FIELD CONSISTENCY AND THE FINITE ELEMENT ANALYSIS OF MULTI-FIELD STRUCTURAL PROBLEMS

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Abstract

We introduce the concept of field consistent representation of continuum problems in structural mechanics which require definition by more than one field variable. In some practical situations, the engineering dimensions of the problem require that the many field variables needed to model the deformation of the continuum may have to be constrained among themselves in a suitable way. If this is not properly accommodated in the finite element model, severe errors can result.

Here, we consider a new terminology for such errors and examine a simple case in which such errors can lead to the 'locking' phenomenon. Techniques, most of which originated on an ad-hoc basis even before the mechanics of field-consistency was established, can now be re-interpreted to show how they successfully introduce a consistent representation of the constrained strain-fields and thus remove the 'spurious constraints' that lead to 'errors of the second kind'

1. INTRODUCTION.

The early developments in the finite element method proceeded from an a posteriori understanding of the behaviour of the structural situation the elements were to model. However, a subsequent invasion of mathematical rigour resulted in an emphasis on the mathematical principles that should ensure the convergence of finite element discretisation. This, we shall examine below.

1.1 Requirements for convergence - the conventional wisdom.

The conventional wisdom emphasises the mathematical precision of the problem statement. The completeness of the polynomial fields used as shape functions is considered to be important. Compatibility, or in other words, the continuity of these functions and their derivatives, where required, across inter-element boundaries is another important criterion. Two other conditions, considered to be of vital importance, are the rigid-body motion requirement, i.e. that rigid-body motion of the element would be strain-free, and the constant strain-rate requirement, i.e. in the limit the constant strains are correctly recovered.

However, it was soon noticed that in some situations, elements that were carefully constructed to satisfy these rigid principles could still behave in dramatically erratic ways. It was clear that the cardinal virtues of compatibility, rigid-body motion requirement and constant strain-rate requirement were not sufficient to ensure the convergence of these elements.

1.2 Requirements for convergence - field consistency in multi-field problems.

It was recognised at this time, that special difficulties arose in problems in structural mechanics that needed a description of its continuum behaviour by more than one field variable, and which required that certain strain fields had to be constrained in particular ways. This meant that the interpolation functions for the field variables should be able to ensure the recovery of only the true constrained strain fields in the limiting situations. This requirement is called 'field consistency'. In this paper, we examine this problem in which the inability of the chosen field interpolations to represent consistently, the constrained strain fields, can result in the very poor behaviour known as 'locking'.

2. DEFINITION OF NEW TERMINOLOGY.

2.1 Field Consistency.

In the development of a finite element, we interpolate the field variables using interpolation functions of a certain order. The number of constants used will depend on the number of nodal variables and any additional nodeless variables that may be introduced. From these definitions, one can compute the strain fields also as interpolations associated with these constants by obtaining the correct derivatives of the field variables. In a multi-field problem, these strain fields will have as coefficients terms from more than one field variable. Depending on the order of derivatives of each field variable appearing in the definition of that strain field and on the order of the interpolation functions used for each contributing field variable, the coefficients of the strain-field interpolations may have constants from all contributing field variable interpolations or from only one or some of these. In some limiting cases of physical behaviour, it will become necessary for these strain fields to be constrained to zero values. These can arise naturally, but indirectly, from classical variational statements (eg. the minimum energy principle in displacement type elements) on which the finite element models are based, in certain geometrical situations, eg. vanishing thickness of shear flexible beams, plates and shells, or they can arise directly in exterior penalty type formulations, eg. the discrete Kirchhoff hypothesis enforcement in thin plates and shells by a large penalty factor.

Where the strain-field is such that all the terms in it (i.e. the constant, linear, quadratic etc.) have, associated with it, coefficients containing contributions from all the independent interpolations of the field variables that appear in the definition of that strain-field, the constraint that appears in the limit can be correctly enforced. Such a representation is said to be field consistent. The constraints thus enforced are called 'true constraints'. Where the strain field has coefficients in which the contributions from some of the field variables are absent, the constraints may incorrectly constrain the contributions from the field variables present. These are called 'spurious constraints'.

2.2 Errors of first and second kind.

We now distinguish the special errors that arise due to the 'spurious constraints' from the other, more familiar errors of discretisation. In an unconstrained continuum problem with only a single field variable, the errors of discretisation are of simple form and these usually vanish rapidly as the mesh size is reduced. However, in a multi-field problem in which at limiting cases, constrained strain fields must be enforced, the 'spurious constraints' can give rise to a form of errors that vanish very slowly with reduction in mesh size, and whose slowness of convergence and magnitude of error is greatly exaggerated by changes in the structural parameters that emphasise the appearance of the limiting physical situations.

Walz et al.[1] recognised the existence of these two kinds of errors. This interpretation in terms of 'errors of the first kind' and 'errors of the second kind' was found useful when Prathap and Bhashyam[2] separated the spurious constraints from the true constraints for an exactly integrated shear flexible element. It was seen that a field consistent element obtained by reduced integration and which had only the true Kirchhoff constraints had errors which vanished rapidly as the mesh size was reduced. These errors were independent of the relevant structural parameter - the thinness of the beam, (L/t) - and these were identified as errors of the first kind. On the other hand, the field inconsistent element which would result from an exact integration and which will have the additional spurious constraints, will have errors due to these constraints which are exaggerated in a $(L/t)^2$ fashion. In the very thin limits, these errors are so large as to make the results virtually meaningless. This phenomenon is known as 'locking', and these locking errors were called 'errors of the second kind'.

2.3 Functional reconstitution after discretisation.

Here, we introduce a procedure which was found to be useful in making quantitative estimates of the errors of the second kind. It was seen that with elements of finite size (i.e. a practical level of discretisation at which reasonably accurate results can be expected) , the errors of a special nature that arise because the strain fields are not consistently interpolated with terms from all the independent field functions that contribute to it, can be predicted by a technique introduced in References 2-4.

The method of functional reconstitution after discretisation has been explained very clearly in Reference 2 to show how shear locking emerges in a simple linear shear flexible beam element in which the normal displacement w and face rotation θ are independently interpolated. We shall elaborate on this when we consider the curved beam element in greater detail in the next section. Here, we shall briefly describe the technique.

We recognise first that the problem of locking arises because of the need to work with a finite size element. Thus, in a finite element idealisation, the mathematical operations of defining a small but non-infinitesimal element, prescribing interpolations for the field variables over this domain to a certain order of polynomial, and of subsequently integrating the functional for the strain energy of the element will result in a discretised estimate for the strain energy of that region in terms of the values of the field variables at the nodes and the element sizes. In many practical situations, it is required that in limiting situations, some of the strain fields must vanish and that the strain energies associated with it must also vanish. This will be possible if the structural parameters multiplying the strain energy terms take large values in a penalty-limiting sense and the energy terms associated with it reflect true constraints. However, because the original interpolations are not 'field consistent', we have terms in which a structural parameter involving an element dimension multiplies a 'spurious constraint'. In an ideal infinitesimal case of vanishing size of the element, this structural parameter vanishes, leaving the 'spurious constraint' unenforced. However, in a practical analysis, we need to work with elements of reasonably finite size. In this case, the structural parameter enforces the 'spurious constraints' and adds an additional 'spurious energy' and therefore an 'additional stiffening' effect. These constraints are physically equivalent to an altered system in the non-penalty regime, which we can obtain by reconstituting a functional for the effective strain energy of an element based on a finite size discretisation. This is done, by carrying out the discretisation operations for a finite size element, re-grouping the true energy terms, the energy terms that should vanish due to the true constraints and the energy terms that do not vanish due to the spurious constraints. The strain energy density of such a discretised element is obtained by dividing this strain energy by the element volume. This strain energy density represents a physical system that contains the 'locking' effect due to field inconsistency. This procedure of obtaining a strain energy functional for the field inconsistent system will be called functional reconstitution and shall be further elaborated when we study the curved beam element later.

2.4 The additional stiffening parameter.

Conventional error analyses norms in the finite element method are based on the percentage error or equivalent in some computed value as compared to the theoretically predicted value. In multi-field problems which are inconsistently modelled, the errors of the second kind can be exaggerated without limit as the structural parameter that acts as a penalty multiplier becomes indefinitely large. The percentage error norms therefore saturate quickly to a value approaching 100% and do not sensibly reflect the relationship between error and the structural parameter even on a logarithmic plot.

Reference 5 introduced a new error norm called the additional stiffening parameter, e . This helps to recognise the manner in which the errors of the second kind can be blown out of proportion by a large variation in the structural parameter. Essentially, this takes into account, the fact that the spurious constraints give rise to a spurious energy term and consequently alters the rigidity of the system being modelled. In many examples, it was seen that the rigidity, I , of the field consistent system and the rigidity, I' , of the inconsistent system, were related to the structural parameters in the form,

$$\frac{I'}{I} - 1 = \alpha \left(\frac{L}{t}\right)^2$$

where L is an element dimension and t is the element thickness. Thus, if w(theory) is the deflection of a reference point as predicted by theory and w(fem) is the deflection predicted by a field inconsistent finite element model, we would expect

$$e = \frac{w(\text{theory})}{w(\text{fem})} - 1 = \alpha \left(\frac{L}{t}\right)^2 .$$

A logarithmic plot of the new error norm against the parameter (L/t) will show a quadratic relationship that will continue indefinitely as (L/t) is increased. This was found to be true of the many multi-field problems reviewed in Reference 5 and this will be further described in the subsequent sections.

3. A SIMPLE PHYSICAL EXAMPLE - THEORY AND ANALYSIS.

In this section, we take up a well known multi-field problem in structural analysis which has exhibited these errors of the second kind i.e. the poor bending response of the curved beam element can be shown to be due to locking.

3.1 Poor bending response of curved elements - membrane locking.

Here, we try to understand the errors that arise in curved finite elements which undergo both flexural and membrane deformations. It is shown that with elements of finite size (i.e. a practical level of discretisation at which reasonably accurate results can be expected) , there can be errors of a special nature that arise because the membrane strain fields are not consistently interpolated with terms from the two independent field functions that characterise such a problem. These lead to errors, described here as of the 'second kind' and a physical phenomenon named as 'membrane locking'.

Recently, the role that reduced integration can play in evaluating the extensional energy of a curved beam was identified[2,6-7]. In Reference 2, Prathap and Bhashyam examined a shallow curved Timoshenko beam and showed that an exact integration of the extensional energy term created a mechanism that they called 'in-plane locking'. This delayed convergence and it was seen that reduced integration of these terms improved the performance of the shallow curved beam. Noor and Peters[6] observed that inextensional (or nearly inextensional) deformations are poorly represented by stiffness models based on exact integration of low-order independent interpolations for displacement and rotational components unless a reduced integration of the displacement model or equivalently, a discontinuous force field mixed model is used.

Stolarski and Belytschko[7] observed that when low-order in-plane displacement fields were used for a Marguerre type shallow shell theory to model a curved beam, an exact integration caused an apparent increase in bending stiffness due to the curvature coupled membrane strains and that this can be alleviated by using reduced integration on the extensional energy terms. This action was called 'membrane locking'

Here, we unify these observations in terms of the field consistency approach. It is seen that when low-order interpolations are used for the in-plane displacement terms, the inconsistency in the number of constants required to define interpolations for the $u_{,y}$ and w/R terms in the extensional strain energy functional produces spurious constraints in the penalty limit of extreme thinness (i.e. nearly inextensional behaviour) and these cause the locking of the solution. These errors of the second kind can be removed by optimally integrating the extensional energy terms so that only the true constraints in the inextensional limit are retained. Thus the general extensional deformation behaviour of a curved beam can be modelled by independently chosen low order polynomial functions and this will still recover the inextensional case in the penalty limit without spurious constraints provided selective integration is used.

In the next section, we re-work the 'cubic in w - linear in u' curved beam/arch/finite ring element (henceforth CL element) to show the presence of 'membrane locking' and its removal by reduced integration.

3.2 The 'cubic in w - linear in u' element.

We start with the simplest shape function representation in the literature that ensures C^0 continuity for the tangential displacement u and C^1 continuity for the normal displacement w corresponding to the thin curved beam/deep arch/finite circular ring model ,

$$
\begin{aligned}
u &= b_1 + b_2 y \\
w &= a_1 + a_2 y + a_3 y^2 + a_4 y^3
\end{aligned}
\qquad (1)
$$

The extensional strain and change of curvature are

$$
\begin{aligned}
\epsilon &= u_{,y} + w/R \\
\chi &= \frac{1}{R} u_{,y} - w_{,yy}
\end{aligned}
\qquad (2)
$$

and the total energy of an element of length 2L is

$$
U = \int_{-L}^{L} \left(\frac{EA}{2} \epsilon^2 + \frac{EI}{2} \chi^2 \right) dy \qquad (3)
$$

3.3 Discretisation of the strain energy functional.

Following Reference 4, we examine the terms arising from the discretised functional U above. We have

$$
U = U_1 + U_2
$$

where

$$U_1 = \frac{Ebt}{2} \int_{-L}^{L} \left(b_2 + \frac{a_1}{R} + \frac{a_2 y}{R} + \frac{a_3 y^2}{R} + \frac{a_4 y^3}{R} \right)^2 dy$$

$$U = \frac{Ebt^3}{24} \int_{-L}^{L} \left(\frac{b_2}{R} - 2a_3 - 6a_4 y \right)^2 dy. \qquad (4)$$

After carrying out the discretisation and regrouping of terms, we have

$$U_1 = \frac{Ebt}{2} .2L \left[\left(b_2 + \frac{a_1}{R} + \frac{a_3 L^2}{3R} \right)^2 + \frac{1}{3} \left(\frac{a_2 L}{R} \right)^2 + \frac{4}{45} \left(\frac{a_3 L^2}{R} \right)^2 + \frac{2}{5} \left(\frac{a_2 L}{R} \right) \left(\frac{a_4 L^3}{R} \right) + \frac{1}{7} \left(\frac{a_4 L^3}{R} \right)^2 \right] \qquad (5)$$

$$U_2 = \frac{Ebt}{24} .2L \left[\left(\frac{b_2}{R} - 2a_3 \right)^2 + 36 \frac{a_4^2 L^2}{3} \right]$$

3.4 Constraints in the inextensional limit.

Comparing the energy terms, and following the arguments of Reference 4, we see that in the penalty limit of extreme thinness, there are penalty terms that tend to enforce the following constraints,

$$b_2 + \frac{a_1}{R} + \frac{a_3 L^2}{3R} = 0$$

$$a_2 = 0 \qquad (6)$$

$$a_3 = 0$$

$$a_4 = 0$$

.

The first constraint has terms from both interpolation functions and therefore enforces a valid constraint that implies the inextensibility condition

$$(u_{,y})_0 - w'/R = 0$$

where w' is the normal deflection of some reference point on the element and the subscript 0 indicates the value at the mid-point of the element. This therefore is a simple measure of the averaged constant membrane strain in the element and in the penalty limit, it is constrained to take a zero value.

On the other hand, $(a_2, a_3, a_4) \rightarrow 0$ imply that $(w_{,y})_0$, $(w_{,yy})_0$ and $(w_{,yyy})_0 \rightarrow 0$ are enforced in the penalty limit and are therefore spurious constraints. These lead to the locking phenomenon which we described as 'in-plane locking' in Reference 2, or as 'membrane locking' in References 4,7. It is quite simple to remove the locking terms by reduced integration. For the orders of the interpolation functions chosen above, a one point Gaussian integration would give only the single true inextensibility constraint,

$$b_2 + \frac{a_1}{R} = 0 . \qquad (7)$$

Any higher order integration rule will introduce the spurious constraints and an order that ensures exact integration will introduce all three spurious constraints. These spurious constraints are responsible for the errors of the second kind as defined in Reference 5 and cause the severe deterioration of results with increasing penalty multiplier value i.e. $(L/t)^2$. It is instructive to see how this emerges.

3.5 Functional reconstitution after discretisation with finite size element.

We consider the strain energy in an elemental arch of length 2L in a truly inextensional limit. We recognise that the extensional energy term due to the true constraint will vanish in this limit, but not the extensional energy contributions from the spurious terms. Thus for the discretised element of finite length 2L , we observe that the strain energy density in an inextensional limit will be of the type (here, we consider only the two leading locking terms in U as these will have the greater effect)

$$U = \frac{Ebt^3}{24}\left[\left(\frac{b_1}{R} - 2a_3\right)^2 + 36\,\frac{a_4^2 L^2}{3}\right] + \frac{Ebt}{2}\left[\frac{1}{3}\left(\frac{a_2 L}{R}\right)^2 + \frac{4}{45}\left(\frac{a_3 L^2}{R}\right)^2\right] \tag{8}$$

This has been obtained by dividing the strain energies of the element in Eqn.(5) by its length. We observe now, the presence of the L term in Eqn.(8). In the real infinitesimal limit, the contributions to the extensional strain energy from the a_2, a_3 and a_4 terms will vanish due to vanishing L without enforcing any constraints on these terms. However, in a real analysis, we need to work with a practical discretisation (i.e. a finite L) and still expect reasonably accurate results to be obtained. This is the crux of the problem. With finite L, in the inextensional limit characterised by $t \gg t^3$ in Eqn.(8), the membrane strain energy contribution is made to vanish by a_2 and a_3 tending to zero. These constraints are physically equivalent to an altered system in the non-penalty regime, which we can obtain by reconstituting a functional for the effective strain energy of an arch element based on a finite size discretisation, as

$$U = \frac{Ebt^3}{24}\int_{-L}^{L} w_{,yy}^2\, dy$$

$$+ \frac{Ebt}{2}\int_{-L}^{L} \frac{1}{3}\left(\frac{L}{R}\right)^2 w_{,y}^2 + \frac{1}{45}\left(\frac{L^2}{R}\right)^2 w_{,yy}^2 \quad dy$$

$$= \frac{EI}{2}\int_{-L}^{L} \left(1 + \frac{4}{15}\left(\frac{L^2}{Rt}\right)^2\right) w_{,yy}^2\, dy + \frac{1}{2}\cdot\frac{EAL^2}{3R^2}\int_{-L}^{L} w_{,y}^2\, dy \qquad (9)$$

Thus when a finite size element of length L is used, the element after discretisation of the functionals, acts as a system with an altered modulus of inertia

$$I' = I\left(1 + \frac{4}{15}\left(\frac{L^2}{Rt}\right)^2\right)$$

and is further stiffened by a spurious in-plane force $F = EAL^2/3R^2$. It is necessary to recognise which of these two terms will play the leading stiffening role.

Consider the simple example of a very shallow circular arch of length l and radius of curvature R, simply-supported at the ends and subtending an angle $\beta = l/R$ at the centre. A uniformly distributed load is assumed to act on it and we may take a simple one term approximation for the normal deflection in the form,

$$w = c \sin\left(\frac{\pi y}{l}\right) \tag{10}$$

so that, when the entire arch is discretised by arch elements of length 2L,

$$U = \frac{1}{2}\int_0^l EI \left(1 + \frac{4}{15}\left(\frac{L^2}{Rt}\right)^2\right) \frac{c^2\pi^4}{l^4} \sin^2\left(\frac{\pi y}{l}\right) dy + \frac{1}{2}\int_0^l \frac{EAL^2}{3R^2} \frac{c^2\pi^2}{l^2} \cos^2\left(\frac{\pi y}{l}\right) dy \tag{11}$$

The total altered stiffness of the model can now be shown to be,

$$\frac{\pi^4}{l^4} EI \left\{1 + \frac{4}{15}\left(\frac{L}{R}\right)^2\left(\frac{L}{t}\right)^2 + \frac{4}{\pi^2}\left(\frac{l}{R}\right)^2\left(\frac{L}{t}\right)^2\right\}$$

Since $l \gg L$ when a sufficiently large number of elements are used, the term from the in-plane stiffening force will be the principal stiffening factor and we can identify the principal locking term to depend on a structural parameter of the type $(\beta L/t)^2$ which will now act as the penalty multiplier term. Numerical experiments in the next section will establish that this is indeed true.

3.6 Numerical experiments with the CL element.

In the computer implementation of the CL element, provision was made to evaluate the stiffness matrix contributions from the bending energy and the membrane energy separately using Gaussian integration orders that could be varied in each case. The order of the energy functionals dictate a 4-point integration rule for the exact evaluation of the membrane energy terms and a 2-point rule for the exact evaluation of the bending energy terms. Options for all rules from 1-point to 4-point were provided.

One half of a clamped-clamped arch with a central concentrated load was studied, with only two elements to model this half of the structure. A shallow and deep arch were defined to have a subtended angle $\beta = 1$ radian and $\pi/2$ radians respectively (See Fig.1). A previous exercise[5] has shown that in cases where the convergence of finite element results is delayed due to errors of the second kind, it is useful to define an additional stiffening factor e in terms of the finite element results w(fem) and the theoretically predicted results w(theory) as

$$e = \frac{w(\text{theory})}{w(\text{fem})} - 1$$

This parameter would now be directly dependent on the structural parameter or parameters that magnify the errors of the second kind in the penalty limits and therefore causes that phenomenon known as "locking'.

Fig.2 shows the variation of Log e vs Log (L/t) for the shallow and deep arches defined above. With a 1-point integration of the membrane energy

(CL1), there was no rapid deterioration of the results with increase in the parameter L/t over a range 3.142 to 314.2 and 20.0 to 2000.0 in each case, and any errors present can be attributed to those described as errors of the first kind in the terminology introduced in this lecture. However, with 2-point and 4-point integrations of the membrane energy (CL2 ,CL4), the locking is nearly identical and in any case, too close to be separated on the Log-Log plot in Fig.2. The locking can be seen to vary almost exactly as $(L/t)^2$,showing that errors of the second kind virtually dominate the behaviour in such limits. With increasing β , both the errors of the first and second kind increase. Our simple shallow circular arch example had predicted a structural multiplier term of the type $(\beta L/t)^2$. Thus a replot of the CL2 and CL4 results in the form Log e vs Log $\beta L/t$ in Fig.3 yields a slope of 2.00 and does indicate a locking term of the kind $(\beta L/t)^2$ is indeed operative unless removed by a 1-point integration of the extensional energy.

3.7 Conclusions.

Here, it has been possible to indicate general principles governing the errors of the second kind associated with the use of low order independent polynomial fields for thin arch and shell structures, and show how the in-plane constraints that arise in the very thin (inextensional) limit can be removed by an optimal application of integration rules. An old, familiar but till now discarded element has been reworked and shown to be useful, and should be a powerful candidate for inclusion in general purpose libraries which deal with elements with few degrees of freedom per node. A shear flexible version of this element would indeed be that used by Ahmed and Peters[6] with linear interpolations for u , w and the face rotations θ and will have a one-point integration rule for the membrane and shear energies. It is clear that the unexpected accuracy of such simple elements in Reference 6 is due to the removal of spurious membrane constraints in addition to the removal of spurious shear constraints.

4. TECHNIQUES TO INTRODUCE FIELD CONSISTENCY.

In this section, we shall survey a few of the techniques that are used to ensure field consistency in multi-field problems. The types of techniques used vary greatly and their number has been growing rapidly. Very often, the successful use of the technique preçedes the actual understanding of the mechanics of its operation. Here, we shall try to examine each in the context of field consistency.

4.1 Unequal order interpolation.

This is perhaps the simplest to understand, as no 'trick' such as reduced integration is required. This proceeds from an understanding of the fact that field inconsistency arises when equal order interpolations are used for field variables which appear in different orders of its derivatives in the strain field that has to be constrained. Thus, if one ensures that the strain field is consistently represented by a proper a priori choice of unequal order interpolations for the contributing field variables, there would be no 'locking'.

A simple example of this is the mixed linear/quadratic beam element considered in Reference 2. This has three nodes for w and two nodes for θ resulting in

$$w = a_1 + a_2x + a_3x^2$$
$$\theta = b_1 + b_2x \tag{12}$$

so that the shear strain field becomes

$$\gamma = \theta - w_{,x}$$
$$= (b_1 - a_2) + (b_2 - 2a_3)x \tag{13}$$

It is clear that in the penalty limit, the two constraints that appear can correctly enforce the Kirchhoff constraint.

An element based on this unequal order interpolation can easily be transformed into a two noded element by statically condensing the additional degree of freedom for w at the mid-side node. It was found in Reference 8 that such an element is identical to a conventional two noded element with equal order interpolations which has had its shear strain energy evaluated with a 1 pt. integration[2,9].

4.2 Reduced integration.

This is perhaps the first of the 'variational crimes' to be discovered[10] and proved to be very effective although at first, difficult to justify. Early attempts to explain its effectiveness were based on the number of constraints introduced at integration points and on its relation to the total number of degrees available in the model. Often, the orders of integration, although lower than the order needed for an exact evaluation of the strain energy functional, were still too high to remove locking. Sometimes, the orders were too low and introduced singularities and zero-energy mechanisms that degraded the element's behaviour. This led to considerable experimentation and often, confusion in the published literature.

Only very recently, did clearer guidelines about the orders of optimal integration for multi-field problems with constrained strain fields begin to emerge. What is important is that the part of the strain energy functional pertaining to the strain fields that must be constrained in the limit must be 'consistently' represented. If the shape function definitions were of equal order for all variables, and these variables appear at differing orders of its derivatives, then an exact integration of this part of the strain energy will produce 'inconsistent' terms where constraints are imposed on quantities from only one or some of the field variables that contribute to the strain field definition. If an optimal order of integration could be found that could correctly integrate all the 'consistent' terms and remove all the 'inconsistent' terms, then, one would have all the true constraints and would have removed all the spurious constraints. If the order of integration is not high enough to cover all the 'consistent' terms, one or more of the true constraints may vanish, leading to a rank-deficiency and therefore a 'zero-energy mechanism' that can degrade the element in certain applications. Again, an order of integration that is high enough to retain even one spurious constraint, is enough to retain 'locking'. We shall discuss these by a survey of a few of the elements studied earlier.

4.2.1 The linear beam element.

Consider a linear shear flexible beam element. The shear strain energy contribution is now for the shear strain (see Eqn.(13)) of the form

$$\gamma = c_1 + c_2 x \tag{14}$$

where c_1 is the consistent part and c_2 is the inconsistent part. An exact evaluation of the shear strain energy for an element of length L, will produce an expression for the shear strain energy which will have the independent contributions

$$(c_1 + c_2 L/2)^2 \quad \text{and} \quad (c_2 L)^2/12 \tag{56}$$

In the very thin limit, these two terms become independent constraints. It is obvious that the latter constraint is the term that causes locking. Fortunately, in this case, an optimal rule exists (a 1 pt. Gaussian integration), which will retain only the 'consistent' contribution and remove the 'inconsistent' contribution to the shear strain energy. This explains the remarkable efficiency of the linear beam element with a 1 pt. integration of the shear strain energy[2,9].

4.3 Addition of incompatible modes.

This is another interesting technique used to restore field consistency in a multi-field problem. It will be instructive here to demonstrate how the poor bending response of a plane stress element can be improved dramatically by the addition of properly chosen incompatible modes[11].

The original field definitions for u and v are augmented by adding two incompatible modes associated with additional internal variables. The simplest way this can be done is

$$u = a_0 + a_1 x + a_2 y + a_3 xy + a_4(1-x^2) + a_5(1-y^2)$$

$$v = b_0 + b_1 x + b_2 y + b_3 xy + b_4(1-x^2) + b_5(1-y^2)$$

The shear strain field now becc..es

$$u_{,y} + v_{,x} = (a_2 + b_1) + (a_3 - 2b_4)x + (b_3 - 2a_5)y$$

Thus , the field is now modelled consistently , as each of the coefficients - ie. associated with constant , linear x , linear y - now comprise terms from both field functions. Thus , in the very thin limit , they give rise to consistent constraints of the type

$$a_2 + b_1 = 0$$

$$a_3 - 2b_4 = 0$$

$$b_3 - 2a_5 = 0$$

There is now no spurious constraint and hence no locking. It is important to recognise that the addition of incompatible modes must be done in in an optimal way An arbitrary addition of polynomial terms that do not lead to a field consistent representation will lock!

5. CONCLUDING REMARKS.

In this paper, we have surveyed an area that has generated considerable interest in recent years. The problem area has been identified and defined - as continuum field problems that need a multi-field characterisation in which some field functions are constrained. The nature of the problem has been given a new name, and it is hoped that the requirement of 'field-consistency' will join the other more well known principles used in constructing efficient finite elements. The concept of 'errors of the second kind' to delineate a special form of discretisation errors' and the 'additional stiffness parameter' will also help to make a posteriori evaluations of such elements - whether there is locking or not.

The extension of these concepts to other similar field problems will be of further interest and the invention of new techniques or 'tricks' that can restore field-consistency will be interesting exercises.

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8. LIST OF FIGURE CAPTIONS.

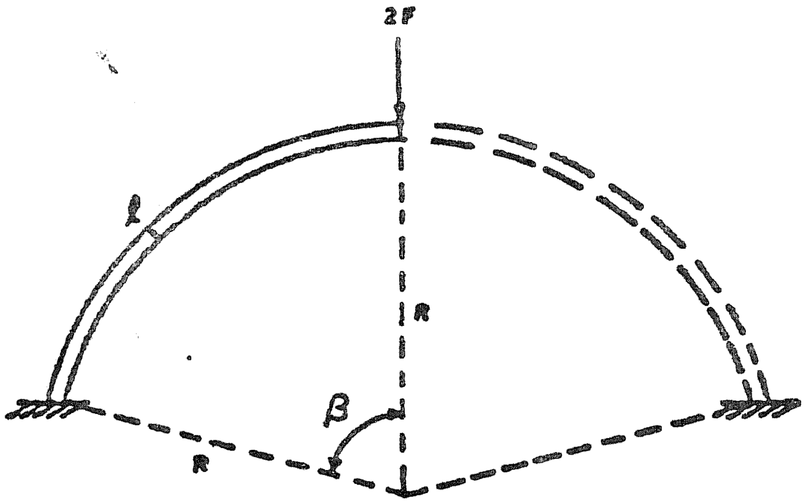

Fig. 1 Clamped-clamped circular arch under central concentrated load.

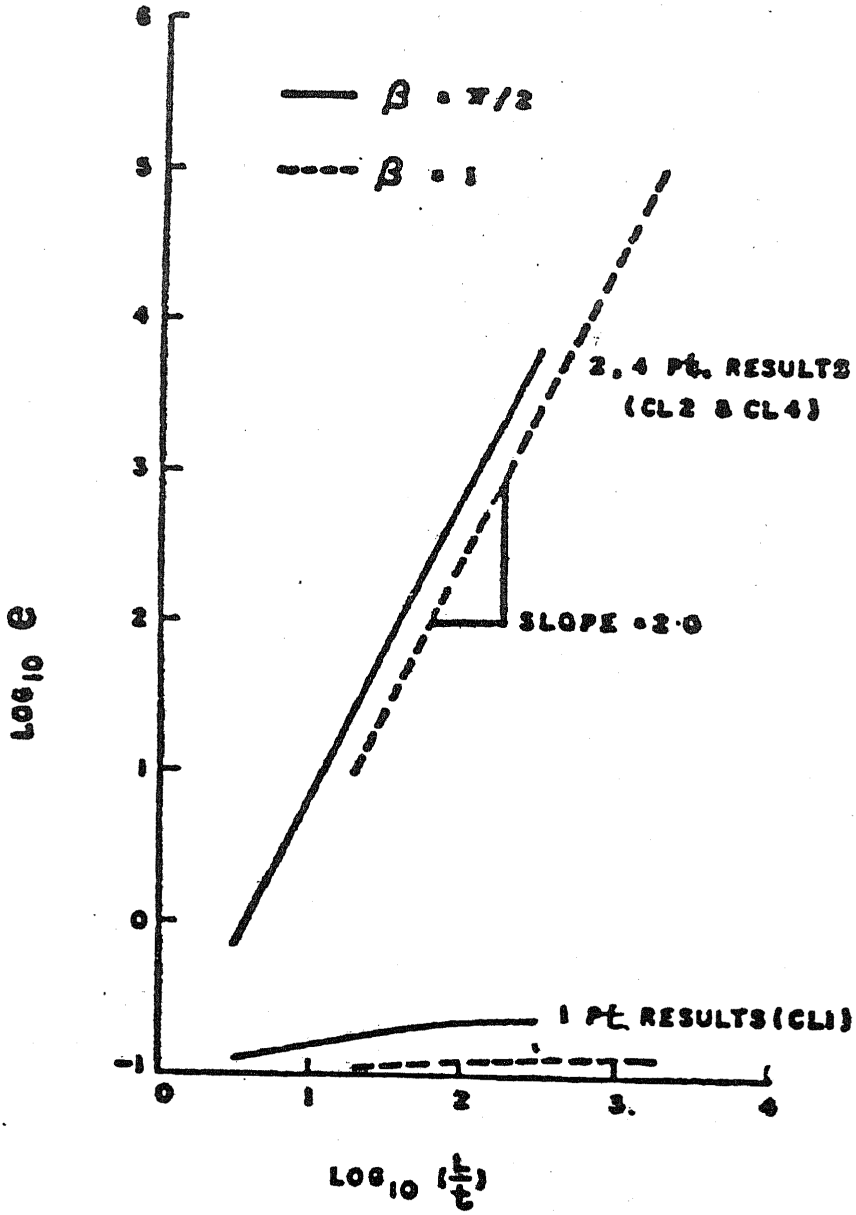

Fig. 2 Additional stiffness parameter for clamped arch under central concentrated load.

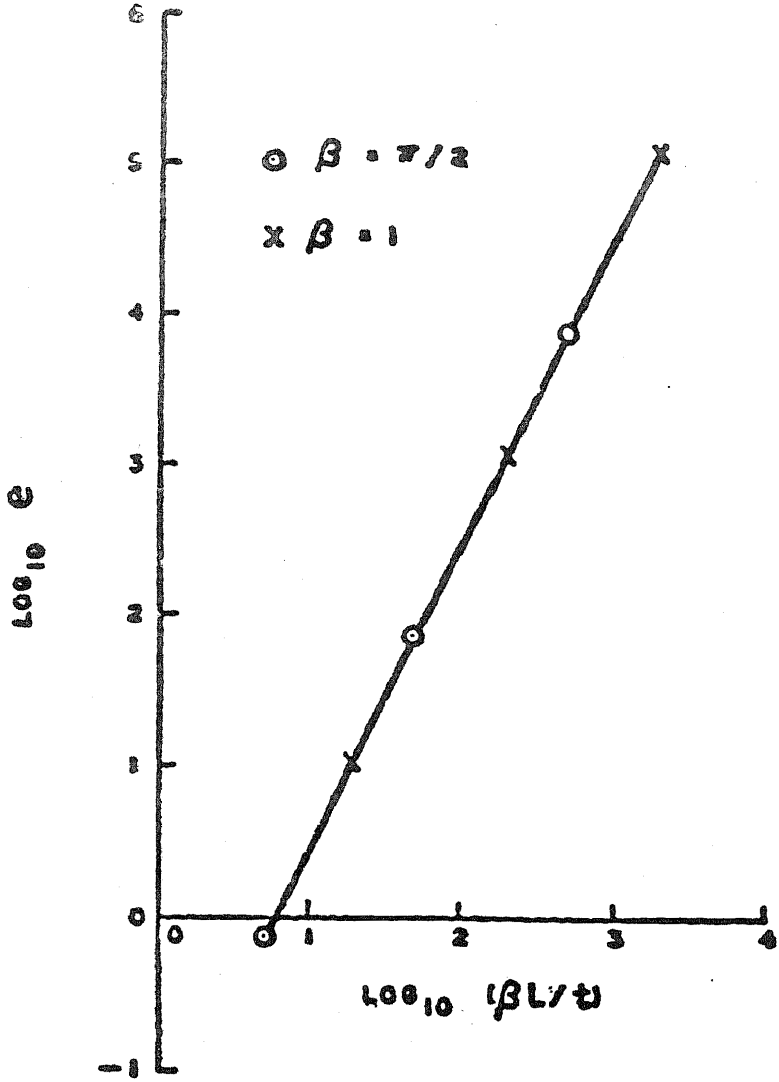

Fig. 3 Additional stiffness parameter for clamped arch under central concentrated load.