field. Clearly, these spectra yield direct estimates of the wavenumber structure, but since the frequency is unknown it is questionable if the observed fluctuations are in the internal wave range. Moored data are not affected with this problem but allow only the estimation of a few parameters characterizing the wavenumber structure.

V.3.2.1 Duality of wavenumber spectrum and coherences

The IWEX data set consists of frequency cross-spectra with separation covering almost three decades. In order to model the wavenumber structure we have to understand how it finds its way into the dependence of coherence and phase on the separation. For simplicity consider an isotropic and symmetric internal wave field. Then the coherence between two sensors separated horizontally by r and vertically by Δz becomes

$$\gamma(\omega, r, \Delta z) = \int_0^\infty d\alpha \, B(\alpha, \omega) \, J_0(\alpha r) \cos \beta \Delta z \qquad (V.27)$$

Strictly speaking γ is the normalized cospectrum of the displacement or rotary components

$$\gamma(\omega, r, \Delta z) = \frac{P_{\nu\nu}^{ij}}{(P_{\nu\nu}^{i} \; P_{\nu\nu}^{j})^{1/2}} \quad , \nu = +, - \text{ or } 0 \qquad (V.28)$$

The quadspectrum vanishes. For the analytical discussion in this section we will approximate the vertical phase Θ_{ij} by $\beta \Delta z$ corresponding to a constant Brunt Väisälä frequency. In the inverse program we use the correct expression (V.2).

The coherence is a combination of cosine and Hankel transform and in principle $A(\alpha,\omega)$ may be obtained by inverting either of them. IWEX was not conceived for such a direct inversion method but rather for a parametrization approach as described in part III. Some parameters of the model as e.g. the bandwidth of $A(\alpha,\omega)$ or its asymptotic behaviour can roughly be estimated by inspection of (V.27). We will use displacement data only because they are less contaminated than the current data.

On the IWEX mooring there are horizontal separations for which the coherence becomes

$$\gamma_h(\omega,\tau) = \int_0^\infty d\alpha\, A(\alpha,\omega)\, J_0(\alpha\tau) \qquad (V.29)$$

There are no purely vertical separations. However, at frequencies not too close to N most slant angles are rather steep compared to the slope of the characteristics

$$\frac{dz}{d\tau} = \pm \frac{(\omega^2 - f^2)^{1/2}}{(N^2-\omega^2)^{1/2}} = \pm \frac{\alpha}{\beta} \qquad (V.30)$$

Hence

$$\beta \Delta z \gg \alpha \tau \qquad (V.31)$$

for most slant separations. For a rough discussion the slant coherence may be replaced by the vertical coherence

$$\gamma_v(\omega, \Delta z) = \int d\alpha\, A(\alpha,\omega) \cos \beta \Delta z \qquad (V.32)$$

V.3.2.2 The model spectrum

Our model of the wavenumber structure $A(\alpha,\omega)$ follows closely the one of Garrett and Munk (1975). It includes two more parameters to allow for more variability.

The GM 75 model involves two parameters to describe the wavenumber structure of the internal wave spectrum, the 'bandwidth' α_* and the high wavenumber slope t. A further parameter corresponding to a low-wavenumber cutoff at the

lowest internal wave mode was discussed in their early GM 72 model but was dropped later to simplify the analysis. Because of evidence in the IWEX-data we have to include such a low-wavenumber cutoff in our model. We also introduce another parameter which is responsible for the shape of the distribution at low wavenumbers. These new parameters allow to model a distribution with a more or less developed peak at a non-zero wavenumber.

The analytical form of the model is given by

$$B(\alpha,\omega) = B(\alpha/\alpha_*\,;\,\alpha_p/\alpha_*\,,\,t\,,\,s)\,/\,\alpha_* \qquad (V.33)$$

with

$$B(\lambda;d,t,s) = \begin{cases} \dfrac{I(t,s)}{[1+(\lambda-d)^s]^{t/s}} & ,\ \lambda \geq d \\ 0 & ,\ \lambda < d \end{cases} \qquad (V.34)$$

and

$$\lambda = \alpha/\alpha_* \quad ,\quad d = \alpha_p/\alpha_* \qquad (V.35)$$

The normalization constant I(t,s) is given by

$$I(t,s) = \frac{s\,\Gamma(t/s)}{\Gamma(1/s)\,\Gamma((t-1)/s)} \qquad (V.36)$$

and listed for some values of t and s in Table V.2. All four parameters

α_* - horizontal scale wavenumber
α_p - peak-wavenumber
t - high-wavenumber slope
s - peak shape parameter

are allowed to depend on ω . Notice that $B(\alpha,\omega)$ does not have the property of self-similarity as the GM models unless α_p/α_*, t and s are independent of ω

In general the moments

$$\langle \lambda^n \rangle = \int_0^\infty d\lambda\, B(\lambda) \qquad (V.37)$$

of the distribution (V.34) do not exist. To bypass this problem we may introduce a high-wavenumber cutoff which, however, can only be determined if data with sufficiently small separations are available.

The wavenumber distributions of GM and Cairns are included in the model

(V.34) for the special values of d, t and s given in Table V.1.

Figure V.3 a and b shows $A(\lambda; d, t, s)$ as a function of λ for d = 0 (the dependence on d is evident) and various t and s. Notice that A develops a pronounced peak at $\lambda = d$ as $s \to 1$ (GM 75) which flattens for large s. For fixed s and increasing t the energy is shifted to smaller scales.

V.3.2.3 The bandwidth

The usual definition of bandwidth in terms of the second moment makes no sense if its existence is only achieved by a high-wavenumber cutoff. A distribution $A(\alpha,\omega)$ with non-existing moments can be characterized by its equivalent bandwidth

$$\alpha_e(\omega) = \frac{\left[\int_0^\infty d\alpha\, A(\alpha,\omega)\right]^2}{\int_0^\infty d\alpha\, \left[A(\alpha,\omega)\right]^2} \qquad (V.38)$$

This quantity is used in power spectral analysis to compare spectral windows of different shape with respect to their widths.

The equivalent bandwidth of our model (V.34) is found to be

$$\alpha_e(\omega) = \gamma(t,s)\, \alpha_*(\omega) \qquad (V.39)$$

where the conversion factor J(t,s) is given by

$$\gamma(t,s) = \frac{\Gamma(1/s)\,\Gamma(2t/s)}{s\,\Gamma((2t-1)/s)} \left\{ \frac{\Gamma((t-1)/s)}{\Gamma(t/s)} \right\}^2 \qquad (V.40)$$

and listed for some values of t and s in Table V.2. For a top-hat box (GM 72) we have

$$\alpha_e = \alpha_* \qquad (V.41)$$

and for s = 1 (GM 75)

$$\alpha_e = \frac{2t-1}{(t-1)^2}\, \alpha_* \qquad (V.42)$$

If the dispersion relation is known an equivalent mode number $j_e(\omega)$ may be defined by

$$\alpha^{(j_e(\omega))}(\omega) = \alpha_e(\omega) \qquad (V.43)$$

Values of j_e for the models of GM and Cairns and Williams are given in Table V.1 using the dispersion relation (V.20). Notice that the equivalent mode numbers j_e of the three models do not differ as drastically as the mode number scales j_*.

The values of the bandwidth is determined by the drop of coherence with separation. Rough estimates may be obtained from the scaling argument

$$\Delta z_{1/2}\, \beta_e = O(1) \qquad (V.44)$$

where $\Delta z_{1/2}$ is the separation at which the vertical coherence drops below 1/2, and

$$\beta_e = \alpha_e \left(\frac{N^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2} = j(t,s)\, \beta_* \qquad (V.45)$$

the vertical bandwidth. To illustrate (V.44) we have evaluated the vertical coherence (V.32) for some special cases of our model distribution $H(\lambda)$:

For the top-hat distribution GM 72

$$\gamma(\Delta z) = \frac{\sin \beta_* \Delta z}{\beta_* \Delta z} \qquad (V.46)$$

and

$$\Delta z_{1/2}\, \beta_e \approx 1.9 \qquad (V.47)$$

For d = 0, s = 2 we have

$$\gamma(\Delta z) = \frac{2^{1-\mu}}{\Gamma(\mu)} (\beta_* \Delta z)^{\mu} K_{\mu}(\beta_* \Delta z) \qquad (V.48)$$

with $\mu = (t-1)/2$ (K_{μ} is the modified Bessel function). Almost independent of t we find

$$\Delta z_{1/2}\, \beta_e \approx 2.1 \qquad (V.49)$$

An exponential decay $H(\lambda) \propto e^{-p\lambda}$ yields

$$\Delta z_{1/2}\, \beta_e \approx 2.0 \qquad (V.50)$$

These examples show that α_e is indeed a sensible definition of a bandwidth: spectra with similar distribution of energy and same α_e produce a similar drop of coherence. This is illustrated in Fig. V.4.

Figure V.5 shows some observed coherences as function of slant separation. Notice that there is no monotonic behaviour of $\Delta\xi_{1/2}$ as function of frequency. This is also visible in Fig. V.13 which displays the normalized cospectrum at some other frequencies and Fig. I.9-14 showing coherences as function of separation and frequency. The coherence at some frequencies, as e.g. the 8th, obviously drops much faster than the coherence at other frequencies as e.g. the 18th. This behaviour cannot be modelled by a frequency independent β_e as proposed by GM's model. Thus already from a rough investigation of the data we expect some variability of the bandwidth with frequency.

The results of the hybrid IWEX model are given in Fig. V.6. There are two curves. The full line represents the equivalent mode number j_e (scale on the left hand side), the dashed line represents the equivalent vertical wavelength (scale on the right hand side). The equivalent mode number j_e has been calculated from the true dispersion relation (shown in Fig. V.7, courtesy of M. Briscoe) of the IWEX profile. For $\omega \ll N$ the long-wave approximation

$$\beta_e = j_e \frac{\pi N}{(bN_0)_{IWEX}} \qquad (V.51)$$

with

$$(bN_0)_{IWEX} \approx 5.500 \text{ cph} \cdot \text{m} \qquad (V.52)$$

is valid and both lines fall together. At frequencies larger than 0.5 cph the long-wave approximation becomes invalid: it overestimates the mode number for a given frequency and horizontal wavenumber. This explains the drop of the j_e-line below the $2\pi/\beta_e$-line at high frequencies.

As expected the equivalent mode number varies considerably with frequency. We find extremely low values near M_2, near the 10th and at the 14th and 18th frequency (the 11th, 14th and 18th are close to subharmonics of M_2). In the continuum the values decrease more or less continuously starting with about 20 modes and dropping to values below 10. The subtraction of the subharmonics of M_2 does not remove all of the variability. In particular the low values at frequency 9 and 10 spoil a simple picture.

V.3.2.4 The high-wavenumber slope

Direct estimates of the high-wavenumber slope t have been obtained from towed and dropped spectra.

For the GM 75 model the dropped spectrum is approximately

$$P_{\zeta\zeta}(\beta, z) \approx \langle \zeta^2 \rangle \frac{t-1}{(1+\beta/k_*)^t} \frac{1}{k_*} \qquad (V.53)$$

with

$$k_* = j_* \frac{\pi}{b N_0} N(z) \qquad (V.54)$$

This representation was used by GM to fit the spectral slope to Millard's spectra (Millard 1974) with the result t = 2.5. Hayes' data (Hayes 1975), which were taken at the IWEX site, are more consistent with t = 2 at vertical wavelengths larger than 50 m. This is demonstrated in Fig. I.4, where we have plotted Hayes' data together with (I.53) for t = 2 and 2.5 ($\langle \zeta^2 \rangle^{1/2} = 6$ m, $k_* = 10^{-2} m^{-1}$ for IWEX).

Towed spectra from the MODE area have been reported by Katz (1975). They show a slope of -2 for horizontal wavelengths from 100 m to 10 km. This is consistent with t = 2 because for $\alpha_1 \gg k_* f/N$ ($\approx 1.6 \cdot 10^{-4} m^{-1}$ or $2.5 \cdot 10^{-2}$ cpkm for IWEX) the GM 75 model yields

$$P_{\zeta\zeta}(\alpha_1, z) \approx \alpha_1^{-t} \left(k_* \frac{f}{N}\right)^{t-1} \langle \zeta^2 \rangle \, Z(t, N/f) \qquad (V.55)$$

which is shown for t = 2 and 3 on Fig. V.8 together with Katz' data. The constant Z(t,N/f) is derived in section V.6.

Moored coherences also reflect the high-wavenumber structure. Both, cosine and Hankel transform, map the asymptotic behavior of the wavenumber spectrum onto the behaviour of the coherence near the origin (and vice versa). For illustration of this feature consider the horizontal and vertical coherence for the GM 75 model (details can be found in section V.6). Near the origin we find

$$\gamma_h(\tau) = 1 - (\alpha_* \tau)^{t-1} C_h + O((\alpha_* \tau)^2) \qquad (V.56)$$

$$\gamma_v(\Delta z) = 1 - (\beta_* \Delta z)^{t-1} C_v + O((\beta_* \Delta z)^2)$$

where

$$C_h = \left[\Gamma^2\left(\frac{t+1}{2}\right) \cos\left(\frac{t-2}{2}\pi\right) \right]^{-1} \frac{(t-1)\pi}{8}$$
$$C_v = (t-1)\,\Gamma(1-t)\cos\left(\frac{t-2}{2}\pi\right) \qquad \text{(V.57)}$$

Notice that C_u and C_v are positive and finite for $1 < t < 3$. Both expansions become invalid if t equals an odd integer. In this case the coherence is regular at the origin. The behaviour of the coherence expressed by (V.56) is found to be a general property of spectra which decay as λ^{-t} for $\lambda \to \infty$. This is shown in section V.6.

No real spectrum extends to infinity. We already mentioned that in order to avoid infinite higher order moments we have to require a high-wavenumber cutoff, say, at λ_1 . If $B(\lambda) \propto \lambda^{-t}$ for $\lambda_0 < \lambda < \lambda_1$ with $\lambda_1 \gg \lambda_0$, we get a two-scale behaviour of the coherence. The vertical coherence, for example, becomes

$$\gamma_v(\Delta z) = 1 - \frac{1}{2}(\beta_* \Delta z)^2 \langle \lambda^2 \rangle + O\left((\lambda_1 \beta_* \Delta z)^4\right) \qquad \text{(V.58)}$$

for $\Delta z \ll (\lambda_1 \beta_*)^{-1}$ while (V.56) still holds for $(\lambda_1 \beta_*)^{-1} \ll \Delta z \ll (\lambda_0 \beta_*)^{-1}$. The horizontal coherence shows a similar behaviour. These results are illustrated in Fig. V.9.

Let us now turn to the observed coherences. Fig. V.10 shows the coherence of displacement as a function of horizontal separation for some representative frequencies (we do not present vertical coherence because of finestructure contamination). An almost linear decrease is visible for separations between 10 m and some hundred meters. To find the power t more precisely we have plotted log $(1 - \gamma)$ against log r in Fig. V.11. Obviously $2 < t < 3$ but the error bars do not allow to determine the value much better. In particular we cannot distinguish between t = 2.5 (GM 75) and t = 2 (Cairns and Williams 1976).

The values obtained from the hybrid IWEX model are shown in Fig. V.12. Here we find the same picture which was already indicated by the few data in Fig. V.11.

V.3.2.5 The peak wavenumber

Generally a cutoff or a sharp peak in $B(\lambda)$ generates oscillations in the coherence, as can easily be seen for the limit $B(\alpha) = \delta(\alpha - \alpha_p)$. In this case the coherences become

$$\gamma_v(\Delta z) = \cos \beta_p \Delta z \qquad \text{(V.59)}$$

and

$$\gamma_h(r) = J_0(\alpha_p r) \qquad (V.60)$$

Oscillations which are due to discontinuities at higher wavenumbers have a smaller period and generally a lower amplitude than those associated with α_p. Hence it is essentially the lowest discontinuity which determines the asymptotic behaviour of the coherence.

There is some evidence in the IWEX data of oscillations in the coherences which can be interpreted in terms of a low-wavenumber cutoff. Some frequencies do not show any zero-crossings in the separation interval covered by the array. For some frequencies the tail of the coherence remains completely within the 95% confidence limit of zero true coherence so that the cutoff α_p is not significantly different from zero (cf. Fig. V.5). Most of the coherences, however, rise again at large separations. This is illustrated in Fig. V.13, which shows the normalized cospectra of displacement (up) for some typical frequencies.

Fig. V.14 displays the results obtained by the hybrid IWEX model. Here j_p denotes the low-modenumber cutoff defined by

$$\alpha^{(j_p(\omega))}(\omega) = \alpha_p(\omega) \qquad (V.61)$$

Notice that for most frequencies (except for the tidal frequency) we find $j_p > 0$, indicating that there is no energy in the barotropic mode. For various frequencies (especially for f) we also find almost no energy in the first baroclinic mode.

These features are also apparent in a modal interpretation of the coherence, where the vertical coherence of displacement takes the form

$$\tilde{\gamma}(\omega, z, z') = \sum_{j=0}^{\infty} H(j)\, \tilde{\varphi}^{(j)}(\omega, z)\, \tilde{\varphi}^{(j)}(\omega, z') \qquad (V.62)$$

Here $\tilde{\varphi}^{(j)}(\omega, z)$ is the eigenfunction of the jth mode (see Fig. V.15, courtesy of F. Schott). In the depth interval covered by IWEX $\varphi^{(0)}$ and $\varphi^{(1)}$ do not change sign. If these modes would have high energy the coherence would be positive for all separations in the array. This is in contrast to the observations. However already $\varphi^{(2)}$ is (for most frequencies) capable of reproducing a zero-crossing.

V.3.2.6 The shape of the peak

The shape of the distribution $H(\lambda)$ at low wavenumbers is determined by the behaviour of the coherence at large separations. We can find relations

between the functions in the two domains which are analogous to those for the determination of the high-wavenumber slope. However, if there is a peak at non-zero wavenumber then the associated oscillations in the coherence will mask completely the contribution of the low-wavenumber shape to the asymptotic behaviour of the coherence. Also the estimates of coherence for large separations are less accurate than those near the origin. Therefore a quick inspection of the data gave no valuable results. At this point the advantage of the inverse technique becomes clear which uses all information simultaneously to estimate the parameter values. In the inverse program the value of the shape parameter s turned out to be reasonably well-defined.

Fig. V.16 shows the results of the hybrid IWEX model. We find s almost constantly near unity with large error bars. A value of s = 2 as suggested by Cairns and Williams (1976) seems less likely.

V.3.3. Angular distribution

In part IV we have discussed tests for the isotropy of the wave field. For one-point measurements the test statistics were coherences of the rotary components of the current vector (e.g. the ellipse stability γ^2_{+-}, Gonella 1972). These coherences only give restricted information about the angular distribution of the energy because the cross-spectral matrix for zero separation only involves the first three (complex) Fourier coefficients of the angular distribution of wave energy. In order to obtain more detailed information about the anisotropy of the wave field we had to consider A^{ij}_{mn} for non-zero separation, i.e. the complete set of consistency relations for an isotropic internal wave field as listed in part II. In this chapter we will discuss the zero separation coherences in relation to our model spectrum in order to illustrate the results of the hybrid model.

The model of the directional distribution of wave energy was chosen to be

$$S(\varphi;\alpha,\omega) = \frac{(p!)^2\, 2^{2p}}{2\pi\,(2p)!}\, \cos^{2p}\left(\frac{\varphi-\varphi_0}{2}\right) \qquad \text{(V.63)}$$

which for each α and ω describes a beam with a mean propagation direction $\varphi_0 = \varphi_0(\alpha,\omega)$ and a beam width parameter $p = p(\alpha,\omega)$. Equivalently we may use

$$q = \frac{p}{p+1} \qquad \text{(V.64a)}$$

which may also be expressed as

$$q e^{i\varphi_0} = \int_0^{2\pi} d\varphi \, S(\varphi) \, e^{-i\varphi} \qquad \text{(V.64b)}$$

Isotropic distribution of energy is obtained for $p \to 0$ ($q \to 0$) and a unidirectional distribution of $p \to \infty$ ($q \to 1$). The beam width $\Delta\varphi$ associated with $S(\varphi)$ is defined by

$$\cos^{2p}\left(\frac{\Delta\varphi}{4}\right) = \frac{1}{2} \qquad \text{(V.65)}$$

The relation between $\Delta\varphi$ and q is almost linear as shown in Fig. V.17.

The Fourier series representation of the directional distribution (V.63) is given by

$$S(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} S_n \, e^{in(\varphi-\varphi_0)} \qquad \text{(V.66)}$$

with

$$\begin{aligned} S_0 &= 1 \\ S_{-n} &= S_n \\ S_{n+1} &= S_n \frac{p-n}{p+n+1} \end{aligned} \qquad \text{(V.67)}$$

The simple analytical structure of (V.63) allows the analytical evaluation of the directional integration appearing in the model cross spectra (V.1) (cf. Schott and Willebrand 1973). For an internal wave field the coherences γ_{+-}, γ_{+0} and γ_{-0} may be expressed in terms of the Fourier coefficients of the energy spectrum (cf. II.82, 83)

$$\begin{aligned} \gamma_{+-} &= \frac{|M_2|}{M_0} \\ \gamma_{+0} = \gamma_{-0} &= \frac{|M_1|}{M_0} \end{aligned} \qquad \text{(V.68)}$$

with

$$\begin{aligned} M_0(\omega) &= E^e(\omega) \\ M_1(\omega) &= E^o(\omega) \int d\alpha \, A(\alpha,\omega) \, S_1(\alpha,\omega) \, e^{-i\varphi_0(\alpha,\omega)} \\ M_2(\omega) &= E^e(\omega) \int d\alpha \, A(\alpha,\omega) \, S_2(\alpha,\omega) \, e^{-i2\varphi_0(\alpha,\omega)} \end{aligned} \qquad \text{(V.69)}$$

The moments (V.69) take a very simple form if either p and φ_0 do not depend on α or - as our model - if $R(\omega,\alpha)$ is strongly peaked. We then find

$$\gamma_{+-} = q\left|\frac{2q-1}{2-q}\right|$$
$$\gamma_{+0} = \gamma_{-0} = q\left|\frac{E^o}{E^e}\right| \qquad (V.70)$$

The model coherence γ_{+-} is shown in Fig. V.18 as a function of the beam width $\Delta\varphi$. We find zero ellipse stability not only for isotropy ($q = o$, $\Delta\varphi = 2\pi$) but also for a beam width $\Delta\varphi = \pi$ ($q = 1/2$). This latter zero appears in general for directional distributions which are symmetric about the mean propagation direction. The ellipse stability is hence a rather poor test of isotropy in the case of (almost) symmetrical angular distributions. It leads to a rejection of isotropy only for very peaked distributions.

The coherence γ_{+0} is affected by another shortcoming because it only measures the anisotropy of the odd part of the energy distribution. Thus, only if the asymmetry E^o/E^e is large and known, then γ_{+0} is a good measure of the beam width.

The observed coherence γ_{+-} is given in Fig. V.19. The figure includes the bias, i.e. the expectation value for zero true coherence. Notice that in the internal wave continuum ($\omega > M_2$) the coherence γ_{+-} is close to the bias so that an isotropic model would be consistent with the data, but also any symmetric $S(\varphi)$ with a beam width $\Delta\varphi$ larger than π. At low frequencies, especially M_2, the coherence is significantly larger than the bias indicating an anisotropic energy distribution; specifically we find $\gamma_{+-} \approx 0.3$ at M_2 which yields $\Delta\varphi \approx 100^o$ or $q \approx 0.7$.

The observed γ_{+0} and γ_{-0} are also shown in Fig. V.19. These coherences are also close to the bias for $\omega > M_2$, indicating isotropy and/or vertical symmetry.

The model phases corresponding to (V.70) are given by

$$\phi_{+-} = \arctan(\sin 2\varphi_0, \cos 2\varphi_0)$$
$$\phi_{+0} = \phi_{-0} = \arctan(-\sigma\sin\varphi_0, -\sigma\cos\varphi_0) \qquad (V.71)$$

Notice that Fig. V.19 displays the phases $\phi_{+\zeta}$ and $\phi_{-\zeta}$. From $u_0 = u_3 = -i\omega\zeta$ (ζ is down!) we find $\phi_{\pm\zeta} = \phi_{\pm 0} + \pi/2$. The observed phases at M_2 and the corresponding directions of propagation are (with $\sigma = +$ from Fig. V.2)

$$\begin{aligned} \phi_{+-} &= -128^\circ \rightarrow \quad \varphi_0 = 116^\circ \pm 180^\circ \\ \phi_{+o} &= 5.5^\circ \rightarrow \quad \varphi_0 = -35^\circ \\ \phi_{-o} &= 168^\circ \rightarrow \quad \varphi_0 = -78^\circ \end{aligned} \tag{V.72}$$

Thus all three estimates yield propagation of tidal waves to the southeast, consistent with the findings of Nobel and Joyce (1977).

The values for q and φ_0 of the hybrid IWEX model are presented in Fig. V.20. At low frequencies ($f < \omega < 2M_2$) we find a wave field with beamwidth less than π propagating to the southeast. The high-frequency part is almost isotropic (φ_0 should be regarded to be undetermined in this part) with a small anisotropy possibly at the very high frequencies (cf. the isotropy test Fig. IV.6).

V.4. Contamination

So far we have discussed the parametrization of the energy spectrum of that part of the observed fluctuations which can be described by internal waves. When illustrating the parametrization we have used data which are least affected by contamination. The complete data set, however, cannot be parameterized by a pure internal wave field. An attempt is shown in Fig. V.21: the model is indeed unable to reproduce all the observed coherences and behaves rather diplomatically, almost fitting none of them. For this reason we introduced a hybrid model which is a combination of internal waves and some contamination fields which will be discussed in this section. The parameter values of the internal wave spectrum obtained with this hybrid model have already been presented above.

V.4.1 Disparity of disparities

We learned already in section V.3.1 that there is a discrepancy between the IWEX data and a WKB internal wave field. A disparity was found between the energy levels of horizontal currents and displacement (up): if the WKB relations (V.23) and (V.24) hold then the total wave energy computed from currents exceeds the one computed from ups by about 10% (cf. Fig. IV.10). This feature will be referred to as energy disparity.

Another much more obvious disparity was already noticed by Briscoe (1975). He found a strong disparity between up coherence and current coherence for the same pair of instruments. For a WKB wave field the coherences $\gamma_{oo}(\omega, \tau, \Delta z)$ and $\gamma_{\nu\nu}(\omega, \tau, \Delta z)$, $\nu = \pm$, are identical. However, as displayed in

Fig. V.22 the observed current coherence decreases with increasing frequency faster than the up coherence. Generally this coherence disparity is small for purely horizontal separations and becomes noticeably larger if the instruments are also vertically separated.

Let us consider the coherence disparity as a function of separation in more detail. Fig. V.23 and V.24 display the current and displacement coherences for a frequency in the internal wave continuum as function of horizontal and slant (on leg A) separation. We notice that both slant coherences ($\zeta\zeta$ and $u_A u_A$) and the horizontal up coherence show a two-scale behaviour: for separations less than a few meters (for $\zeta\zeta$ about 3 m; for $u_A u_A$ less than the smallest separation $\sim$ 7 m) they drop rapidly; for larger separations they decrease linearly in the same way as the horizontal up coherence, but at a lower level. The linear decrease is in agreement with a $t \approx 2$ high-wavenumber slope of the internal wave energy spectrum. At separations larger than a few meters the discrepancy between current and up coherence is about 10% for horizontal separations and considerably larger - about 30% - for slant separations.

Because of this strong spatial structure Briscoe (1975) excluded instrumental noise in the current measurements as a possible source of the coherence disparity. This is also confirmed by the consistency tests in part IV. As a possible explanation he suggested either contamination by nonlinearities in the wave field or by finestructure in the current profile.

At first sight contamination of the current measurements by some small scale velocity finestructure seems to be a satisfactory explanation of the coherence disparity because there is also an excess of energy in the current autospectra. However, an energy excess of 10%, as observed, would only yield a coherence disparity of the same amount if the contamination has a much smaller correlation scale than the internal wave field (cf. IV.87). To explain the coherence disparity of 30% we have to look for a more complicated contamination pattern.

A two-scale behaviour of the coherence must not necessarily be associated with a contamination process. The rapid drop at small separations of each single coherence may of course be modelled by a corresponding hump of internal wave energy at high wavenumbers. From the smooth behaviour of the horizontal up coherence we can however exclude such a high-wavenumber hump. Therefore the coherence drop must be associated with non-internal wave energy. In fact, the spatial structure of the coherence may only be explained by at least two separate contamination processes: a contamination in the current measurements which degrades the coherence vertically and horizontally, and a contamination in the displacement estimates which lowers only the vertical coherence.

Since we have estimated the displacement from moored temperature measurements its contamination may be understood as due to finestructure of the temperature profile as discussed e.g. by Garrett and Munk (1972b), McKean (1974) and others.

V.4.2 Contamination by temperature finestructure

The contamination of displacement (up) by temperature finestructure is modelled after McKean (1974) (cf. Joyce and Desaubies 1977). The total cross-spectrum $\hat{A}^{ij}_{\zeta\zeta}(\omega)$ is a sum of a mean gradient part $A^{ij}_{\zeta\zeta}(\omega)$ (which must be identified with our model (V.1) of the internal wave cross-spectrum) and a finestructure cross-spectrum $G^{ij}(\omega)$:

$$\hat{A}^{ij}_{\zeta\zeta}(\omega) = A^{ij}_{\zeta\zeta}(\omega) + G^{ij}(\omega) \qquad (V.73$$

For a true displacement spectrum

$$P_{\zeta\zeta}(\omega) = \langle \zeta^2 \rangle \frac{f}{\omega^2} \qquad (V.74)$$

McKean obtains

$$G^{ij}(\omega) = \delta_f(\omega) \left[P^i_{\zeta\zeta}(\omega) P^j_{\zeta\zeta}(\omega) \right]^{1/2} \Lambda(\omega, \Delta z) \qquad (V.75)$$

The finestructure ratio

$$\delta_f(\omega) = \frac{G^i(\omega)}{P^i_{\zeta\zeta}(\omega)} \qquad (V.76)$$

is explicitly given by

$$\delta_f(\omega) = \frac{1}{2} \frac{\overline{\Theta^2}}{(\partial_3 \overline{T})^2} \frac{(\omega/f)^{1/2}}{h \langle \zeta^2 \rangle^{1/2}} \qquad (V.77)$$

where $\overline{\Theta^2}$ is the mean square temperature jump over a mean layer thickness h and $\partial_3 \overline{T}$ the mean temperature gradient. The finestructure coherence (i.e. normalized cospectrum) is explicitly given by

$$\Lambda(\omega, \Delta z) = e^{-\Delta z/a} \left\{ \cos \frac{\Delta z}{a} + \sin \frac{\Delta z}{a} \right\} \qquad (V.78)$$

where the correlation scale $a(\omega)$ is given by

$$a(\omega) = \langle \zeta^2 \rangle^{1/2} \left(\frac{f}{\omega} \right)^{1/2} \qquad (V.79)$$

Rough estimates of δ_f and a may be obtained from the relation between the hybrid and the true coherence (i.e. normalized cospectrum)

$$\hat{\gamma}_{33} = \frac{\gamma_{33} + \delta_f \Lambda}{1+\delta_f} \approx \gamma_{33} \frac{1}{1+\delta_f} \qquad \text{for } \Delta z \gg a \tag{V.80}$$

Comparison with Fig. V.24 yields $\delta_f = O(0.1)$ and $a = O(2m)$.

The results of the hybrid IWEX model (using only up-data) are given in Fig. V.25 and V.26. Included in the figures are the theoretical predictions (V.77) and (V.79), taking $\bar{\Theta}^2/(\partial_3 \bar{T})$ from Joyce and Desaubies (1977). The values for δ_f are well represented by the theory. There is also reasonable agreement in case of the correlation scale a. At low frequencies this latter parameter could not be determined since δ_f is almost zero.

V.4.3 The current contamination

The current contamination was modelled similarly to the displacement contamination. Writing the total cross-spectrum of the horizontal current components as

$$\hat{H}^{ij}_{\alpha\beta}(\omega) = H^{ij}_{\alpha\beta}(\omega) + F^{ij}_{\alpha\beta}(\omega) \tag{V.81}$$

the contamination part $F^{ij}_{\alpha\beta}$ must produce coherences which drop very rapidly to zero with vertical separation and to some finite value with horizontal separation (less than the smallest available separation of the IWEX array).

We took the simple model

$$F^{ij}_{\alpha\beta}(\omega) = \delta_{\alpha\beta} E_{cont}(\omega) f(\omega, \Delta z) g(\omega, r) \tag{V.82}$$

where E^{cont} denotes the energy of the current contamination. The correlation functions $f(\omega, \Delta z)$ and $g(\omega, r)$ model the spatial structure of the current contamination. For simplicity we took $f(\omega, \Delta z)$ equal to the displacement finestructure coherence $\Lambda(\omega, \Delta z)$ and

$$g(\omega, r) = \begin{cases} (1-\varepsilon)\, e^{-(r/a_h)^2} & \text{for } r > D \\ 1 & \text{for } r \le D \end{cases} \tag{V.83}$$

Here ϱ denotes the horizontal coherence drop of the contaminating field. D is a small scale and a_h is a large scale horizontal coherence length. These two length scales could not be determined from the data as both lie outside the separation interval covered by the mean data set. Thus we took $a_h = \infty$ and D smaller than the smallest separation of the IWEX array.

The hybrid coherence then becomes

$$\hat{\gamma}_{vv} = \frac{\gamma_{vv} + \gamma_c f(\Delta z) g(\tau)}{1 + \gamma_c} \qquad (V.84)$$

with

$$\gamma_c = \frac{E_{cont}}{P_{vv}} \qquad (V.85)$$

The limiting cases

$$\Delta z = 0: \quad \hat{\gamma}_{vv} = \begin{cases} \gamma_{vv} & \text{for } \tau < D \\ \gamma_{vv} \dfrac{1 + \gamma_c (1 - \varrho)}{1 + \gamma_c} & \text{for } D < \tau \ll a_h \end{cases} \qquad (V.86)$$

$$\tau = 0: \quad \hat{\gamma}_{vv} = \gamma_{vv} (1 + \gamma_c) \qquad (V.87)$$

may then be used to estimate the parameters E^{cont} and ϱ from the observed coherence drops in Fig. V.23 and V.24. We find $\gamma_c = 0.7$ and $\varrho = 0.4$ at ω_{21}.

The results of the hybrid IWEX model (from horizontal currents alone) are shown in Fig. V.27 and V.28. The energy of the current contamination E^{cont} (normalized by GM's frequency distribution $B^{GM}(\omega)$) increases from zero at $\omega < M_2$ to values comparable to the internal wave energy at larger frequencies. The coherence drop ϱ is almost constant (~ 0.3) for frequencies larger than M_2 and essentially zero below. Thus at low frequencies the current contamination is small but horizontally completely coherent; with increasing frequency the contamination becomes larger and less coherent.

So far no physical process has been associated with the current contamination. In Müller, Olbers and Willebrand (1977) it is argued that the current contamination is presumably due to the superposition of two processes: the passive advection of current finestructure past the sensors by the internal wave field (in analogy to McKean's (1974) theory) and small scale three dimensional

turbulence in the layers of the density stratification.

V.4.4 Coherent signal contamination

The parameters of the contaminating fields were adjusted in order to reproduce the coherence disparity correctly. This is demonstrated in Fig. V.29, which compares the hybrid IWEX model with a pure internal wave model. The energy disparity has not been removed; on the contrary we are now faced with an even larger disparity in the energies. The situation is illustrated in Fig. V.30. As the finestructure ratio of the displacement field is about 0.1 the decontaminated displacement spectrum P_{ss} is about 10% less than the observed one. Similarly the decontaminated current spectrum P_{--} must be about 40% less than the observed $\hat{P}_{--}$. What remains is a 20% excess of energy in the displacement.

Accurate values are given in Fig. V.31, which displays the wave energy as obtained separately from the displacement and horizontal currents with inclusion of the corresponding contaminations discussed above. Notice that at all frequencies larger than M_2 the wave energy in the displacements is almost constantly larger than the one in the currents. Since the coherences are now modelled correctly this excess energy in the displacements must be at least as coherent as the wave field itself. In other words, if we impose an additional coherent signal on the displacements we are able to model the observations correctly.

V.5. The IWEX model

In chapter V.3 and V.4 we have introduced the parametrization of the IWEX cross-spectra by a combination of a random internal wave field and some contamination processes. This chapter will present a summary of the final model and will discuss its consistency and uniqueness.

V.5.1 Summary of the parametrization

Our final model is given by

$$\hat{A}^{ij}_{\alpha\beta} = A^{ij}_{\alpha\beta} + F^{ij}_{\alpha\beta}$$

$$\hat{A}^{ij}_{ss} = A^{ij}_{ss}\left(1 + \frac{E_c}{E^e}\right) + G^{ij} \qquad \text{(V.88)}$$

$$\hat{A}^{ij}_{\alpha\zeta} = A^{ij}_{\alpha\zeta}$$

Here the internal wave part A^{ij}_{mn} is given by (V.1), the current contamination $F^{ij}_{\alpha\beta}$ by (V.82), the finestructure contamination of displacement by (V.75) and the coherent contamination of displacement by a coherent signal with energy E_c. This model has been called the hybrid IWEX model. Its parameters have been shown in the course of this part. A summary is given in Fig. V.32a-c. Notice that the wave energies E^e and E^o are scaled by GM's frequency distribution $B^{GM}(\omega)$. The energies of the contaminations are presented in form of the energy ratios $\delta_f(\omega)$ (cf. V.76), $E_c(\omega)/E^e(\omega)$ and $E_{cont}(\omega)/E^e(\omega)$.

The parameters show a large variability in the frequency domain (e.g. the equivalent mode number and the peak shape parameter). We have demonstrated in this report that this variability corresponds closely to the data but we were unable to provide a physical explanation. Some features can be attributed to subharmonics of the tides, but in addition to those frequencies there are others which also show a completely different behaviour than their next neighbours. The most prominent representatives of this group are the 10th and 26th frequency.

Apart from this variability the magnitudes of the error bars allow simple parametrizations in the frequency domain which for this purpose should be divided into the low frequency part ($\omega < 0.1$ cph) and the internal wave continuum ($\omega > 0.1$ cph). The behaviour in the low frequency part is dominated by inertial and tidal waves which must be treated separately.

In the continuum the wave spectrum is similar to the GM model:

(1) the distribution of energy is horizontally isotropic and vertically symmetric
(2) the wavenumber structure is characterized by an equivalent mode number $j_e = 0$ (10-15) with a slight tendency to decrease with frequency
(3) the slope of the spectrum at high wavenumbers is well described by $t = 0$ (2-2.5)
(4) there is a sharp peak at low wavenumbers corresponding to a mode number $j_p = 0$ (1-2)
(5) the frequency distribution of the total energy per unit surface area is well-represented by a ω^{-2}-law.

The low frequency part of the wave field is found to be anisotropic and asymmetric. We find a propagation of energy to the southeast with a beamwidth around $\pi/2$. The vertical asymmetry is characterized by an excess of downward propagating energy which is about 20% larger than the upward propagating energy.

The tide M_2 is characterized by an extremely small bandwidth: only about 3 equivalent modes are excited. However, it should be pointed out that our model at this frequency is not consistent with the data (see section 5.2.1).

Inertial waves show a peak of energy in the frequency domain which is well above the one of GM's model (by a factor of about 5). The bandwidth is about 15 equivalent modes with a peak at the third mode.

The parameters which describe the contamination processes have a rather simple frequency dependence. The relative energies of the current contamination and of the displacement contamination by temperature finestructure are negligibly small in the low frequency part and increase in the internal wave continuum. The displacement contamination by temperature finestructure is in good agreement with McKean's theory (McKean 1974) though a smaller vertical correlation scale is found ($\sim$ 1.8 m). The energy of the current finestructure becomes comparable with the wave energy for $\omega > 0.5$ cph. The horizontal coherence drop can be considered as constant in the wave continuum. The relative energy of the coherent signal imposed on the displacement estimates (ups) is about 20% of the wave energy, almost independent of frequency. Marked peaks are only found at the inertial and tidal frequency.

The sub-buoyancy range covering the last three to four frequency points shows some peculiarities. Here many of the parameters do not follow the general trend indicated in the continuum at lower frequencies. Also, as shown below (section V.5.2.1), the model is not completely consistent with the data. The reason is obviously the poor WKB approximation near the turning point. Nevertheless our model does not fail completely, which must be attributed to the inclusion of the contaminations. The hybrid IWEX model is complex enough to fit to some degree the sub-buoyancy humps in energy and coherence (Desaubies 1975).

V.5.2. Consistency and uniqueness of the hybrid model

In the previous chapters of this part we have developed some characteristic features of the hybrid IWEX model and specified its parameters. We will now discuss the following questions:

i) Do these parameters give a consistent description of the full data set?
ii) Are they determined uniquely?

V.5.2.1 Consistency

To find out whether or not our model cross-spectra coincide with the observed cross-spectra, we may use the technique described in part III. The appropriate statistical quantity is the deviation between observed and model data

$$\varepsilon^2 = (\underset{\sim}{y} - \hat{\underset{\sim}{y}})^T \underset{\sim}{W} (\underset{\sim}{y} - \hat{\underset{\sim}{y}}) \qquad (V.89)$$

(For the notation see part III).

In Fig. V.33 we show for all frequencies the actual value of ε^2, the expectation value $\langle \varepsilon^2 \rangle$ under the hypothesis that $\langle \underset{\sim}{y} \rangle = \langle \hat{\underset{\sim}{y}} \rangle$ and the 95% confidence limit. All these quantities are normalized by $\varepsilon_0^2 = \underset{\sim}{y}^T \underset{\sim}{W} \underset{\sim}{y}$, so that the absolute magnitude gives some idea to what percentage $\underset{\sim}{y}$ and $\hat{\underset{\sim}{y}}$ coincide. The somewhat irregular dependence of $\langle \varepsilon^2 \rangle$ on the frequency is due to two opposing influences. Since the majority of all coherences decreases with increasing frequency, this should lead to an increase of $\langle \varepsilon^2 \rangle$. On the other hand, the increasing number of degrees of freedom leads to a decrease in $\langle \varepsilon^2 \rangle$, which is stepwise at low, but continuous at high frequencies due to the frequency averaging as described in section I.8.1. Near $\omega = 0.5$ cph we find a jump because for higher frequencies the instruments on level 14 have been omitted (For the inverse fit we omitted sensors already at frequencies far before the turning point, whereas the consistency tests take all sensors i with $\omega < N(z_i)$.

By and large the actually obtained value for ε^2 scatters around the expected value. Except for M_2 and frequencies larger than 1 cph, the hybrid IWEX model gives - on the 95% probability level - a consistent description of the IWEX data set. The discrepancy at high frequencies could be easily removed by using instead of the WKB-solution for the vertical wave functions the more accurate Airy functions (Desaubies 1973), or a modal representation.

At M_2, we must conclude that no random internal wave model, even if modified as described above, gives a consistent representation of the observed data.

Fig. V.34 shows the normalized quantity $\varepsilon^2 / \langle \varepsilon^2 \rangle$ which allows a direct comparison with the results of the consistency tests, in particular Fig. IV.23. The two pictures agree satisfactorily; deviations must be attributed to the fact that the results of the fitting procedure depend on all data while the results of the tests only depend on some linear combinations.

V.5.2.2 Correlations between the parameters

The parameters are estimated from random data, and must hence be regarded as random as well. Their statistical variability is described by the parameter covariance $R_{\alpha\beta} = \mathrm{COV}[x_\alpha, x_\beta]$ which can be calculated according to (III.66). The diagonal elements of $\underset{\sim}{R}$ determine the standard deviations $\sigma_\alpha = R_{\alpha\alpha}^{1/2}$ which have been shown together with the corresponding parameters. If the matrix $\underset{\sim}{R}$ is nondiagonal the different parameters are correlated.

From eq. (III.66), one finds explicitly

$$\underset{\sim}{R} = \underset{\sim}{M}^{-1} \underset{\sim}{A}^T \underset{\sim}{W} \underset{\sim}{S} \underset{\sim}{W} \underset{\sim}{A} \underset{\sim}{M}^{-1} \qquad \text{(V.90)}$$

For the special metric $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$, which is optimal in the sense discussed in

part III, (V.90) reduces to

$$\underset{\sim}{R} = \underset{\sim}{M}^{-1} = (\underset{\sim}{A}^T \underset{\sim}{W} \underset{\sim}{A})^{-1} \qquad (V.91)$$

Though we did not use $\underset{\sim}{W} = \underset{\sim}{S}^{-1}$, but rather the diagonal metric defined by (III.37), we always found a strong similarity between $\underset{\sim}{R}$ and $\underset{\sim}{M}^{-1}$, i.e.

$$\underset{\sim}{R} \approx \underset{\sim}{M}^{-1} \qquad (V.92)$$

Usually more than 90% of the corresponding matrix elements had the same sign. Also the scaling of eigenvalues and the eigenvector systems were similar. This is not surprising since the metric (III.37) was chosen to resemble $\underset{\sim}{S}^{-1}$ as much as possible. As $\underset{\sim}{M}$ reflects the properties of the model matrix $\underset{\sim}{A} = \{\partial y_\ell / \partial x_\alpha\}$, we expect that a high parameter correlation should be related in a simple way to some feature in our model.

Let us consider a simple example with two parameters. Assume that the 2 by 2-matrix $\underset{\sim}{M}$ has the eigenvectors (1,1) and 1,-1), the corresponding eigenvalues being λ_1 and λ_2. Then

$$\underset{\sim}{M} \propto \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix} \qquad (V.93)$$

and

$$\underset{\sim}{R} \propto \begin{pmatrix} \lambda_1^{-1} + \lambda_2^{-1} & \lambda_1^{-1} - \lambda_2^{-1} \\ \lambda_1^{-1} - \lambda_2^{-1} & \lambda_1^{-1} + \lambda_2^{-1} \end{pmatrix} \qquad (V.94)$$

The correlation between x_1 and x_2 is given by

$$\rho_{12} = \frac{R_{12}}{(R_{11} R_{22})^{1/2}} = \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} \qquad (V.95)$$

Consider first the case $\lambda_1 \gg \lambda_2 > 0$. This means that our model depends mainly on the parameter combination $(x_1 + x_2)$ and much more weakly on $(x_1 - x_2)$. Then $\rho_{12} < 0$, so that a strong <u>negative</u> correlation is found if both parameters describe essentially the <u>same</u> features in the data.

Consider next the case $\lambda_2 \gg \lambda_1 > 0$ which means that $x_1 - x_2$

is the more important parameter. In this case we obtain $\rho_{12} > 0$. Thus a strong positive correlation occurs if both parameters have opposing influences.

Note that this sign rule does not hold if the matrix $\underset{\sim}{M}$ is nearly singular, i.e. if λ_1 or λ_2 are smaller than the threshold discussed in section III.6.4. In this case the correlation would be ± 1 with the opposite sign than discussed above.

For more than two variables the situation is more complex, and the above interpretation must be used with caution. As discussed in section III.6.4 for the variances, the correlations are also very sensitive to the value of the threshold. Usually a correlation between two well-determined parameters will be dominated by their mutual correlations with the poorest determined parameters.

In the following the correlations for some parameter pairs are discussed. The list is not complete, mainly because of the large number (~ 100) of possible combinations. The 95% confidence level for zero true correlation has not been calculated exactly; a rough estimate leads to a value 0.1 - 0.2.

$E^{\uparrow} - E^{\downarrow}$ (Fig. V.35 a) The correlation is positive for all frequencies. This is due to the fact that both $E^{\uparrow}$ and $E^{\downarrow}$ are correlated in exactly the same way with most of the weakly determined parameters. If only $E^{\uparrow}$ and $E^{\downarrow}$ were determined, their correlation would turn out to be negative, since $E^{\uparrow} + E^{\downarrow}$ is obviously the more important parameter than $E^{\uparrow} - E^{\downarrow}$ (see below).

$E^{\uparrow} - \delta_t$, $E^{\downarrow} - \delta_t$ (Fig. V.35 a) The correlation is negative (increasing both parameters increases the energy in the displacement spectra) but small for most frequencies (the parameters are nearly independent).

$E^{\uparrow} - E_{cont}$, $E^{\downarrow} - E_{cont}$ (Fig. V.35 b) The correlation is negative for all frequencies: all parameters increase the energy in horizontal current spectra.

$E - j_e$ (Fig. V.35 c) The correlation is positive throughout ($E^{\downarrow} - j_e$ shows a similar behaviour). Both parameters have opposite effects on all cross-spectra between spatially separated instruments, which roughly are proportional to E/j_e. This correlation could be avoided by fitting complex coherences instead of cross-spectra.

$E_{cont} - j_e$ (Fig. V.35 c) The correlation is negative throughout. Both parameters, when increasing, lower the coherence between current cross-spectra.

$j_e - j_p$ (Fig. V.35 d) These parameters are negatively correlated. Both parameters, when increasing, lower coherences between separated instruments (j_e generally, j_p for most IWEX separations).

$j_e - t$ (Fig. V.35 d) The correlation is significant, but no clear trend is visible.

$E^{\uparrow} - q$, $E^{\downarrow} - q$ (Fig. V.35 e) Both correlations have almost the same

magnitude but opposite sign. This can be understood since the current-displacement cross-spectra which dominate the determination of q are approximately proportional to $(E^{\uparrow} - E^{\downarrow})q$. One hence expects q to be correlated with $E^{\uparrow} - E^{\downarrow}$, but not with $E^{\uparrow} + E^{\downarrow}$.

$q - \varphi_0$, $j_e - q$, $E_{cont} - t$ (Fig. V.35 f) These represent examples of unsignificant correlations.

V.5.2.3 Statistically uncorrelated parameters

An obvious way to overcome the difficulties connected with the interpretation of correlated parameters is to construct statistically uncorrelated parameters. Conceptually, this is straightforward, requiring the diagonalization of the covariance matrix $\underset{\sim}{R}$. There are no computational problems involved, and so we have always performed this calculation. The interpretation of the new parameters, however, represents a severe problem. We found ourselves unable to attribute a distinct physical meaning to linear combinations of so different quantities as energies, shape parameters, directions etc.

It is nevertheless possible to define physically meaningful parameters which are at least much less correlated than the original ones. For example, from our experience it would have been more optimal to choose the total energy $E = E^{\uparrow} + E^{\downarrow}$ and the ratio $(E^{\uparrow} - E^{\downarrow})/(E^{\uparrow} + E^{\downarrow})$ instead of $E^{\uparrow}$ and $E^{\downarrow}$. Also, the use of the rotary instead of Cartesian representation would reduce parameter correlations since then different parameter groups (isotropy, symmetry etc.) are essentially determined from different subsets of the data.

Finally, we ask for the relative importance of the different parameters (or parameter combinations). Formally, this is found by ordering the uncorrelated parameters with increasing variance, provided the original parameters are scaled properly to have the same dimension and order of magnitude. It was found that the original parameters fell essentially into three groups:

1) $E^{\uparrow}$, $E^{\downarrow}$, E_{cont}, E_c These parameters were most strongly determined. From these, the combination $E^{\uparrow} + E^{\downarrow}$ ($+ E_{cont}$) was found to be the most important parameter in almost all runs, but even $E^{\uparrow} - E^{\downarrow}$ was usually more important than the parameters of the following groups.

2) j_e, j_p, s, q, ξ, (φ_0) These parameters (or some linear combinations of them) were on intermediate positions; usually they could be well determined.

3) t, σ_f, a, (φ_0) These parameters were determined least of all. Sometimes they caused the least-square problem to be ill-conditioned.

These results are supported by Fig. V.36 where ε^2 is plotted as a function of each of the parameters, all other parameters held constant. The curvature of these curves at the minimum characterizes the standard deviations of the parameters.

V.6. Some analytical features of spectra and coherences

Here we present some analytical results for spectra and coherences which we used in this part. A detailed discussion of these quantities for the spectrum $A(\lambda) \propto (1+\lambda^2)^{-1}$ (i.e. s = 2, t = 2) has been reported by Desaubies (1977). Our analysis will mainly be done for GM 75 (i.e. s = 1 and t arbitrary).

V.6.1 Towed spectrum (derivation of V.55)

The one-sided towed spectrum is given by (Garrett and Munk 1972)

$$P_{ss}(\alpha_1,z) = \frac{2}{\pi} \int_f^N d\omega\, E(\omega,z) \frac{\omega^2-f^2}{\omega^2(N^2-f^2)} \int_{\alpha_1}^{\infty} d\alpha \frac{A(\alpha,\omega)}{(\alpha^2-\alpha_1^2)^{1/2}} \qquad (V.96)$$

For the model spectrum V.34 we find

$$P_{ss}(\alpha_1,z) = \frac{2}{\pi} (N^2-f^2)^{-1} \int_f^N d\omega\, E(\omega,z) \frac{\omega^2-f^2}{\omega^2 \alpha_*} \int_{\alpha_1/\alpha_*}^{\infty} d\lambda \frac{A(\lambda)}{[\lambda^2-(\alpha_1/\alpha_*)^2]^{1/2}} \qquad (V.97)$$

For $\alpha_1/\alpha_*(N) = \alpha_1/k_* \gg 1$ this becomes

$$P_{ss}(\alpha_1,z) \approx \frac{2}{\pi} \frac{I(t,s)}{N^2-f^2} B(\tfrac{t}{2},\tfrac{1}{2})\, \alpha_1^{-t} \int_f^N d\omega\, E(\omega,z) \frac{\omega^2-f^2}{\omega^2} \alpha_*^{t-1} \qquad (V.98)$$

or

$$P_{ss}(\alpha_1,z) \approx \alpha_1^{-t}\, \hat{\alpha}^{t-1} \langle \zeta^2 \rangle \frac{8}{\pi^2} I(t,s)\, B(\tfrac{t}{2},\tfrac{1}{2}) \int_1^{N/f} dx\, x^{-3} (x^2-1)^{t/2} \qquad (V.99)$$

where $B(.,.)$ is the Beta function and

$$\hat{\alpha} = k_* f/N = j_* \frac{\pi f}{b N_0} \qquad (V.100)$$

The integral in (V.99) can be expressed by a Beta function for $t < 2$. For $t > 2$ we did not find a general representation (note that the integral depends on the upper limit N/f). For the specific values t = 2 and t = 3 we obtain

$$\int_0^{N/f} dx\, x^{-3}(x^2-1)^{t/2} = \begin{cases} \frac{1}{2} + \ln\frac{N}{f} & , t = 2 \\ \frac{3}{2}\frac{N}{f} & t = 3 \end{cases} \qquad (V.101)$$

which has been used in Fig. V.8.

V.6.2 Vertical coherence

V.6.2.1 GM 75

For the GM 75 model spectrum the vertical coherence (V.32) becomes

$$\gamma(y) = (t-1)\int_0^\infty d\lambda \frac{\cos y\lambda}{(1+\lambda)^t} \qquad (V.102)$$
$$= (t-1)\, y^{t-1}\left\{\cos y\, C(1-t,y) + \sin y\, S(1-t,y)\right\}$$

where $y = \beta_* \Delta z$ and C and S are the generalized Fresnel integrals

$$C(a,y) = \int_y^\infty dx\, x^{a-1}\cos x$$
$$S(a,y) = \int_y^\infty dx\, x^{a-1}\sin x \qquad (V.103)$$

After some algebra we get the more useful representation

$$\gamma(y) = (t-1)\, y^{t-1}\,\Gamma(1-t)\cos\left\{y + \frac{(t-1)\pi}{2}\right\}$$
$$- (t-1)\sum_{m=0}^{\infty}\frac{(-y)^m}{m!\,(m+1-t)}\cos\left\{y + \frac{m\pi}{2}\right\} \qquad (V.104)$$

which has been used in (V.56).

V.6.2.2 Spectrum with s = 2 and t arbitrary

The vertical coherence becomes in this case

$$\gamma(y) = I(t,2)\int_0^\infty d\lambda \frac{\cos y\lambda}{(1+\lambda^2)^{t/2}} \qquad (V.105)$$

and may be expressed in terms of the modified Bessel function K_μ

$$\gamma(y) = \frac{2^{1-\mu}}{\Gamma(\mu)}\, y^{\mu} K_{\mu}(y) \qquad \text{(V.106)}$$

with $\mu = (t-1)/2$. For t = 2,3,4 and 5 we find

$$\gamma(y) = \begin{cases} e^{-y} & , \ t = 2 \\ y K_1(y) & , \ t = 3 \\ e^{-y}(1+y) & , \ t = 4 \\ \frac{1}{2} y^2 K_2(y) & , \ t = 5 \end{cases} \qquad \text{(V.107)}$$

K_1 and K_2 are tabulated (Abramowitz and Stegun 1964); $\gamma(y)$ is plotted in Fig. V.4.

V.6.2.3 Asymptotic tail

An asymptotic tail $\propto \lambda^{-t}$ determines the behaviour of $\gamma(y)$ near the origin. If $A(\lambda) = A_0 \lambda^{-t}$ for $\lambda > \lambda_0$ we have

$$\gamma(y) = \int_0^{\lambda_0} d\lambda\, A(\lambda) \cos \lambda y \; + A_0 \int_{\lambda_0}^{\infty} d\lambda\, \lambda^{-t} \cos \lambda y \qquad \text{(V.108)}$$

The second integral is a generalized Fresnel integral (cf. V.103). After some algebra we find for t > 0

$$\gamma(y) = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} \left\{ \int_0^{\lambda_0} d\lambda\, \lambda^{2m} A(\lambda) - A(\lambda_0) \lambda_0 \frac{\lambda_0^{2m}}{2m+1-t} \right\} \qquad \text{(V.109)}$$

$$+ A(\lambda_0) \lambda_0 (\lambda_0 y)^{t-1} \Gamma(1-t) \cos\left[\frac{\pi}{2}(1-t)\right]$$

V.6.3 Horizontal coherence

V.6.3.1 GM 75

For the GM 75 model the horizontal coherence (V.29) becomes

$$\gamma(x) = (t-1) \int_0^\infty d\lambda \, \frac{J_0(\lambda x)}{(1+\lambda)^t} \qquad \text{(V.110)}$$

where $x = \alpha_* r$. For $t > -1/2$ this reduces to (Gradshteyn and Ryzhik 1965, eq. 6.563)

$$\gamma(x) = \frac{\pi(t-1)}{\Gamma(t)\sin[\pi(2-t)]} \sum_{m=0}^{\infty} \left(\frac{x}{2}\right)^{2m} \frac{1}{m!} \cdot$$

$$\cdot \left\{ (-1)^m \frac{\Gamma(2m+1)}{\Gamma(m+1)\Gamma(2m+2-t)} - \frac{\Gamma(t+m)\sin\left[\frac{\pi}{2}(2-t-m)\right]}{\Gamma^2\left(\frac{t+m+1}{2}\right)} \right\} \qquad \text{(V.111)}$$

which has been used in (V.56).

V.6.3.2 Asymptotic tail

If $H(\lambda) = H_0 \lambda^{-t}$ for $\lambda > \lambda_0$ we have

$$\gamma(x) = \int_0^{\lambda_0} d\lambda \, H(\lambda) J_0(\lambda x) + H_0 \left(\frac{2}{\pi}\right)^{1/2} x^{-1/2} \int_{\lambda_0}^\infty d\lambda \, \lambda^{-t-1/2} \cos\left(\lambda x - \frac{\pi}{2}\right) \qquad \text{(V.112)}$$

using the asymptotic representation for $J_0(\lambda x)$. For $x \to 0$ the second integral produces the singular term

$$\delta\gamma(x) \approx \left(\frac{2}{\pi}\right)^{1/2} H(\lambda_0) \lambda_0 (x\lambda_0)^{t-1} \Gamma\left(\tfrac{1}{2} - t\right) \cos\left(t\frac{\pi}{2}\right) \qquad \text{(V.113)}$$

if $t < 3/2$.

VI. CONCLUSIONS

The IWEX experiment provided a unique data set to study the structure and energy distribution of oceanic motions in the internal wave frequency band. The simultaneous measurement of horizontal current and vertical displacement and the space and time scales covered by the experiment were sufficient for the determination of the kinematic structure and the essential parameters of the energy spectrum.

The analysis was basically performed in two steps. We defined as a model class the kinematic structure of a motion field which is independent of the specific distribution of energy in wavenumber-frequency space, and as model a model class with a parameterized functional form of the energy distribution. First we tested whether or not a given model class is able to represent the observed fluctuations consistently. Having found a consistent model class we then determined the energy distribution by parametrization and least square fit. The combination of these two methods - the consistency tests and the least square fit - was found to be a powerful tool in the interpretation of spectral data of a vector time series. Parametrization methods alone are also able to find consistent models. For a data set as large as IWEX's, this would however be extremely (man and computer) time consuming.

The computer time to be used for the testing of model classes by means of consistency relations was negligible ($\sim 1\ ^{o}/_{oo}$) in comparison with the least square fit of the final model. Therefore consistency tests are an efficient method when searching for a consistent picture of the data. By successive splitting of complex tests one is able to locate those structures of the model class which are least consistent with the data and thereby comes to the construction of better model classes.

The least square fitting procedure, though formally reducing to the diagonalization of a small matrix, turned out to be a highly elaborate task. Both the large amount of data and the complicated kinematic structure of the model which was necessary to parameterize the data were responsible for this fact. In order to keep the computer time within reasonable limits and still keep the model as variable as necessary (and of course meet the required numerical accuracy) we had to construct a rather specialized least square fitting program, which was specifically tailored to our data organization and available computer facilities.

It should be mentioned that the least square fitting program could in no way be used as black box where one specifies a model puts in data and gets out parameters. The highly nonlinear models we had to use required a careful

choice of the start values for the parameters. Even then the minimum search had to be watched conscientiously. Start values were found by investigating the analytical properties of model classes and of models of the energy spectrum in close relation to corresponding features in the data. The least square fit was then used to make the conceptions obtained by simple analysis of only few data as precise as possible using all data. Beyond that the least square fit enabled us to decide whether or not a consistent model was found.

The model that gives a consistent representation of the IWEX measurements is a sum of internal waves and three kinds of contamination signals: a current contamination, a displacement contamination, by temperature finestructure and a coherent signal in the displacement estimate.

The internal wave part of the observed fluctuations could be fitted by a spectrum that differs from Garrett and Munk's (1975) spectrum only slightly but significantly. Horizontal isotropy and vertical symmetry is found except at low frequencies where we find energy propagation to the southeast and an energy excess propagating downwards through the water column. The wavenumber structure of the spectrum differs in so far from GM's that there is a marked peak at non-zero wavenumber corresponding approximately to the first or second mode, and that the bandwidth is not constant with frequency but decreases from about 20 modes at 0.1 cph to about 10 at high frequencies. The tide and its harmonics tend to have a much smaller bandwidth.

The current contamination and the displacement contamination by temperature had to be included to fit the observed two-scale behaviour of the coherences correctly. Current as well as temperature coherence drop rapidly within a few meters and then show the smooth decrease associated with the wave field. That this small scale drop is indeed associated with non-internal-wave energy can be learned from the fact that the horizontal coherence of temperature behaves smoothly at small separations.

The contamination of the displacement field by temperature finestructure was found to be in good agreement with McKean's theory (McKean 1974) as well as with the results of Joyce and Desaubies (1977), who used the temperature difference records of IWEX in their analysis.

The kinematic structure of the current contamination can be described as a horizontal motion with a vertical coherence scale of a few meters and a two-scale behaviour of the horizontal coherence: a 30 - 40% drop of coherence within a few meters and a second coherence scale larger than the resolution of our array. Physically this behaviour may be explained by a superposition of two processes: current finestructure advected past the sensors by the internal

wave field and small scale three dimensional turbulence in the layers of the density stratification (Müller, Olbers and Willebrand 1977). The energy in the current contamination is zero at low frequencies and increases continuously to values comparable to the energy in the wave field at higher frequencies.

The coherent contamination in the displacement field had to be introduced in order to model the energy correctly. Having fitted the coherences of current and displacement correctly there remains a disparity between the wave energies in the current and the displacement. This disparity may be described as a coherent signal of about 10 - 20% of the wave energy imposed on the displacement field. It is suggested that this contamination is not associated with a real oceanic field but represents a systematic bias of the displacement spectra due to an inconsistent value for the Brunt Väisälä frequency.

Acknowledgements

Thanks are expressed to all who made IWEX possible. We are particularly grateful to M. Briscoe, who made the cross-spectral data set available to us. We have benefitted greatly from discussions with M. Briscoe, Cl. Frankignoul, K. Hasselmann, T. Joyce, M. Noble and F. Schott. Numerical computations were carried out at the Rechenzentrum of the Kiel University, which compliantly made almost the entire core available for many nights. We would like to thank Susan Euteneuer for typing the various manuscripts. Two of the authors (P. M. and D. J. O.) are supported through the Sonderforschungsbereich 'Meeresforschung Hamburg' by the Deutsche Forschungsgemeinschaft.

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