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Analytical solution for an orthotropic elastic plate containing cracks

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Abstract. The problem of estimating the bending stress distribution in the neighborhood of a crack located on a single line in an orthotropic elastic plate of constant thickness subjected to bending moment or twisting moment is examined. Using classical plate theory and integral transform techniques, the general formulae for the bending moment and twisting moment in an elastic plate containing cracks located on a single line are derived. The solution is obtained in a closed form for the case in which there is a single crack in an infinite plate and the results are compared with those obtained from the literature.

Key words: Bending, crack, integral transform, orthotropic plate, stress intensity factor, twisting.

1. Introduction

Many relatively thin structures such as airplane fuselage skins can be subjected to bending loads and the study of crack tip stress state is important for the design and construction of safe structures. The solution of the thin plate-bending problem was pioneered by Williams (1961), who made use of the eigen function expansion technique and determined the stress distribution in the neighborhood of a crack. Sih et al. (1962) applied a complex variable method for evaluating the strength of stress singularities at crack tips in plate extension and bending problems. A straightforward and accurate analytical method for the determination of crack-tip stress fields in pure bending and twisting problems for thin plates is proposed by Jones and Subramonian (1983), where the three dimensional equations of equilibrium have been reduced to an equivalent two dimensional set by using the simplifications valid for pure bending of thin plates. A study of plate-tearing mode of fracture from the simplified analytical approach and finite element approach is validated through photoelastic results in (Jones and Subramonian (1983)). The general solution for finite number of cracks using anisotropic elasticity is presented by Krenk (1975). Alwar and Ramachandran (1983) showed that the stress intensity factor is nearly linear through the thickness for thin plates, in the absence of crack closure. Using finite element method, Mark et al. (1995), Alberto Zucchini et al. (2000) computed stress intensity factors for thin cracked plates. Approximate weight functions are applied to investigate the influence of the orthotropy of the material on the fracture behavior of double cantilever beam in Massabò et al. (2003). Using complex variable method Zehnder and Hui (1994) calculated stress intensity factor for a finite crack in an infinite isotropic plate. The present method uses an integral transform technique and does not assume any symmetry about the co-ordinate axis and hence it differs from the other methods used for solving plate bending problems containing cracks in the literature. Also the constants appearing in the solution of the governing differential equations are obtained from the displacement boundary conditions by defining the curvature discontinuities on the crack surfaces apart from the moment boundary conditions and continuity conditions. The mechanical behavior near the crack tip is modeled in a more simple approach, using classical plate theory in the case of an isotropic plate by the author in Chattopadhyay (2003). In the present study, the general formulae for the stress distribution in an infinite elastic orthotropic plate containing cracks are derived and the stress intensity factor is determined in a closed form in the case of a single crack when the plate is subjected to bending or twisting moments and the results are compared with those from the literature.

2. Formulation of the problem

Let us consider the cases of bending or twisting actions of an infinite plate by moments that are uniformly distributed along the edges of the plate containing collinear cracks. We take xy-plane to coincide with the middle plane of the plate before deformation. The z-axis is assumed to be perpendicular to the middle plane. We denote the bending moment per unit length about x-axis by M_{yy} and about y-axis by M_{xx} and the twisting moment per unit length by M_{xy}. The constant thickness of the plate is h and we consider it to be small in comparison with other dimensions. Let us assume that during bending, the plate undergo the displacement w perpendicular to xy-plane. In the present analytical method, we consider the problem in which an infinite orthotropic elastic plate whose material principle axes are aligned with respect to the co-ordinate axes, contains cracks located on a single line is acted upon by applied moments. Let the co-ordinate system be so chosen that the x-axis coincide with the line on which the cracks are located. Let L denote the union of intervals occupied by the cracks on the x-axis and M is the interval not occupied by the cracks. Suppose that a thin plate containing a crack is subjected to uniform bending or twisting moments at infinity. The boundary conditions for pure bending and twisting of thin cracked plates may be expressed in terms of the moment boundary conditions. Since the crack surface is traction-free, the boundary conditions along the crack surface permitting all of the free edge conditions for pure bending and twisting of thin plate is given by the following equations:

$$M_{xy}(x,0)=0, \quad x\in L, \tag{1}$$

$$M_{yy}(x,0)=0, \quad x\in L. \tag{2}$$

We note that these boundary conditions are expressed in terms of surface stresses by Jones et al. (1983). The solution to the present problem may be obtained by judiciously superposing the simple solution of an uncracked plate under uniform bending moment or twisting moment to that of a cracked plate with bending or twisting moment applied to the crack surfaces. That is, the solution may be obtained by using standard superposition technique and thus for the purpose of evaluating the crack tip singular stresses it is sufficient to consider the problem in which self-equilibrating crack surface loads are the only external loads. Thus, it suffices to solve the problem of specifying uniform bending and twisting moment on the crack segment of the plate. Let the desired system be composed of two parts, one the uniform moment field and the other a perturbation field due to the crack which dies out at infinity. While the boundary conditions along the free edges of the crack require traction free conditions, it is possible to formulate the problem as one of finding solutions for the perturbation solutions satisfying the filed equations and the boundary conditions

$$M_{xy}(x,0) = \frac{G^*(x)}{2}, \quad x \in L, \tag{3}$$

$$M_{yy}(x,0) = \frac{H^*(x)}{2}, \quad x \in L. \tag{4}$$

and $G^*(x)$ and $H^*(x)$ are the known prescribed functions on the crack surfaces. The required solution will now be determined in two parts, one for $y > 0$, the upper half plane and the other for $y < 0$, the lower half plane and subsequently they will be matched to insure continuity of the solution for the segments outside the crack.

The displacements in the x direction and y direction at any point are given by the following expressions:

$$u_x = -z\frac{\partial w}{\partial x}, \tag{5}$$

$$u_y = -z\frac{\partial w}{\partial y}. \tag{6}$$

The moment–curvature relations in an orthotropic plate (Timoshenko, 1959) are given by

$$M_{xx} = -\frac{h^3}{12}\left(C_{11}\frac{\partial^2 w}{\partial x^2} + C_{12}\frac{\partial^2 w}{\partial y^2}\right), \tag{7}$$

$$M_{yy} = -\frac{h^3}{12}\left(C_{22}\frac{\partial^2 w}{\partial y^2} + C_{12}\frac{\partial^2 w}{\partial x^2}\right), \tag{8}$$

$$M_{xy} = -\frac{h^3}{6}C_{66}\frac{\partial^2 w}{\partial x \partial y}. \tag{9}$$

where $C_{11},\ C_{12},\ C_{22},\ C_{66}$ are elastic constants of the material and are defined as follows:

$$\begin{aligned} &C_{11} = \frac{1}{E_y\Delta_0}; \quad C_{22} = \frac{1}{Ex\Delta_0}; \quad C_{66} = G_{xy} \\ &C_{12} = \frac{\nu_{yx}}{E_y\Delta_0} = \frac{\nu_{xy}}{E_x\Delta_0}; \quad \Delta_0 = \frac{1-\nu_{yx}\nu_{xy}}{E_xE_y} \end{aligned} \tag{10}$$

where E_x and E_y are Young's moduli in the directions of the x and y axes, respectively. G_{xy} is the shear modulus for plane parallel to the xy-plane. ν_{xy} is the Poisson ratio characterizing the contraction in the direction of y-axis when the tension is applied in the direction of x-axis. Likewise, ν_{yx} is the Poisson ratio characterizing the contraction in the direction of x-axis when the tension is applied in the direction of y-axis.

The maximum magnitude of the stress components located at the top or bottom at $z = h/2$ of the plate are functions of (x, y) and given by the following expressions:

$$\sigma_{xy} = \frac{6M_{xy}}{h^2}; \quad \sigma_{yy} = \frac{6M_{yy}}{h^2}; \quad \sigma_{xx} = \frac{6M_{xx}}{h^2} \tag{11}$$

Using the definitions of in-plane strain components, these strain components in terms of the curvatures are given by,

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \tag{12}$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}, \tag{13}$$

$$\gamma_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}. \tag{14}$$

We define the boundary conditions in terms of the derivative of the displacements (in-plane strain) as given by

$$A(x) = 0, \quad x \in M, \tag{15}$$

$$B(x) = 0, \quad x \in M, \tag{16}$$

where the in-plane strain components on the crack segment are defined by the functions $A(x)$, $B(x)$

$$A(x) = \frac{\partial}{\partial x}\left[u_x^{(1)}(x,0) - u_x^{(2)}(x,0)\right], \quad x \in L, \tag{17}$$

$$B(x) = \frac{\partial}{\partial x}\left[u_y^{(1)}(x,0) - u_y^{(2)}(x,0)\right], \quad x \in L, \tag{18}$$

and the superscripts (1) and (2) denote the components in the upper half plane $y > 0$ and lower half plane $y < 0$ respectively.

3. Solution of crack problem

In this section, we consider an elastic plate subjected to bending loads and containing cracks located on a single line. Using Fourier transform we solve the equations of equilibrium, and from the prescribed boundary conditions and continuity conditions, the solution of the present crack problem reduces to that of solving singular integral equations. If Q_x and Q_y are the shearing forces per unit length parallel to the y and x axes then the governing equations of bending effect are given in Timoshenko (1959),

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0, \tag{19}$$

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0, \tag{20}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0. \tag{21}$$

From equation (19) and the boundary condition (2), the Kirchhoff boundary condition given by the equation

$$Q_y - \frac{\partial M_{yx}}{\partial x} = 0, \quad y = 0, \quad x \in L$$

is also satisfied on free edge of the crack surfaces.

The strain compatibility equation is given by,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{22}$$

If we define the moment resultants in terms of the Airy's function $\varphi(x, y)$ as given by

$$M_x = \frac{\partial^2 \varphi}{\partial y^2}; \quad M_y = \frac{\partial^2 \varphi}{\partial x^2}; \quad M_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y}, \tag{23}$$

then the governing Equations (19–21) are satisfied. Also from the compatibility Equation (22) and from moment curvature relations (7–9), the present problem reduces to that of solving the bi-harmonic equation in $\varphi(x, y)$

$$\nabla^4 \varphi = 0, \tag{24}$$

where,

$$\nabla^4 \varphi = \frac{\partial^4 \varphi}{\partial y^4} + 2\Delta_1 \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \Delta_2 \frac{\partial^4 \varphi}{\partial x^4}, \tag{25}$$

$$\Delta_1 = \frac{C_{11}C_{22} - C_{12}^2 - 2C_{12}C_{66}}{2C_{22}C_{66}}, \tag{26}$$

$$\Delta_2 = \frac{C_{11}}{C_{22}}. \tag{27}$$

Let $\varphi^{(1)}(x, y)$ denote $\varphi(x, y)$ in the upper half plane $y > 0$ and $G^{(1)}(\xi, y)$ be the Fourier transform of $\varphi(x, y)$ for $y > 0$. Then

$$G^{(1)}(\xi, y) = \int_{-\infty}^{\infty} \varphi^{(1)}(x, y) e^{i\xi x} \, \mathrm{d}x, \quad y > 0 \tag{28}$$

Taking Fourier transformation of the bi-harmonic equation we get the ordinary differential equation in $G(\xi, y)$ as given by

$$\frac{\mathrm{d}^4 G(\xi, y)}{\mathrm{d}y^4} - 2\Delta_1 \xi^2 \frac{\mathrm{d}^2 G(\xi, y)}{\mathrm{d}y^2} + \Delta_2 \xi^4 G(\xi, y) = 0 \tag{29}$$

The solutions for the above differential equation are given by the following expressions

$$G^{(1)}(\xi, y) = P_1(\xi) e^{-t_1|\xi|y} + Q_1(\xi) e^{-t_2|\xi|y}, \quad y > 0, \tag{30}$$

$$G^{(2)}(\xi, y) = P_2(\xi) e^{t_1|\xi|y} + Q_2(\xi) e^{t_2|\xi|y}, \quad y < 0, \tag{31}$$

where the superscripts (1) and (2) indicate the upper and lower half planes respectively.

The constants t_1 and t_2 are the roots with the real parts of the quartic equation

$$t^4 - 2\Delta_1 t^2 + \Delta_2 = 0 \tag{32}$$

$G^{(1)}$ and $G^{(2)}$ are the Fourier transforms of $\varphi(x, y)$ for $y > 0$ and $y < 0$, $P_1(\xi), P_2(\xi)$, $Q_1(\xi), Q_2(\xi)$ are the unknown functions to be determined. From the moment boundary conditions (3–4) and the continuity conditions outside the crack segment, we have the following equations,

$$M_y^{(1)}(x, 0) - M_y^{(2)}(x, 0) = 0, \quad \forall x, \tag{33}$$

$$M_{xy}^{(1)}(x, 0) - M_{xy}^{(2)}(x, 0) = 0, \quad \forall x. \tag{34}$$

The bending and twisting moments in terms of $G^{(1)}(\xi, y)$ for $y > 0$ are given by

$$M_x^{(1)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2 G^{(1)}(\xi, y)}{\partial y^2} e^{-i\xi x} \, \mathrm{d}\xi, \quad y > 0, \tag{35}$$

$$M_y^{(1)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 G^{(1)}(\xi, y) e^{-i\xi x} \, \mathrm{d}\xi, \quad y > 0, \tag{36}$$

$$M_{xy}^{(1)}(x, y) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \xi \frac{\partial G^{(1)}(\xi, y)}{\partial y} e^{-i\xi x} \, \mathrm{d}\xi, \quad y > 0. \tag{37}$$

The bending and twisting moments for $y < 0$ in terms of $G^{(2)}(\xi, y)$ are given by

$$M_x^{(2)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2 G^{(2)}(\xi, y)}{\partial y^2} e^{-i\xi x} \, \mathrm{d}\xi, \quad y < 0, \tag{38}$$

$$M_y^{(2)}(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 G^{(2)}(\xi,y) e^{-i\xi x} \, \mathrm{d}\xi, \quad y<0, \tag{39}$$

$$M_{xy}^{(2)}(x,y) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} \xi \frac{\partial G^{(2)}(\xi,y)}{\partial y} e^{-i\xi x} \, \mathrm{d}\xi, \quad y<0. \tag{40}$$

Taking Fourier transforms of the boundary conditions (17–18), moment–curvature relations (7–9), the strain–curvature relations (12–14), from (33) to (34) and from the Fourier transforms of the moment components from (35) to (37), we get the four algebraic equations for solving the four unknowns $P_1(\xi), P_2(\xi), Q_1(\xi), Q_2(\xi)$ appearing in $G^{(1)}$ and $G^{(2)}$ in terms of $\bar{A}(\xi)$ and $\bar{B}(\xi)$, the Fourier transforms of $A(x)$ and $B(x)$. Substituting these values into the Equations (36), we get the bending moment resultants in the upper half plane $y>0$, as given by,

$$M_y^{(1)}(x,y) = \frac{\Delta_0}{4\pi(t_2^2-t_1^2)} \int_{-\infty}^{\infty} \left\{ \left[\bar{A}\,|\xi| - \frac{i\bar{B}\,\mathrm{sgn}(\xi)}{t_1} \right] e^{-t_1|\xi|y} - \left[\bar{A}\,(\xi) - \frac{i\bar{B}\,\mathrm{sgn}(\xi)}{t_2} \right] e^{-t_2|\xi|y} \right\} e^{-i\xi x} \mathrm{d}\xi, \quad y>0, \tag{41}$$

where $\Delta_0 = C_{11} - \frac{C_{12}^2}{C_{22}}$.

Performing the inner integral in terms of $A(s)$ and $B(s)$ we get the bending moment resultants in the upper half-plane $y>0$, in terms of functions $A(s)$ and $B(s)$ as given by

$$M_y^{(1)}(x,y) = \frac{\Delta_0}{2\pi(t_2^2-t_1^2)} \int_{-\infty}^{\infty} A(s) y \left\{ \frac{t_1}{[(x-s)^2+t_1^2y^2]} - \frac{t_2}{[(x-s)^2+t_2^2y^2]} \right\} \mathrm{d}s - \int_{-\infty}^{\infty} \frac{B(s)(x-s)}{t_1 t_2} \left\{ \frac{t_1}{[(x-s)^2+t_1^2y^2]} - \frac{t_2}{[(x-s)^2+t_2^2y^2]} \right\} \mathrm{d}s, \quad y>0. \tag{42}$$

Similarly, substituting the values of the $P_2(\xi), Q_2(\xi)$ in terms of $\bar{A}(\xi)$ and $\bar{B}(\xi)$ into the Equations (39), we get the bending moment resultants in the lower half plane $y<0$, as given by,

$$M_y^{(2)}(x,y) = \frac{-\Delta_0}{4\pi(t_2^2-t_1^2)} \int_{-\infty}^{\infty} \left\{ \left[\bar{A}(\xi) + \frac{i\bar{B}(\xi)\,\mathrm{sgn}(\xi)}{t_1} \right] e^{t_1|\xi|y} - \left[\bar{A}\,(\xi) + \frac{i\bar{B}\,\mathrm{sgn}(\xi)}{t_2} \right] e^{t_2|\xi|y} e^{-i\xi x} \, \mathrm{d}\xi, \quad y<0, \tag{43}$$

Performing the inner integral in terms of $A(s)$ and $B(s)$ we get the bending moment resultants in the lower half-plane $y<0$, in terms of the unknown displacement functions $A(s)$ and $B(s)$ as given by

$$M_y^{(2)}(x,y)=\frac{\Delta_0}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}A(s)y\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s$$
$$-\int_{-\infty}^{\infty}\frac{B(s)(x-s)}{t_1t_2}\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s,\quad y<0,\quad (44)$$

Combining (42) and (44) we get the bending moment resultant as given by,

$$M_y(x,y)=\frac{\Delta_0}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}A(s)y\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s$$
$$-\int_{-\infty}^{\infty}\frac{B(s)(x-s)}{t_1t_2}\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s,\quad y\neq 0,\quad (45)$$

Similarly the expression for the bending moment $M_x(x,y)$ is given by

$$M_{xx}(x,y)=-\frac{\Delta_0}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}A(s)y\left\{\frac{t_1^3}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2^3}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s$$
$$-\int_{-\infty}^{\infty}B(s)(x-s)\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s,\quad y\neq 0,\quad (46)$$

The expression for the twisting moment $M_{xy}(x,y)$ is given by

$$M_{xy}(x,y)=\frac{\Delta_0}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}\left[A(s)(x-s)+yB(s)\right]$$
$$\times\left\{\frac{t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s,\quad y\neq 0,\quad (47)$$

The curvature terms are given by

$$\frac{\partial^2 w}{\partial x^2}=-\frac{1}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}A(s)y\left\{\frac{m_1t_1}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{m_2t_2}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s$$
$$-\int_{-\infty}^{\infty}\frac{B(s)(x-s)}{t_1t_2}\left\{\frac{m_1t_2}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{m_2t_1}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s,\quad (48)$$

$$\frac{\partial^2 w}{\partial x\partial y}=\frac{1}{2\pi(t_2^2-t_1^2)}\int_{-\infty}^{\infty}\left[A(s)(x-s)+yB(s)\right]\left\{\frac{t_1m_2}{\left[(x-s)^2+t_1^2y^2\right]}-\frac{t_2m_1}{\left[(x-s)^2+t_2^2y^2\right]}\right\}\mathrm{d}s\quad (49)$$

$$m_1 = t_1^2 + \frac{C_{12}}{C_{22}}; \quad m_2 = t_2^2 + \frac{C_{12}}{C_{22}}$$

where $A(s)$ and $B(s)$ are the unknown functions to be determined. The limiting values as $y \to 0+$ and $y \to 0-$ of the bending and twisting moments along the crack line are given by,

$$M_y(x,0) = -\frac{\Delta_0}{2\pi t_1 t_2 (t_1 + t_2)} \int_{-\infty}^{\infty} \frac{B(s)}{x-s}\, \mathrm{d}s, \tag{50}$$

$$M_{xy}(x,0) = -\frac{\Delta_0}{2\pi\,(t_1 + t_2)} \int_{-\infty}^{\infty} \frac{A(s)}{x-s}\, \mathrm{d}s, \tag{51}$$

$$M_x(x,0) = -\frac{\Delta_0}{2\pi\,(t_1 + t_2)} \int_{-\infty}^{\infty} \frac{B(s)}{x-s}\, \mathrm{d}s. \tag{52}$$

By using the conditions (3–4) and (15–16) in the above expressions, the interval of integration reduces to L. From the boundary conditions (3–4) and the above relations we get the singular integral equations

$$\int_{-L}^{L} \frac{A(s)}{x-s}\, \mathrm{d}s = \frac{-\pi(t_1 + t_2)}{\Delta_0} G^*(x), \quad x \in L, \tag{53}$$

$$\int_{-L}^{L} \frac{B(s)}{x-s}\, \mathrm{d}s = \frac{-\pi t_1 t_2 (t_1 + t_2)}{\Delta_0} H^*(x), \quad x \in L, \tag{54}$$

for the determination of the functions A and B on the interval L.

4. The single crack problem

In this section, we consider the problem of determining the distribution of stress and moment in the vicinity of a Griffith crack of length $2c$, occupying the interval $(-c, c)$ on the x-axis in an infinite orthotropic elastic plate subjected to the moment about x-axis $H^*(x)$ as given in Figure 1. In this case the interval $L = (-c, c)$ on the x-axis and we have the moment boundary condition as given by,

$$M_y(x,0) = \frac{1}{2} H^*(x), \quad |x| < c \tag{55}$$

and the singular integral Equations (54) reduce to the following equations,

$$\int_{-c}^{c} \frac{B(s)}{x-s}\, \mathrm{d}s = \frac{-\pi t_1 t_2 (t_1 + t_2)}{\Delta_0} H^*(x), \quad |x| < c \tag{56}$$

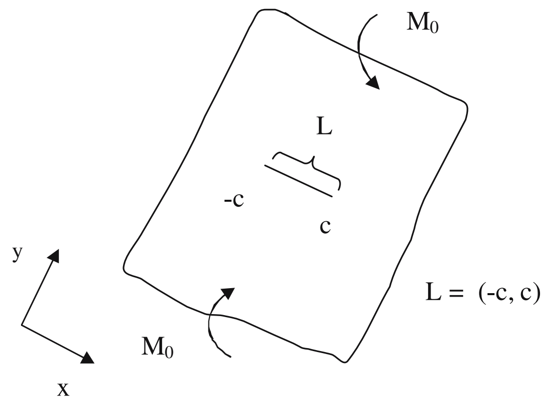

Figure 1. Crack in a plate subject to bending load.

Solution of this integral equation is given by the following expressions

$$B\,(s) = \frac{-t_1 t_2 (t_1 + t_2)}{\pi\,\Delta_0 \sqrt{c^2 - s^2}} \int_{-c}^{c} \frac{H^*(x)\sqrt{(c^2 - x^2)}}{x - s}\, dx + c_1, \quad |s| < c \tag{57}$$

where the arbitrary constant c_1 is determined from the condition

$$\int_{-c}^{c} B(s)\mathrm{d}s = 0 \tag{58}$$

From the above conditions (58) and the expressions (57) we find that

$$c_1 = 0 \tag{59}$$

Substituting the value of and $B(s)$ from (57) into (50), the bending moment resultants for a single crack problem is given by the following expression,

$$M_y(x, 0) = \frac{\operatorname{sgn}(x)}{2\pi\sqrt{(x^2 - c^2)}} \int_{-c}^{c} \frac{H^*(x)\sqrt{(c^2 - t^2)}}{t - x}\mathrm{d}t, \quad |x| > c \tag{60}$$

5. Particular cases of loadings along the edges of the plate

To illustrate the above procedure, we consider the infinite plate subject to (a) bending and (b) twisting as shown in the Figures 1 and 2 respectively for the determination of stress intensity factors.

Case (a): We consider an infinite elastic plate containing a crack opened by the moment M_0 acting along its edges (Figure 1). In this case the function $H^*(x)$ is given by

$$H^*(x) = M_0 \tag{61}$$

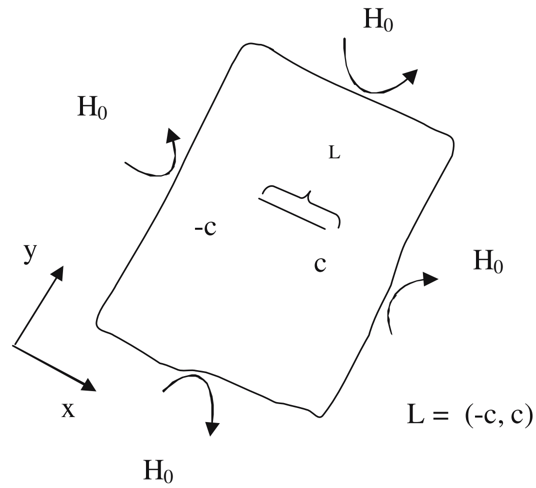

Figure 2. Crack in a plate subject to twisting load.

The bending moment resultant along the crack line from the Equations (60) is given by,

$$M_y(x,0) = \frac{\operatorname{sgn}(x)}{2\pi\sqrt{(x^2-c^2)}} M_0 \int_{-c}^{c} \frac{\sqrt{(c^2-t^2)}}{t-x}\,\mathrm{d}t, \quad |x|>c, \tag{62}$$

The maximum bending stress at $z = h/2$, along the crack line from Equations (11) and (62) is given by

$$\sigma_{yy}(x,0) = \frac{6M_0\ \operatorname{sgn}(x)}{2\pi h^2\sqrt{(x^2-c^2)}} \int_{-c}^{c} \frac{\sqrt{(c^2-t^2)}}{t-x}\mathrm{d}t, \quad |x|>c, \tag{63}$$

Stress intensity factor K_I due to bending moment in this case is given by

$$K_I = \underset{x\to c}{Lt}\left\{\sqrt{[2(x-c)]}\sigma_{yy}(x,0)\right\} = \frac{6M_0\sqrt{c}}{h^2} \tag{64}$$

The above stress intensity factor is the same as that of given by Sih et al. (1962), Jones and Subramonian (1983), Zehnder and Hui (1994), Murakami (1987) and Lalitha Chattopadhyay (2003) for isotropic material.

Case (b): In this case we assume that the plate containing a crack is subjected to the twisting moment H_0 (Figure 2) along its edges.

From the curvature term (Equation 49) the twisting moment along the crack line $(y = 0)$ is given by

$$\frac{h^3 C_{66}}{6}\left[\frac{\partial^2 w}{\partial x \partial y}\right]_{y=0} = \frac{(t_1 m_2 - t_2 m_1)}{2\pi (t_2^2 - t_1^2)} \int_{-c}^{c} \frac{A(s)}{(x-s)}\,\mathrm{d}s \tag{65}$$

From the boundary condition we have the following condition on the crack lone ($y=0$),

$$\frac{h^3 C_{66}}{6}\left[\frac{\partial^2 w}{\partial x \partial y}\right]=H_0, \quad |x| \le c \tag{66}$$

Since the twisting moment is given by,

$$M_{xy}=\frac{h^3 C_{66}}{6}\left[\frac{\partial^2 w}{\partial x \partial y}\right] \tag{67}$$

$$\frac{\partial M_{xy}}{\partial x}=\frac{h^3}{6}C_{66}\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x \partial y}\right]=\frac{(t_1 m_2 - t_2 m_1)}{2\pi(t_2^2-t_1^2)}\int_{-c}^{c}\frac{A(s)}{(x-s)^2}\,\mathrm{d}s=0 \tag{68}$$

the twisting moment M_{xy} along the crack line ($y=0$) is non-singular
Stress intensity factor K_{II} due to twisting moment M_{xy} in this case is given by,

$$K_{II}=\underset{x\to c}{\mathrm{Lt}}\left\{\sqrt{[2(x-c)]}\sigma_{xy}(x,0)\right\}=0 \tag{69}$$

The above stress intensity factor is the same as that of given Zehnder and Hui (1994) for isotropic material.

6. Conclusion

A simple method for determining the analytical expression is explained for the bending stress distribution, the bending moment and twisting moment resultant in the vicinity of a crack in an infinite orthotropic elastic thin plate of constant thickness. An elastic plate containing a single crack is examined in detail and the stress intensity factor is calculated for the cases when the plate is subjected to two loading cases namely (i) bending and (ii) twisting. These stress intensity factors are independent of material constants for orthotropic material with axes aligned along the x-axis and y-axis. The stress intensity factors are compared and the results agree closely with the literature results.

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