Method for System Parameter Estimation

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In this paper the theory of Equation Decoupling method for parameter estimation of unstable or augmented dynamical system is presented. It is also shown that the method is a generalisation of the so called Total Least Squares method in which errors in measurements as well as states are taken into account.

Keywords: Asymptotic, Decoupling, Numerical Simulation, Total Least Square

INTRODUCTION

The output method (OEM)' is the most widely used technique for estimation of parameters" of stable dynamical system including aircraft [1-6]. However the method poses severe difficulties when applied to inherently unstable or augmented systems[7]. When the system is unstable, numerical integration leads to diverging solutions. A method called equation decoupling (ED) has been recently presented for parameter estimation of unstable systems'. This method uses measured states to decouple the state equations and integrate the system of differential equations independent of each other. The decoupling of the equations may change the unstable system to the stable one. Asymptotic theory of ED output error method (EDOEM) and generalisation of the total least squares (TLS) solution are presented. Such studies are limited in open literature'-'.

EQUATION DECOUPLING OUTPUT ERROR METHOD

The dynamics of the system are given as

$$\dot{x} = Ax + Bu \quad with\ x(0) = x_0 \tag{1}$$

$$y = Cx + Du \tag{2}$$

$$z(k) = y(k) + v(k) \quad k = 1, 2, \ldots, N \tag{3}$$

Here, N is the number of data points and v is the mensurement noise assumed to be Gaussian with zero mean and covariance matrix R. The $\Theta\{A, B, C, D\}$ represents the parameter vector to he estimated. The cost function to be minimised is defined as

$$E(\Theta) = \frac{1}{2}\sum_{k=1}^{N} [z(k) - y(k)]^T R^{-1} [z(k) - y(k)] + \frac{N}{2} \ln |R| \tag{4}$$

Minimisation of the above cost function with respect to Θ yields the estimates of Θ as

$$\hat{\Theta}_{l+1} = \hat{\Theta}_l + \mu \Delta \Theta_l \tag{5}$$

$$\Delta\Theta_l = \left\{ \sum_k \left(\frac{\partial y(k)}{\partial \Theta}\right)^T R^{-1} \left(\frac{\partial y(k)}{\partial \Theta}\right) \right\}^{-1} \left\{ \sum_k \left(\frac{\partial y(k)}{\partial \Theta}\right)^T R^{-1} (z_m(k) - y(k)) \right\} \tag{6}$$

Here, l stands for iteration number. The sensitivity matrix $\frac{\partial y(k)}{\partial \Theta}$ needs to be computed. The instability caused due to numerical divergence can be overcome by incorporating stabilisation into the OEM by using measured states, x_m. The manner of utilising the measured states to stabilise the system equations leads to EDOEM[8] wherein the slates pertaining to the off diagonal elements are replaced by tlie corresponding measured states. The system matrix A is partitioned into two submatrices denoted by A_D (diagonal matrix) and A_{OD} (with offdiagonal elements of A). Equation (1) can be rewritten with $x = x_m$ as

$$\dot{x} = A_D x + [B \mid A_{OD}] \begin{bmatrix} u_m \\ x_m \end{bmatrix} \tag{7}$$

The only integrated variables entering the differential equation are in the first term. Thus each differential equation can be integrated independently of the others. The cost function to be minimised is given by equation (4). However, the computation of the sensitivity function involves the decoupled materices A_D and A_{OD} and the state measurements augmenting the control input measurements.

ASYMPTOTIC THEORY OF EDOEM

In this section the implications of the use of measured states, in terms of sensitivity matrix computation arc studied thereby providing analytical basis to the working of the EDOM. For the case where there is no process noise, equation (1) can be discretised as

$$x(k+1) = \phi x(k) + \psi Bu(k) \quad \text{and} \quad x(0) = x_0 \tag{8}$$

$$y(k) = Cx(k) + Du(k) \tag{9}$$

where ϕ denotes the state transition matrix and ψ its integral.

$$\phi = e^{A\Delta t} = I + A\Delta t + A^2 \frac{\Delta t^2}{2!} + \ldots \tag{10}$$

$$\psi = \int e^{A\tau} d\tau \approx I\Delta t + A\frac{\Delta t^2}{2!} + A^2 \frac{\Delta t^3}{3!} + \ldots \tag{11}$$

where $\Delta t = t(k+1) - t(k)$ is the sampling interval.

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Computation of the parameter increment vector, $\Delta\Theta$, requires the computation of the sensitivity equations, which are obtained by partial differentiation of the system equations with respect to each element of the unknown parameter vector Q. Since the sensitivity equations have the matrix A as the system equations, the same transition matrix, equation (10), can be used to solve them. By differentiating equation (8) and (9) with request to Θ, the discrete form of the sensitivity equations are obtained as[3]

$$\frac{\partial x(k+1)}{\partial\Theta} = \phi\frac{\partial x(k)}{\partial\Theta} + \frac{\partial\phi}{\partial\Theta}x(k) + \psi\frac{\partial B}{\partial\Theta}u(k) + \psi B\frac{\partial u(k)}{\partial\Theta} + \frac{\partial\psi}{\partial\Theta}Bu \quad (12)$$

$$\frac{\partial y(k)}{\partial\Theta} = C\frac{\partial x(k)}{\partial\Theta} + \frac{\partial C}{\partial\Theta}x(k) + \frac{\partial D}{\partial\Theta}u(k) \quad (13)$$

For simplicity second order longitudinal dynamics of an aircraft are considered. The dynamics, in continuous and discrete form are given by

$$\begin{bmatrix}\dot w\\ \dot q\end{bmatrix} = \begin{bmatrix}Z_w & u_0+Z_q\\ M_w & M_q\end{bmatrix}\begin{bmatrix}w\\ q\end{bmatrix} + \begin{bmatrix}Z\delta_e\\ M\delta_e\end{bmatrix}\delta \quad (14)$$

$$\begin{bmatrix}W_m\\ q_m\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}w\\ q\end{bmatrix} \quad (15)$$

$$\begin{bmatrix}w(k+1)\\ q(k+1)\end{bmatrix} = \phi\begin{bmatrix}w(k)\\ q(k)\end{bmatrix} + \psi\begin{bmatrix}Z\delta_e\\ M\delta_e\end{bmatrix}\delta_e \quad (16)$$

The matrices ϕ, ψ (to first order approximation) and the system matrices are given by

$$\phi = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}Z_w & u_0+Z_q\\ M_w & M_q\end{bmatrix}\Delta t\ ;\ \psi = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Delta t\ ;$$

$$A = \begin{bmatrix}Z_w & u_0+Z_q\\ M_w & M_q\end{bmatrix};\ B = \begin{bmatrix}Z\delta_e\\ M\delta_e\end{bmatrix} \quad (17)$$

The sensitivity matrix computation can be made using equation (12). The parameter vector to be estimated is given by $\Theta = [Z_w, M_w, M_q, Z\delta_e, M\delta_e]$, the elements of A and B. To illustrate the computations involved in the sensitivity matrix, the partial differentiation of the states with respect to the derivative M_w is considered, first for OEM and then for EDOEM.

Using equation (12) and (16) and the fact that the last three terms in equation (12) vanish, one obtains,

$$\begin{bmatrix}\frac{\partial w(k+1)}{\partial M_w}\\ \frac{\partial q(k+1)}{\partial M_w}\end{bmatrix} = \left[\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}Z_w & u_0+Z_q\\ M_w & M_q\end{bmatrix}\Delta t\right]\begin{bmatrix}\frac{\partial w(k)}{\partial M_w}\\ \frac{\partial q(k)}{\partial M_w}\end{bmatrix} + \Delta t\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}w(k)\\ q(k)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\partial w(k)}{\partial M_w}\\ \frac{\partial q(k)}{\partial M_w}\end{bmatrix} + \begin{bmatrix}Z_w\frac{\partial w(k)}{\partial M_w}\\ M_q\frac{\partial q(k)}{\partial M_w}\end{bmatrix}\Delta t + \begin{bmatrix}0\\ w(k)\end{bmatrix}\Delta t + \begin{bmatrix}(u_0+Z_q)\frac{\partial q(k)}{\partial M_w}\\ M_w\frac{\partial w(k)}{\partial M_w}\end{bmatrix}\Delta t \quad (18)$$

In equation (6) the second term represents the first gradient of the cost function. Expanding this first gradient and using equation (18), (subscript I stands for integrated variable), one obtains;

$$\nabla E_O(\Theta) = \sum_{k=1}^{N}\begin{bmatrix}(w_m(k)-w_I(k))\\ (q_m(k)-q_I(k))\end{bmatrix}^T\left\{\begin{bmatrix}\frac{\partial w_I(k-1)}{\partial M_w}\\ \frac{\partial q_I(k-1)}{\partial M_w}\end{bmatrix} + \begin{bmatrix}Z_w\frac{\partial w_I(k-1)}{\partial M_w}\\ M_q\frac{\partial q_I(k-1)}{\partial M_w}\end{bmatrix}\Delta t + \begin{bmatrix}0\\ w_I(k-1)\end{bmatrix}\Delta t + \begin{bmatrix}(u_0+Z_q)\frac{\partial q_I(k-1)}{\partial M_w}\\ M_w\frac{\partial w_I(k-1)}{\partial M_w}\end{bmatrix}\Delta t\right\} \quad (19)$$

For the problem described by equation (14), the Equation Decoupling formulation is given by

$$\begin{bmatrix}\dot w\\ \dot q\end{bmatrix} = \begin{bmatrix}Z_w & 0\\ 0 & M_q\end{bmatrix}\begin{bmatrix}w\\ q\end{bmatrix} + \begin{bmatrix}Z\delta_e & 0 & u_0+Z_q\\ M\delta_e & M_w & 0\end{bmatrix}\begin{bmatrix}\delta_e\\ w_m\\ q_m\end{bmatrix} \quad (20)$$

The corresponding transition matrix and system matrices are given by

$$\phi = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}Z_w & 0\\ 0 & M_q\end{bmatrix}\Delta t\ ;\ \psi = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Delta t\ ;$$

$$A = \begin{bmatrix}Z_w & 0\\ 0 & M_q\end{bmatrix};\ B = \begin{bmatrix}Z\delta_e & 0 & (u_0+Z_q)\\ M\delta_e & M_w & 0\end{bmatrix} \quad (21)$$

Using equation (21) in equation (12) for the sensitivity equation with respect to M_w and considering the fact that terms 2nd and 5th in equation (12) vanish, thus..

$$\begin{bmatrix}\frac{\partial w(k+1)}{\partial M_w}\\ \frac{\partial q(k+1)}{\partial M_w}\end{bmatrix} = \left[\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}Z_w & 0\\ 0 & M_q\end{bmatrix}\Delta t\right]\begin{bmatrix}\frac{\partial w(k)}{\partial M_w}\\ \frac{\partial q(k)}{\partial M_w}\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Delta t\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\times\begin{bmatrix}\delta_e(k)\\ w_m(k)\\ q_m(k)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Delta t\begin{bmatrix}Z\delta_e & 0 & (u_0+Z_q)\\ M\delta_e & M_w & 0\end{bmatrix}\frac{\partial}{\partial M_w}\begin{bmatrix}\delta_e(k)\\ w_m(k)\\ q_m(k)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\partial w(k)}{\partial M_w}\\ \frac{\partial q(k)}{\partial M_w}\end{bmatrix} + \Delta t\begin{bmatrix}Z_w\frac{\partial w(k)}{\partial M_w}\\ M_q\frac{\partial q(k)}{\partial M_w}\end{bmatrix} + \begin{bmatrix}0\\ w_m(k)\end{bmatrix}\Delta t + \begin{bmatrix}(u_0+Z_q)\frac{\partial q_m(k)}{\partial M_w}\\ M_w\frac{\partial w_m(k)}{\partial M_w}\end{bmatrix}\Delta t \quad (22)$$

Fur the fist gradient (*wrt* M_w) for EDOEM

$$\nabla E_E(\Theta) = \sum_{k=1}^{N}\begin{bmatrix}(w_m(k)-w_I(k))\\ (q_m(k)-q_I(k))\end{bmatrix}^T\left[\begin{bmatrix}\frac{\partial w(k-1)}{\partial M_w}\\ \frac{\partial q(k-1)}{\partial M_w}\end{bmatrix} + \begin{bmatrix}Z_w\frac{\partial w(k-1)}{\partial M_w}\\ M_q\frac{\partial q(k-1)}{\partial M_w}\end{bmatrix}\Delta t + \begin{bmatrix}0\\ w_m(k-1)\end{bmatrix}\Delta t + \begin{bmatrix}(u_0+Z_q)\frac{\partial q_m(k-1)}{\partial M_w}\\ M_w\frac{\partial w_m(k-1)}{\partial M_w}\end{bmatrix}\Delta t\right] \quad (23)$$

To derive the asymptotic equivalence between EDOEM and OEM,

$$w_m = w_t + w_n \, ; q_m = q_t + q_n \, ; \; w_I = w_1 + w_i \qquad (24)$$

where subscript t denotes true values, n the measurement noise, I the integrated value, and i the errors due to integration. Substituting equation **(24)** into equation **(19)** and observing the fact that the integration errors (w_i) tend to zero **as** the iterations are increased. Here, the convergence of the OE algorithm .is assumed which.is generally guaranteed, for OEM,

$$\frac{\nabla E_o(\Theta)}{N-1} = \frac{1}{N-1}\sum_{k=1}^{N}\begin{bmatrix} w_n(k) \\ q_n(k)\end{bmatrix}^T \left[\begin{bmatrix} \dfrac{\partial w_t(k-1)}{\partial M_w} \\ \dfrac{\partial q_t(k-1)}{\partial M_w}\end{bmatrix} + \begin{bmatrix} Z_w \dfrac{\partial w_t(k-1)}{\partial M_w} \\ M_q \dfrac{\partial q_t(k-1)}{\partial M_w}\end{bmatrix}\Delta t + \begin{bmatrix} 0 \\ w_t(k-1)\end{bmatrix}\Delta t + \begin{bmatrix} (u_0 + Z_q)\dfrac{\partial q_t(k-1)}{\partial M_w} \\ M_w \dfrac{\partial w_t(k-1)}{\partial M_w}\end{bmatrix}\Delta t\right] \qquad (25)$$

Substituting equation (24) into equation **(23)** and the fact that $w_i \rightarrow 0$ for EDOEM. one, gets,

$$\frac{\nabla E_E(\Theta)}{N-1} = \frac{1}{N-1}\sum_{k=1}^{N}\begin{bmatrix} w_n(k) \\ q_n(k)\end{bmatrix}^T \left[\begin{bmatrix} \dfrac{\partial w_t(k-1)}{\partial M_w} \\ \dfrac{\partial q_t(k-1)}{\partial M_w}\end{bmatrix} + \begin{bmatrix} Z_w \dfrac{\partial w_t(k-1)}{\partial M_w} \\ M_q \dfrac{\partial q_t(k-1)}{\partial M_w}\end{bmatrix}\Delta t + \begin{bmatrix} 0 \\ w_t(k-1) + w_n(k-1)\end{bmatrix}\Delta t + \begin{bmatrix} (u_0 + Z_q)\dfrac{\partial q_t(k-1) + q_n(k-1)}{\partial M_w} \\ M_w \dfrac{\partial w_t(k-1) + w_n(k-1)}{\partial M_w}\end{bmatrix}\Delta t\right] \qquad (26)$$

In equation (26) since the measurement noise $[w_n \, , q_n]$ is independent of the parameters to be estimated, the partials of q_n and w_n wrt M_w vanish. Next, the following term involving $\{w_n \, , q_n\}$ are obtained, in equation (26)

$$\frac{\nabla E_N(\Theta)}{N-1} = \frac{1}{N-1}\sum_{k=1}^{N}\begin{bmatrix} w_n(k) \\ w_n(k)\end{bmatrix}^T \begin{bmatrix} 0 \\ w_n(k-1)\end{bmatrix} \Delta t \qquad (27)$$

which tends to zero since the measurement noise q_n and w_n are uncorrelated. Hence, in the light of the aboveobservations, comparing equation **(25)** and (26), one gets.

$$\frac{\nabla E_E(\Theta)}{N-1} \approx \frac{\nabla E_o(\Theta)}{N-1} \qquad (28)$$

asymptotically for the two-state model.

Thus, the asymptotic behaviow of the EDOEM is similar to that of OEM. However, the OEM does not work directly for unstable systems **as** noted earlier. In EDOEM, since the measured states, obtained from the unstable plant operating in closed loop are stable,their use in the estimation process, tries to arrest this divergence and at the same time enables parameter estimation of the basic unstable plant directly.

GENERALISATION THEORY OF EDOEM

The EDOEM can be viewrd **as a** generalisation of the **TLS** approach for parameter estimation. The TLS approach' is used to account for the errors in the measurements of X, the variables in the regression equation, in addition to the errors due to noise in Y. The general regression equation is given by

$$Y = X\theta + \varepsilon \qquad \textbf{(29)}$$

When the measurements X are noisy, the LS method does not explicitly account for these errors. The TLS method addresses this problem **as** follows:

The equation **(29)** is restated **as**

$$[X \mid Y] \; [\Theta^T \mid -1]^T \approx 0 \qquad (30)$$

Due to the measurement errors, the compound data matrix [XI Y] is full rank and there is no nonzero solution vector $[\Theta^T \mid -1]^T$. To get a solution, the rank of the measured data [XI Y] is reduced with **an** estimate of data errors $[\Delta X \mid \Delta Y]$

$$([X \mid Y] - [\hat{\Delta X} \mid \hat{\Delta Y}])\,[\Theta^T \mid -1]^T = 0 \qquad (31)$$

subject to the constraint of minimal approximation effort

$$\| [\Delta \hat{X} \mid \Delta \hat{Y}] \, C^{-1} \|_F^2 \text{ is minimal} \qquad (32)$$

where C is the square root of the convariance matrix of the row vectors of the dataerror matrix $[\Delta X \mid \Delta Y]$. The solution of equation (31) is given in terms of the singular value decomposition of the matirx $[X \mid Y] \, C^{-1}$:

$$C\,[\hat{\theta}_{TLS} \mid -1]^T = \lambda v_{n+1} \qquad (33)$$

The trasformed solution vector $C[\theta^T \mid -1]$ is found **as** the kernel of matrix $[\hat{X} \mid \hat{Y}]\,C^{-1}$, which equals the last column vector v_{n+1} of the singular matrix V. Equation **(31)** is solved for $[\hat{\theta}_{TLS} \mid -1]^T$, λ is chosen such that the last element equals -1. In case of EDOEM

$$\dot{x} = A_D x_I + [B \mid A_{OD}] \begin{bmatrix} u_m \\ x_m \end{bmatrix} \qquad (34)$$

$$y = C x_I + n_m \qquad (35)$$

where x_I stands for the integrated state generated from the integration of the state equations and n_m stands for the measurement noise. If $C = I$, the identity matrix,

$$y = x_I + n_m \qquad (36)$$

Introducing the integrated variables from equation **(34)** into the equation **(35)**,

$$y_j = \phi_D x_{I_{j-1}} + [B \, A_{OD}] \begin{bmatrix} u_m \\ x_m \end{bmatrix}_{j-1} \Delta t + n_{mj} \qquad (37)$$

which is rewritten **as** equation **(38)** and in compact form **as** equation (39)

$$y_j^T = \left[x_j^T \ u_m^T \ \Delta t \ x_m^T \ \Delta t\right] \begin{bmatrix} \phi D^T \\ B^T \\ A_{OD}^T \end{bmatrix}_{j-1} + n_{mj}^T \quad (38)$$

$$Y = X\theta + n_m \quad (39)$$

where X is the expanded matrix containing the integrated state, the measured states and control inputs. θ is the parameter vector to be estimated. Equation (39) has the same general form as the regression equation (29) for the TLS problem. There are measurement errors in Y. The x_j contains errors due to integration caused by incorrect initial conditions, round off errors *etc.* Also the measurement errors in the states x_m are present in general. Thus the EDOEM formulation of the estimation problem is such that it generalised the TLS problem which is, in itself, the generalisation of the LS problem. Thus, the analogy and the generalisation of the TLS problem have been established in terms of the EDOEM for which the asymptotic theory was developed in the previous sections.

NUMERICAL SIMULATION

Short period data of an aircraft are simulated. The data is generated with a doublet input to the pilot stick with a sampling time of 0.05 s.

State equations:

$$\dot{w} = Z_w w + (u_o + Z_q)\, q + Z_{\delta_e} \delta_e \quad (40)$$

$$\dot{q} = M_w w + M_q q + M_{\delta_e} \delta_e \quad (41)$$

Observation equations:

$$A_z = Z_w w + M_q q + Z_{\delta_e} \delta_e$$

$$W_m = w$$

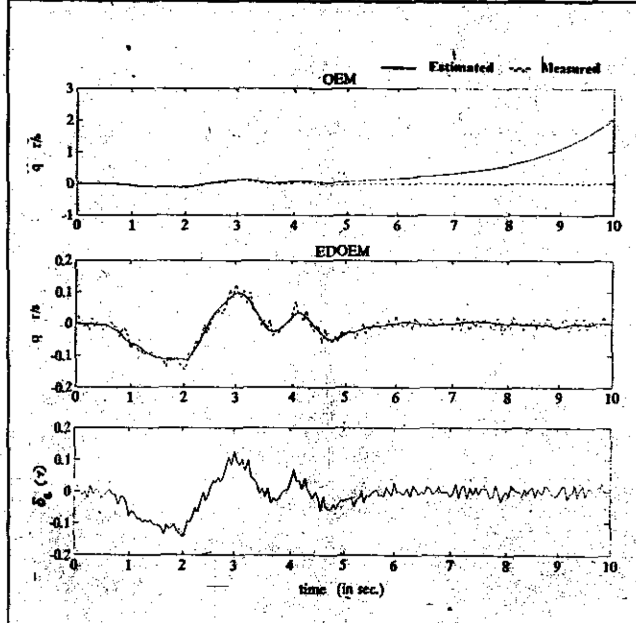
$$q_m = q \quad (42)$$

where w is the vertical velocity, u_0 the stationary forward speed, q the pitch rate, A_z the vertical acceleration and δ_e is the elevator deflection. Since for M_w with positive numerical value of system is generally unstable, w is feedback with a gain K to stabilise the system as follows:

$$\delta_p = \delta_e + K w \quad (43)$$

where δ_p denotes the pilot input. The direct identification between δ_e and output measurements is attempted. The OEM and EDOEM were used to analyse the data. Fig 1 shows the time history match for q. This numerical divergence caused due to integration of the inherently unstable plant is evident when OEM is used. When EDOEM is used, the measured state arrests the numerical divergence and the time history match is satisfactory. The theoretical and application results indicate that EDOEM can be used successfully for estimation of aerodynamic derivatives of unstable or augmented systems.

FIGURE 1
TIME HISTORY MATCH FOR OEM/EDOEM



CONCLUSIONS

The asymptotic theory of EDOEM which uses measured states and enables parameter estimation of unstable or augmented systems is presented and validated for simulated data. Also it is shown that the EDOEM can be viewed as a generalisation of the TLS solution to parameter estimation.

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