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Consistency relations for internal waves*

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Abstract—A complete set of linearly independent relationships among the different cross spectral components obtained from pairs of moored instruments is derived which can be utilized to test whether or not the observed fluctuations within the internal wave frequency band represent a field of propagating internal waves. A further complete set of relationships is derived which enables to test whether or not the internal wave field is horizontally isotropic and (or) vertically symmetric. These relations are compared with corresponding relations for alternative models (standing internal wave modes, three-dimensional isotropic turbulence) and their capability to discriminate between the various models is investigated. The tests are applied to a set of data for which it is found that the observed fluctuations are consistent with both propagating and standing internal waves whereas isotropic turbulence must be rejected for the most part of the internal wave frequency band.

1. INTRODUCTION

THE EXISTENCE of internal waves and the distribution of internal wave energy among the different wavenumbers and frequencies need to be established but cannot rigorously be derived from existing observations. The present kind of measurement techniques only provide weighted projections of the four-dimensional energy density spectrum on to one-dimensional cuts in wavenumber–frequency space.

As regards the existence of internal waves it was realized by FOFONOFF (1969) that there exist relationships among the different cross spectral components obtained from time series of moored instruments which must be satisfied if the fluctuations represent a superposition of free linear internal waves. He found that the ratio of the horizontal and vertical kinetic energy density spectrum and the rotary coherence between the horizontal velocity components do not depend on the energy density spectrum of the internal wave field but on the frequency only. These relationships can be utilized to test whether or not observed fluctuations can be ascribed to linear internal waves. Applying these tests to site D data, Fofonoff concluded that the observed fluctuations in the internal wave frequency band were in fact consistent with internal wave motions. Later GOULD (1971) applied the same tests to measurements in the eastern North Atlantic and arrived at the same conclusion.

As a measure of the horizontal anisotropy of the internal wave field FOFONOFF (1969) introduced the collinear coherence and GONELLA (1972) the ellipse stability. Both quantities have been calculated from internal wave observations by various authors (cf. FRANKIGNOUL, 1974; SIEDLER, 1974a) but revealed variable results.

The first attempt to reconstruct the complete energy density spectrum of the internal wave field from the observed projections was made by GARRETT and MUNK (1972). Combining measurements from different locations, depths, and instruments they proposed an energy distribution which is consistent with most observations and which is believed to reflect the principal features

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of the mean internal wave field in the deep ocean. An updated version of their spectrum and a classification of the various measurement techniques can be found in GARRETT and MUNK (1975).

Deriving from inadequate observations a model spectrum is a standard problem of statistical inference and involves solving the 'inverse problem'. Various methods (e.g. least squares, maximum likelihood) exist. Given a class of model spectra described by a set of parameters, these methods allow the determination of the parameters by minimizing a prescribed error functional and the determination of the statistical significance of the parameters. Although conceptually straightforward, the inverse problem usually requires considerable computational effort and a large data set in order to yield reasonably significant results.

Before trying to determine the complete kinematical structure of fluctuations by inverse techniques, general hypotheses concerning their structure may be tested. Examples are the internal wave and isotropy tests of Fofonoff and Gonella. Such tests require less effort and are also applicable if the data set is too small to pose a meaningful inverse problem. These tests also provide insight into the structure of the fluctuations which might be utilized in solving the inverse problem.

This paper derives such tests for internal wave observations. It generalizes the concept of Fofonoff by deriving complete sets of tests and by considering the statistical variability of the data.

These tests originated from an examination of the basic assumptions of the Garrett and Munk model. Although Garrett and Munk start from vertically standing modes, they take locally defined vertical averages and smear out the modal structure into a continuum. This corresponds to a WKBJ approximation. Within the WKBJ framework the basic assumptions of the Garrett and Munk spectrum are:

(i) the observed fluctuations within the internal wave range represent realizations of a statistically stationary and homogeneous process;

(ii) the observed fluctuations represent a superposition of free linear internal waves;

(iii) the internal wave field is horizontally isotropic, i.e. the energy distribution is independent of the direction of the horizontal wavenumber vector;

(iv) the internal wave field is vertically symmetric, i.e. the energy distribution does not depend on the sign of the vertical wavenumber.

The last assumption is a consequence of the modal approach.

The stationarity and homogeneity of the data can be tested by classical statistical methods (cf. LEHMANN, 1959). For testing the other assumptions a complete set of linearly independent relationships among the different cross spectral conponents obtained from pairs of moored instruments is derived which enables to test whether or not the observed fluctuations represent linear internal waves. These tests include those given by Fofonoff. A further complete set of linear relationships is derived which enables to test whether or not the internal wave field is horizontally isotropic and (or) vertically symmetric.

The tests only state whether or not the observed fluctuations are consistent with the assumptions above and do not preclude any alternative assumptions. In order to investigate the potentiality of the tests they are compared with corresponding tests for alternative models and applied to a set of data.

2. LINEAR THEORY IN THE WKBJ APPROXIMATION

For our purposes the internal wave field is most conveniently described by its rotary velocity components

$$u_{\pm}(\mathbf{x}, t) = u_1(\mathbf{x}, t) \pm i\, u_2(\mathbf{x}, t)$$
$$u_0(\mathbf{x}, t) = u_3(\mathbf{x}, t), \qquad (2.1)$$

where u_1 describes the eastward, u_2 the northward, and u_3 the upward component of the velocity vector as functions of space, $\mathbf{x}$, and time, t. In

$$\Gamma^{AB}_{\nu\mu}(\omega) = \Lambda^{AB}_{\nu\mu}(\omega) - i\,\Psi^{AB}_{\nu\mu}(\omega)$$

$$= \frac{1}{2\pi}\int d\tau < u^A_\nu(t)\, u^B_\mu(t+\tau) > e^{-i\omega\tau} \quad (3.2)$$

between time series $u^A_\nu(t)$ and $u^B_\mu(t)$ at locations $\mathbf{x}^A$ and $\mathbf{x}^B = \mathbf{x}^A + \mathbf{r}$ is evaluated for $\omega \geq 0$ as

$$\Gamma^{AB}_{\nu\mu}(\omega) = \sum_{S=\pm 1}\int d^2\alpha\, \tfrac{1}{2}\, E(\mathbf{q})\, U_\nu(\mathbf{q}) U^*_{-\mu}(\mathbf{q}) e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (3.3)$$

Cross spectra obtained from moored sensors represent weighted projections of the energy density spectrum $E(\mathbf{q})$ on to the frequency axis. Spectra obtained by other measurement techniques are found to represent weighted projections on to other cuts in $\mathbf{q}$-space. For details, see GARRETT and MUNK (1975).

Cross spectra may also be defined by the relationship

$$\hat{\Gamma}^{AB}_{\nu\mu}(\omega)\,\delta(\omega - \omega') = < [u^A_\nu(\omega)]^*\, u^B_\nu(\omega') >,$$

where $\mathbf{u}(\omega)$ is the Fourier Transform of $\mathbf{u}(t)$. Since the components $u_\nu(t)$, $\nu = +, -$, are complex functions the two definitions are not equivalent. We have instead

$$\hat{\Gamma}^{AB}_{\nu\mu} = \Gamma^{AB}_{-\nu\mu} .$$

The cross spectral matrix for negative frequencies is determined by the relationship

$$\Gamma^{AB}_{\nu\mu}(-\omega) = [\Gamma^{AB}_{-\nu-\mu}(\omega)]^*, \quad (3.4)$$

which follows from the 'reality condition', $u_\nu(t) = u^*_{-\nu}(t)$. The statistical stationarity of the wave field implies

$$\Gamma^{BA}_{\mu\nu}(\omega) = \Gamma^{AB}_{\nu\mu}(-\omega), \quad (3.5)$$

which determines the cross spectral matrix if the order of the time series is interchanged. The cross spectral matrix $\Gamma^{AB}_{\nu\mu}(\omega)$ ($\nu, \mu = +, -, 0$) hence contains 18 different real functions of $\omega \geq 0$. If $\mathbf{r} = 0$, this number reduces to 9 because of the additional relationships

$$\Gamma_{\nu\mu}(\omega) = [\Gamma_{-\mu-\nu}(\omega)]^*. \quad (3.6)$$

If only the horizontal velocity components are measured the corresponding numbers are 8 and 4, respectively. A comprehensive listing of these numbers is part of Table 2.

4. CONSISTENCY RELATIONS

It was first noted by FOFONOFF (1969) that—besides the relationships (3.4) and (3.5)—there exist other linear relations which must be satisfied if the fluctuations represent linear internal waves. In order to generalize his findings we ask if there exist any linear relationships

$$\sum_{\nu,\mu} a_{\nu\mu}(\omega, \mathbf{r})\, \Gamma^{AB}_{\nu\mu}(\omega) + \sum_{\nu,\mu} b_{\nu\mu}(\omega, \mathbf{r})\, [\Gamma^{AB}_{\nu\mu}(\omega)]^* = 0 \quad (4.1)$$

which are satisfied for arbitrary energy density spectra $E(\mathbf{q})$. Explicitly (4.1) is given by

$$\sum_{s=\pm 1}\int d^2\alpha \tfrac{1}{2} E(\mathbf{q})\, \{e^{-i\mathbf{k}\cdot\mathbf{r}} \sum_{\nu,\mu} \alpha_{\nu\mu} U_\nu\, U^*_{-\mu} + e^{i\mathbf{k}\cdot\mathbf{r}} \sum_{\nu,\mu} b_{\nu\mu} U^*_\nu\, U_{-\mu}\} = 0. \quad (4.2)$$

Since $E(\mathbf{q})$ is an arbitrary function of $\mathbf{q}$, (4.1) is valid if and only if the expression in parenthesis vanishes. Since $\sum_{\nu,\mu} a_{\nu\mu}(\omega, \mathbf{r}) U_\nu(\omega, \varphi, s)\, U^*_{-\mu}(\omega, \varphi, s)$ and $\sum_{\nu,\mu} b_{\nu\mu}(\omega, \mathbf{r})\, U^*_\nu(\omega, \varphi, s)\, U_{-\mu}(\omega, \varphi, s)$ do not depend on the modulus of the horizontal wavenumber vector, whereas $e^{\pm i\mathbf{k}\cdot\mathbf{r}}$ does, this expression vanishes if and only if both terms vanish individually. Hence we have as a necessary condition for the validity of (4.1)

$$\sum_{\nu,\mu} a_{\nu\mu} U_\nu\, U^*_{-\mu} = 0 \quad (4.3a)$$

and

$$\sum_{\nu,\mu} b_{\nu\mu} U^*_\nu\, U_{-\mu} = 0. \quad (4.3b)$$

We can confine ourselves to the condition (4.3a) since the complex conjugate condition (4.3b) does not yield any additional linearly independent relationships.

Explicitly the matrix $B_{\nu\mu} = U_\nu U^*_{-\mu}$ is given by

$$(B_{\nu\mu}) = \frac{\omega^2 - f^2}{N^2 - f^2}$$

$$\begin{pmatrix} F(\omega)F(-\omega)e^{2i\varphi} & F^2(\omega) & -sF(\omega)e^{i\varphi} \\ F^2(-\omega) & F(\omega)F(-\omega)e^{-2i\varphi} & -sF(-\omega)e^{-i\varphi} \\ -sF(-\omega)e^{i\varphi} & -sF(\omega)e^{-i\varphi} & 1 \end{pmatrix}. \quad (4.4)$$

The terms proportional to $e^{im\varphi}$ ($m = -2, -1, 0, 1, 2$) transform differently under rotation about the x_3-axis. They represent the irreducible constituents of the two-dimensional rotation group. Hence the conditions (4.3) must be valid individually for terms within a single class m. If there are n terms in a class, there will exist $n - 1$ linearly independent relationships.

Specifically, the following set of linearly independent relationships is obtained

$$m = 0: \quad \Delta_1^{AB} = \Gamma_{-+}^{AB} + \Gamma_{+-}^{AB} - \{F^2(-\omega) + F^2(\omega)\}\,\Gamma_{00}^{AB} = 0 \quad (4.5a)$$

$$\Delta_2^{AB} = F^2(-\omega)\,\Gamma_{+-}^{AB} - F^2(\omega)\Gamma_{-+}^{AB} = 0 \quad (4.5b)$$

$$m = 1: \quad \Delta_3^{AB} = F(\omega)\,\Gamma_{0+}^{AB} - F(-\omega)\,\Gamma_{+0}^{AB} = 0 \quad (4.5c)$$

$$m = -1: \quad \Delta_4^{AB} = F(-\omega)\,\Gamma_{0-}^{AB} - F(\omega)\,\Gamma_{-0}^{AB} = 0. \quad (4.5d)$$

These relationships define 8 consistency relations (counting the real and imaginary part separately) which can be utilized to test whether or not observed fluctuations represent internal waves. If only the horizontal velocity components are measured the number of consistency relations reduces to 2. If $\mathbf{r} = 0$, the number of consistency relations is 4 and 1, respectively. The relations found by FOFONOFF (1969) correspond to $Re\{\Delta_1\} = 0$ and $Re\{\Delta_2\} = 0$.

Although some of the consistency relations can be interpreted in suggestive terms (e.g. the relation $Re\{\Delta_1\} = 0$ compares the horizontal and vertical kinetic energy density) they follow from the specific structure of the amplitude factors (2.4). Cross spectra $\Gamma_{\nu\mu}^{AB}(\omega)$ describe the amplitude and phase relations of the frequency components in the series $u_{-\nu}^A(t)$ and $u_\mu^B(t)$. For all internal waves which contribute to a certain frequency component equation (2.4) or (4.4) states that

(i) some of the phase differences between velocity components are equal (e.g. the phase lag between U_- and U_0 equals the phase lag between U_0 and U_+);
(ii) the amplitudes of the velocity components can be made equal by multiplication with the known functions $F(\omega)^{-1}$, $F(-\omega)^{-1}$, and 1, respectively.

Hence some of the cross spectra become equal when appropriately normalized. This fact is expressed in the consistency relations.

5. INDEPENDENT MOMENTS

If the observed fluctuations represent internal waves, only 10 linearly independent cross spectral components are measured, namely the real and imaginary part of the normalized moments

$$M_m^{AB}(\omega) = \Sigma \int d^2\alpha \tfrac{1}{2} E(\mathbf{q}) s^m e^{im\varphi} e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (5.1)$$

where

$$\begin{aligned} \Gamma_{--}^{AB} &\sim M_{-2}^{AB} \\ \Gamma_{0-}^{AB},\ \Gamma_{-0}^{AB} &\sim M_{-1}^{AB} \\ \Gamma_{00}^{AB},\ \Gamma_{+-}^{AB},\ \Gamma_{-+}^{AB} &\sim M_0^{AB} \\ \Gamma_{0+}^{AB},\ \Gamma_{+0}^{AB} &\sim M_1^{AB} \\ \Gamma_{++}^{AB} &\sim M_2^{AB} \end{aligned} \quad (5.2)$$

The significance of these moments can easily be interpreted for $\mathbf{r} = 0$. The spectrum $E(\omega, \alpha, \varphi, s)$ can be decomposed into its *even* and *odd* component with respect to the sign of the vertical wavenumber

$$E^{e,o}(\omega, \alpha, \varphi) = \tfrac{1}{2}\{E(\omega, \alpha, \varphi, s = +1) \pm E(\omega, \alpha, \varphi, s = -1)\} \qquad (5.3)$$

and the φ-dependence can be expanded into a Fourier series

$$E^{e,o}(\omega, \alpha, \varphi) = \sum_{m=-\infty}^{\infty} A_m^{e,o}(\omega, \alpha)\, e^{im\varphi} \quad . \qquad (5.4)$$

[Here $E(\omega, \alpha, \varphi, s)$ is normalized according to

$$\sum_s \int d^2\alpha E(\omega, \alpha, s) = \frac{1}{2\pi} \sum_s \int d\alpha d\varphi\, E(\omega, \alpha, \varphi, s)].$$

The moments M_m can now be expressed in terms of the Fourier coefficients $A_m^{e,o}(\omega, \alpha)$ as

$$M_0(\omega) = \int d\alpha\, A_0^e(\omega, \alpha)$$
$$M_{\pm 1}(\omega) = \int d\alpha\, A_{\mp 1}^o(\omega, \alpha)$$
$$M_{\pm 2}(\omega) = \int d\alpha\, A_{\mp 2}^e(\omega, \alpha). \qquad (5.5)$$

Hence, M_0 measure the energy in the isotropic even part of the spectrum whereas only energy from the anisotropic odd and anisotropic even part contributes to $M_{\pm 1}$ and $M_{\pm 2}$, respectively. No information about the isotropic odd part, A_0^o, can be inferred from measurements at a single instrument.

6. ISOTROPY AND SYMMETRY RELATIONS

Further linear relationships hold among the independent moments if the energy density spectrum $E(\mathbf{q})$ is vertically symmetric, i.e. independent of s, or horizontally isotropic, i.e. independent of φ.

The analysis will only be presented for the case that the instruments are vertically separated, $\mathbf{r} = (0, 0, r_3)$. The results for slant, horizontal and no separation are listed in Table 1.

If $E(\mathbf{q})$ is vertically symmetric, the normalized moments reduce to

$$M_m^{AB} = \int d^2\alpha \tfrac{1}{2} E(\omega, \boldsymbol{\alpha}) e^{im\varphi} \sum_s s^m \exp(-is|k_3|r_3) \qquad (6.1)$$
$$= \int d^2\alpha \tfrac{1}{2} E(\omega, \boldsymbol{\alpha}) e^{im\varphi} \begin{cases} 2\cos|k_3|r_3 \\ -2i\sin|k_3|r_3. \end{cases}$$

Hence the following 5 relationships are satisfied for an arbitrary but symmetric internal wave field

$$Im\{M_0^{AB}\} = 0$$
$$M_1^{AB} + [M_{-1}^{AB}]^* = 0 \qquad (6.2)$$
$$M_2^{AB} - [M_{-2}^{AB}]^* = 0.$$

If $E(\mathbf{q})$ is horizontally isotropic the normalized moments reduce to

$$M_m^{AB} = \sum_s \int d\alpha \tfrac{1}{2} E(\omega, \alpha, s)\; s^m \exp(-is|k_3|r_3) \cdot \frac{1}{2\pi} \int d\varphi e^{im\varphi}$$
$$= \sum_s \int d\alpha \tfrac{1}{2} E(\omega, \alpha, s) s^m \exp(-is|k_3|r_3) \cdot \begin{cases} 0 & m = \pm 1, \pm 2 \\ 1 & m = 0 \end{cases} \quad . \qquad (6.3)$$

yielding the 8 linearly independent relationships

$$M_{-2}^{AB} = M_{-1}^{AB} = M_1^{AB} = M_2^{AB} = 0. \qquad (6.4)$$

Finally, if the wave field is both isotropic and symmetric, 9 independent relations

$$M_{-2}^{AB} = M_{-1}^{AB} = Im\{M_0^{AB}\} = M_1^{AB} = M_2^{AB} = 0 \qquad (6.5)$$

are obtained. In this case the only non-vanishing moment is

Table 1. *Symmetry and isotropy relations for various separations of the instruments.*

Separation:	$\mathbf{r} \neq 0$, $\mathbf{r} = (d\cos\psi, d\sin\psi, r_3)$ *Slant separation* $d, r_3 \neq 0$	*Horizontal separation* $r_3 = 0$	*Vertical separation* $d = 0$	$\mathbf{r} = 0$ $(M_m = M_{-m}^*)$
Relations satisfied by a vertically symmetric internal wave field		$M_1^{AB} = 0$ $M_{-1}^{AB} = 0$	$Im\{M_0^{AB}\} = 0$ $M_1^{AB} + [M_{-1}^{AB}]^* = 0$ $M_2^{AB} - [M_{-2}^{AB}]^* = 0$	$M_1 = 0$
Relations satisfied by a horizontally isotropic internal wave field	$e^{-i\psi}M_1^{AB} - e^{i\psi}M_{-1}^{AB} = 0$ $e^{-2i\psi}M_2^{AB} - e^{2i\psi}M_{-2}^{AB} = 0$	$Im\{M_0^{AB}\} = 0$ $M_1^{AB} + [M_{-1}^{AB}]^* = 0$ $Re\{e^{-i\psi}M_1^{AB}\} = 0$ $M_2^{AB} - [M_{-2}^{AB}]^* = 0$ $Im\{e^{-2i\psi}M_2^{AB}\} = 0$	$M_1^{AB} = 0$ $M_{-1}^{AB} = 0$ $M_2^{AB} = 0$ $M_{-2}^{AB} = 0$	$M_1 = 0$ $M_2 = 0$
Relations satisfied by a symmetric and isotropic internal wave field	$Im\{M_0^{AB}\} = 0$ $M_1^{AB} - [M_{-1}^{AB}]^* = 0$ $Im\{e^{-i\psi}M_1^{AB}\} = 0$ $M_2^{AB} - [M_{-2}^{AB}]^* = 0$ $Im\{e^{-2i\psi}M_2^{AB}\} = 0$	$Im\{M_0^{AB}\} = 0$ $M_1^{AB} = 0$ $M_{-1}^{AB} = 0$ $M_2^{AB} - [M_{-2}^{AB}]^* = 0$ $Im\{e^{-2i\psi}M_2^{AB}\} = 0$	$Im\{M_0^{AB}\} = 0$ $M_1^{AB} = 0$ $M_{-1}^{AB} = 0$ $M_2^{AB} = 0$ $M_{-2}^{AB} = 0$	$M_1 = 0$ $M_2 = 0$
Non-vanishing moments in case of symmetry and isotropy	$Re\{M_0^{AB}\}$ $Re\{e^{-i\psi}M_1^{AB}\}$ $Re\{e^{-2i\psi}M_2^{AB}\}$	$Re\{M_0^{AB}\}$ $Re\{e^{-2i\psi}M_2^{AB}\}$	$Re\{M_0^{AB}\}$	$Re\{M_0\}$

$$Re\{M_0^{AB}\} = \int d\alpha\, E(\omega,\alpha)\, J_0(\alpha d) \cos|k_3| r_3$$

$$Re\{e^{-i\psi}M_1^{AB}\} = -\int d\alpha\, E(\omega,\alpha)\, J_1(\alpha d) \sin|k_3| r_3$$

$$Re\{e^{-2i\psi}M_2^{AB}\} = -\int d\alpha\, E(\omega,\alpha)\, J_2(\alpha d) \cos|k_3| r_3$$

J_n = Bessel function of order n

Table 2. *Number of independent moments and relations. The first column refers to the case where all three velocity components are measured. The second column refers to the case where only the horizontal components are measured.*

	Slant separation		*Horizontal separation*		*Vertical separation*		*No separation*	
Different cross spectral components	18	8	18	8	18	8	9	4
Independent consistency relations	8	2	8	2	8	2	4	1
Independent internal wave moments	10	6	10	6	10	6	5	3
Independent relations satisfied by a symmetric and isotropic internal wave field	7	4	8	4	9	5	4	2
Non-vanishing moments in case of symmetry and isotropy	3	2	2	2	1	1	1	1

$$Re\,\{M_0^{AB}\} = \int d\alpha\, E(\omega,\alpha) \cos|k_3|r_3, \qquad (6.6)$$

from which the ω and α-dependence of $E(\omega, \alpha)$ may be inferred. All numbers given above and in Table 2 represent the maximum number of linearly independent relationships as may again be inferred from group theoretical arguments.

7. CARTESIAN REPRESENTATION

The consistency, symmetry and isotropy relations are derived and listed for the case that the velocity field is given by its rotary velocity components. The rotary representation is most convenient for the general analysis since it is only in this representation that the different cross spectral components represent irreducible constituents of the two-dimensional rotation group and the number of independent relations can easily be determined. For applications it might be more convenient to use different representations. The usual Cartesian representation is obtained by the linear transformation

$$\Gamma_{\nu\mu}^{AB} = \sum_{i,j=1}^{3} H_{\nu i}\, H_{\mu j}\, \Gamma_{ij}^{AB}\,, \qquad (7.1)$$

with

$$(H_{\nu i}) = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.2)$$

The consistency relations for $\mathbf{r} = 0$ then take the form

$$\Delta_1 = \Lambda_{11} + \Lambda_{22} - f(\omega)\Lambda_{33} = 0 \qquad (7.3a)$$

$$\Delta_2 = g(\omega)(\Lambda_{11} + \Lambda_{22}) + 2\Psi_{12} = 0 \qquad (7.3b)$$

$$\Delta_3 = \Lambda_{23} + h(\omega)\Psi_{13} = 0 \qquad (7.3c)$$

$$\Delta_4 = \Lambda_{13} - h(\omega)\Psi_{23} = 0, \qquad (7.3d)$$

with

$$f(\omega) = \frac{N^2 - \omega^2}{\omega^2 - f^2}\,\frac{\omega^2 + f^2}{\omega^2}, \qquad g(\omega) = 2\,\frac{\omega f}{\omega^2 + f^2},$$

$$h(\omega) = \frac{\omega}{f}. \qquad (7.4)$$

The isotropy and symmetry tests $M_{+2} = 0$ and $M_{+1} = 0$ become

$$\begin{aligned} \Delta_5 &= \Lambda_{11} - \Lambda_{22} = 0 \\ \Delta_6 &= \Lambda_{12} = 0 \end{aligned} \qquad (7.5a)$$

$$\begin{aligned} \Delta_7 &= \Lambda_{13} = \Psi_{23} = 0 \\ \Delta_8 &= \Lambda_{23} = \Psi_{13} = 0. \end{aligned} \qquad (7.5b)$$

It might also be more convenient to use nonlinear combinations of the relations presented here as e.g. the collinear coherence (FOFONOFF, 1969) and ellipse stability (GONELLA, 1972).

8. CONFIDENCE INTERVALS

Time series of finite length only provide estimates $C_{ij}^{AB}(\omega) = P_{ij}^{AB}(\omega) - i\, Q_{ij}^{AB}(\omega)$ of the cross spectral components. These estimates can be regarded as realizations of estimators $\hat{C}_{ij}^{AB}(\omega)$ which are random variables. Instead of the linear combinations $\Delta = \sum_{i,j}(a_{ij}\, \Gamma_{ij}^{AB} + b_{ij}\, [\Gamma_{ij}^{AB}]^*)$, we have to consider random variables

$$\hat{D} = \sum_{ij} (a_{ij}\, \hat{C}_{ij}^{AB} + b_{ij}\, [\hat{C}_{ij}^{AB}]^*) \qquad (8.1)$$

and have to decide whether or not realizations of these random variables differ significantly from zero. For this we compare $\hat{D}$ with its standard deviation and consider the normalized random variable

$$\hat{T} = \hat{D}/\hat{S}, \qquad (8.2)$$

where $\hat{S}$ is an estimator of the standard deviation of $\hat{D}$. The covariance matrices of the cross spectral estimators $\hat{C}_{ij}^{AB}$ can be approximated by (cf. GOODMAN, 1957; JENKINS and WATTS, 1968)

$$\begin{aligned} \mathrm{COV}[\hat{C}_{ij}^{AB}, \hat{C}_{kl}^{AB}] &\approx \frac{2}{n}\, \Gamma_{il}^{AB}\Gamma_{kj}^{AB} \\ \mathrm{COV}[\hat{C}_{ij}^{AB}, [\hat{C}_{kl}^{AB}]^*] &\approx \frac{2}{n}\, \Gamma_{ik}^{AA}\, \Gamma_{lj}^{BB}, \end{aligned} \qquad (8.3)$$

where n is the equivalent number of degrees of freedom. Accordingly

$$\hat{A}_{ijkl} = \frac{2}{n} \hat{C}^{AB}_{il} \quad \hat{C}^{AB}_{kj}$$
$$\hat{B}_{ijkl} = \frac{2}{n} \hat{C}^{AA}_{ik} \quad \hat{C}^{BB}_{lj} \tag{8.4}$$

may be used as estimators of the covariance matrices and the standard deviation of $\hat{D}$ may be estimated by

$$\hat{S} = [\sum_{i,j,k,l} (a_{ij}a_{kl}\hat{A}_{ijkl} + a_{ij}b_{kl}\hat{B}_{ijkl} + b_{ij}a_{kl}\hat{B}_{ijkj} + b_{ij}b_{kl}\hat{A}^{*}_{ijkl})]^{\frac{1}{2}}. \tag{8.5}$$

Given the probability distribution functions of the estimators $\hat{C}^{AB}_{ij}$, $\hat{A}_{ijkl}$ and $\hat{B}_{ijkl}$ the probability distribution function $f_\Delta(T)$ of $\hat{T}$ can be determined and appropriate confidence intervals can be constructed. For simplicity we assume that $f_\Delta(T)$ can be approximated by a normal distribution $N_{\Delta,1}(T)$ with mean Δ and variance 1. If the number of degrees of freedom is sufficiently large this approximation may be justified by the Central Limit Theorem.

Confidence intervals are defined by

$$Pr\{-T_\alpha < T - \Delta \leq T_\alpha\} = \int_{\Delta - T_\alpha}^{\Delta + T_\alpha} dT f_\Delta(T) = 1 - \alpha, \tag{8.6}$$

where α usually is a small number, say 0·05 or 0·01. Values of T_α are tabulated for the normal distribution.

In our case we want to test whether or not $\Delta = 0$. Hence, whenever the observed realization T falls outside the $1 - \alpha$ confidence interval, i.e.

$$|T| \geq T_\alpha \tag{8.7}$$

we reject the hypothesis $\Delta = 0$ since there is only a small probability α that a realization satisfies (8.7) when the hypothesis $\Delta = 0$ is true. Whenever the observed realization falls inside the confidence interval, $|T| < T_\alpha$, no definite statement can be made except that the hypothesis $\Delta = 0$ is consistent with the observed realization. There exists no sensible test which may lead to a rejection of $\Delta \neq 0$ and hence to a 'statistical verification' of the hypothesis $\Delta = 0$.

The usual way out of this dilemma is either to consider alternative models of the fluctuations and test whether or not they must be rejected or to enlarge the hypothesis $\Delta = 0$ to $|\Delta| < \Delta_0$ with $\Delta_0 > 0$. In the latter case there does exist a sensible test which may lead to a rejection of $|\Delta| \geq \Delta_0$ and hence to a statistical verification of the hypothesis $|\Delta| < \Delta_0$ (cf. LEHMANN, 1959). However, in order to give any physical meaning to the number Δ_0, say in the consistency relations, one has to relate Δ_0 to the amount of non internal-wave energy. This again requires a model of non internal-wave motion. Without specifying alternative models the relations derived can only be used to test whether or not the observed fluctuations are consistent with the assumed internal wave structure but do not preclude any alternative models.

9. ALTERNATIVE MODELS

In order to investigate the potentiality of the consistency relations we compare them with corresponding relations for alternative models.

Isotropic turbulence

The basic question is whether the observed fluctuations in the internal wave frequency band do represent internal waves or must be ascribed to turbulent motions. As the simplest model of turbulence we choose three-dimensional isotropic turbulence—hence neglecting the effect of stratification and rotation on the flow field. This might be reasonable for frequencies (much) larger than the Brunt–Väisäla frequency. Also, noise introduced into the measurement may be modeled in this way.

For isotropic turbulence the cross spectral matrix is of the form

$$\Gamma^{AB}_{ij} = A(r^2)\,\frac{r_i r_j}{r^2} + B(r^2)\,\delta_{ij}, \tag{9.1}$$

which follows from the invariance of the flow field under arbitrary rotations and reflections (cf. BATCHELOR, 1953). For $\mathbf{r} = 0$ isotropic turbulence may thus be characterized by the relations

$$\Lambda_{11} + \Lambda_{22} - 2\Lambda_{33} = 0 \qquad \Psi_{12} = 0 \tag{9.2a}$$

$$\Lambda_{11} - \Lambda_{22} = \Lambda_{12} = 0 \qquad \Lambda_{13} = \Psi_{13} = \Lambda_{23} = \Psi_{23} = 0\,. \tag{9.2b}$$

Similar relations exist for $\mathbf{r} \neq 0$.

Isotropic turbulence satisfies the consistency relations $\Delta_3 = 0$ and $\Delta_4 = 0$. Conversely, a symmetric and isotropic internal wave field satisfies the relations (9.2b). The corresponding tests do not discriminate between isotropic turbulence and a vertically symmetric and horizontally isotropic internal wave field. To what extent the remaining tests discriminate can be inferred by rewriting the relations (9.2a) as

$$\Delta_1 = \Lambda_{11} + \Lambda_{22} - f(\omega)\Lambda_{33} = [2 - f(\omega)]\,\Lambda_{33}$$
$$\Delta_2 = g(\omega)(\Lambda_{11} + \Lambda_{22}) + 2\psi_{12} = g(\omega)(\Lambda_{11} + \Lambda_{22}). \tag{9.3}$$

Hence, if the fluctuations represent isotropic turbulence the probability distribution of the normalized estimators $\hat{T}_{1,2} = \hat{D}_{1,2}/\hat{S}_{1,2}$ is given by a normal distribution with variance 1 and mean

$$\mu_1 = \frac{[2 - f(\omega)]\Lambda_{33}}{\mathrm{VAR}^{\frac{1}{2}}[\hat{P}_{11} + \hat{P}_{22} - f(\omega)\hat{P}_{33}]} = \frac{[2 - f(\omega)]\,\Lambda_{33}}{\left\{\frac{2}{n}(\Lambda_{11}^2 + \Lambda_{22}^2) + f^2(\omega)\Lambda_{33}^2\right\}^{\frac{1}{2}}} = \left(\frac{n}{2}\right)^{\frac{1}{2}} \frac{2 - f(\omega)}{\{2 + f^2(\omega)\}^{\frac{1}{2}}}$$

$$\mu_2 = \frac{g(\omega)\,(\Lambda_{11} + \Lambda_{22})}{\mathrm{VAR}^{\frac{1}{2}}[g(\omega)(\hat{P}_{11} + \hat{P}_{22}) + 2\hat{Q}_{12}]} = \frac{g(\omega)\,(\Lambda_{11} + \Lambda_{22})}{\left\{\frac{2}{n}[g^2(\omega)(\Lambda_{11}^2 + \Lambda_{22}^2) + 2\,\Lambda_{11}\Lambda_{22}]\right\}^{\frac{1}{2}}} = \left(\frac{n}{2}\right)^{\frac{1}{2}} \frac{2g(\omega)}{\{2 + 2g^2(\omega)\}^{\frac{1}{2}}}\,. \tag{9.4}$$

The tests T_1 and T_2 discriminate between internal waves and isotropic turbulence (in the sense that the acceptance of one model implies the rejection of the other model) if the confidence interval $(-T_\alpha, T_\alpha)$ for internal waves and the confidence interval $(\mu_{1,2} - T_\alpha, \mu_{1,2} + T_\alpha)$ for isotropic turbulence do not overlap.

Standing internal wave modes

Even if isotropic turbulence (or any other turbulent model) must be rejected there remains the question whether the observed fluctuations represent vertically propagating waves or vertically standing modes. In the modal representation the internal wave field is given by (cf. SCHOTT and WILLEBRAND, 1973)

$$u_\nu(\mathbf{x}, t) = \int_f^N d\omega \int d^2\alpha\, [a(\omega,\boldsymbol{\alpha})\tilde{U}_\nu(\omega,\boldsymbol{\alpha},x_3)e^{-i(\boldsymbol{\alpha}\cdot\mathbf{x}-\omega t)} + a^*(\omega,\boldsymbol{\alpha})\tilde{U}^*_{-\nu}\,(\omega,\boldsymbol{\alpha},x_3,)e^{i(\boldsymbol{\alpha}\cdot\mathbf{x}-\omega t)}], \tag{9.5}$$

with amplitude factors

$$\tilde{U}_+(\omega,\boldsymbol{\alpha},x_3) = \frac{\omega + f}{\omega}\, e^{i\varphi}\, \frac{1}{\alpha}\, \psi'(\omega,\alpha,x_3)$$

$$\tilde{U}_-(\omega,\boldsymbol{\alpha},\mathbf{x}_3) = \frac{\omega - f}{\omega}\, e^{-i\varphi}\, \frac{1}{\alpha}\, \psi'(\omega,\alpha,x_3)$$

$$\tilde{U}_0(\omega,\boldsymbol{\alpha},x_3) = -\,i\,\psi\,(\omega,\alpha,x_3). \tag{9.6}$$

The vertical eigenfunctions $\psi(\omega,\alpha,x_3)$ ($\psi' = \partial/\partial x_3\psi$) have to be determined by solving the eigenproblem

$$\frac{\partial^2}{\partial {x_3}^2}\,\psi + \alpha^2\, \frac{N^2(x_3) - \omega^2}{\omega^2 - f^2}\,\psi = 0, \tag{9.7}$$

with appropriate boundary conditions. The cross spectral matrix becomes

$$\tilde{\Gamma}^{AB}_{\nu\mu} = \int d^2\alpha \tfrac{1}{2}\tilde{E}(\omega,\boldsymbol{\alpha})\tilde{U}_\nu(\omega,\boldsymbol{\alpha},x_3^A) \tilde{U}^*_{-\mu}(\omega,\boldsymbol{\alpha},x_3^B)\, e^{-i\boldsymbol{\alpha}\cdot\mathbf{r}}, \tag{9.8}$$

where the matrix $\tilde{B}_{\nu\mu} = \tilde{U}_\nu \tilde{U}^*_{-\mu}$ is explicitly given by

$$(\tilde{B}_{\nu\mu}) = \frac{1}{\alpha^2}\,\frac{1}{\omega^2} \tag{9.9}$$

$$\begin{pmatrix} (\omega+f)(\omega-f)\,\psi'(x_3^A)\,\psi'(x_3^B)\,e^{2i\varphi}, & (\omega+f)^2\,\psi'(x_3^A)\,\psi'(x_3^B), & i\,\omega(\omega+f)\,\psi'(x_3^A)\,\alpha\,\psi(x_3^B)\,e^{i\varphi} \\ (\omega-f)^2\,\psi'(x_3^A)\,\psi'(x_3^B), & (\omega+f)(\omega-f)\,\psi'(x_3^A)\,\psi'(x_3^B)\,e^{-2i\varphi}, & i\,\omega(\omega-f)\,\psi'(x_3^A)\,\alpha\,\psi(x_3^B)\,e^{-i\varphi} \\ -i\,\omega(\omega-f)\,\alpha\,\psi(x_3^A)\,\psi'(x_3^B)\,e^{i\varphi}, & -i\,\omega(\omega+f)\,\alpha\,\psi(x_3^A)\psi'(x_3^B)\,e^{-i\varphi}, & \omega^2\alpha^2\,\psi(x_3^A)\,\psi(x_3^B) \end{pmatrix}$$

The modal spectrum $\tilde{\Gamma}^{AB}_{\nu\mu}$ represents an ensemble mean wherein the internal wave field is assumed to be statistically stationary and horizontally homogeneous whereas the WKBJ spectrum $\Gamma^{AB}_{\nu\mu}$ represents an ensemble mean wherein the internal wave field is additionally assumed to be vertically homogeneous. Hence $\tilde{\Gamma}^{AB}_{\nu\mu}$ depends on x_3^A and x_3^B whereas $\Gamma^{AB}_{\nu\mu}$ depends only on the difference $r_3 = x_3^B - x_3^A$. Furthermore, the vertical homogeneity implies that upward and downward propagating waves are uncorrelated [cf. equation (3.1)] whereas in the modal representation upward and downward propagating waves have the same amplitude and a fixed phase relationship in order to form standing modes. This statistical difference between the modal and WKBJ approach changes the phase relationships between the velocity components and hence the consistency relations.

Arguing the same way as in Section 4 we obtain the following relation for vertically standing modes

$$\tilde{\Delta}_2^{AB} = (\omega-f)^2\,\tilde{\Gamma}^{AB}_{+-} - (\omega+f)^2\tilde{\Gamma}^{AB}_{-+} = 0. \tag{9.10}$$

Additionally, we find for purely horizontal separation

$$\begin{aligned} (\omega-f)\tilde{\Gamma}^{AB}_{+0} + (\omega+f)\,\tilde{\Gamma}^{AB}_{0+} &= 0 \\ (\omega+f)\tilde{\Gamma}^{AB}_{-0} + (\omega-f)\,\tilde{\Gamma}^{AB}_{0-} &= 0, \end{aligned} \tag{9.11}$$

and for purely vertical separation

$$Im\,\{\tilde{\Gamma}^{AB}_{00}\} = 0$$

$$Im\,\{\tilde{\Gamma}^{AB}_{+-}\} = Im\,\{\tilde{\Gamma}^{AB}_{-+}\} = 0$$

$$\begin{aligned} \tilde{\Gamma}^{AB}_{++} - [\tilde{\Gamma}^{AB}_{--}]^* &= 0 \\ (\omega-f)\,\tilde{\Gamma}^{AB}_{+0} + (\omega+f)\,[\tilde{\Gamma}^{AB}_{-0}]^* &= 0 \\ (\omega+f)\,\tilde{\Gamma}^{AB}_{0+} + (\omega-f)\,[\tilde{\Gamma}^{AB}_{0-}]^* &= 0. \end{aligned} \tag{9.12}$$

The relation (9.10) is identical to the consistency relation $\Delta_2^{AB} = 0$ for vertically propagating waves. The consistency relations $\Delta_1^{AB} = 0$, $\Delta_3^{AB} = 0$ and $\Delta_4^{AB} = 0$ are not satisfied by standing modes. However, evaluating Δ_1^{AB} for standing modes we find

$$\tilde{\Delta}_1^{AB} = \tilde{\Gamma}^{AB}_{+-} + \tilde{\Gamma}^{AB}_{-+} - [F^2(-\omega) + F^2(\omega)]\,\tilde{\Gamma}^{AB}_{00}$$

$$= \left\{\left(\frac{\omega+f}{\omega}\right)^2 + \left(\frac{\omega-f}{\omega}\right)^2\right\} \int d^2\alpha \tfrac{1}{2}\tilde{E}(\omega,\boldsymbol{\alpha})e^{-i\boldsymbol{\alpha}\cdot\mathbf{r}} \cdot \frac{1}{\alpha^2}\left\{\psi'\psi' - \frac{N^2-\omega^2}{\omega^2-f^2}\,\alpha^2\psi\,\psi\right\}, \tag{9.13}$$

which may be arbitrarily small depending on the energy density spectrum $\tilde{E}$ and the eigenfunctions ψ. The same is true for Δ_3^{AB} and Δ_4^{AB}. Hence, the acceptance of propagating waves cannot safely be interpreted as a rejection of standing modes.

On the other hand, a vertically symmetric field of propagating waves satisfies all relations for standing modes. Hence we cannot discriminate between a vertically symmetric field of propagating waves and standing modes.

10. APPLICATION TO A SET OF DATA

The consistency, symmetry and isotropy tests for propagating internal waves and the corresponding tests for isotropic turbulence and standing modes are applied to a set of data obtained from an experiment described in detail by SIEDLER (1974a). Two moorings with current meters and temperature sensors were set near Site D ($f \approx 2\pi/18\ \mathrm{h}^{-1}$) at a depth of 2600 m with a horizontal separation of 920 m only. The instrument array was located in the main thermocline ($N \approx 2\pi\ \mathrm{h}^{-1}$). A schematic representation of the instrument configuration is shown in Fig. 1. For our analysis only the instruments 4182, 4184, 4185 and 4193 will be used. It should be noted that this data set, although of high interest because of the small horizontal separation, is of unusually short duration of 6 days only. The sampling interval is 15 min.

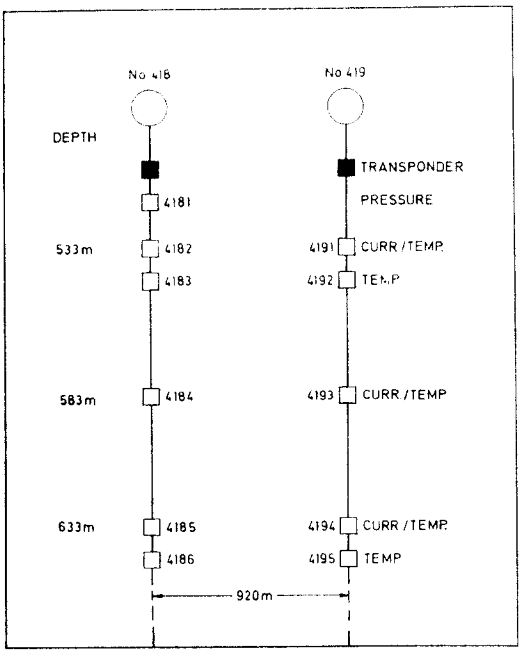

Fig. 1. Instrument configuration.

Time series of the vertical velocity are estimated from temperature time series $T(t)$ and the mean temperature gradient $\partial\bar{T}/\partial x_3$ using the relationship

$$\frac{\partial}{\partial t} T + u_3 \frac{\partial \bar{T}}{\partial x_3} = 0. \tag{10.1}$$

Contamination by microstructure and by horizontal advection is neglected (cf. SIEDLER, 1974b).

The time series are divided into 4 non-overlapping pieces of 36 h. The Fourier coefficients for each piece are computed using Fast Fourier Transform. Cross spectra obtained from the Fourier coefficients are averaged over the 4 pieces and over two adjacent frequency bands yielding $n = 16$ equivalent degrees of freedom.

Let us first consider the consistency, isotropy, and symmetry tests for $\mathbf{r} = 0$ as given by (7.3) and (7.5). Note that the consistency relations $\Delta_1 = 0$ and $\Delta_2 = 0$ involve the isotropic even part of the energy spectrum whereas the relations $\Delta_3 = 0$ and $\Delta_4 = 0$ involve the anisotropic odd part.

The relation $\Delta_1 = 0$ corresponds to the ratio of horizontal and vertical kinetic energy. For all 4 instruments the estimates T_1 are shown in Fig. 2. Also included are the 95% confidence intervals ($T_{0\cdot05} \approx 2$) for propagating internal waves (solid lines) and isotropic turbulence (dashed lines). Focussing on the internal wave test the figure shows that up to about $\omega = 0{\cdot}8\ N$ (with the exception at $\omega = f$) the estimates are within the 95% confidence interval and are hence consistent with our zero order WKBJ model. The inconsistency at $\omega = f$ may partly be due to our incorrect counting of the equivalent number of degrees of freedom. When adding up two adjacent estimates we have simply doubled the number of degrees of freedom. This is correct only for a white spectrum. In case of a pronounced inertial peak, the equivalent number of degrees of freedom and hence the estimate T_1 is reduced. Furthermore, the short piecelength of 36 h and the frequency averaging

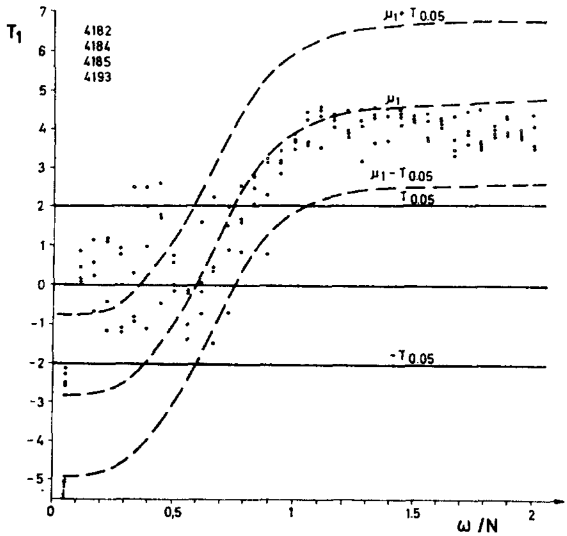

Fig. 2. Consistency test T_1 versus normalized frequency ω/N for the instruments indicated by the numbers. The solid lines represent the 95% confidence interval for propagating internal waves. The dashed lines represent the 95% confidence interval for isotropic turbulence. The arrow indicates the inertial frequency.

implies that the estimate at $\omega = f$ contains considerable energy from frequencies less than f.

Beyond $\omega = 0{\cdot}8\ N$ the estimates are outside the 95% confidence interval and are hence inconsistent with the assumed internal wave structure. The inconsistency within the interval $0{\cdot}8\ N \lesssim \omega \leq N$ can presumably be ascribed to the limitations of our WKBJ model. As the frequency approaches N, the zeroth order WKBJ solutions become less appropriate and break down at $\omega = N$ where they predict an infinite vertical wavelength. Using the more appropriate Airy function approximation near the turning point, DESAUBIES (1973) found that, instead of (7.3a), the ratio of the vertical and horizontal kinetic energy density spectrum becomes near $\omega = N$

$$\frac{\Lambda_{33}}{\Lambda_{11} + \Lambda_{22}} \approx \frac{\omega^2 - f^2}{\omega^2 + f^2} \frac{\omega^2}{N^2} 0{\cdot}4 \left(\frac{N\mu}{-N'}\right)^{2/3}, \quad (10.2)$$

where μ is the α-bandwidth of the internal wave spectrum. Using the Airy function approximation it can also be shown that in an exponentially stratified ocean our asymptotically valid zeroth order solutions are accurate within 10% for

$$\omega \leq \omega_0 = \left\{1 - \frac{3}{8}\left(\frac{2}{j_i\pi} \frac{N(x_3 = 0)}{N}\right)^{2/3}\right\}N \quad (10.3)$$

provided that $1 - \omega^2/N^2 \ll 1$. Here j_i denotes the equivalent mode number at $\omega = f$ (GARRETT and MUNK, 1972). Taking a representative value of $j_i = 10$ and

$$\frac{N(x_3 = 0)}{N} = 3$$

significant deviations from the zeroth order WKBJ relationships are expected for $\omega \geq \omega_0 = 0{\cdot}87N$ which seems to be consistent with our data.

Considering the confidence interval for isotropic turbulence we find that for frequencies beyond $\omega = N$ the estimates are consistent with isotropic turbulence. However, no effort has been made to resolve the question to what extent this result must be ascribed to noise in the measurements. The estimates are also consistent with isotropic turbulence for part of the internal wave frequency band. In this range we cannot reject isotropic turbulence. Although we cannot avoid the zero crossing of $\mu_1(\omega)$ the non-discriminating range can be decreased by increasing the equivalent number of degrees of freedom.

The relationship $\Delta_2 = 0$ corresponds to the rotary coherence. The estimates T_2 and the confidence intervals for internal waves and isotropic turbulence are shown in Fig. 3. For all frequencies resolved by the experiment the estimates are consistent with both internal waves and isotropic turbulence. In order that this test discriminates between internal waves and isotropic turbulence a much higher number of degrees of freedom is necessary. The confidence intervals $(-T_{0{\cdot}05}, T_{0{\cdot}05})$ for internal waves and $(\mu_2 - T_{0{\cdot}05}, \mu_2 + T_{0{\cdot}05})$ for isotropic turbulence do not overlap within the internal wave frequency band if

$$n \geq n_0 = 4\,\frac{N^2}{f^2}, \quad (10.4)$$

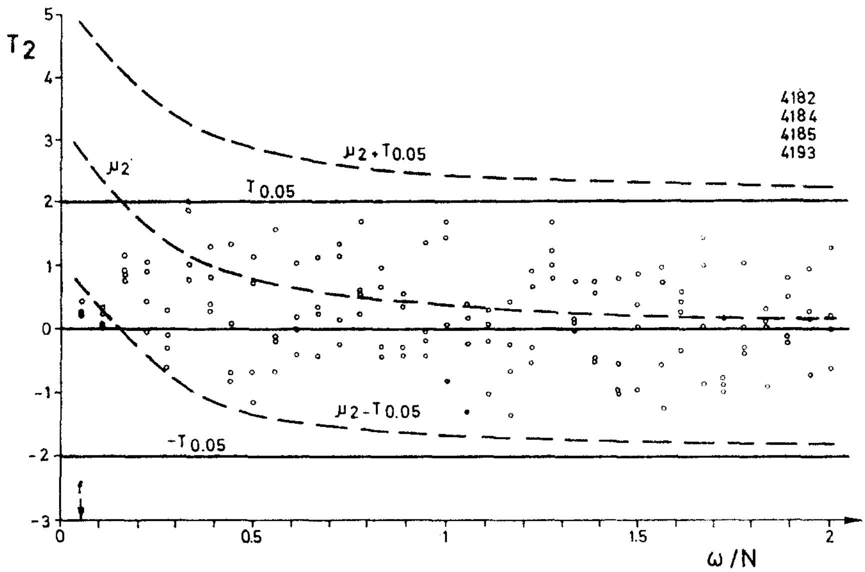

Fig. 3. Consistency test T_2 versus normalized frequency ω/N. Notation as in Fig. 2.

which is about $n \geq 1300$ in the main thermocline at Site D.

In our case the equivalent number of degrees of freedom can be increased by averaging cross spectra from the four instruments. This does not considerably improve the capability to discriminate but requires considerable computational effort since the measurements at the different instruments are not completely independent.

Figure 4 shows the estimates $|T_3|$ and $|T_4|$ corresponding to the relations $\Delta_3 = 0$ and $\Delta_4 = 0$. For all frequencies the estimates are well below the 95% confidence level and are hence consistent with both internal waves and isotropic turbulence for these tests do not discriminate between the two models. This result is not surprising because these tests only involve cross spectra (and no autospectra) and every motion with arbitrary autospectra but vanishing cross spectra will satisfy these tests.

Having convinced ourselves that the data are consistent with our zeroth order WKBJ model of the internal wave field for most of the internal wave frequency band the isotropy and symmetry relations (7.5a) and (7.5b) may be tested. The estimates $|T_{5,6}|$ are shown in Fig. 5. From this

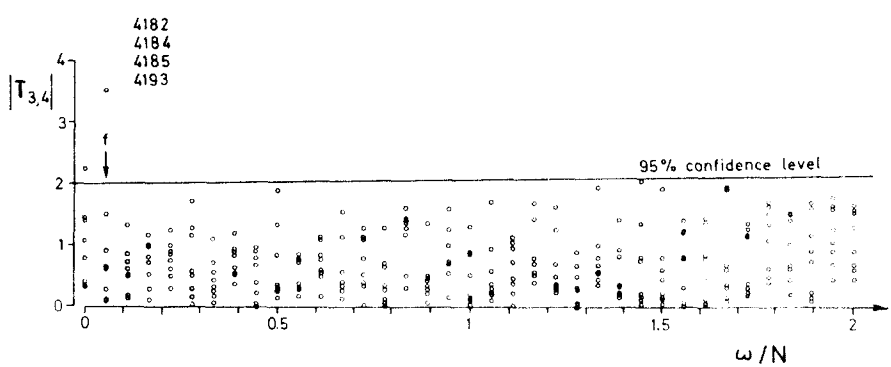

Fig. 4. Consistency tests $|T_{3,4}|$ versus normalized frequency ω/N.

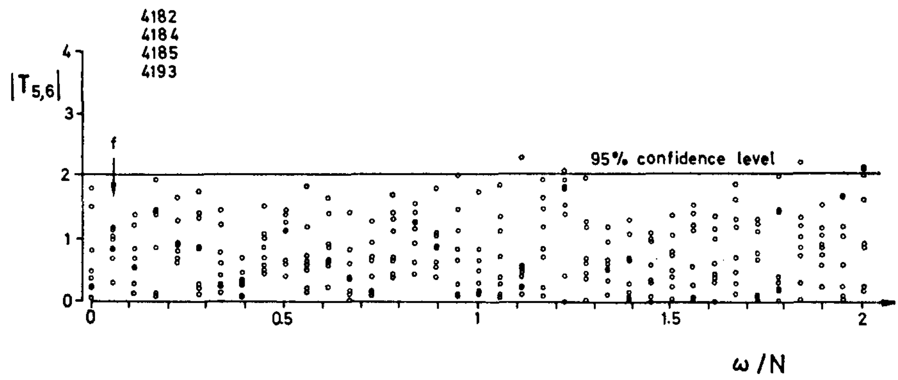

Fig. 5. Symmetry and isotropy tests $|T_{5,6}|$ versus normalized frequency ω/N.

figure and from the estimates $T_{7,8}$ (not shown) it can be concluded that the internal wave field is consistent with the assumption of isotropy and symmetry. More precisely, these tests only show that there is no energy in the Fourier coefficients $A^{o}_{\pm1}$ and $A^{e}_{\pm2}$ defined by (5.4). Since the isotropy and symmetry relations (7.5) are identical with the relations (9.2b) for isotropic turbulence we cannot reject isotropic turbulence on the basis of these tests. Furthermore, since we found a symmetric field of propagating waves we cannot reject standing modes.

The consistency tests for $\mathbf{r} \neq 0$ equally support the statement that our data are consistent with internal wave motion. As an illustration the test corresponding to the relation $Re\{\Delta_1^{AB}\} = 0$ is shown in Fig. 6. However, except for the first three or four internal wave frequencies the estimates of Γ_{+-}^{AB}, Γ_{-+}^{AB} and Γ_{00}^{AB} are found not to differ significantly from zero. This is consistent with the generally observed feature that—except near local N—the coherence drops toward higher frequencies (cf. SIEDLER, 1974a; WEBSTER, 1972). These estimates also satisfy the relations for isotropic turbulence. Rejection of isotropic turbulence is only found for the very low frequencies where the cross spectral estimates differ significantly from zero.

To summarize, the application of all tests for isotropic turbulence, propagating and standing waves leads to the following conclusion. The data are consistent with an isotropic and symmetric field of propagating internal waves with the exception of frequencies near $\omega = N$. This inconsistency is presumably due to the inaccuracy of our zeroth order WKBJ model near the turning point. The alternative model of isotropic turbulence must only be rejected for the first half of the internal wave frequency band, mainly due to our small number of degrees of freedom.

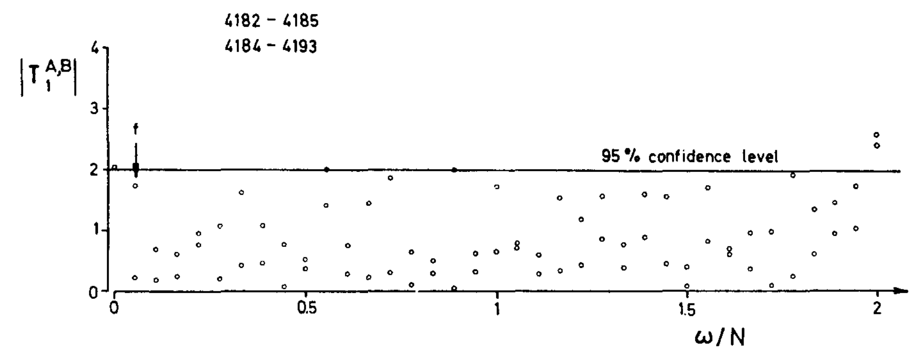

Fig. 6. Consistency test $|T_1^{AB}|$ versus normalized frequency ω/N for the instrument pairs indicated by the numbers.

Furthermore, no definite statement can be made whether propagating or standing modes are more appropriate to describe the observed fluctuations.

11. CONCLUSIONS

The complete set of consistency relations derived here may be utilized to test whether or not observed fluctuations in the internal wave frequency band are consistent with the assumption of the fluctuations representing a superposition of linear internal waves, and similar tests are given for the isotropy and symmetry of the wave field.

However, these tests not only provide insight into the kinematical structure of the internal wave field; they may also be utilized to determine its dynamics. As pointed out by WUNSCH (1975) only deviations from the mean state of the wave field lead to the detection of dynamical processes. Regarding the internal wave field as an equilibrium state wherein generation processes are balanced by transfer and dissipation processes, any variations in the external forcing fields lead to modulations of the internal wave field. These modulations can be detected by applying the tests presented here.

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