Lecture 4
Special Topics of FEA

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Lecture 4
Special Topics of FEA

Chapters

1. **Shear Locking in Timoshenko beam elements.**
2. **Error Analysis in Computational Elastodynamics.**
3. **Rank Deficiency in elements.**

Lecture 4
Special Topics of FEA

Chapter 1

Shear Locking in Timoshenko beam elements

(A Pathological Problem)

1.1 The Pathological problem of locking

Locking is a pathological problem encountered in formulating a certain class of elements for structural analysis, although these elements satisfy completeness and continuity requirements.

- **Locking causes slow convergence even for very fine mesh.**
- **Locking is manifested as Spurious Stiffening and Stress Oscillations.**

Explanations:

(1) Locking is caused by ill conditioning of the stiffness matrix due to the very large magnitude of the shear stiffness terms as compared to the those of bending stiffness (Tessler and Hughes).

(2) Locking occurs due to coupling between the shear deformation and bending deformation, and that it can be eliminated by appropriate de-coupling (Carpenter *et al).*

(3) Elements lock because they inadvertently enforce spurious constraints that arise from inconsistencies in the strains developed from the assumed displacement functions. (Prathap et al*).*

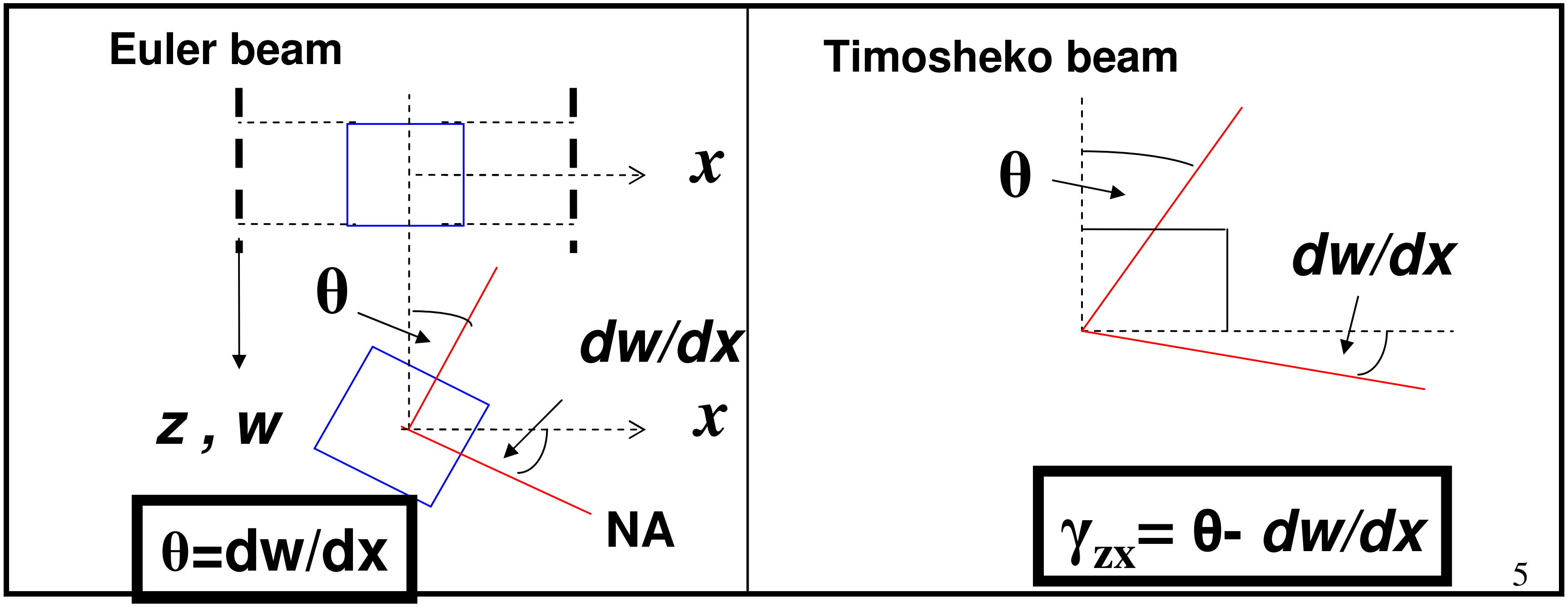
1.2 The Shear-flexible beam (Timoshenko)

In the classical Euler beam (meant only for thin beams), it has been shown that despite the presence of shear stress in the beam sections, the shear strain is ignored.

The Euler beam is of infinite shear rigidity (!)

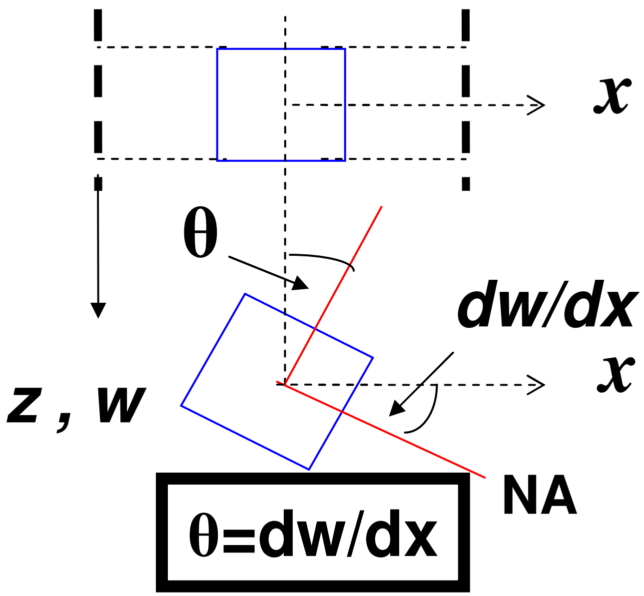
For thick beams (of wider webs), the Euler beam theory is not valid. Shear deformation of the web requires shear-flexible formulations.

Prof S P Timoshenko



Elementary beam theory as constrained media problem

Euler beam model



The Euler beam has infinite shear rigidity κ

But the practice of using a large shear rigidity κ for thin beams creates a problem called Shear Locking in shear-flexible beam elements.

$$\delta\Pi = \delta\left[\int_0^L \frac{1}{2} EI\left(\frac{d\theta}{dx}\right)^2 dx - \int_0^L qw\,dx + \int_0^L \frac{1}{2}\kappa\left(\theta - \frac{dw}{dx}\right)^2\right] = 0$$

Equilibrium Equations

$$EI\frac{d^2\theta}{dx^2} - \kappa\left(\theta - \frac{dw}{dx}\right) = 0 \qquad ....(i)$$

$$\kappa\left(\frac{d\theta}{dx} - \frac{d^2 w}{dx^2}\right) = q \qquad i.e. \quad \frac{d}{dx}\kappa\left(\theta - \frac{dw}{dx}\right) = q \qquad ....(ii)$$

Combining (i) & (ii)

$$\frac{d}{dx} EI \frac{d^2\theta}{dx^2} - q = 0 \qquad ...(iii)$$

Boundary conditions at $x = 0$ & $x = L$

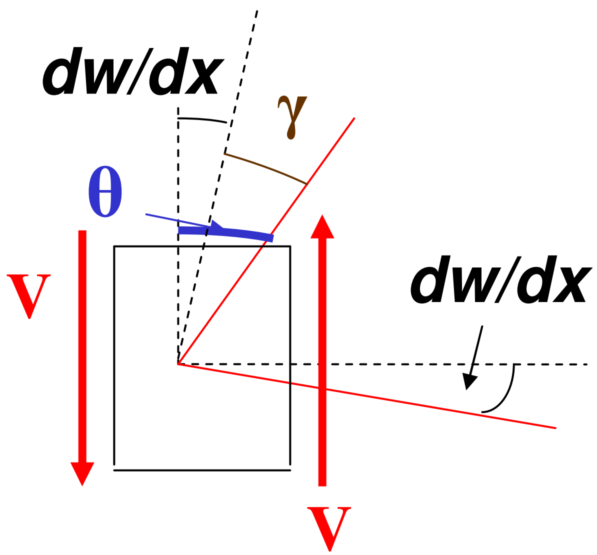
$$Either \quad EI\frac{d\theta}{dx} = 0 \quad or \quad \delta\theta = 0$$

$$Either \quad \kappa\left(\theta - \frac{dw}{dx}\right) = 0 \quad or \quad \delta w = 0$$

$$As \quad \kappa \to \infty \quad \theta \to \frac{dw}{dx}$$

$$Equation(iii) \quad reduces \quad to \quad \left(EI\frac{d^4 w}{dx^4} - q\right) \to 0$$

Equilibrium equations of the Shear flexible (deep) beams



Shear strain

γ= θ- *dw/dx*=V/(kGA)

(1.1)

Shear rigidity: kGA

$$\delta\Pi = \delta\left[\int_0^L \frac{1}{2} EI\left(\frac{d\theta}{dx}\right)^2 dx + \int_0^L \frac{1}{2} kGA\left(\theta - \frac{dw}{dx}\right)^2 - \int_0^L qw dx\right] = 0$$

Equilibrium Equations

$$EI\frac{d^2\theta}{dx^2} - kGA\left(\theta - \frac{dw}{dx}\right) = 0 \qquad ....(i)$$

-dM/dx-V=0

(1.2)

$$kGA\left(\frac{d\theta}{dx} - \frac{d^2w}{dx^2}\right) = q \qquad i.e. \quad \frac{d}{dx}kGA\left(\theta - \frac{dw}{dx}\right) = q \qquad ....(ii)$$

Combining (i) & (ii)

dV/dx=q

(1.3)

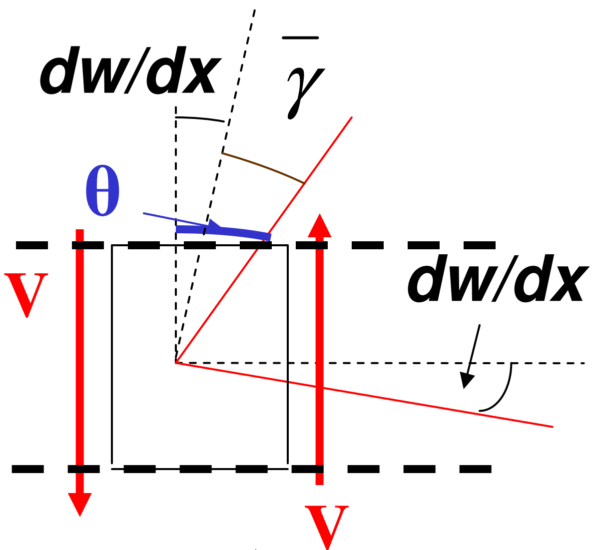
$$\frac{d}{dx}EI\frac{d^2\theta}{dx^2} - q = 0 \qquad ...(iii)$$

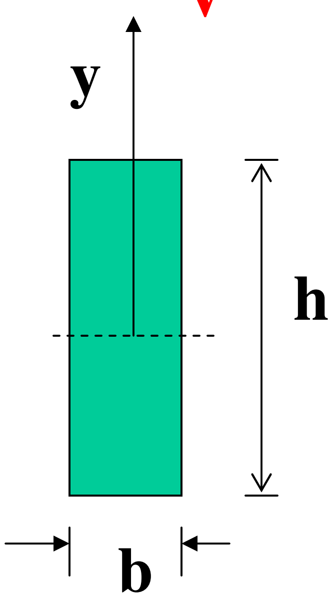
Boundary conditions at $x = 0$ & $x = L$

Either $EI\frac{d\theta}{dx} = 0$ *or* $\delta\theta = 0$

Either $kGA\left(\theta - \frac{dw}{dx}\right) = 0$ *or* $\delta w = 0$

Shear rigidity of deep beams





$$\tau = \frac{VQ}{Ib} \qquad \gamma = \frac{\tau}{G} = \frac{VQ}{GIb}$$

$$U_{shear} = \int_V \frac{1}{2}.\tau.\gamma.dV = \int_0^L \int_{y=-h/2}^{h/2} .\frac{\tau^2}{2G}.(bdy)dx = \int_0^L \int_{y=-h/2}^{h/2} .\frac{V^2Q^2}{2G(Ib)^2}.(bdy)dx$$

$$U_{shear} = \int_0^L \frac{1}{2} V\left(\frac{V}{kGA}\right)dx \qquad \frac{1}{kGA} = \frac{1}{G}\int_{-h/2}^{h/2} \frac{Q^2}{I^2 b}dy$$

$$k = \frac{1}{\frac{A}{I^2}\int_{-h/2}^{h/2} \frac{Q^2}{b}dy}$$

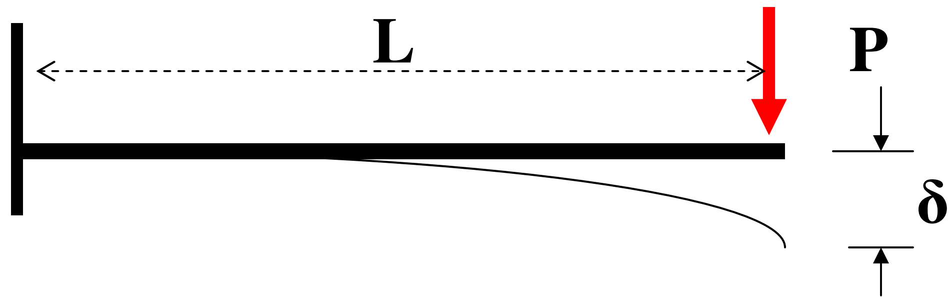
$$U_{shear} = \int_0^L \frac{1}{2} V\left(\frac{V}{kGA}\right)dx = \int_0^L \frac{1}{2} V\bar{\gamma}dx \qquad \bar{\gamma} = \frac{V}{kGA} = \theta - \frac{dw}{dx}$$

$$U_{shear} = \int_0^L \frac{1}{2} V\bar{\gamma}dx = \int_0^L \frac{1}{2} kGA(\bar{\gamma})^2 dx = \frac{1}{2}kGA\int_0^L \left(\theta - \frac{dw}{dx}\right)^2 dx \qquad \textbf{(1.4)}$$

***k* is called the shear correction factor**

***k*=5/6 for a rectangular section**

Example 1. **Find the tip deflection of a cantilever subjected to a concentrated tip load P. (Include shear deformation)**



Deflection at the free end :

$$\delta = \frac{PL^3}{3EI} + \bar{\gamma}L = \frac{PL^3}{3EI} + \frac{P}{kGA}L = \frac{PL^3}{3EI}\left(1 + \frac{3EI}{kGAL^2}\right)$$

For thin beams,

$$\frac{kGAL^2}{EI} \to \infty, \qquad \frac{EI}{kGAL^2} \to 0$$

1.3 Formulation of the two-noded Timoshenko Beam Element (Using Linear Lagrangian C^0 Shape Functions)

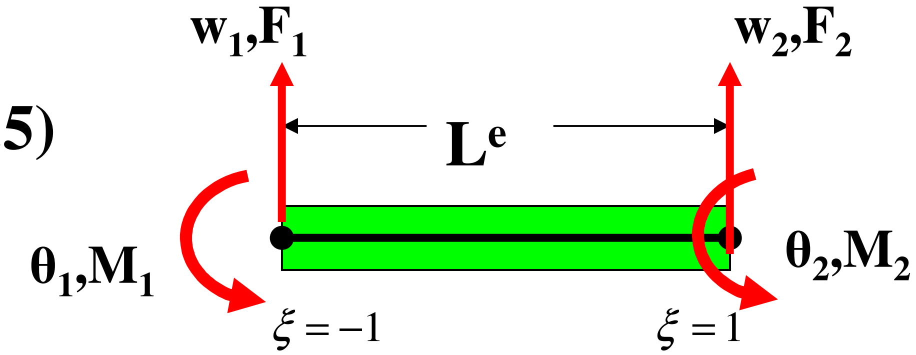
Element displacement and geometry (iso-parametric):

$$w^h = N_1 w_1 + N_2 w_2 \qquad \theta^h = N_1\theta_1 + N_2\theta_2 \qquad N_1 = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2} \qquad -1 \le \xi \le 1$$

$$x = N_1 x_1 + N_2 x_2 \qquad with \qquad x_1 = 0, \quad x_2 = L^e \qquad x = \frac{L^e}{2}(\xi + 1), \qquad \xi = \frac{2x}{L^e} - 1$$

$$dx = \frac{L^e}{2} d\xi$$

$$\begin{Bmatrix} w^h \\ \theta^h \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = [N]\{\delta^e\} \qquad \mathbf{(1.5)}$$



Element Strain vector:

$$\left(\varepsilon^h\right) = \begin{pmatrix} d\theta^h/dx \\ \theta^h - dw^h/dx \end{pmatrix} = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & (1-\xi)/2 & -1/L & (1+\xi)/2 \end{bmatrix} \{\delta^e\}$$

$$\{\varepsilon^h\} = [B]\{\delta^e\} \qquad \mathbf{(1.6)}$$

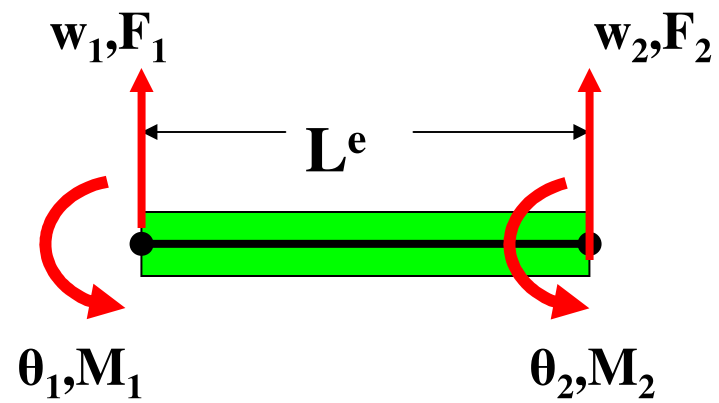
Element stress resultants :

$$\begin{Bmatrix} M \\ V \end{Bmatrix} = \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \begin{Bmatrix} d\theta / dx \\ \theta - dw/dx \end{Bmatrix} = [D][B]\{\delta^e\} \qquad \textbf{(1.7)}$$

Element potential energy:

$$\Pi = \left[\int_0^L \frac{1}{2} EI \left(\frac{d\theta}{dx} \right)^2 dx + \int_0^L \frac{1}{2} kGA \left(\theta - \frac{dw}{dx} \right)^2 - \int_0^L qwdx - \{\delta^e\}^T R^e \} \right]$$

$$\Pi = \frac{1}{2} \{\delta^e\}^T [K^e] \{\delta^e\} - \{\delta^e\}^T \left(\{F^e\} + \{R^e\} \right)$$



Equilibrium

$$\delta\Pi = 0 \qquad [K^e]\{\delta^e\} = \{F^e\} + \{R^e\} \qquad \textbf{(1.8)}$$

Element Stiffness matrix

$$[K^e] = \int_{-1}^{1} [B]^T [D][B] \frac{L^e}{2} d\xi \qquad \textbf{(1.9)}$$

Element Force vector

$$\{F^e\} = \int_{-1}^{1} [N]^T q \frac{L^e}{2} d\xi \qquad \textbf{(1.10)}$$

Using a 2 point Gauss integration the stiffness matrix is

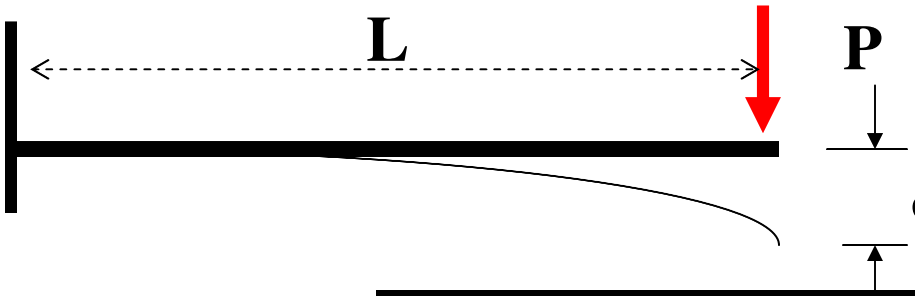
$$[K^e] = \int_{-1}^{1} [B]^T [D][B] \frac{L^e}{2} d\xi = [K^e{}_b] + [K^e{}_s]$$

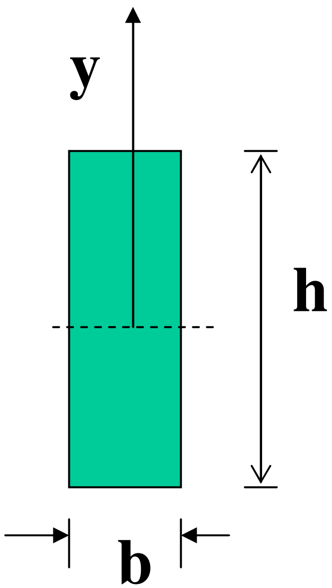
$$[K^e] = [K^e{}_b] + [K^e{}_s] = \frac{EI}{L^e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} + \frac{kGA}{L^e}\begin{bmatrix} 1 & L^e/2 & -1 & L^e/2 \\ L^e/2 & (L^e)^2/3 & -L^e/2 & (L^e)^2/6 \\ -1 & -L^e/2 & 1 & -L^e/2 \\ L^e/2 & (L^e)^2/6 & -L^e/2 & (L^e)^2/3 \end{bmatrix}$$

(1.11)

FE results of analysis of deep beam cantilever beam under tip load

E=1000
G=375
b=1, h=1
L=4





Locked results

Observations

- Large errors
- A pattern in the error.
- Slow convergence

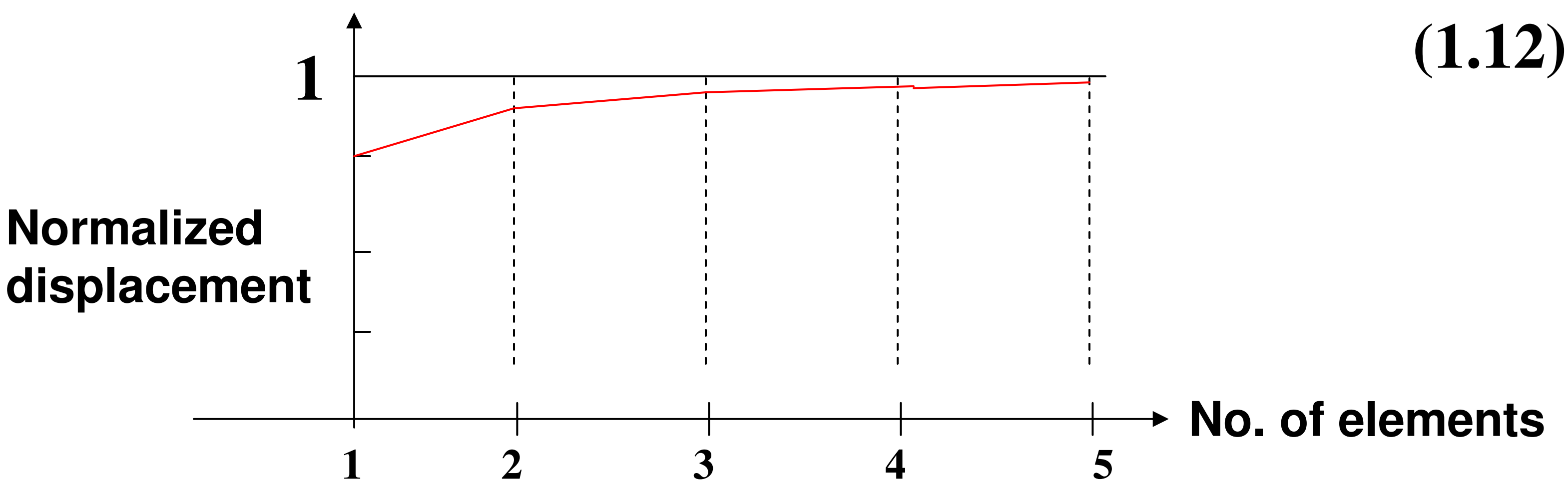
No of elements	Normalized tip Displacement (Locked)
1	$0.2\,(10)^{-5}$
2	$0.8\,(10)^{-5}$
4	$0.32\,(10)^{-4}$
8	$0.128\,(10)^{-3}$
16	$0.512\,(10)^{-3}$

Antidote for shear locking.

Use a 1 point (instead of 2 point) Gauss integration scheme for the stiffness matrix is

$$[K^e*]=[K^e{}_b]+[K^e{}_s*]=\frac{EI}{L^e}\begin{bmatrix}0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&-1\end{bmatrix}+\frac{kGA}{L^e}\begin{bmatrix}1&L^e/2&-1&L^e/2\\L^e/2&(L^e)^2/4&-L^e/2&(L^e)^2/4\\-1&-L^e/2&1&-L^e/2\\L^e/2&(L^e)^2/4&-L^e/2&(L^e)^2/4\end{bmatrix}$$

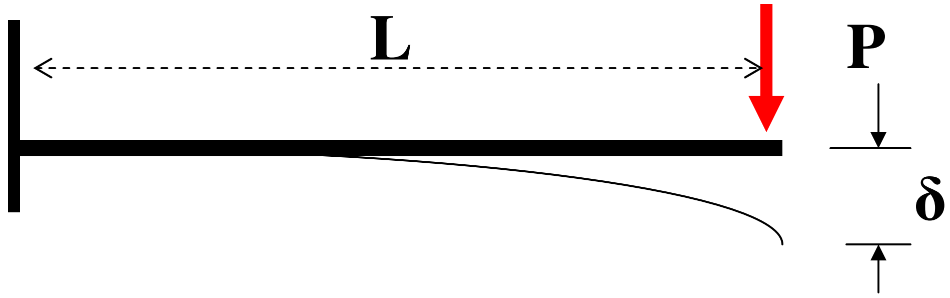
(1.12)

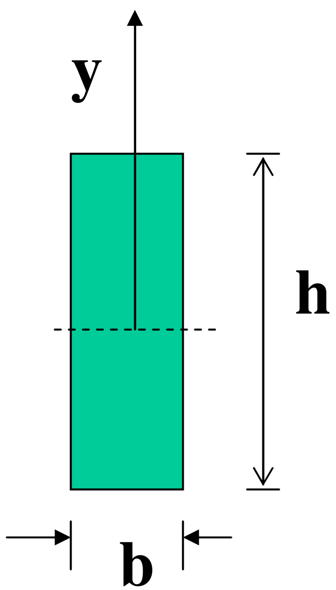


Magic! An error in the integration eliminates locking! WHY ?

FE results of analysis of deep beam cantilever beam under tip load

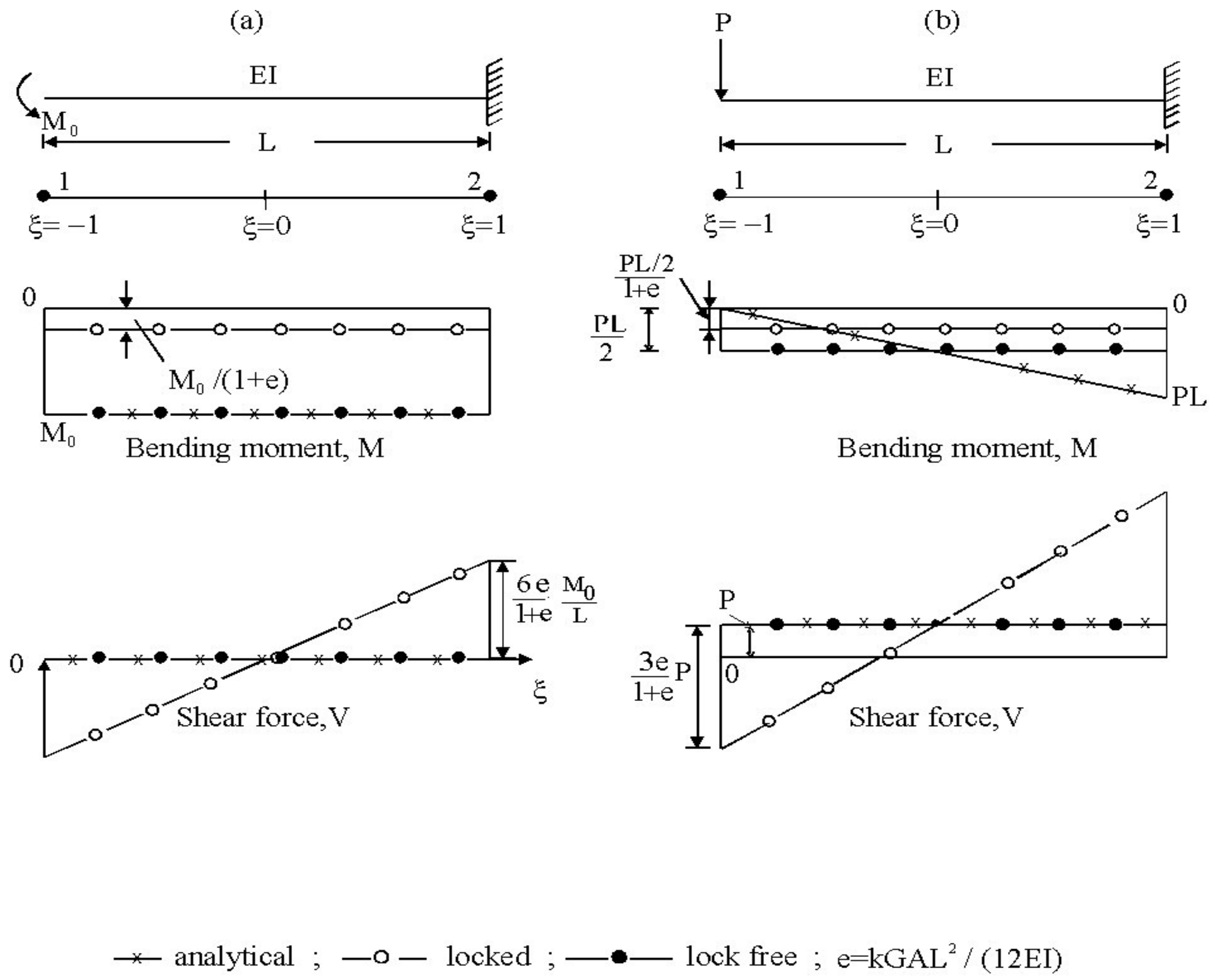
E=1000
G=375
b=1, h=1
L=4





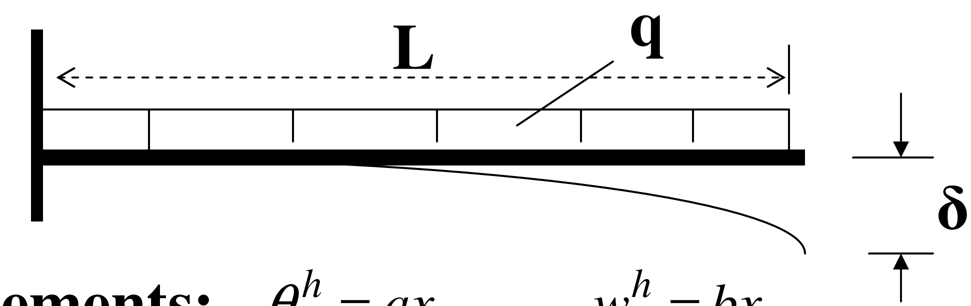
Number of elements	Normalized tip Displacement (Locked)	Normalized tip Displacement (Lock-free, Reduced Int.)
1	$0.2\ (10)^{-5}$	0.75
2	$0.8\ (10)^{-5}$	0.75+0.75/4 =0.9375

Example problems solved using a single Timoshenko beam element



Observations: Spurious shear oscillations and bending stiffening for the locked case.

1.4 Explanations for the origin of locking
(The field-consistency paradigm)



Linear displacements: $\theta^h = ax$ $\quad w^h = bx$

Shear strain $\gamma = \theta^h - \dfrac{dw^h}{dx} = ax - b$ **Bending strain** $\dfrac{d\theta^h}{dx} = a$

Rayleigh-Ritz procedure

$$\left\{ EI\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \kappa \begin{bmatrix} L^3/3 & -L^2/2 \\ -L^2/2 & L \end{bmatrix} \right\} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 0 \\ qL^2/2 \end{Bmatrix}$$

$$a = \frac{-3qL^2}{12EI + \kappa L^2} \qquad b = -\frac{qL}{2\kappa} - \frac{1.5qL^3}{12EI + \kappa L^2}$$

$$As \quad \kappa \to \infty, \quad a \to 0, \quad b \to 0 \quad \Rightarrow \quad \frac{d\theta^h}{dx} \to 0$$

Element locks when shear rigidity κ is increased indefinitely.

The parameter *a* effects both bending and shear strains. It is a spurious constraint that stiffens bending as well as shear strains

1.5 Explanation of shear locking in the element by Field-Consistency Theory

The shear strain in the element is $\theta^h - dw^h/dx = \alpha + \beta\xi$
where $\alpha = (\theta_2 + \theta_1)/2 - (w_2 - w_1)/\mathrm{L}$ and $\beta = (\theta_2 - \theta_1)/2$

- For **thin** beams, the shear strain energy term vanishes, leading to two constraints: $\alpha \to 0 \qquad \beta \to 0$
(First constraint is physically meaningful in terms of the equivalent Euler beam model, but the **second constraint is a spurious one**.

The spurious term β effectively enhances the element's bending stiffness to $EI^*=EI+kGA(L^e)^2/12$, where EI and kGA are the bending and shear rigidities respectively of the actual beam, leading to locking.

$$w_{LF}/w_L = I^*/I = 1 + kGA\mathrm{L}^2/(12EI) = 1 + e$$

$e=kGA(L^e)^2/(12EI)=K/n2$, (l=total beam length, n=total number of equal elements, L^e= element length=l/n).

The parameter *e* becomes larger for thinner beams, leading to spuriously high bending stiffness, and spurious shear strain oscillations in the elements.

1.6 How shear locking is eliminated by reduced integration

The integrand in the element stiffness matrix [K^e] is quadratic, so we need a 2 point Gauss rule for exact integration. This element suffers shear locking.

$$[K^e]=\int_{-1}^{1}[B]^T[D][B]\frac{L^e}{2}d\xi=[K^e{}_b]+[K^e{}_s]$$

$$[K^e]=[K^e{}_b]+[K^e{}_s]=\frac{EI}{L^e}\begin{bmatrix}0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&-1\end{bmatrix}+\frac{kGA}{L^e}\begin{bmatrix}1&L^e/2&-1&L^e/2\\L^e/2&(L^e)^2/3&-L^e/2&(L^e)^2/6\\-1&-L^e/2&1&-L^e/2\\L^e/2&(L^e)^2/6&-L^e/2&(L^e)^2/3\end{bmatrix}$$

A reduced integration actually eliminates (ignores) the spurious term β of the shear strain (associated with linear variation in ξ) so that only constant terms are needed to be integrated. This elimination of the spurious constraint is done by a 1 point Gaussian rule for integration.

$$[K^e*]=[K^e{}_b]+[K^e{}_s*]=\frac{EI}{L^e}\begin{bmatrix}0&0&0&0\\0&1&0&-1\\0&0&0&0\\0&-1&0&-1\end{bmatrix}+\frac{kGA}{L^e}\begin{bmatrix}1&L^e/2&-1&L^e/2\\L^e/2&(L^e)^2/4&-L^e/2&(L^e)^2/4\\-1&-L^e/2&1&-L^e/2\\L^e/2&(L^e)^2/4&-L^e/2&(L^e)^2/4\end{bmatrix}$$

If one uses a *Reduced Integration* scheme with a *one-point rule of Gauss Quadrature,* instead of the *two-point rule* necessary for accurate integration in the shear strain energy, it leads to

- **Elimination of shear locking by releasing the stiffening constraint β.**

- **Elimination of spurious shear stress oscillations.**

Reduced integration effectively drops the Second Legendre Polynomial from the shear strain,

$$\alpha + \beta\xi \rightarrow \alpha$$

The Function Space Approach to Locking Problems

1.7 Definition of the Inner product

The inner product for the Timoshenko beam element is defined through the symmetric bilinear forms:

$$a(u^h, u)^e = \int_e \left\{\varepsilon^h\right\}^T \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \left\{\varepsilon\right\} dx$$

$$= \int_{-1}^{1} \left\{\varepsilon^h\right\}^T \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \left\{\varepsilon\right\} \frac{L^e}{2} d\xi = < \varepsilon^h, \varepsilon >$$

$$a(u^h, u^h)^e = \int_e \left\{\varepsilon^h\right\}^T \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \left\{\varepsilon^h\right\} dx$$

$$= \int_{-1}^{1} \left\{\varepsilon^h\right\}^T \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \left\{\varepsilon^h\right\} \frac{L^e}{2} d\xi = < \varepsilon^h, \varepsilon^h > \qquad \textbf{(1.13)}$$

EI=Flexural Rigidity, kGA=Shear Rigidity

1.8 The B Subspace

The B subspace is the space in which the column vectors of the strain-displacement matrix [B] lie.

$$[B]=\begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & (1-\xi)/2 & -1/L & (1+\xi)/2 \end{bmatrix} \qquad (1.14)$$

The Gram-Schmidt Algorithm for getting the orthogonal basis vectors spanning the B Space:

$$\{v_1\}=\{b_1\}$$
$$\{v_{k+1}\}=\{b_{k+1}\}-\sum_{j=1}^{k}\frac{\langle b_{k+1},v_j\rangle}{\langle v_j,v_j\rangle}\{v_j\} \qquad (1.15))$$

After scaling, only TWO NON-ZERO orthogonal basis vectors are obtained that span the B Space (of 2 dimensions, m=N-R=4-2=2)

$$\{v_1\}=\begin{Bmatrix}0\\1\end{Bmatrix},\qquad \{v_2\}=\begin{Bmatrix}2/L\\ \xi\end{Bmatrix} \qquad (1.16)$$

$$B\subset P_{n=2}^{r=2} \quad ; \quad P_2^2=\left\{\{p\}:\{p\}=\sum_{i=1}^{2}\{a_i\}\xi^{i-1}, -1\le\xi\le 1,\ \{a_i\}\in R^2\right\}$$

$$\dim(B)=2<\dim P_{n=2}^{r=2}=2\times 2=4 \qquad (1.17)$$

1.9 Strain projections on the B Subspace; Shear Locking

Orthogonal Projection of the Analytical Strain onto the B Subspace yields the FEA computed element strains (best-fits).

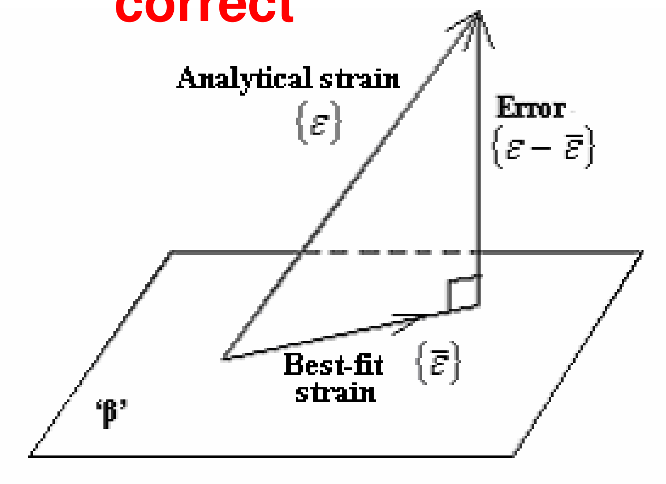
$$\{\varepsilon^h\}=\{\bar{\varepsilon}\}=\sum_{j=1}^{m=2}\frac{<\varepsilon,v_j>}{<v_j,v_j>}\{v_j\},\qquad <v_1,v_2>=0 \tag{1.18}$$

However, we have **problems for thin beams**:

1. The bending strain is a lot smaller than the analytical one, showing that **spurious bending stiffness** has been introduced through FEA **.**
2. There is **spurious shear strain oscillation** in FEA results.
3. **Slow Convergence** even with many elements.

These are the symptoms of locking

Locked FEA solutions agree with the best-fit strain vector at the element level. Thus locked solutions are variationally correct



$$\varepsilon^h = \bar{\varepsilon} = Best - fit$$

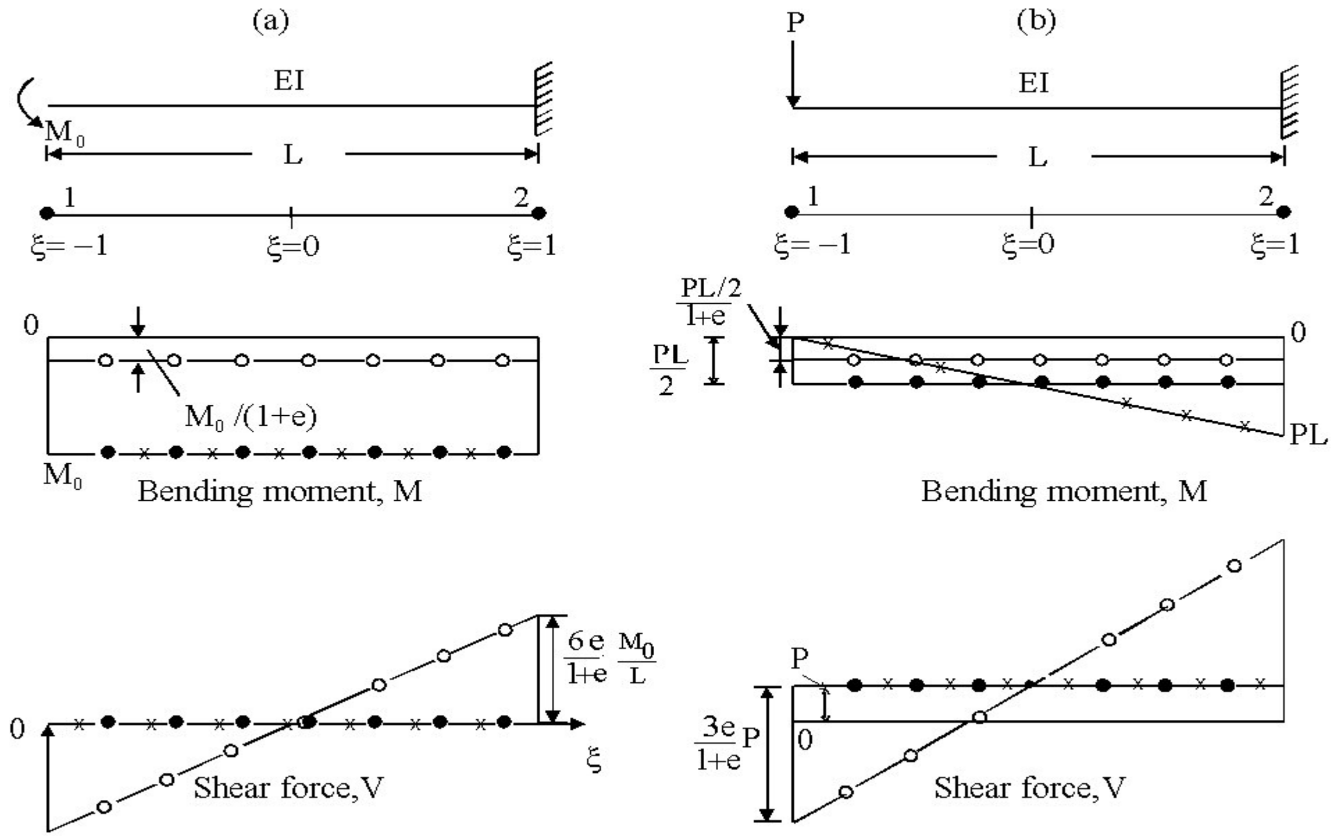
$$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{j=1}^{m=2} \frac{<\varepsilon, v_j>}{<v_j, v_j>}\{v_j\}, \qquad <v_1, v_2> = 0$$

A best fit satisfies the Projection Theorem (Pythagoras)

$$\left\|\varepsilon - \bar{\varepsilon}\right\|^2 = \left\|\varepsilon\right\|^2 - \left\|\bar{\varepsilon}\right\|^2$$

Thus $$\left\|\varepsilon - \varepsilon^h\right\|^2 = \left\|\varepsilon\right\|^2 - \left\|\varepsilon^h\right\|^2$$

i.e. **The Energy of the Error= Error of the Energies**



—×— analytical ; —o— locked ; —●— lock free ; $e=kGAL^2/(12EI)$

TABLE 1

Analytical strains and their locked projections as finite element strains
e=kGAL2/(12EI).

	Cantilever with tip moment M_0	Cantilever with tip load P
Analytical strain vector	$\{\varepsilon\} = \begin{Bmatrix} M_0/EI \\ 0 \end{Bmatrix}$	$\{\varepsilon\} = \begin{Bmatrix} PL(1+\xi)/(2EI) \\ P/kGA \end{Bmatrix}$
Locked strain vector	$\{\bar{\varepsilon}\} = \begin{Bmatrix} (M_0/EI)/(1+e) \\ \dfrac{6e}{(1+e)}\dfrac{M_0\xi}{LkGA} \end{Bmatrix}$	$\{\bar{\varepsilon}\} = \begin{Bmatrix} (PL/2EI)/(1+e) \\ \dfrac{P}{kGA}(1+\dfrac{3e\xi}{1+e}) \end{Bmatrix}$

$$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{j=1}^{m=2} \frac{<\varepsilon, v_j>}{<v_j, v_j>}\{v_j\}, \qquad <v_1, v_2> = 0$$

$$\{v_1\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \qquad \{v_2\} = \begin{Bmatrix} 2/L \\ \xi \end{Bmatrix}$$

FE Strain vectors exactly agree with these orthogonal projections of analytical strains

TABLE 2

Error norm square for locked strain projections with the linear two noded Timoshenko beam element. e=kGAL²/(12EI)

$$\|q\|^2 = \frac{L}{2}\int_{-1}^{1}\{q\}^T\,[D]\{q\}\,d\xi \qquad \{q\}=\{\varepsilon\}-\{\bar{\varepsilon}\}$$

Case	*Locked Solution*
Cantilever with tip moment, M_o	$\|q\|^2 = \frac{L}{2}\frac{2M_0^2}{EI}\cdot\frac{e}{1+e}$
Cantilever with tip transverse load P	$\|q\|^2 = \frac{L}{2}\cdot\frac{(P\mathcal{L})^2}{2EI}\left(\frac{e}{1+e}+\frac{1}{3}\right)$

$$\{\varepsilon^h\} = \{\bar{\varepsilon}\}$$

$$\|q\|^2 = \left\|\varepsilon-\varepsilon^h\right\|^2 = \|\varepsilon\|^2 - \left\|\varepsilon^h\right\|^2$$

1.10 The Function Space explanation of locking and its elimination

The original field-inconsistent [B] matrix is

$$[B]=\begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & (1-\xi)/2 & -1/L & (1+\xi)/2 \end{bmatrix}$$

Locking occurs because the 2-dimensional **B subspace is field-inconsistent,** which **cannot** be spanned by the **standard basis vectors of its 4-dimensional parent space** P_2^2 (linear in ξ),

$\{L_1\} = [0, 1]^T$, $\{L_2\} = [1, 0]^T$, $\{L_3\} = [0, \xi]^T$, $\{L_4\} = [\xi, 0]^T$., (1.19)

Actually, the field-inconsistent B space is spanned by non-standard basis vectors,

$$\{v_1\}=\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \qquad \{v_2\}=\begin{Bmatrix} 2/L \\ \xi \end{Bmatrix}$$

1.11 Elimination of shear locking

Reduced Integration effectively sets the highest order Legendre Polynomial ξ in the [B] matrix to zero.

It replaces [B] by a (modified) [B*].

Lock-free strain vector is expressed as,

$$\{\varepsilon^h *\} = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & 1/2 & 1/L & 1/2 \end{bmatrix} \{\delta *^e\} = [B*]\ \{\delta *^e\} \qquad (1.20)$$

A new field-consistent space B* emerges from [B*]. This **lockfree, field-consistent space B* is two-dimensional, and can be spanned by the standard orthogonal basis vectors,**

$$\{v_1 *\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \qquad \{v_2 *\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad (1.21)$$

$$B^* \subset P_{n=1}^{r=2} \quad ; \quad P_1^2 = \left\{ \{p\} : \{p\} = \{a_i\}\ ,\ \{a_i\} \in R^2 \right\}$$

$$\dim(B) = 2 = \dim P_n^r = 2 \times 1 = 2 \qquad (1.22)$$

Lockfree stiffness matrix for the Timoshenko beam is obtained from the field-consistent (lockfree) strain-displacement matrix [B*] with exact integration

$$[K^e*] = \int_{-1}^{1} [B*]^T [D][B*] \frac{L^e}{2} d\xi = [K^e{}_b] + [K^e{}_s*]$$

$$[K^e*] = [K^e{}_b] + [K^e{}_s*] = \frac{EI}{L^e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} + \frac{kGA}{L^e}\begin{bmatrix} 1 & L^e/2 & -1 & L^e/2 \\ L^e/2 & (L^e)^2/4 & -L^e/2 & (L^e)^2/4 \\ -1 & -L^e/2 & 1 & -L^e/2 \\ L^e/2 & (L^e)^2/4 & -L^e/2 & (L^e)^2/4 \end{bmatrix} \quad (1.23)$$

1.12 Orthogonal Projection on B* space

In general, Reduced Integrated FEA results are NOT variationally correct. (RI is a variational crime !)

Reduced Integrated FEA strains will agree with the best-fit solution, provided the following rule holds good,

$$\{F^e{}_E\} = -\int_e [[B]-[B^*]]^T [D]\{\varepsilon\} dx = 0 \qquad (1.24)$$

Then:

$$\{\varepsilon^h *\} = \{\overline{\varepsilon *}\} = \sum_{i=1}^{m} \frac{<\varepsilon, \nu_i *>}{<\nu_i *, \nu_i *>}\{\nu_i *\}, \quad <\nu_i *, \nu_j *> = 0 \quad for \quad i \neq j \qquad (1.25)$$

When this extraneous force $\{F^e{}_E\}$ does not vanish, then the best-fit solution (on the B* space) will suffer additional strain from this extraneous force vector, over the lockfree (reduced integrated) FEA solution.

TABLE 3

Analytical strains and their locked and lockfree projections as finite element strains $e=kGAL^2/(12EI)$.

	Cantilever with tip moment M_0	Cantilever with tip load P
Analytical strain vector	$\{\varepsilon\}=\begin{Bmatrix} M_0/EI \\ 0 \end{Bmatrix}$	$\{\varepsilon\}=\begin{Bmatrix} PL(1+\xi)/(2EI) \\ P/kGA \end{Bmatrix}$
Locked strain vector	$\{\bar{\varepsilon}\}=\begin{Bmatrix} (M_0/EI)/(1+e) \\ \dfrac{6e}{(1+e)}\dfrac{M_0\xi}{LkGA} \end{Bmatrix}$	$\{\bar{\varepsilon}\}=\begin{Bmatrix} (PL/2EI)/(1+e) \\ \dfrac{P}{kGA}(1+\dfrac{3e\xi}{1+e}) \end{Bmatrix}$
Lockfree strain vector	$\{\bar{\varepsilon}^*\}=\begin{Bmatrix} M_0/EI \\ 0 \end{Bmatrix}$	$\{\bar{\varepsilon}^*\}=\begin{Bmatrix} PL/(2EI) \\ P/kGA \end{Bmatrix}$

$$\{\varepsilon^h\}=\{\bar{\varepsilon}\}=\sum_{j=1}^{m=2}\frac{<\varepsilon,v_j>}{<v_j,v_j>}\{v_j\},\qquad <v_1,v_2>=0$$

$$\{v_1\}=\begin{Bmatrix}0\\1\end{Bmatrix},\qquad \{v_2\}=\begin{Bmatrix}2/L\\ \xi\end{Bmatrix}$$

B

$$\{\varepsilon^h*\}=\{\bar{\varepsilon}*\}=\sum_{j=1}^{m=2}\frac{<\varepsilon,v_j>}{<v_j,v_j>}\{v*_j\},\qquad <v_1*,v_2*>=0$$

$$\{v_1*\}=\begin{Bmatrix}0\\1\end{Bmatrix},\qquad \{v_2*\}=\begin{Bmatrix}1\\0\end{Bmatrix}$$

B*

TABLE 4

Error norm square for strain projections with the linear two noded Timoshenko beam element. $e=kGAL^2/(12EI)$

$$\|q\|^2 = \frac{L}{2}\int_{-1}^{1}\{q\}^T[D]\{q\}\,d\xi \qquad \{q\}=\{\varepsilon\}-\{\bar{\varepsilon}\}$$

Case	Locked Solution	Lockfree Solution
Cantilever with tip moment, M_o	$\|q\|^2 = \frac{L}{2}\frac{2M_0^2}{EI}\cdot\frac{e}{1+e}$	$\|q^*\|^2 = 0$
Cantilever with tip transverse load P	$\|q\|^2 = \frac{L}{2}\cdot\frac{(PL)^2}{2EI}\left(\frac{e}{1+e}+\frac{1}{3}\right)$	$\|q^*\|^2 = \frac{L}{2}\cdot\frac{(PL)^2}{6EI}$

$$\{\varepsilon^h\}=\{\bar{\varepsilon}\}$$
$$\{\varepsilon^h\}=\{\overline{\varepsilon^*}\}$$

$$\|q\|^2 = \|\varepsilon-\varepsilon^h\|^2 = \|\varepsilon\|^2 - \|\varepsilon^h\|^2,$$
$$\|q^*\|^2 = \|\varepsilon-\varepsilon^{h*}\|^2 = \|\varepsilon\|^2 - \|\varepsilon^{h*}\|^2$$

A case of variational incorrectness through reduced integration

A cantilever beam with uniformly distributed loading ρ

FI: Field inconsistent, Locked, but variationally correct FE results.

FC: Field consistent, Lock free, Reduced Integrated FE results.

Note that FC (by FEA) deviates from the field-consistent best-fit results.

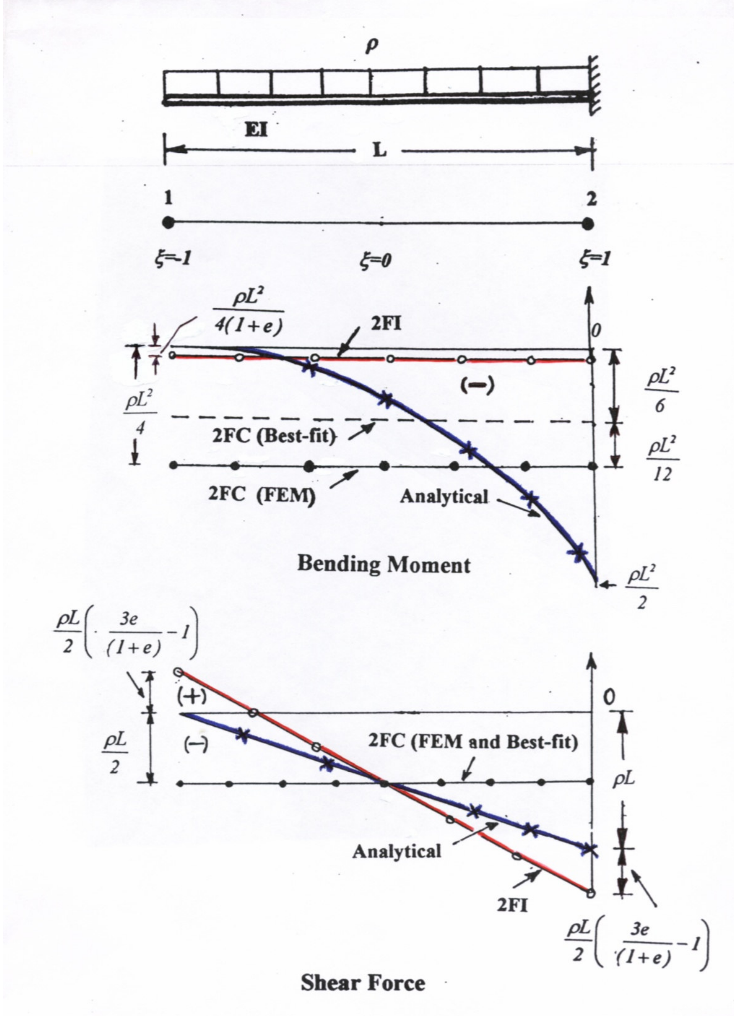
For this case :

The extraneous force vector (a non-zero vector) from Reduced Integration consists of self-equilibrating moments, that shift the FC Best-fit from the FC-FEM results.

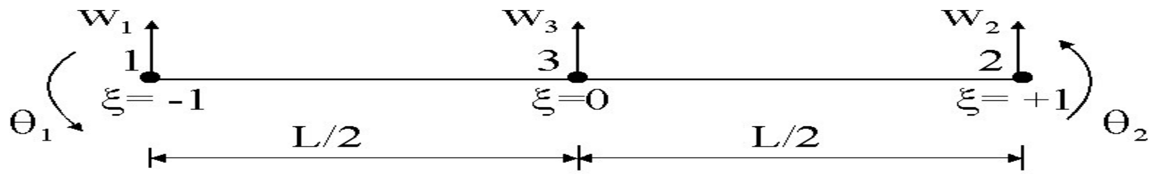
$$\{F^e{}_E\} = -\int_e [[B]-[B^*]]^T [D]\{\varepsilon\}dx$$

$$\{F_E^e\} = \begin{Bmatrix} 0 \\ \rho L^2/12 \\ 0 \\ -\rho L^2/12 \end{Bmatrix}$$

$$\{\varepsilon^h *\} = \{\overline{\varepsilon *}\} + \delta\varepsilon *$$



1.13 Lockfree an-isoparametric formulation (quadratic transverse displacement and linear rotation)



Displacements: $$w^h = \sum_{i=1}^{3} \bar{N}_i w_i \qquad \theta^h = \sum_{i=1}^{2} \bar{N}_i \theta_i \qquad (1.26)$$

Shape functions $$N_1 = \frac{1-\xi}{2} \qquad N_2 = \frac{1+\xi}{2} \qquad N_3 = 1-\xi^2$$

Strain vector: $$\{\varepsilon^h\} = \begin{pmatrix} d\theta/dx \\ \theta - dw/dx \end{pmatrix} = [B]\{\delta^e\}$$

$$\{\varepsilon^h\} = \begin{bmatrix} 0 & -1/L & 0 & 1/L & 0 \\ -(2\xi-1)/L & (1-\xi)/2 & -(2\xi+1)/L & (1+\xi)/2 & 4\xi/L \end{bmatrix} \{\delta^e\} \qquad (1.27)$$

Standard basis vectors spanning 3-dimensional B space:

$$\{v_1\} = [0, \xi]^T, \quad \{v_2\} = [1, 0]^T \text{ and } \{v_3\} = [0, 1]^T \qquad (1.28)$$

Summary

Shear locking in Timoshenko's Shear Flexible beam element occurs from spurious constraints that arise from reducing the discretized domain into an Euler beam (of infinite shear rigidity).

Shear locking is displayed through slow convergence, Spurious bending stiffening and shear oscillations.

The field consistency paradigm identifies the spurious constraints related to locking, and suggests methods to eliminate field inconsistency by eliminating the spurious constraints (thereby enforcing field consistency).

Reduced integration (RI) eliminates shear locking by eliminating the spurious constraint in the strain.

The function space approach shows that locked strain vector in an element (through FEA) is actually the orthogonal projection of the analytical strain vector onto a field-inconsistent subspace B, arising from a field-inconsistent [B] matrix (strain-displacement matrix). B cannot be spanned by standard orthogonal basis vectors.

FEA through reduced integration (RI) effectively projects the analytical strain vector onto a field-consistent subspace B*. However, RI is variationally incorrect in general, and the FE strain vector agrees with the orthogonal projection on B* only when the spurious extraneous force vector vanishes.

Lecture 4
Special Topics of FEA

Chapter 2

Error Analysis in Computational Elastodynamics

A comedy of errors…

2.1 Finite Element Elastodynamic Equations using the Principle of Least Action

Action $I = \int_1^2 L(q,\dot{q},t)\,dt$

Lagrangian $L = T - V$

Hamilton's Principle $\delta I = 0$ **for** $\delta q(t)$, $\delta q(t_1) = \delta q(t_2) = 0$

Lagrange's Equation for motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

$Q_i = Non-conservative \quad generalised \quad force$

In elastodynamics, the equations of motion are generally derived in a global sense (with element assembly)

$$L = T - V = \sum_{e=1}^{N} T^e - \left(\sum_{e=1}^{N} U^e - \sum_{e=1}^{N} W^e \right) = \sum_{e=1}^{N} \frac{1}{2}.\{\dot{\delta}^e\}^T [M^e]\{\dot{\delta}^e\} - \left(\sum_{e=1}^{N} \{\delta^e\}^T [K^e]\{\delta^e\} - \sum_{e=1}^{N} \{\delta^e\}^T \{F^e\} \right)$$

Element Stiffness Matrix: $[K^e] = \int_e [B]^T [D][B] dV$

Element Consistent Mass Matrix: $[M^e] = \int_e [N]^T [\rho][N] dV$

Element Generalized Force Vector (time dependent) : $\{F^e\} = \int_e [N]^T \{f(t)\} dV$

With element assembly, we get the global form

$$L = T - V = \frac{1}{2}\{\dot{\delta}^G\}^T [M^G]\{\dot{\delta}^G\} - \left(\frac{1}{2}\{\delta^G\}^T [K^G]\{\delta^G\} - \{\delta^G\}^T \{F^G\} \right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Equation of motion

$$[M^G]\{\ddot{\delta}^G\} + [K^G]\{\delta^G\} = \{F^G\} \quad \textbf{(2.1)}$$

2.2 Free Vibration Analysis

$$[M^G]\{\ddot{\delta}^G\} + [K^G]\{\delta^G\} = \{0\} \qquad \textbf{(2.2)}$$

Let

$$\left\{\delta^G\right\} = \{\phi\}.\sin(\omega_n t) \qquad \textbf{(2.3)}$$

$$\left\{[K^G] - \omega_n{}^2[M^G]\right\}\{\phi\} = 0 \qquad \textbf{(2.4)}$$

$$\det\left\{[K^G] - \omega_n{}^2[M^G]\right\} = 0$$

Eigenvalue $\omega_i{}^2$, *Eigenmode* $\{\phi_i\}$,

ω_i *is natural circular frequency* $(rad/\sec)$

Orthogonality of the Eigen-modes (Normal modes)

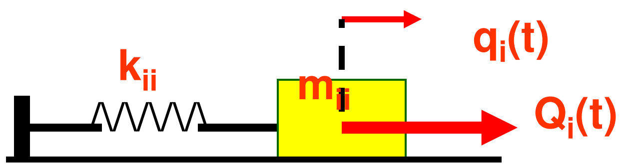
$$\{\phi_i\}^T[K^G]\{\phi_j\}=0 \qquad i\neq j, \quad \{\phi_i\}^T[K^G]\{\phi_i\}=k_{ii}$$

$$\{\phi_i\}^T[M^G]\{\phi_i\}=0 \qquad i\neq j, \quad \{\phi_i\}^T[M^G]\{\phi_i\}=m_{ii} \tag{2.5}$$

Natural Frequencies (rad/sec)

$$\omega_i=\sqrt{\frac{k_{ii}}{m_{ii}}}$$

k_{ii} : generalized modal stiffness for mode *i*
$m_{ii:}$: generalized modal mass for mode *i*

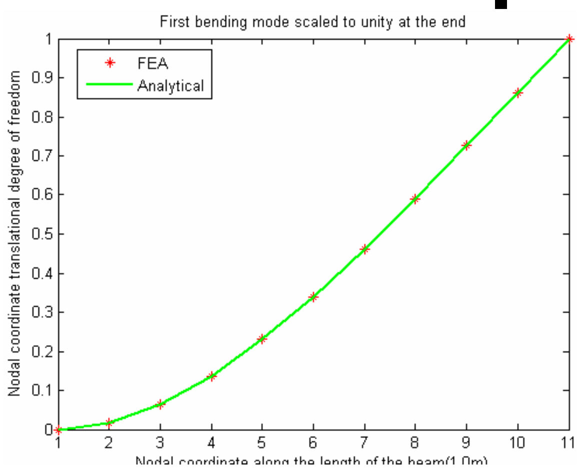


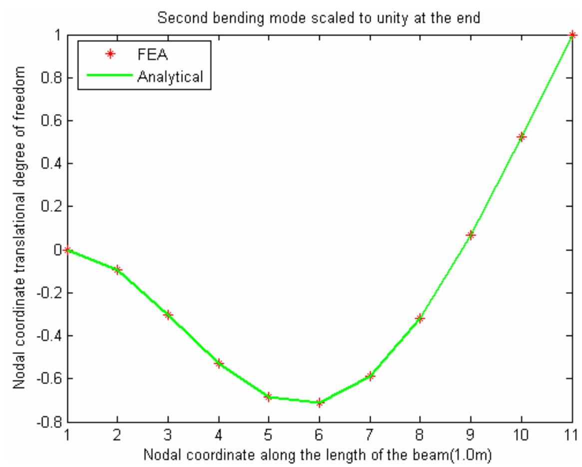
Example 1. Free vibration analysis of a simple cantilever beam using 10 Euler beam elements.

L=1m, b=0.1m, t=0.001m I=2.5×10-7m^4, A=3×10-4m^2

Density ρ=2722.77 kg/m^3, *M*ass per unit length of the beam is *ρA=0.816 kg/m*

E=7.1 ×*1010 N/m*^{2}*,*





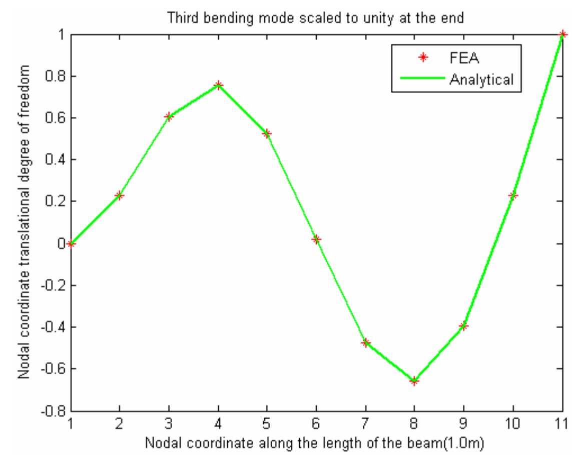


Table 3.1Comparison of the natural frequencies in bending of the uniform cantilever beam obtained by different methods

Different methods	Natural Frequency f_n (Hz)	Natural Circular Frequency ω_n (rad/sec)
Classical solution	82.4915	518.31
FE result	82.4836	518.26

Different methods	Natural Frequency f_n (Hz)	Natural Circular Frequency ω_n (rad/sec)
Classical solution	516.935	3.248×10^3
FE result	516.935	3.248×10^3

Different methods	Natural Frequency f_n (Hz)	Natural Circular Frequency ω_n (rad/sec)
Classical solution	1447.51	9.095×10^3
FE result	1447.83	9.097×10^3

Example 2. Free vibration analysis of an aircraft wing using Euler beam elements.

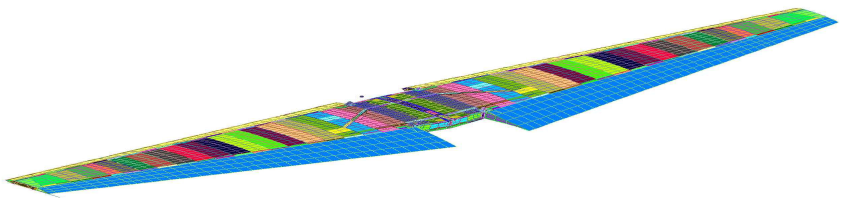


Fig (a) A typical subsonic aircraft wing

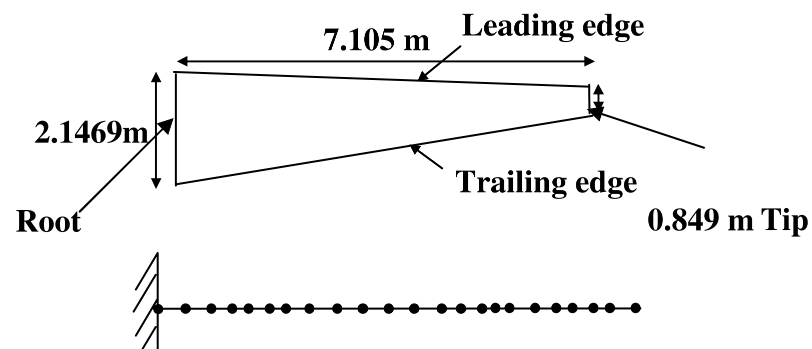


Fig (b) Length of the wing divided in 22 elements for FE formulation

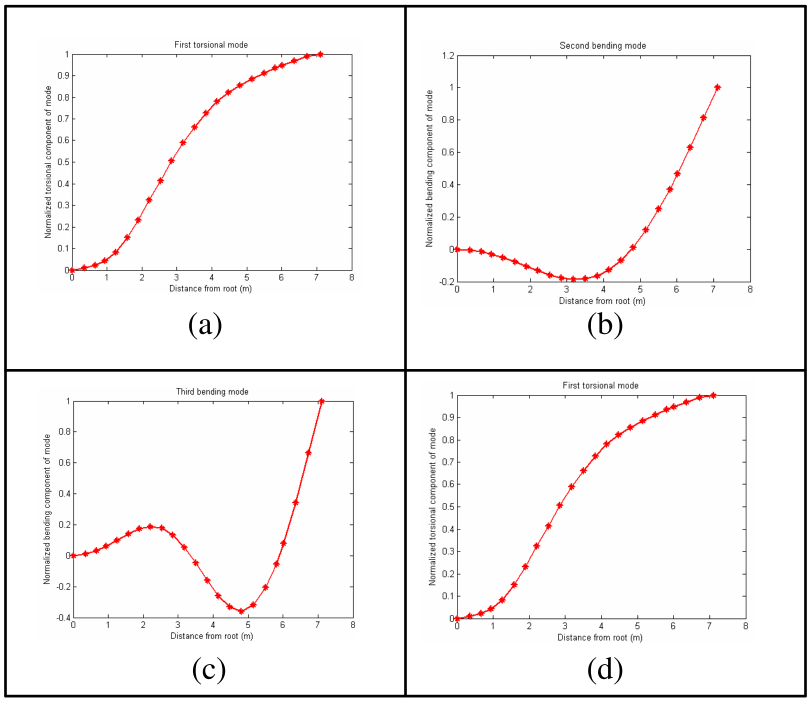
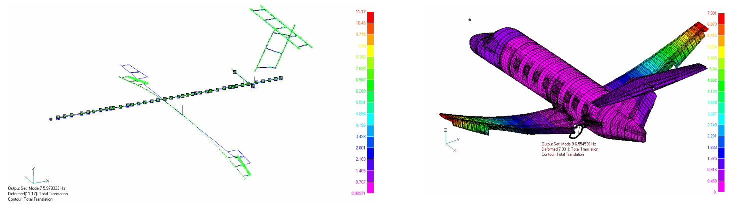
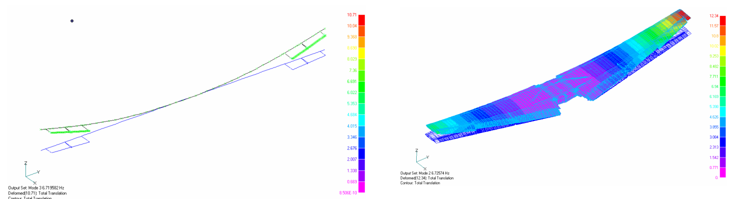


Fig 4.3 Normal mode shapes of the wing of the aircraft
(a) First bending mode in y direction (frequency 7.2165 Hz)
(b) Second bending mode in y direction (frequency 21.138 Hz)
(c) Third bending mode in y direction (frequency 50.405 Hz)
(d) First torsional mode (frequency 56.8296 Hz)

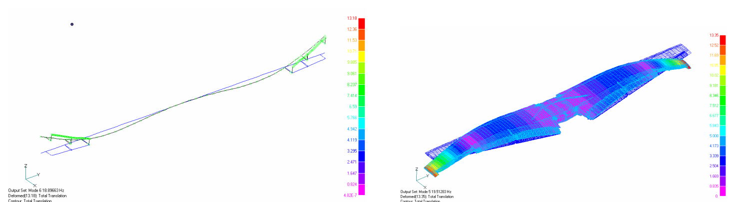
Example 3. Dynamic Characterization of an aircraft using a Stick Model

- Beam Model with provision for Bending-Torsion Coupling (Shear Center offset).
- Results for components from in-house code benchmarked with those from detailed FE model in NASTRAN.

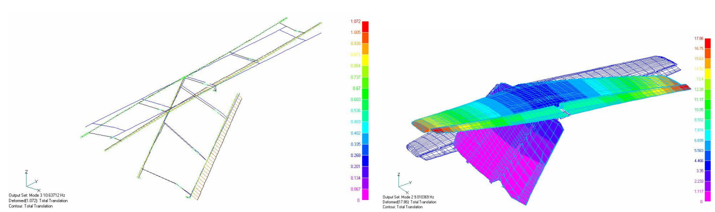




Wing First Symmetric Mode 6.71 Hz (by Stick Model) and 6.72 Hz (by detailed FE model in NASTRAN)



Wing Second Symmetric Mode 18.89 Hz (by Stick Model) and 19.51 Hz (by detailed FE model in NASTRAN)



HT Anti-Symmetric Mode 10.4 Hz (by Stick Model) and 9.1 Hz (by detailed FE model in NASTRAN)

2.3 Definitions of Inner Products in Elastodynamics

[D]= element elastic rigidity matrix

[ρ]= element inertia density matrix.

Stiffness-inner product

$$< a,b >= \sum_{ele=1}^{N^e} \int_{ele} \{a\}^T [D]\{b\} dx \qquad (2.6)$$

***Stiffness-norm squared* value of the vector *{a}* is given as**

$$\|a\|^2 =< a,a > \qquad (2.7)$$

Inertia-inner product

$$(c,d) = \sum_{ele=1}^{N^e} \int_{ele} \{c\}^T [\rho]\{d\} dx \qquad (2.8)$$

***Inertia-norm squared* value of the vector *{c}* is given as**

$$|c|^2 = (c,c) \qquad (2.9)$$

2.4 The Rayleigh Quotient

Free vibration of a system in a given mode can be expressed as

$$\{U(x,t)\} = \{u(x)\}e^{i\omega t} \tag{2.10}$$

Rayleigh Quotient from exact solutions for displacement and strain modes *u* and ε

$$\omega^2 = \frac{\|\varepsilon\|^2}{|u|^2} \tag{2.11}$$

Let u^h and ε^h be the approximate modal vector and the strain vector. **Rayleigh Quotient from FEA solution**

$$(\omega^h)^2 = \frac{\|\varepsilon^h\|^2}{|u^h|^2} \tag{2.12}$$

But interestingly (!) for a variationally correct solution,

$$\omega^2 = \frac{<\varepsilon^h, \varepsilon>}{(u^h, u)} \tag{2.13}$$

2.5 The Error Statements of Elastodynamics

Combining equations (5.6) and (5.7), we get one rule

$$\|\varepsilon\|^2 - \|\varepsilon^h\|^2 = \omega^2 |u|^2 - (\omega^h)^2 |u^h| \qquad (2.14)$$

Error of global strain energy = Error of global kinetic energy

Combining equations (2.12) and (2.13), we get another rule, valid for variationally correct solutions only,

$$< \varepsilon^h, \varepsilon - \varepsilon^h >= (u^h, \omega^2 u - (\omega^h)^2 u^h) \qquad (2.15)$$

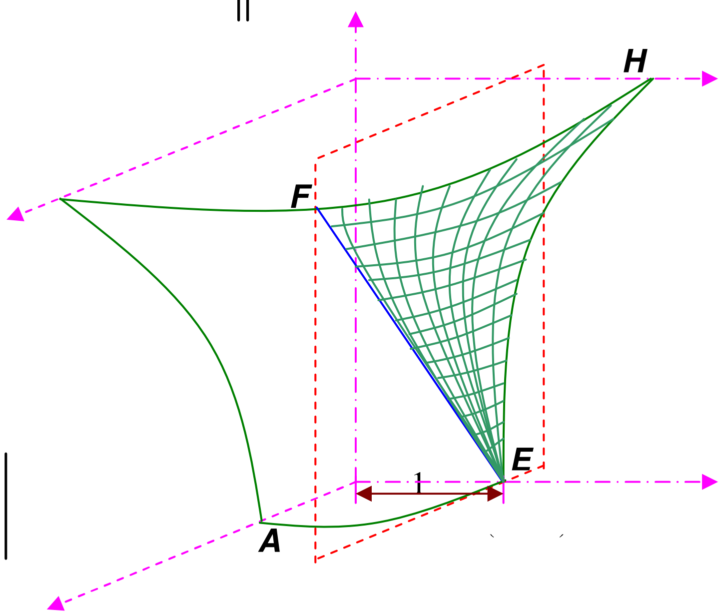
Observation: ***The Errors in Elastodynamics are decided by both displacements and strains.***

2.6 The Frequency-Error Hyperboloid

It can be shown that for variationally correct formulations,

$$\frac{\left|u-u^h\right|^2}{\left|u^h\right|^2}+\frac{(\omega^h)^2}{\omega^2}-\frac{\left\|\varepsilon-\varepsilon^h\right\|^2}{\omega^2\left|u^h\right|^2}=1$$

$$Z=\left\|\varepsilon-\varepsilon^h\right\| \qquad (2.16)$$



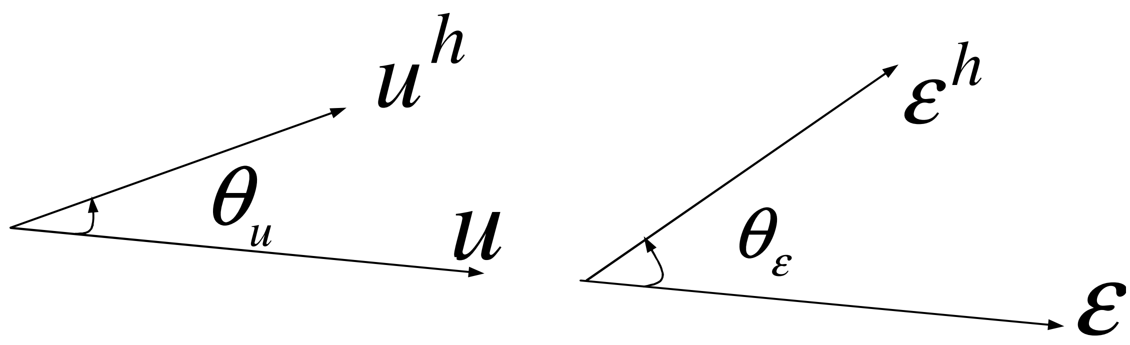
$$X=\left|u-u^h\right|$$

$$Y=\frac{\omega^h}{\omega}$$

Fig 1. Geometric interpretation of eigenvalue analysis of the variationally correct formulation using Frequency-Error Hyperboloid. Approximate eigenvalues obtained form a variationally correct formulation lie in the shaded portion of the Hyperboloid.

2.7 Why is the approximate Rayleigh Quotient higher than the Exact one ?

(Valid only for variationally correct formulations)



$$\cos(\theta_u) = \frac{(u, u^h)}{|u||u^h|}, \qquad \cos(\theta_\varepsilon) = \frac{<\varepsilon, \varepsilon^h>}{\|\varepsilon\|\|\varepsilon^h\|}$$

$$\frac{\cos\theta_u}{\cos\theta_\varepsilon} = \frac{(u, u^h)}{|u||u^h|}\frac{\|\varepsilon\|\|\varepsilon^h\|}{<\varepsilon, \varepsilon^h>} = \frac{1}{\omega^2}\frac{\|\varepsilon\|\|\varepsilon^h\|}{|u||u^h|}$$

Hence for variationally correct formulations

$$\frac{\cos^2\theta_u}{\cos^2\theta_\varepsilon} = \frac{1}{\omega^4}\frac{\|\varepsilon\|^2\|\varepsilon^h\|^2}{|u|^2|u^h|^2} = \frac{1}{\omega^4}\omega^2(\omega^h)^2 = \frac{(\omega^h)^2}{\omega^2} \tag{2.17}$$

- **Geometrically, the modal displacement vector suffers less *deviation* than that of the modal strain vector. Hence** $\frac{(\omega^h)^2}{\omega^2} > 1$

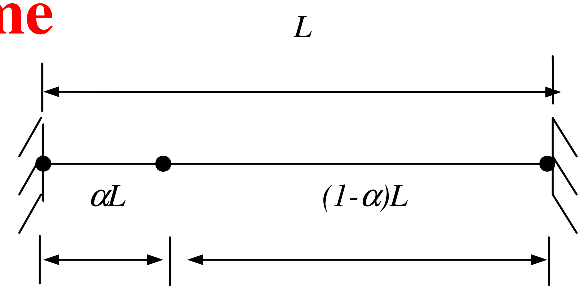
Example 4: Free Vibration of a Simply Supported Beam

	Analytical	Approximate
Modal Disp.	$w = a\sin(\pi x/L)$	$w^h = b\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)$
Modal Strain	$\varepsilon = \left(-d^2w/dx^2\right) = a\left(\frac{\pi}{L}\right)^2 \sin(\pi x/L)$	$\varepsilon^h = \left(-\frac{d^2w^h}{dx^2}\right) = \frac{2b}{L^2}$
Eigenvalue	$\omega^2 = \pi^4 EI/(\rho AL^4)$	$(\omega^h)^2 = 120 * EI/(\rho AL^4)$

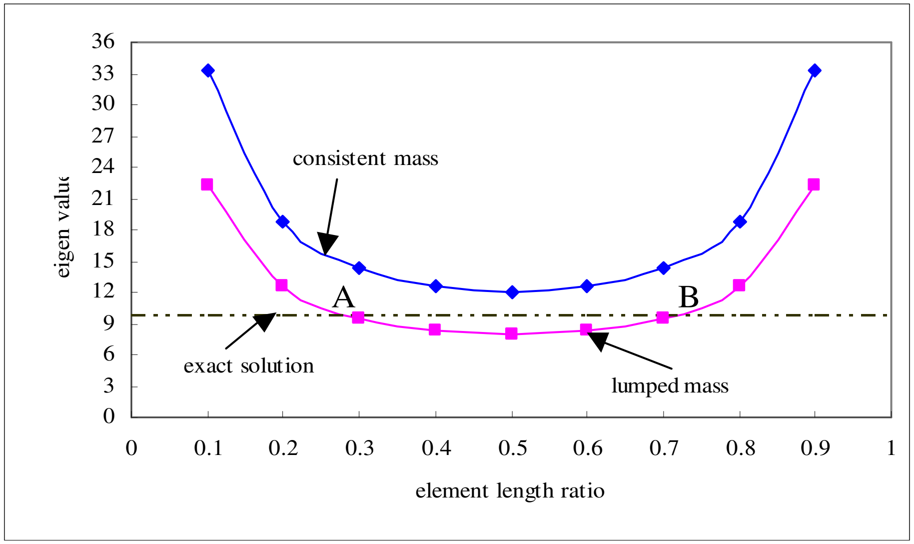
$$\|\varepsilon\|^2 - \|\varepsilon^h\|^2 = \omega^2|u|^2 - (\omega^h)^2|u^h| = \frac{EI}{L^3}\left\{\left(\frac{\pi^4}{2}\right)a^2 - 4b^2\right\}$$

$$<\varepsilon^h, \varepsilon - \varepsilon^h> = (u^h, \omega^2 u - (\omega^h)^2 u^h) = 4\frac{EI}{L^3}(\pi ab - b^2)$$

2.8 Replacement of Consistent Mass by Lumped Mass; A variational crime



Example 5. Free Vibration Analysis of a Fixed-Fixed Bar using 2 elements (First Mode Only shown)



Any variationally incorrect formulation (with Lumped Mass, Reduced Integration etc.) that does not conform exactly to the Weak form is variationally incorrect.

Variationally incorrect formualtions - Do not satisfy the Hyperboloid Rule

- Cannot guarantee and upper bound of the frequency.

Lecture 4
Special Topics of FEA

Chapter 3

Rank Deficiency in elements

3.1 What is rank deficiency ?

The rank of the stiffness matrix is the dimension of the B subspace that emerges from the strain-displacement matrix [B], i.e.

$$\text{Rank } [K^e] = \dim (B) \tag{3.1}$$

In the dimension of the B subspace is given by

$$\dim (B) = N - R \tag{3.2}$$

N= Number of degrees of freedom of the element
R= Number of rigid body motions

To eliminate locking, a reduced order integration effectively converts the Field-inconsistent [B] matrix into a Field-consistent [B*] matrix, by simply removing the highest Legendre Polynomial in the field-inconsistent spurious term of [B].

Using Gram Schmidt algorithm for orthogonal basis vector spanning B* it can be shown that for some elements

$$\dim (B^*) < \dim (B) \quad \text{or} \quad (N-R^*) < (N-R), \quad \text{i.e.} \quad R^* > R \tag{3.3}$$

$\text{Rank } [K^{e*}] < \text{Rank } [K^e]$ because of introduction of spurious rigid body motions

Reduced integration may introduce rank deficiency

Rank deficiency of the plane stress Quad 4 element

1,2,3 are rigid body modes

4,5,6 are constant strain modes

7,8 are bending modes, but cannot be sensed by a 1x1 reduced integration

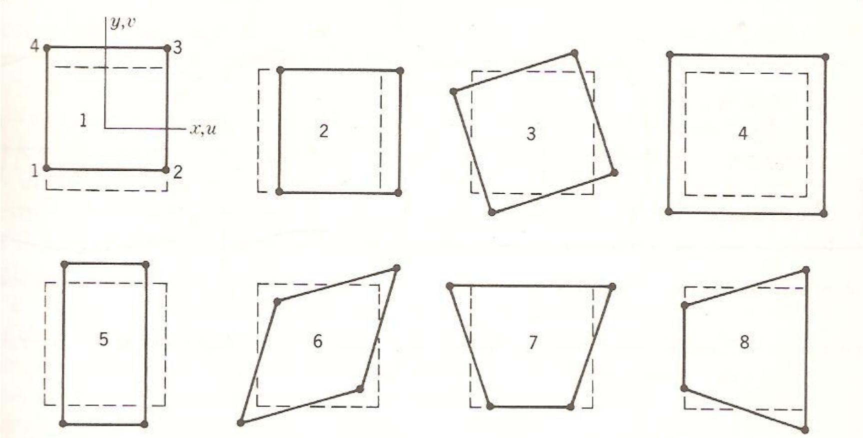


Figure 6.12-1. Independent displacement modes of a bilinear element.

1x1 reduced integration (with sampling point at element center of zero strain) effectively considers modes 7 and 8 as zero energy hour-glass modes (spurious rigid body motions)

Rank deficiency of this plane stress Quad 4 element is thus 2.

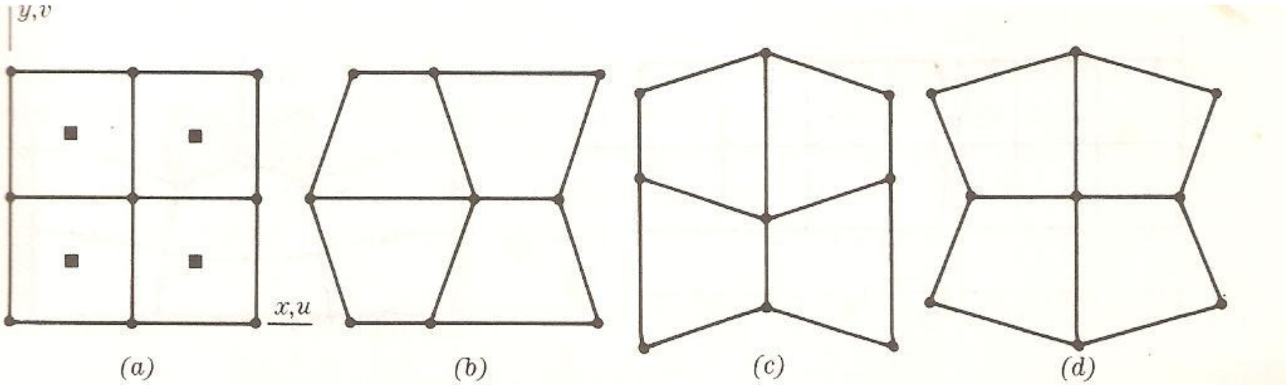
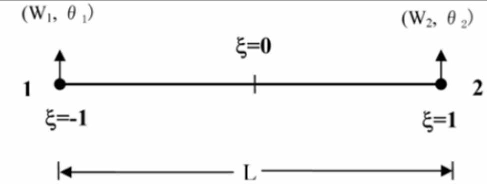


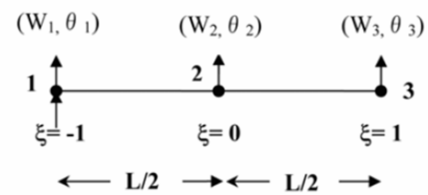
Figure 6.12-2. (*a*) Mesh of four bilinear elements, showing Gauss points of an order 1 rule in each element (squares). (*b,c,d*) Possible mechanisms ("hourglass" modes).



Isoparametric two-noded Timoshenko beam element

Two-noded Timoshenko beam element ($N_f = 4$, $N_r^p = 2$, $N_r = 2$, *Rank deficiency* = 0, $dimB = dimB^* = 2$)

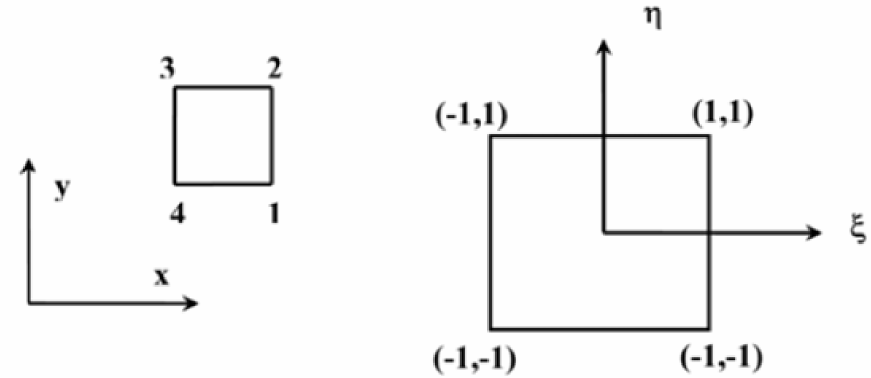
Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
two noded (4 d.o.f)	Full	2	0	2
	Reduced	2	0	2



Isoparametric three-noded Timoshenko beam element.

Three-noded Timoshenko beam element ($N_f = 6$, $N_r^p = 2$, $N_r = 2$, *Rank deficiency* = 0, $dimB = dimB^* = 4$)

Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
three noded (6 d.o.f)	Full	4	0	4
	Reduced	4	0	4



Isoparametric 4-noded rectangular element

Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
 Four noded (8 d.o.f)	Full	5	0	5
	Reduced	3	2	3

The Mindlin Plate element

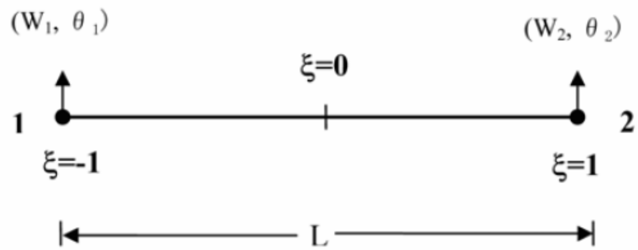
Element type	Integration rule			Rank deficiency = No. of mechanisms	Rank of K_e (no. of nonzero eigen values)	Dimension of B space	Dimension of B^* space
	Type	$[k_b]$	$[k_s]$				
	Full	2×2	2×2	0	9	9	9
	Reduced	1×1	1×1	4	5	9	5
Four noded (12 d.o.f)	Selective	2×2	1×1	2	7	9	7
	Shear Selective	2×2	2×1 1×2	0	9	9	9

Appendix

The basis vectors for spanning the B subspaces of the elements

For the simple Timoshenko beam element (Fig. 1a) the element strain vector is given by

$$\{\varepsilon^{he}\} = [B]\{\delta^e\} = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & (1-\xi)/2 & -1/L & (1+\xi)/2 \end{bmatrix}\{\delta^e\}$$



Isoparametric two-noded Timoshenko beam element

Here L is the element length and $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2]^T$ is the nodal displacement vector. The space ***B*** is evidently a subspace of the polynomial space P_2^2 (linear in ξ). Applying the Gram-Schmidt process on the column vectors of $[B]$, we get the normalized orthogonal basis vectors $\{u_i\}$ for the subspace ***B*** (of two dimensions) as

$$\{u_1\} = [0 \quad 1]^T \quad \text{and} \quad \{u_2\} = [2/L \quad \xi]^T$$

The function space $\boldsymbol{B}^*$ is a subspace of the space P_1^2 which is actually the space R^2. It is obtained from $[B]$, by dropping the highest Legendre polynomial, i.e., the ξ term. Thus,

$$[B^*] = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & 1/2 & -1/L & 1/2 \end{bmatrix}$$

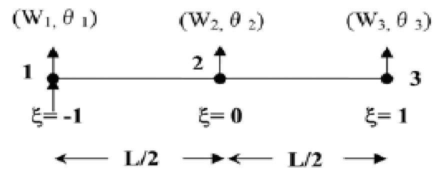
The normalized basis vectors for the subspace $\boldsymbol{B}^*$ (again of two dimensions) are given by

$$\{u_1^*\} = [0 \quad 1]^T \quad \text{and} \quad \{u_2^*\} = [2/L \quad 0]^T$$

So, in this example, using a lower order integration does not bring in a change in the dimension of the $[B]$ matrix.

The three noded Timoshenko beam element (Fig. 1b) uses quadratic Lagrangian interpolation functions for displacement and geometry. The element strain vector is given by

$$\{\varepsilon^{hc}\} = [B]\{\delta^e\} = \begin{bmatrix} 0 & (2\xi-1)/L & 0 & -4\xi/L & 0 & (2\xi+1)/L \\ -(2\xi-1)/L & -\xi(1-\xi)/2 & 4\xi/L & (1-\xi^2) & -(2\xi+1)/L & \xi(1+\xi)/2 \end{bmatrix}\{\delta^e\}$$



Isoparametric three-noded Timoshenko beam element.

Here L is the element length and $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2, w_3, \theta_3]^T$ is the nodal displacement vector. Using the Gram-Schmidt procedure on the column vectors of the above matrix, the four orthogonal basis vectors spanning the four dimensional subspace $\boldsymbol{B}(B \subset P_3^2)$ are determined as

$$\{u_1\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \{u_2\} = \begin{Bmatrix} 0 \\ \xi \end{Bmatrix}, \quad \{u_3\} = \begin{Bmatrix} (2\xi-1)/L \\ (3\xi^2-1)/6 \end{Bmatrix} \quad \text{and} \quad u_4 = \begin{Bmatrix} (2\xi+\kappa)/L \\ (3\xi^2-1)/6 \end{Bmatrix}$$

where $\kappa = \dfrac{4(e+5)}{15}$, $e = \dfrac{kGAL^2}{12EI}$

The strain displacement matrix $[B^*]$ that emerges from using a two-point Gaussian quadrature rule instead of the necessary three point rule for integration for the stiffness matrix is obtained by first expressing ξ^2 in terms of the Legendre quadratic polynomial as

$$\xi^2 = (3\xi^2-1)/3 + 1/3 = P_3 + 1/3$$

and then dropping the Legendre polynomial $P_3 = 3\xi^2 - 1$. Thus the matrix $[B^*]$ is obtained from the $[B]$ matrix by replacing ξ^2 by (1/3) as follows

$$\boldsymbol{B}^* = \begin{bmatrix} 0 & (2\xi-1)/L & 0 & -4\xi/L & 0 & (2\xi+1)/L \\ -\dfrac{(2\xi-1)}{L} & \dfrac{\{\xi-(1/3)\}}{2} & \dfrac{4\xi}{L} & \dfrac{2}{3} & -\dfrac{(2\xi+1)}{L} & \dfrac{\{\xi+(1/3)\}}{2} \end{bmatrix}$$

The normalized basis vectors for subspace $\boldsymbol{B}^*$ (of dimension 4), as obtained by the Gram-Schmidt process are

$$\{u_1^*\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \{u_2^*\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \{u_3^*\} = \begin{Bmatrix} \xi \\ 0 \end{Bmatrix}, \quad \{u_4^*\} = \begin{Bmatrix} 0 \\ \xi \end{Bmatrix}$$

So, in this example too, using a lower order integration does not bring in a change in the dimension of the $[B]$ matrix.

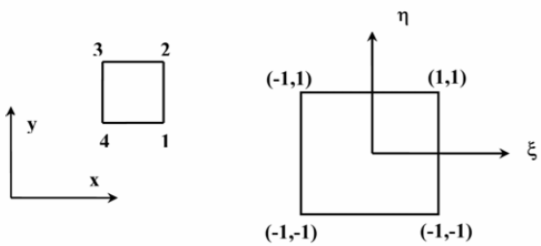
For the QUAD4 element (Fig. 2) for plane stress/strain the element strain vector is given by

$$\{\varepsilon^{he}\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T = [B]\{\delta^e\}$$

$$\{\varepsilon^{he}\} = [B]\{\delta^e\}$$

$$= \begin{bmatrix} \frac{(\eta-1)}{4a} & 0 & \frac{(1-\eta)}{4a} & 0 & \frac{(1+\eta)}{4a} & 0 & -\frac{(1+\eta)}{4a} & 0 \\ 0 & \frac{(\xi-1)}{4b} & 0 & -\frac{(1+\xi)}{4b} & 0 & \frac{(1+\xi)}{4b} & 0 & \frac{(1-\xi)}{4b} \\ \frac{(\xi-1)}{4b} & \frac{(\eta-1)}{4a} & -\frac{(1+\xi)}{4b} & \frac{(1-\eta)}{4a} & \frac{(1+\xi)}{4b} & \frac{(1+\eta)}{4a} & \frac{(1-\xi)}{4b} & -\frac{(1+\eta)}{4a} \end{bmatrix} \{\delta^e\}$$

Here $2a$ and $2b$ are the sides of the rectangle and $\{\delta^e\} = \{u_x \;\; v_y \;\; u_y + v_x\}^T$. The space $\boldsymbol{B}$ is evidently a sub-space of the space of polynomials (linear in ξ and η). Applying the Gram-Schmidt process on the column vectors of $[B]$, we get the normalized orthogonal basis vectors $\{u_i\}$ for subspace $\boldsymbol{B}$ as



Isoparametric 4-noded rectangular element

$$u_1 = \begin{Bmatrix} \frac{\eta-1}{a} \\ 0 \\ \frac{\xi-1}{b} \end{Bmatrix} \quad u_2 = \begin{Bmatrix} b(\eta-1) \\ \frac{\xi-1}{b} t_1 \\ \frac{\eta-1}{a} t_1 - a(\xi-1) \end{Bmatrix} \quad u_3 = \begin{Bmatrix} (80a^2 - 11b^2)(\eta-1) \\ (80b^2 - 11a^2)(\xi-1) \\ \frac{t_2}{3a} \end{Bmatrix}$$

$$u_4 = \begin{Bmatrix} 9bt_3 \\ \frac{t_4}{2b} \\ 3at_5 \end{Bmatrix} \quad u_5 = \begin{Bmatrix} 35a^2 + b^2(273\eta + 100) \\ -\frac{3}{2}(7a^2 + 20b^2) \\ 273ab\xi \end{Bmatrix}$$

where

$$t_1 = -\frac{(80b^2 + 28a^2)}{39}, \qquad t_2 = 240(a^4\xi + b^4\eta) - 33a^2b^2(\xi + \eta) + 80a^4 + 875a^2b^2 + 80b^4$$

$$t_3 = (\eta - 1)(20a^2 + 7b^2), \quad t_4 = a^4(420\xi + 140) + a^2b^2(1649 + 4476\xi) + 560b^4$$

$$t_5 = a^2(70\eta + 60\xi) + b^2(746\eta + 21\xi)$$

The function space B^* is a subspace of the space B_1^2, which is actually the space R^2. It is obtained from $[B]$, by dropping the highest Legendre polynomials, i.e., the ξ and η terms. Note that this must strictly be the higher order term. Equivalently, this means that the number of points required for optimal integration is reduced by one. Thus

$$\boldsymbol{B}^* = \begin{bmatrix} -1/4a & 0 & 1/4a & 0 & 1/4a & 0 & -1/4a & 0 \\ 0 & -1/4b & 0 & -1/4b & 0 & 1/4b & 0 & 1/4b \\ -1/4b & -1/4a & -1/4b & 1/4a & 1/4b & 1/4a & 1/4b & -1/4a \end{bmatrix} \{\delta^e\}$$

The normalized basis vectors for the subspace B^* are given by

$$\{u_1^*\} = \begin{Bmatrix} a \\ 0 \\ b \end{Bmatrix} \quad \{u_3^*\} = \begin{Bmatrix} b \\ -1/b \\ -1/a + r_1 a \end{Bmatrix} \quad \{u_3^*\} = \begin{Bmatrix} (10a^2 - 3b^2) \\ (10b^2 - 3a^2) \\ -26ab \end{Bmatrix}$$

where

$$r_1 = \frac{13}{4} * \left(\frac{b}{2b^2 + 7a^2}\right)$$

So, in this example, using a lower order integration reduces the number of nonzero vectors by 2, as is reflected in the dimension of the $[B^*]$ matrix.

For the Mindlin plate element (Fig. 2) the element strain vector is given by

$$\{\varepsilon^{he}\} = \{\theta_{x,x} \;\; \theta_{y,y} \;\; \theta_{x,y}+\theta_{y,x} \;\; \theta_{y}-w_{,y} \;\; \theta_{x}-w_{,x}\}^{T} = [B]\{\delta^{e}\}$$

$$\{\varepsilon^{he}\} = [B]\{\delta^{e}\} =$$

$$\begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1-\xi}{4b} & 0 & \frac{(1-\xi)(1-\eta)}{4} & \frac{1+\xi}{4b} & 0 & \frac{(1+\xi)(1-\eta)}{4} & -\frac{1+\xi}{4b} & 0 & \frac{(1+\xi)(1+\eta)}{4} & \frac{\xi-1}{4b} & 0 & \frac{(1-\xi)(1+\eta)}{4} \\ \frac{1-\eta}{4a} & \frac{(1-\xi)(1-\eta)}{4} & 0 & \frac{\eta-1}{4a} & \frac{(1+\xi)(1-\eta)}{4} & 0 & -\frac{1+\eta}{4a} & \frac{(1+\xi)(1+\eta)}{4} & 0 & \frac{1+\eta}{4a} & \frac{(1-\xi)(1+\eta)}{4} & 0 \end{bmatrix} \{\delta^{e}\}$$

Here $2a$ and $2b$ are the sides of the rectangle nd $\{\delta^{e}\} = \{w_1 \;\; \theta_{x1} \;\; \theta_{y1} \; \ldots \; w_4 \;\; \theta_{x4} \;\; \theta_{y4}\}^{T}$. When the stiffness matrix is evaluated with full integration, the number of basis vectors of the $[B]$ matrix is 9. Using a selective integration strategy (2×2 for bending and 1×1 for shear) to evaluate the stiffness matrix, is equivalent to replacing the $[B]$ matrix by the following $[B^{*}]$ matrix in Eq. (9).

$$[B^{*}] = \begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1}{4b} & 0 & \frac{1}{4} & \frac{1}{4b} & 0 & \frac{1}{4} & -\frac{1}{4b} & 0 & \frac{1}{4} & \frac{1}{4b} & 0 & \frac{1}{4} \\ \frac{1}{4a} & \frac{1}{4} & 0 & -\frac{1}{4a} & \frac{1}{4} & 0 & -\frac{1}{4a} & \frac{1}{4} & 0 & \frac{1}{4a} & \frac{1}{4} & 0 \end{bmatrix}$$

The subspace B^{*}, spanned by the column vectors of the $[B^{*}]$ matrix, has 7 basis vectors so that this integration rule reduces the dimension of the $\boldsymbol{B}^{*}$ space and hence is not optimal. A shear selective integration rule corresponds to the following $[B^{*}]$ matrix,

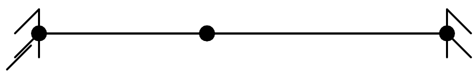
$$[B^{*}] = \begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1-\xi}{4b} & 0 & \frac{(1-\xi)}{4} & \frac{1+\xi}{4b} & 0 & \frac{(1+\xi)}{4} & -\frac{1+\xi}{4b} & 0 & \frac{(1+\xi)}{4} & \frac{\xi-1}{4b} & 0 & \frac{(1-\xi)}{4} \\ \frac{1-\eta}{4a} & \frac{(1-\eta)}{4} & 0 & \frac{\eta-1}{4a} & \frac{(1-\eta)}{4} & 0 & -\frac{1+\eta}{4a} & \frac{(1+\eta)}{4} & 0 & \frac{1+\eta}{4a} & \frac{(1+\eta)}{4} & 0 \end{bmatrix}$$

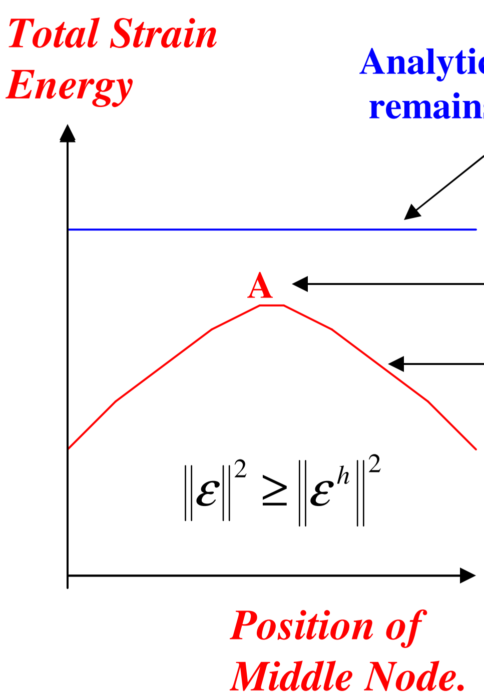
The corresponding $\boldsymbol{B}^{*}$ space is 9-dimensional, which is equal to the dimension of the $\boldsymbol{B}$ space used to evaluate the stiffness matrix in Eq. (9) by full integration. Thus, a shear selective integration strategy eliminates locking, without reducing the dimension of the $\boldsymbol{B}^{*}$ space.

Some Thoughts

A burning question:

Does Mesh Optimisation Maximize Numerical Entropy?





Analytical Strain Energy with the analytical solution u remains Invariant (Maximum entropy)

Optimized mesh corresponds to maximum FEA Strain Energy (with highest possible entropy with the approximations made).

FEA Strain Energy with the approximate solution u^h depends on meshing (the position of the middle node). Lower entropy than at A.

Cui bono ? (For whose good ?)

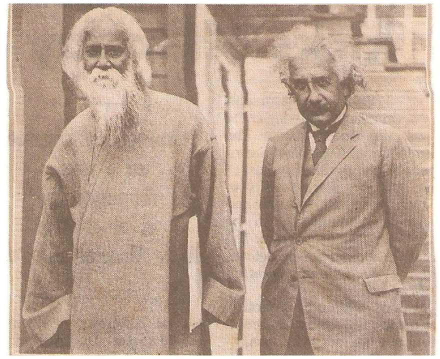
How the best-fit paradigm helps

(a) Gives the exact, but hidden, mechanism of the way the Finite Element Method works. It shows that computations in FEM are actually determined in a best-fit manner of the strains (and stresses), instead of the existing myth that they are based on displacements.

(b) Helps one to make a priori error estimates for bench mark problems easily.

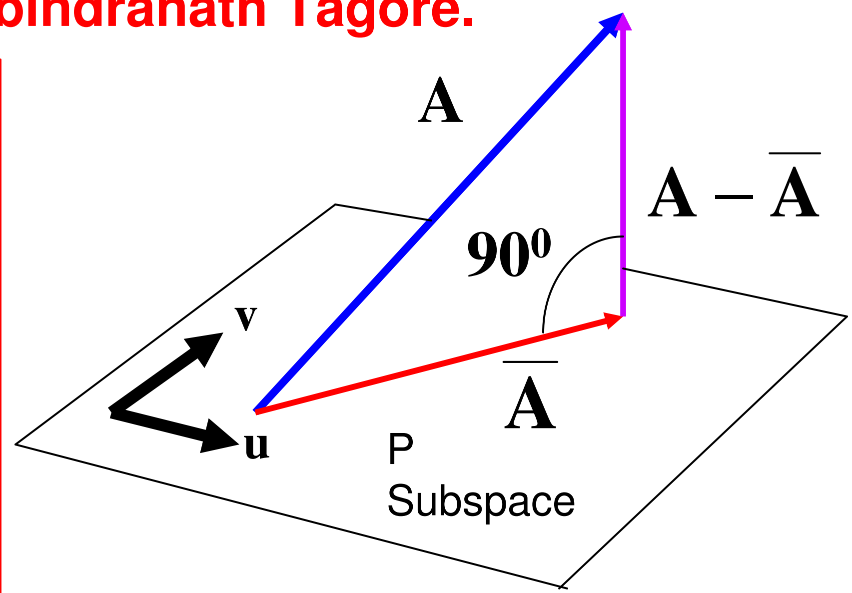
(c) Helps one to evaluate the quality of the element that he/she develops. The origins of the pathological problems of elements can now be understood, diagnosed and eliminated by appropriate methods.

When Arts and Science met at the crossroads…



An extract from “ *Sanchaita* ” by Rabindranath Tagore.

আমারই চেতনার রঙে পান্না হল সবুজ,
চুনী উঠল রাঙা হয়ে।
আমি চোখ মেললুম আকাশে —
জ্বলে উঠল আলো
পূবে পশ্চিমে।
গোলাপের দিকে চেয়ে বললুম, সুন্দর —
সুন্দর হল সে।
তুমি বলবে, এ যে তত্ত্বকথা,
এ কবির বাণী নয়।
আমি বলব, এ সত্য,
তাই এ কাব্য।



– "সঞ্চয়িতা" – "আমি"
কবিগুরু রবীন্দ্রনাথ ঠাকুর
১৫ই জ্যৈষ্ঠ, ১৩৪৩
শান্তিনিকেতন

Bibliography

Variational Principles of Classical & Computational Solid Mechanics

1 **J R Taylor**. *Classical Mechanics*. University Science Books.
2 **L Meirovitch**. *Analytical Mechanics*. Dover.
3 **C Lanczos**. *The Variational Principles of Mechanics*. Dover.
4 **C L Dyms, I H Shames**. *Solid Mechanics- A Variational Approach*. McGraw Hill.
5 **T H Richards**. *Energy Methods in Stress Analysis*. Ellis Horwood.
6 **K Washizu.** *Variational Methods in Elasticity and Plasticity*. Pergamon Press.
7 **S Rajashekaran, G Sankarasubramanian**. *Computational Structural Mechanics.* Prentice Hall of India.
8 **R Solecki, J R Conant**. *Advanced Mechanics of Materials*. Oxford University Press.

Linear Algebra

9 **L H Edwards, D E Penny**. *Elementary Linear Algebra*. Prentice Hall.
10 **G Strang.** *Linear Algebra.* Thomson Brooks/Cole.

Bibliography (Finite Element Method)

Elementary FEM

1. **T R Chandrapatla and A D Belegundu**. *Finite Elements in Engineering.* Eastern Economy Edition.

2. **R D Cook, D S Malkus, M E Plesha**. *Concepts and Applications of Finite Element Analysis*. John Wiley & Sons.

3. **J N Reddy.** *An introduction to the Finite Element Method*. McGraw Hill.

4. **K H Huebner, D L Dewherst, D E Smith, T G Byrom**. *The Finite Element Method for engineers*. John Wiley & sons.

5. **O C Zienkiewicz**. *The Finite Element Method.* McGraw Hill

Advanced FEM

6. **G Strang, G J Fix**. *An analysis of the Finite element Method*. Prentice Hall, NJ. .

7. **G Prathap**. *The Finite Element Method in Structural Mechanics*. Kluwer Academic Press, Dordrecht.

8. **T J R Hughes**. *The Finite Element Method*. Dover.
9. **O C Zienkiewicz, R L Taylor, J Z Hu**. *The Finite Element Method; Its basis and fundamentals, 6-th edition.* Butterworth Heinemann.

Recent Publications from NAL

1. **S. Mukherjee and G. Prathap** 2001 17 (6), pp 385-393. *Communications in Numerical Methods in Engineering.* Analysis of Shear Locking in Timoshenko beam elements using a function space approach.

2. **S. Mukherjee and G. Prathap** 2002 *Sadhana*.27(5) 507-526. Analysis of delayed convergence in the three noded isoparametric Timoshenko beam element using the function space approach.

3. **G. Prathap and S. Mukherjee** 2003 *Current Science,* 85(17), pp 989-994. The engineer grapples with Theorem 1.1 and Lemma 6.3 of Strang and Fix.

4. **H. Mishra and S. Mukherjee** 2004 *Sadhana* 29(6), pp 573-588. Examining the best-fit paradigm in FEM at element level.

5. **S. Mukherjee, P. Jafarali and G. Prathap** 2005 *Journal of Sound and Vibration. Vol 285(3), pp 615-635*. A variational basis for error analysis in finite element elastodynamics.

6. **K. Sangeeta , Somenath Mukherjee, and Gangan Prathap** 2005 *Structural Engineering and Mechanics Vol 21(5), pp 539-551.* A function space approach to study rank deficiency and spurious modes in finite elements.

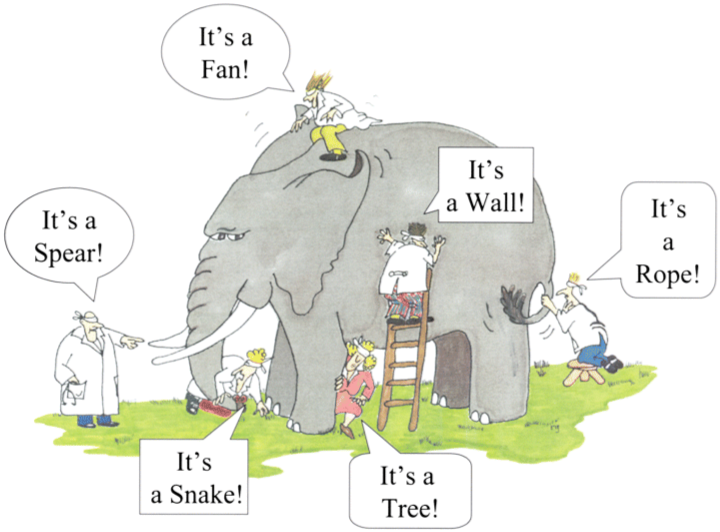
7. **K. Sangeeta, Somenath Mukherjee and Gangan Prathap** 2006 *International Journal for Computational Methods in Engineering Science & Mechanics, Vol 7, pages 1-12.* Conservation of the Best-Fit Paradigm at Element Level.

8. **P. Jafarali, M Ammen, S. Mukherjee, G. Prathap, 2007** *Journal of Sound and Vibration,* Vol 299(2), pp 196-211. Variational Correctness in Timoshenko beam finite element elastodynamics.

9. **S. Mukherjee, P. Jafarali,** (Accepted in October 2008 by *Communications in Numerical Methods in Engineering; available on-line*). Prathap's best-fit paradigm and optimal strain recovery points in indeterminate tapered bar analysis using linear element.

Thank You

The Blind Men and the Elephant



And so these men of Indostan,
Disputed loud and long,
Each in his own opinion,
Exceeding stiff and strong,
Though each was partly in the right,
And all were in the wrong!

- *John Godfrey Saxe (1816-1887)*