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On the Small-Amplitude Waves in an Inhomogeneous Moving Medium

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Along with a conventional Eulerian representation of equations of hydrodynamics (see [1, Chapter 1]), the Lagrangian or combined Eulerian–Lagrangian description is often used in studies of waves in flows (see, for example, [2, 3]). In this paper, a special version of the Eulerian–Lagrangian representation is suggested; this results in a considerable simplification of the boundary conditions and especially of the equations that define both the acoustic and the acoustic–gravity internal and surface waves of small amplitude against the background of the three-dimensionally nonuniform flow. A specific choice of dependent variables for the description of the wave field was presented as a result of an analysis of the acoustic quantities that are invariant [4–6] with respect to the interchange of the detector and the source of sound under the condition of the simultaneous and global reversal of the direction for the velocity vector of the flow unperturbed by the wave. A crucial feature of the proposed approach is the concept of the oscillatory displacement of the fluid particles, which is introduced below.

1. THE OSCILLATORY DISPLACEMENT OF PARTICLES

Let a fluid particle at the initial moment t_0 be located at a point $\mathbf{r}_0$ and at the moment $t > t_0$ be located at a point $\mathbf{r}(t)$. The Lagrangian characteristic of the fluid motion, i.e., the displacement of the particle $\tilde{\mathbf{a}}(t) \equiv \mathbf{r}(t) - \mathbf{r}_0$ is treated as a function of the Eulerian coordinates $\mathbf{r}$ and time t. We expand the displacement $\tilde{\mathbf{a}}$ and the velocity of particles

$$\tilde{\mathbf{v}} = \left(\frac{\partial \tilde{\mathbf{a}}}{\partial t}\right)_{\mathbf{r}_0} = \left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\tilde{\mathbf{a}}(\mathbf{r}, t) \qquad (1)$$

in powers of a dimensionless small parameter ε proportional to the wave amplitude: $\tilde{\mathbf{v}} = \mathbf{v}_0 + \mathbf{v} + O(\varepsilon^2)$, and $\tilde{\mathbf{a}} = \mathbf{a}_0 + \mathbf{a} + O(\varepsilon^2)$. Here, $\mathbf{a}_0$ and $\mathbf{v}_0$ are the displacement and velocity of a particle in the absence of the wave and $\mathbf{v}$ and $\mathbf{a}$ are proportional to ε. We consider the flow not disturbed by the wave as a steady-state one: $\dfrac{\partial \mathbf{v}_0}{\partial t} = 0$. By equating the terms of the same order in ε in (1), we obtain

$$\mathbf{v}_0 = \frac{d\mathbf{a}_0}{dt}, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_0\nabla; \qquad (2)$$

$$\mathbf{v} = \frac{d\mathbf{a}}{dt} + (\mathbf{v}\nabla)\mathbf{a}_0. \qquad (3)$$

In particular, if the fluid-particle velocity in the flow not disturbed by the wave does not vary along the flow line, i.e., if $\dfrac{d\mathbf{v}_0}{dt} = 0$, then $\mathbf{a}_0 = \mathbf{v}_0(t - t_0)$ and (3) takes the form

$$\mathbf{v} = \frac{d\mathbf{a}}{dt} + (t - t_0)(\mathbf{v}\nabla)\mathbf{v}_0. \qquad (4)$$

It follows from (3) and (4) that the particle-displacement disturbance $\mathbf{a}$ induced by the wave has both periodic and aperiodic (the latter increases with time) components if the flow is nonuniform and the wave features a periodic time dependence of the oscillatory velocity $\mathbf{v}$. In particular, these components expressed as complex quantities vary with time, according to (4), as $\exp(-i\omega t)$ and $(t - t_0)\exp(-i\omega t)$, respectively, for a harmonic wave of frequency ω under the condition that $\dfrac{d\mathbf{v}_0}{dt} = 0$.

We represent $\mathbf{a}$ as the sum of the periodic ($\mathbf{w}$) and secular ($\mathbf{W}$) components: $\mathbf{a} = \mathbf{w} + \mathbf{W}$. Relationship (3) is valid if

$$\mathbf{v} = \frac{d\mathbf{w}}{dt} - (\mathbf{w}\nabla)\mathbf{v}_0; \qquad (5)$$

$$\mathbf{W} = -(\mathbf{w}\nabla)\mathbf{a}_0. \qquad (6)$$

When the state of the medium not disturbed by the wave is specified, $\mathbf{a}_0$ is the known function of the coordinates and time, which makes it possible to separate $\mathbf{a}(\mathbf{r}, t)$ uniquely into the components $\mathbf{w}$ and $\mathbf{W}$ at any point $\mathbf{r}$. Thus, the vector function $\mathbf{w}$ has a clear physical

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meaning: it is an observable quantity, and, generally speaking, can be measured directly.

Equation (5) allows one to calculate easily the oscillatory velocity $\mathbf{v}$ provided the oscillatory displacement $\mathbf{w}$ of the particles is known. This equation admits an obvious interpretation, namely, the disturbances of the fluid flow, to a first-order approximation in the wave amplitudes, are caused by the variation of oscillatory displacement with time and also by periodic displacement of particles from the undisturbed trajectory into the region with a differing magnitude of $\mathbf{v}_0$ provided the undisturbed flow velocity varies in the direction of $\mathbf{w}$.

2. THE LINEARIZED EQUATIONS OF HYDRODYNAMICS

We now consider the small-amplitude oscillations of a multicomponent compressible fluid with respect to an undisturbed steady state, which is characterized by the flow velocity $\mathbf{v}_0$, pressure p_0, density ρ_0, and the sound velocity c_0. The fluid is placed into the uniform gravity field and is subjected to no other extraneous forces than those related to the wave sources. We ignore the diffusion of impurities and treat all the thermodynamic processes in the medium (both in the undisturbed state and in the presence of waves) as adiabatic. In other words, the fluid is assumed to be perfect (see [1, § 2]). We do not impose any restrictions on the spatial dependence of $\mathbf{v}_0$, p_0, ρ_0 and c_0 apart from the assumption that these functions obey the relevant nonlinear equations of hydrodynamics.

The Euler and continuity equations linearized with respect to the wave amplitude are of the form (see [4, § 4.1]):

$$\frac{d\mathbf{v}}{dt} + (\mathbf{v}\nabla)\mathbf{v}_0 = -\frac{\nabla p}{\rho_0} + \frac{\rho}{\rho_0^2}\nabla p_0 + \frac{\mathbf{F}}{\rho_0}, \tag{7}$$

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\mathbf{v}_0 + \mathrm{div}(\rho_0\mathbf{v}) = \rho_0 A. \tag{8}$$

Here, p and ρ are the wave-induced disturbances of the pressure and density in the medium, and A and $\mathbf{F}$ are the volume densities for the sources of the volume velocity and of the extraneous wave-generating forces [formally speaking, the quantities A and $\mathbf{F}$ should be considered to have the order of $O(\varepsilon)$]. It is noteworthy that the gravity force affects equations (7) and (8) only indirectly, i.e., via the parameters of the undisturbed medium.

Expressing $\mathbf{v}$ in terms of oscillatory displacements $\mathbf{w}$ and using the continuity equation

$$\frac{d\rho_0}{dt} + \rho_0\,\mathrm{div}\,\mathbf{v}_0 = 0$$

for the undisturbed flow and the identities that follow from (5)

$$\mathrm{div}\,\mathbf{v} = \left(\frac{d}{dt}\right)\mathrm{div}\,\mathbf{w} - (\mathbf{w}\nabla)\mathrm{div}\,\mathbf{v}_0, \tag{9}$$

$$\mathbf{v}\nabla Q = \left(\frac{d}{dt}\right)\mathbf{w}\nabla Q - (\mathbf{w}\nabla)\frac{dQ}{dt}, \tag{10}$$

where $Q(\mathbf{r}, t)$ is an arbitrary smooth function, we can easily rewrite equations (7) and (8) in the form

$$\frac{d^2\mathbf{w}}{dt^2} - (\mathbf{w}\nabla)\frac{d\mathbf{v}_0}{dt} = -\frac{\nabla p}{\rho_0} + \frac{\rho}{\rho_0^2}\nabla p_0 + \frac{\mathbf{F}}{\rho_0}, \tag{11}$$

$$\frac{d}{dt}\left[\mathrm{div}\,\mathbf{w} + \frac{\rho + \mathbf{w}\nabla\rho_0}{\rho_0}\right] = A. \tag{12}$$

The total pressure $\tilde{p} = p_0 + p + O(\varepsilon^2)$ in a multicomponent medium is a function of the density $\tilde{\rho} = \rho_0 + \rho + O(\varepsilon^2)$ of the medium, the entropy density $\tilde{S} = S_0 + S + O(\varepsilon^2)$, and the impurity concentrations $\tilde{K} = K_0 + K + O(\varepsilon^2)$:

$$\tilde{p} = \Phi(\tilde{\rho}, \tilde{S}, \tilde{K}), \tag{13}$$

with $p_0 = \Phi(\rho_0, S_0, K_0)$. It follows from (13) that

$$\nabla p_0 = c_0^2\nabla\rho_0 + \alpha\nabla S_0 + \beta_j\nabla K_{0j}; \tag{14}$$

$$p = c_0^2\rho + \alpha S + \beta_j K_j, \tag{15}$$

where $c_0^2 = \left(\frac{\partial p_0}{\partial\rho_0}\right)_{S_0, K_0}$ is the sound velocity squared,

$$\alpha = \left(\frac{\partial p_0}{\partial S_0}\right)_{\rho_0, K_0}, \quad \beta_j = \left(\frac{\partial p_0}{\partial K_{0j}}\right)_{\rho_0, S_0, K_{0i}, i \neq j}.$$

Hereafter, summation over the repeating indices is implied.

By virtue of the assumption that the processes are adiabatic, the entropy density does not vary in the fluid particles:

$$\left(\frac{\partial}{\partial t} + \tilde{\mathbf{v}}\nabla\right)\tilde{S} = 0 \tag{16}$$

and, as a result,

$$\frac{dS_0}{dt} = 0, \quad \frac{dS}{dt} + \mathbf{v}\nabla S_0 = 0. \tag{17}$$

It follows from (17) and (10) that $\frac{d}{dt}(\mathbf{w}\nabla S_0 + S) = 0$.

In what follows, we assume that $\frac{dB}{dt} \equiv 0$, if $B \equiv 0$ for any characteristic B of the field, which is linear with respect to the wave amplitude. The physical sense of this assumption lies in the fact that the frequency of the wave in the coordinate frame related to the fluid particle does not vanish on any flow line. This assumption is undoubtedly valid provided that there are no points of synchronism in the medium (see [7, § 9.4]) in the vicinity of which, a resonant interaction between the wave and the flow is possible.

Thus, due to the adiabaticity condition, we can write

$$\mathbf{w}\nabla S_0 + S = 0. \tag{18}$$

According to the condition of the absence of diffusion of impurities and using reasoning similar to that employed in the derivation of (18), we obtain

$$(\mathbf{w}\nabla)K_0 + K = 0. \tag{19}$$

Equalities (14), (15), (18), and (19) yield

$$p + \mathbf{w}\nabla p_0 = c_0^2(\rho + \mathbf{w}\nabla\rho_0). \tag{20}$$

To a linear approximation with respect to the wave amplitude, relationship (20) may be treated as the equation of state of the fluid. In contrast to forms of the linearized equation of state for a three-dimensional inhomogeneous compressive moving fluid, which have been used in previous publications (see [7, § 1; 8, § 4; 13; 9, § 8.6; 10, Chapter 1]; *etc.*), relationship (20) does not contain any new [as compared to the Euler and continuity equations (11) and (12)] unknown values or additional thermodynamic characteristics of the medium, apart from the adiabatic sound velocity. A direct examination demonstrates that equation of state (20) is consistent with all available published results in special cases considered previously. These are the plane-laminar or cylindrical-laminar flow in a three-dimensionally nonuniform liquid [11]; the geometry-related acoustic and caustic asymptotic characteristics of the acoustic field in an arbitrary inhomogeneous moving medium [4, § 5.1 and § 6.2; 12]; and propagation of sound along a smoothly irregular waveguide in a moving medium (see [4, § 7.3]).

When the parameters $\mathbf{v}_0$, c_0, ρ_0, and p_0 of the undisturbed state of the flow are specified, relationships (11), (12), and (20) form a closed system of equations in p, ρ and $\mathbf{w}$ that are the characteristics of the wave field. This system admits further simplification. Eliminating ρ by employing (20) and the Euler equation for the undisturbed flow, we obtain

$$\rho_0\frac{d^2\mathbf{w}}{dt^2} + \nabla p - \frac{p + \mathbf{w}\nabla p_0}{\rho_0 c^2}\nabla p_0 + (\mathbf{w}\nabla)\nabla p_0 = \mathbf{F}, \tag{21}$$

$$\frac{d}{dt}\left[\operatorname{div}\mathbf{w} + \frac{p + \mathbf{w}\nabla p_0}{\rho_0 c_0^2}\right] = A. \tag{22}$$

If there is no source of volume velocity ($A \equiv 0$), equation (22) makes it possible to express p in terms of $\mathbf{w}$ and to use (21) to obtain a vector wave equation in terms of oscillatory displacement:

$$\rho_0\frac{d^2\mathbf{w}}{dt^2} + (\operatorname{div}\mathbf{w} + \mathbf{w}\nabla)\nabla p_0 - \nabla(\mathbf{w}\nabla p_0 + \rho_0 c_0^2\operatorname{div}\mathbf{w}) = \mathbf{F}. \tag{23}$$

It is noteworthy that the flow velocity $\mathbf{v}_0$ appears in (21)–(23) only via the operator d/dt.

It should be emphasized that all the mathematical manipulations with the linearized equations of hydrodynamics, which were described in this Section, remain valid also in the case of an unsteady moving medium only if the dependent variable $\mathbf{w}$ continues to be related by formula (5) to the wave-induced disturbance of the fluid flow.

3. LINEARIZED BOUNDARY CONDITIONS

Let the flow be bounded by a surface, which is immovable when there is no wave, impermeable for the fluid, and, in the general case, deformable. Let the surface Γ of the boundary not disturbed by the wave be defined by the equation $f(\mathbf{r}) = 0$, where f is a smooth function. Furthermore, let $f > 0$ in the vicinity of Γ and outside the volume occupied by the fluid. We will demonstrate that the wave-induced normal displacement η of the boundary and the normal component of the oscillatory displacement of fluid particles, which was introduced in Section 1, are equal to the first-order approximation in the wave amplitude:

$$\eta = \mathbf{w}\mathbf{N}, \quad \mathbf{r} \in \Gamma. \tag{24}$$

Here, $\mathbf{N} = \mathbf{n}/n$ is the unit outward normal to Γ and $\mathbf{n} = \nabla f$.

For substantiation, we resort to the kinematic relationships [4, § 7.3.2]:

$$\mathbf{v}_0\mathbf{N} = 0, \quad \mathbf{v}\mathbf{N} = \frac{d\eta}{dt} - \eta\mathbf{N}(\mathbf{N}\nabla)\mathbf{v}_0, \quad \mathbf{r} \in \Gamma, \tag{25}$$

which express the equality of the normal (to the boundary) components of the flow velocity and the boundary velocity, respectively, in the presence of the wave and outside of the presence. According to (5) and (25),

$$\left[\frac{d}{dt} - \mathbf{N}(\mathbf{N}\nabla)\mathbf{v}_0\right]\tau = D, \quad \tau \equiv \mathbf{w}\mathbf{N} - \eta, \tag{26}$$

where

$$D = \mathbf{w}(\mathbf{v}_0\nabla)\mathbf{N} + \mathbf{v}_0(\mathbf{w}\mathbf{N})(\mathbf{N}\nabla)\mathbf{N} - \mathbf{v}_0(\mathbf{w}\nabla)\mathbf{N}.$$

Taking into account that $\mathbf{N}(\mathbf{v}_0\nabla)\mathbf{N} = 0$ and introducing the vector $\mathbf{q} = \mathbf{w} - \mathbf{N}(\mathbf{w}\mathbf{N})$, we obtain

$$D = n^{-1}[\mathbf{q}(\mathbf{v}_0\nabla)\mathbf{n} - \mathbf{v}_0(\mathbf{q}\nabla)\mathbf{n}] = 0.$$

In the general case, equation (26) with a nonzero right-hand side has the nontrivial solutions $\tau(\mathbf{r}, t)$. However, for $\tau \neq 0$, the projection of the phase retardation of the wave on the direction $\mathbf{v}_0$ must be equal to the reciprocal of the flow velocity at the boundary. At the same time, the logarithmic derivative of the amplitude should be proportional to $\mathbf{N}(\mathbf{N}\nabla)\mathbf{v}_0$. Again, as in Section 2, by excluding from consideration the cases of synchronism between the wave and the flow, we use (26) to obtain $\tau \equiv 0$, i.e. equality (24).

We will ignore the surface tension. We assume that the boundary is of a locally responsive (impedance) type, and, in the case of the small-amplitude oscillations, the normal velocity is proportional to the variations of the fluid pressure on the surface:

$$\frac{\partial \eta}{\partial t} = -\frac{\tilde{p}(\mathbf{r}+\eta\mathbf{N}, t) - p_0(\mathbf{r})}{\zeta(\mathbf{r})} + O(\varepsilon^2), \quad \mathbf{r} \in \Gamma. \quad (27)$$

Here, ζ has the meaning of the boundary impedance. In the case of monochromatic waves, the quantity ζ may be considered as a function of frequency ω. In the case $\nabla p_0 = 0$, which is typically considered in acoustics, relationship (27) is reduced to a conventional definition of the impedance boundary [4, § 7.3.2].

Linearizing (27) with respect to ε, taking into account that $\frac{\partial \mathbf{N}}{\partial t} = 0$, and using (24), we obtain the boundary condition on the locally responsive surface:

$$\zeta \mathbf{N}\frac{\partial \mathbf{w}}{\partial t} + p + (\mathbf{w}\mathbf{N})(\mathbf{N}\nabla p_0) = 0, \quad \mathbf{r} \in \Gamma. \quad (28)$$

In the limiting cases as $\zeta \to 0$ and $\zeta \to \infty$, equation (28) generates the following conditions:

$$p + (\mathbf{w}\mathbf{N})(\mathbf{N}\nabla p_0) = 0, \quad \mathbf{r} \in \Gamma; \quad (29)$$

$$\mathbf{w}\mathbf{N} = 0, \quad \mathbf{r} \in \Gamma, \quad (30)$$

for, respectively, free and perfectly rigid boundaries. These conditions can be easily derived directly from the definition of such boundaries.

The linearized boundary conditions on the interface between two moving fluids can be derived using (24) similarly to the derivation of relationship (28). These conditions correspond to the requirement for the functions $\mathbf{w}\mathbf{N}$ и $p + (\mathbf{w}\mathbf{N})(\mathbf{N}\nabla p_0)$ to be continuous on the surface Γ.

In [13, 14], the boundary condition was derived for monochromatic acoustic waves on the impedance surface of an arbitrary shape under the assumption that $\nabla p_0 = 0$. It is easy to verify using (5) that this boundary condition, within its domain of applicability, is consistent with a more general result (28). However, this result does not contain any spatial derivatives of the characteristics of the wave and, as such, is more convenient in the application-oriented studies.

The above-presented new version of the mixed Eulerian–Lagrangian description of small-amplitude oscillations of an inhomogeneous moving flow makes possible a marked advance in studies of the general properties of waves. In particular, it presents an easy derivation of the reciprocity relationships and of the law of conservation of the pseudoenergy of waves under conditions of arbitrary flow.

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REFERENCES

1. Landau, L.D. and Lifshits, E.M., *Gidrodinamika* (Hydrodynamics), Moscow: Nauka, 1986.
2. Eckart, S., *Phys. Fluids*, 1963, vol. 6, no. 8, pp. 1037–1041.
3. Andrews, D.G. and McIntyre, M.E., *J. Fluid. Mech.*, 1978, vol. 89, no. 4, pp. 609–646 and 647–664.
4. Brekhovskikh, L.M. and Godin, O.A., *Acoustics of Layered Media, vol. 2: Point Sources and Bounded Beams*, Berlin: Springer-Verlag, 1992.
5. Godin, O.A., *Dokl. Akad. Nauk SSSR*, 1992, vol. 322, no. 1, pp. 79–85 [*Sov. Phys.–Dokl.* (Engl. transl.), vol. 37, no. 1, pp. 44–47].
6. Godin, O.A., *Dokl. Akad. Nauk*, 1994, vol. 334, no. 1, pp. 37–40 [*Phys.–Dokl.* (Engl. transl.), vol. 39, no. 1, pp. 16–19].
7. Brekhovskikh, L.M. and Godin, O.A., *Akustika sloistykh sred* (Acoustics of Layered Media), Moscow: Nauka, 1989.
8. Blokhintsev, D.I., *Akustika neodnorodnoi dvizhushcheisya sredy* (Acoustics of an Inhomogeneous Moving Medium), Moscow: Nauka, 1981.
9. Pierce, A.D., *Acoustics: An Introduction to Its Physical Principles and Applications*, New York: McGraw-Hill, 1981.
10. Grigor'eva, N.S., *Asimptoticheskie metody v zadachakh o rasprostranenii zvuka v neodnorodnoi dvizhushcheisya srede* (Asymptotic Methods in the Problems of the Propagation of Sound in an Inhomogeneous Moving Medium), Leningrad: Leningr. Gos. Univ, 1991.
11. Godin, O.A., *Dokl. Akad. Nauk SSSR*, 1991, vol. 316, no. 6, pp. 1378–1382 [*Sov. Phys.–Dokl.* (Engl. transl.), vol. 36, no. 2, pp. 168–170].
12. Godin, O.A., *Dokl. Akad. Nauk*, 1995, vol. 341, no. 2, pp. 185–189 [*Phys.–Dokl.* (Engl. transl), vol. 40, no. 3, pp. 102–106].
13. Taylor, K., *J. Sound Vibr.*, 1979, vol. 65, no. 1, pp. 125–136.
14. Myers, M.K., *J. Sound Vibr.*, 1980, vol. 71, no. 3, pp. 429–434.

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