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STRESS OSCILLATIONS AND SPURIOUS LOAD MECHANISMS IN VARIATIONALLY INCONSISTENT ASSUMED STRAIN FORMULATIONS

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convergence and spurious stress oscillations if the assumed strain fields are not variationally correct, i.e. they do not satisfy an important orthogonality condition emerging from the equivalence sought between assumed strain displacement procedures and mixed procedures based on the Hellinger–Reissner theorem. Failure to ensure variational correctness introduces errors which can be equated to the presence of spurious loading mechanisms that cause stress oscillations. In this paper, we use the Timoshenko beam element to demonstrate that field-consistency and variational consistency are two complementary but mutually exclusive principles—one does not imply the other and that both are necessary to successfully implement a displacement type finite element for constrained media.

INTRODUCTION

Assumed strain formulations of the displacement type finite element procedure (also described as mixed interpolation methods, etc.) have gained importance in recent years in eliminating problems such as shear locking, membrane locking, parasitic shear, etc. which arise because of inconsistencies in the concerned strain fields if these are derived directly from the original kinematically admissible displacement fields. The guiding principle here is called field-consistency—i.e. the 'assumed' constrained strain field must be consistently constituted from the kinematically admissible displacement fields so that no spurious constraints emerge in the constrained regime. However, the re-constituted 'assumed' strain field interpolations cannot be chosen arbitrarily. There is a pre-determined variationally correct form that conforms to an exact equivalence of the assumed strain displacement approach to the mixed approaches based on the Hellinger-Reissner or Hu–Washizu variational theorems. If the orthogonality condition that represents this equivalence is violated while designing the strain field to satisfy the field-consistency criteria alone, the resulting elements have poorer efficiency and are also plagued by undesirable strain and stress oscillations. We demonstrate these aspects using, as an example, various orthogonally correct and incorrect field-consistent shear strain fields for the one dimensional quadratic and cubic shear flexible beam elements.

FIELD-CONSISTENCY AND THE VARIATIONALLY CORRECT WAY TO DERIVE ASSUMED STRAIN FIELDS OVER AN ELEMENT DOMAIN

Several problems in structural mechanics need to be described by more than one strain field and it is also required that, in certain physical regimes, one or more of these strain fields must vanish in

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the penalty limit. Conventional displacement type procedures in such instances will lead to difficulties known variously as shear locking, parasitic shear, membrane locking, etc. The field-consistency paradigm[1] offers a conceptual scheme to describe the importance of ensuring that the interpolations for the constrained or nearly-constrained strain fields are 'field-consistent'—i.e. they are made up of contributions from the constituent field-variable interpolations in a balanced way so that only true constraints are enforced in the penalty limits. The field-consistency requirements may be met in several possible ways—e.g. reduced integration, selective integration, mixed and hybrid methods using Hu-Washizu or Hellinger-Reissner theorems, etc. One procedure that has been gaining prominence in recent years is the so-called mixed interpolation or assumed strain strategy. In this approach, field-consistency is sought to be achieved by the use of special strain interpolations for the constrained strain fields as distinct from the form that will obtain if the usual strain gradient operators are applied to the kinematically admissible displacement interpolations.

The question that arises now is whether the well-known variational theorems of structural mechanics will offer guidelines as to how these independent 'assumed' strain fields have to be chosen so that both field-consistency and variational correctness (or variational consistency, as it is sometimes called) are achieved. Two recent studies[2,3] have established that a variational basis can indeed be found for the assumed strain interpolation approach and Reference 3 addresses directly the question of how field-consistency requirements are met while ensuring the variational correctness of the assumed strain formulation.

The argument is made out as follows. The field-consistency paradigm[1] recommends the form of the assumed strain functions in terms of the independent space variables such as x, y or ξ and η as the case may be, so that they are 'field-consistent'—i.e. no spurious constraints develop in the constraining limits. References 2 and 3 show how an orthogonality condition can be derived from the Hellinger-Reissner or Hu-Washizu theorem which determines the constants for the 'consistent' assumed strain field from the constants of the strain fields derived directly from the displacement fields. For a Timoshenko beam element, this can be stated in equation form as

$$\int \delta\bar{\gamma}^{\mathrm{T}}(\bar{\gamma} - \gamma)\,\mathrm{d}x = 0 \tag{1}$$

where γ is the kinematically derived shear strain field and $\bar{\gamma}$ is the field-consistent shear strain field whose constants must be constituted from the constants of γ using the orthogonality relations. For variational correctness or equivalence, the orthogonality condition demands that the assumed strain field must be orthogonal to the difference between the 'assumed' strain field and the strain field resulting from the chosen displacement field. The re-distribution procedure used to determine an admissible assumed shear strain field in the case of isotropic Mindlin plate theory is orthogonally correct if a least squares smoothing over the element domain is performed to arrive at the constants of the assumed strain function. However, if the assumed strain functions have been generated without reference to this requirement, as is the case with various recent formulations, the variational correctness is lost and spurious stress oscillations appear.[4]

The spurious oscillations appearing in this manner can be traced to the fact that such elements were based on assumed strain procedures that did not strictly achieve a variational equivalence to the Hellinger-Reissner or Hu-Washizu forms of the problems. To explore this aspect in detail, we shall investigate variationally correct (i.e. orthogonal) and incorrect (i.e. non-orthogonal) field-consistent assumed strain forms of the quadratic and cubic shear deformable beam elements and show that the non-orthogonal formulations can lead to reasonably accurate displacement solutions but have spurious stress oscillations which can be related to the presence of artificially created spurious load mechanisms. Such mechanisms can be traced to the non-orthogonality of these assumed strain fields and to the subsequent loss of an exact variational equivalence to the corresponding Hellinger–Reissner or Hu–Washizu principles.

For a quadratic beam, the artificial load is equivalent to a load system comprising a spurious uniformly distributed load and end shear forces so that the whole system is self-equilibrating. This leads to additional spurious linear oscillations in the shear force and bending moment computed by these finite element models. In a cubic beam, a distributed couple and end moments which are again self-equilibrating are generated and result in a spurious quadratic variation in bending moment but no change in the shear forces over the element.

THE QUADRATIC SHEAR FLEXIBLE BEAM ELEMENT

Reference 5 examines a shear flexible (i.e. Timoshenko) quadratic isoparametric beam element with three nodes and two degrees of freedom per node (i.e. transverse displacement w and section rotation θ) from the point of view of field-consistency alone. The conventional exactly integrated element (henceforth BM3.0) had slow convergence of displacement predictions and violent spurious quadratic oscillations in shear force predictions, and these were traced to the inconsistency in the shear strain interpolations derived directly from the displacement and section rotation interpolations using a functional re-constitution procedure. The field-consistent version (BM3.1) was free of these difficulties. This element was based on a shear strain field which was a linear interpolation derived as a least squares smoothed fit of the inconsistent quadratic interpolation. It is now clear that, in this instance, this least squares alternative is the variationally (i.e. orthogonally) correct form having an exact equivalence with the mixed theorems of Hellinger-Reissner and Hu–Washizu.[2,3]

It will now be educative to observe the results if the assumed shear strain interpolations had not satisfied the orthogonality condition. We therefore use the functional re-constitution procedure to examine various forms of the quadratic and cubic beam elements to show the effects of non-orthogonality of the assumed constrained strain fields. For completeness and continuity of presentation, a brief account of BM3.0 and BM3.1 elements, is also given here although a very elaborate description is already available.[5]

BM3.0—Field-inconsistent element

The quadratic isoparametric element is based on the following interpolations:

$$N_1 = \xi(\xi - 1)/2 \quad N_2 = (1 - \xi^2) \quad N_3 = \xi(\xi + 1)/2 \tag{2}$$

It is possible to expand the interpolations for the kinematically admissible displacement fields in terms of the nodal values and the Legendre polynomials as follows:

$$\begin{aligned} \theta &= a_0 + a_1\xi - a_2/3(1 - 3\xi^2) \\ w &= b_0 + b_1\xi - b_2/3(1 - 3\xi^2) \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_0 &= (\theta_1 + 4\theta_2 + \theta_3)/6 \quad & b_0 &= (w_1 + 4w_2 + w_3)/6 \\ a_1 &= (\theta_3 - \theta_1)/2 & b_1 &= (w_3 - w_1)/2 \\ a_2 &= (\theta_1 - 2\theta_2 + \theta_3)/2 & b_2 &= (w_1 - 2w_2 + w_3)/2 \end{aligned} \tag{4}$$

The shear strain field used in the conventional displacement type approach is obtained directly from these displacement fields by introducing the strain gradient operators as

$$\gamma^{(0)} = (a_0 - b_1/l) + (a_1 - 2b_2/l)\xi - a_2/3\,(1 - 3\xi^2) \tag{5}$$

where $2l$ is the length of the beam element.

In the Kirchhoff limit of thin beam behaviour, the constants associated with the constant and linear Legendre terms above are 'consistently' balanced and lead only to true Kirchhoff constraints. However, that associated with the quadratic Legendre polynomial comprises a single term from the θ field alone, and Reference 5 has demonstrated that it is this term that causes the poor convergence and triggers off a spurious quadratic shear force oscillation in exactly the same manner as this quadratic Legendre term. We shall briefly re-capitulate that argument here.

The strain energy of a uniform beam of length $2l$ is

$$U = 1/2 \int_{-l}^{l} EI\,\theta_{,x}^2\,dx + 1/2 \int_{-l}^{l} kGA(\theta - w_{,x})^2\,dx \tag{6}$$

where £ and G are the Young's modulus and shear modulus, / and A are the moment of inertia and the cross-sectional area and k is the shear correction factor.

After discretization, we have for the total strain energy in the element,

$$U = U_B + U_S$$

with

$$U_B = EI/l\,(a_1^2 + 4/3\,a_2^2)$$

$$U_S = kGAl\{(a_0 - b_1/l)^2 + 1/3\,(a_1 - 2b_2/l)^2 + 4/45\,a_2^2\} \tag{7}$$

It is the presence of the a_2 terms in U_S that causes inconsistency. Note that, in a consistent formulation, the a_2 term should appear only in the bending energy term. Thus, it was possible to argue in Reference 5 that, in the penalty limit of a thin beam, the shear related a_2 term acts as a spurious constraint, activating a shear related energy term that compromises the role of the a_2 term in the bending energy.

To see how the presence of the spurious a_2 term (i.e. the unwanted shear related term) disturbs the solution, we group the energy terms dependent on a_2 from equation (7) as

$$4EI/3l\,a_2^2 + 4kGAl/45\,a_2^2 \tag{8}$$

It can be shown (see Reference 5 for details) that, by taking a variation with respect to a_2, we can estimate the quantitative degree by which field-inconsistency will modify the term a_2 that should correctly describe the bending moment.

Thus, if we may write the interpolation for the bending moment in a true solution as

$$M = EI/l\,(a_1 + 2a_2\xi) \tag{9}$$

the field-inconsistent element BM3.0 will produce an answer $a_2^{(0)}$ that will depart from the correct estimate by

$$a_2^{(0)} = a_2/(1 + kGAl^2/15EI) \tag{10}$$

The bending moment sensed by BM3.0 will therefore be

$$M^{(0)} = EI/l\,(a_1^{(0)} + 2a_2^{(0)}\xi) \tag{11}$$

For a very thin beam, $kGAl^2/15EI \gg 1$, and therefore $a_2^{(0)} \rightarrow 0$; the net effect is to flatten out the linear variation and the element acts as if it can sense only a constant bending moment variation in the element. This accounts for the poor convergence (i.e. BM3.0 will converge only at the same rate as the field-consistent 2-noded linear element).

BM3.1—Orthogonally correct field-consistent element

The approach above reveals immediately what remedial action is to be taken. It is obvious that the field-consistent shear strain interpolation that is most appropriate is one that is free of the quadratic Legendre term.

Thus,

$$\gamma^{(1)} = (a_0 - b_1/l) + (a_1 - 2b_2/l)\xi \tag{12}$$

will produce an element with true quadratic convergence, as was demonstrated in Reference 5. It is also easy to demonstrate that equation (12) is the least squares fit of equation (5). Thus, the form of the equation, i.e. that it should comprise only constant and linear terms, is dictated from field-consistency considerations, and the actual values of the multipliers are derived from the constants of the original shear strain fields by using the orthogonality condition in equation (1).

It is convenient also to argue that equation (12) can be obtained by using a substitute interpolation for the section rotation 9 which is taken as the linear least squares correct fit of the original quadratic shape functions in equation (2). These are of the form

$$N_1^{(1)} = 1/6 - \xi/2 \quad N_2^{(1)} = 2/3 \quad N_3^{(1)} = 1/6 + \xi/2 \tag{13}$$

Alternatively, we can think of this element as one that can be obtained directly from the original element by using a 2-point Gaussian integration strategy for the shear strain energy and this can also be interpreted as the use of a collocation strategy sampling the substitute function 6 or 7 at $\zeta = \pm 1/\sqrt{3}$.

It is clear that the BM3.1 element will have an a_2 term appearing only in the bending energy part, unlike as seen in equation (8). There will therefore be no disturbance as far as the computed value of a_2 is concerned. In order to see clearly how a_2 will be determined for the BM3.1 element and also correlate this with the disturbances that will be seen in the stress predictions of the other inconsistent or non-orthogonal elements, it would be appropriate to include here the manner in which the field-consistent element BM3.1 will assign linear bending moment and shear force fields over each element length, and also how the fields are related to each other and to the intensity of the distributed load q acting on it.

We can write

$$\begin{aligned} M &= M_0 + M_1\xi = EI/l\,(a_1 + 2a_2\xi) \\ V &= V_0 + V_1\xi = kGA\{(a_0 - b_1/l) + (a_1 - 2b_2/l)\xi\} \end{aligned} \tag{14}$$

We also have, from considerations of equilibrium, the following statements,

$$\partial M/\partial x = V \quad \text{and} \quad \partial V/\partial x = q$$

from which we can establish the equivalences

$$M_1/l = EI/l^2\, 2a_2 = kGA\,(a_0 - b_1 l) = V_0 \tag{15a}$$

$$V_1/l = kGA/l\,(a_1 - 2b_2/l) = q \tag{15b}$$

The value of a_2 determined above gives the true slope of the bending moment curve for the BM3.1 element. In comparison, for the BM3.0 element, this equilibrium is disturbed owing to the presence of the spurious shear related a_2 term (see equation (8)), and so the slope of the predicted bending moment curve will no longer be V_0 but will be diminished by the factor seen in equation (10).

BM3.2—Non-orthogonal smoothing, collocation at $\xi = \pm 1/3$

Let us base this element on a substitute interpolation for the section rotation to be used to evaluate the shear strain energy on a collocation using the points $\xi = \pm 1/3$. The following interpolations result:

$$N_1^{(2)} = 1/18 - \xi/2 \quad N_2^{(2)} = 8/9 \quad N_3^{(2)} = 1/18 + \xi/2 \tag{16}$$

This can be expanded in terms of the nodal values and the Legendre polynomials as follows:

$$\theta^{(2)} = a_0 + a_1\xi - 2a_2/9 \tag{17}$$

where a_0, a_1 and a_2 remain as described in equation (3). We recognize the $2a_2/9$ term as the term that violates the orthogonality condition.

The assumed shear strain field is now

$$\gamma^{(2)} = (a_0 - b_1/l - 2a_2/9) + (a_1 - 2b_2/l)\xi \tag{18}$$

Consistency has not been disturbed and we would not expect to see the quadratic oscillations in the shear force found for the BM3.0 element.

The shear strain energy of a uniform beam of length $2l$ is

$$U_S = kGAl\{(a_0 - b_1/l - 2a_2/9)^2 + 1/3(a_1 - 2b_2/l)^2\}$$

Unlike the orthogonally correct and consistent BM3.1, we note that, for the BM3.2 element, the a_2 term appears again in the shear strain energy in addition to that which will come from the bending energy term. We can therefore expect that this shear related term will disturb the equilibrium in some manner. To trace this effect, we again group the energy terms dependent on a_2 as

$$4EI/3l\,a_2^2 + kGAl(a_0 - b_1/l - 2a_2/9)^2 \tag{19}$$

By taking a variation with respect to a_2, we see that the discretized equation of equilibrium corresponding to this generalized degree of freedom takes the form

$$\delta a_2\,\{(4EI/3l\,a_2 \sim 2kGAl/9(a_0 - b_1/l - 2a_2/9)\} = 0 \tag{20}$$

Comparing equation (18) with equations (12) and (14), we can establish that

$$V_0^{(2)} = kGA(a_0 - b_1/l - 2a_2/9) \tag{21}$$

We can argue from equation (15a) that, if the shear related term had not been present in equation (20), the a_2 in an undisturbed case will have sensed the equilibrium correctly as

$$a_2 = V_0 l^2/2EI = 1/2\;kGAl^2/EI\,(a_0 - b_1/l - 2a_2/9)$$

However, equation (20) now suggests that this equilibrium has actually been disturbed to the extent shown by equation (20), and the new equilibrium is therefore

$$a_2^{(2)} = V_0 l^2/2EI + 1/3\;1/2\,kGAl^2/EI\,(a_0 - b_1/l - 2a_2/9)$$
$$= (V_0 + 1/3\,V_0)l^2/2EI \tag{22}$$

This means that the non-orthogonal BM3.2 element will now predict a bending moment slope that varies not as V_0 but as 4/3 V_0! We shall see later when we take up the numerical experiments that this is indeed exactly so.

BM3.3—Non-orthogonal smoothing, collocation at ξ = + 1

The substitute interpolation for the section rotation to be used to evaluate the shear strain energy is derived from sa collocation based on the points ξ — ± 1. Therefore

$$\gamma^{(3)} = (a_0 - b_1/l - 2a_2/3) + (a_1 - 2b_2/l)\xi \qquad (23)$$

Omitting further details of the derivation, as they closely follow those described in equations (19) to (22), we can observe that field-consistency is preserved but the orthogonality condition is now violated by a term $2a_2/3$ in BM3.3 instead of the $2a_2/9$ term seen in equation (17). We can then obtain for BM3.3 the relation

$$V_0^{(3)} = kGA\,(a_0 - b_1/l + 2a_2/3) \qquad (24)$$

The superimposition of the disturbance in the equilibrium conditions caused by the presence of the non-orthogonal term on to the true slope of the bending moment yields

$$\begin{aligned} a_2^{(3)} &= V_0 l^2/2EI - 1/2\,kGAl^2/EI(a_0 - b_1/l + 2a_2/3) \\ &= (V_0 - V_0)l^2/2EI \end{aligned} \qquad (25)$$

The non-orthogonal BM3.3 element therefore predicts a bending moment slope that varies not as V_0 but as 0! We shall again see later when we take up the numerical experiments that this is indeed exactly so.

THE CUBIC SHEAR FLEXIBLE BEAM ELEMENT

The same exercise was repeated with a cubic isoparametric beam element with four nodes. We have now the conventional exactly integrated element (henceforth BM4.0), which will have cubic oscillations in shear force predictions traceable to the inconsistency in the shear strain interpolations. The field-consistent version (BM4.1) is free from this. To complete our understanding of how non-orthogonal assumed strain formulations lead to spurious load mechanisms, we shall also examine two non-orthogonal assumed strain formulations BM4.2 and BM4.3.

BM4.0—Field-inconsistent element

We directly state the interpolations for *w* and *6* in terms of the nodal values and the Legendre polynomials as

$$\begin{aligned} 0 &= \text{flo} + \text{fli*} + a_2(1 - 3\xi^2) + a_3(3\xi - 5\xi^3) \\ w &= b_0 + b_1\xi + b_2(1 - 3\xi^2) + b_3(3\xi - 5\xi^3) \end{aligned} \qquad (26)$$

where

$$a_0 = [\theta_1 + 3(\theta_2 + \theta_3) + \theta_4]/8$$

$$\text{fl}_t = [11(04 - \theta_1) + 27(\theta_3 - \theta_2)]/40$$

$$a_2 = 3[(\theta_2 + \theta_3) - (\theta_1 + \theta_4)]/16$$

$$\text{fl}_3 = 9[3(\theta_3 - \theta_2) - (\theta_4 - \theta_1)]/80$$

etc.

The shear strain derived from these definitions is

$$\gamma^{(0)} = [a_0 - (b_1 - 2b_3)/l] + (a_1 + 6b_2/l)\xi + (a_2 - 5b_3/l)(1 - 3\xi^2) + a_3(3\xi - 5\xi^3) \tag{27}$$

Only the constant associated with the cubic Legendre polynomial originates solely from the 0 field and this triggers off a spurious cubic shear force oscillation. It is this term that would appear in the non-orthogonal formulations as well.

BM4.1—Orthogonally correct field-consistent element

The orthogonally correct field-consistent shear strain interpolation can be obtained by discarding the cubic Legendre term.

Thus,

$$\gamma^{(1)} = [a_0 - (b_1 - 2b_3)/l] + (a_1 + 6b_2/l)\xi + (a_2 - 5b_3/l)(1 - 3\xi^2) \tag{28}$$

After discretization, we have for the total strain energy in the element,

$$U = U_B + U_S$$

with

$$U_B = EI/l\{(a_1 - 2a_3)^2 + 12a_2^2 + 20a_3^2\}$$
$$U_S = kGAl\{[a_0 - (b_1 - 2b_3/l)]^2 + 1/3(a_1 + 6b_2/l)^2 + 4/5(a_2 - 5b_3/l)^2\} \tag{29}$$

Note that a_3 appears only in the bending energy term. To see how terms like a_3 are resolved in the context of a finite element representation, we can write

$$M = M_0 + M_1\xi + M_2(1 - 3\xi^2) = EI/l[(a_1 - 2a_3) - 6a_2\xi + 5a_3(1 - 3\xi^2)]$$
$$V = V_0 + V_1\xi + V_2(1 - 3\xi^2) = kGA\{[a_0 - (b_1 - 2b_3)/l] + (a_1 + 6b_2/l)\xi + (a_2 - 5b_3/l)(1 - 3\xi^2)\} \tag{30}$$

These are the descriptions of quadratic order that are possible with the cubic beam element.

From the statements of equilibrium,

$$\partial M/\partial x = V \quad \text{and} \quad \partial V/\partial x = q$$

we can establish the equivalence between those constants that will play a crucial role later as

$$-6M_2/l = -30EI/l^2\, a_3 = kGA(a_1 + 6b_2/l)$$
$$V_1/l = kGA/l(a_1 + 6b_2/l) = q \tag{31}$$

and therefore

$$a_3^{(1)} = -l^3/30EI\,q \tag{32}$$

This gives the value of the quadratic variation in the bending moment over the length of a beam element acted on by a uniformly distributed load of intensity q. It is this term that will change dramatically if the assumed strain function had not been consistent (as in BM4.0) or meets the consistency requirements but does not satisfy the orthogonality conditions (as we will see for BM4.2 and BM4.3 later).

BM4.2 — Non-orthogonal smoothing, collocation at $\xi = 0, \pm 1/2$

Following the same procedure adopted for the quadratic elements earlier, the shear strain derived from these definitions is

$$\gamma^{(2)} = [(a_0 - (b_1 - 2b_3)/l] + (a_1 + 7a_3/4 + 6b_2/l)\xi \\ + (a_2 - 5b_3/l)(1 - 3\xi^2) \tag{33}$$

and the shear strain energy is

$$U_S = kGAl\{[a_0 - (b_1 - 2b_3/l)]^2 \\ + 1/3(a_1 + 7a_3/4 + 6b_2/l)^2 + 4/5(a_2 - 5b_3/l)^2 \tag{34}$$

Thus, for the BM4.2 element, the a_3'term appears again in the shear strain energy in addition to that which will come from the bending energy term. As before for the BM3.2 element, we can therefore expect that this shear related term will disturb the equilibrium. By grouping together the energy terms dependent on a_3 and taking a variation with respect to a_3, we can see that

$$_{fl}<_3 2> = _{fl}u> - 1\, ql^3/240EI \\ = -ql^3/30EI - 1\, ql^3/240EI \tag{35}$$

Thus, the non-orthogonal term introduces an additional spurious quadratic variation in the bending moment prediction which can be described in terms of the true bending moment sensed by the BM3.1 element as

$$M^{(2)} = M^{(1)} - 7\,ql^2/48(1 - 3\xi^2) \tag{36}$$

We shall show that this accounts for the behaviour of this element very accurately when we take up some numerical exercises later.

BM4.3 — Non-orthogonal smoothing, collocation at $\xi = 0, \pm 1$

A collocation using the points $\xi = 0, \pm 1$ will lead to shear strains

$$\gamma^{(3)} = [(a_0 - (b_1 - 2b_3)/l] + (a_1 - 2a_3 + 6b_2/l)\xi \\ + (a_2 - 5b_3/l)(1 - 3\xi^2) \tag{37}$$

and an $a_3^{(3)}$ which will change due to violation of orthogonality by

$$a_3^{(3)} = a_3^{(1)} + ql^3/30EI \\ = -ql^3/30EI + ql^3/30EI \tag{38}$$

Thus, the non-orthogonal term introduces an additional spurious quadratic variation in the bending moment prediction which can be described in terms of the true bending moment sensed by the BM3.1 element as

$$M^{(3)} = M^{(1)} + ql^2/6(1 - 3\xi^2) \tag{39}$$

Note that equation (38) predicts that the true quadratic variation is neutralized by this spurious load and therefore equation (39) will give only a linear variation for $M^{(3)}$. We shall show that this accounts for the behaviour of this element very accurately when we take up some numerical exercises later.

NUMERICAL EXPERIMENTS

We shall choose the simplest test cases that can demonstrate the essential principles involved behind our studies here. For this purpose, the cantilever beam of length L, cross-sectional area A, moment of inertia /, is considered to be acted upon by either a concentrated tip shear force Q or a uniformly distributed load of intensity q. The beam is divided into 1, 2 or 4 elements of equal length.

Experiments with BM3 elements

Test case 1a. Cantilever beam under tip shear force Q. Figure 1 shows the bending moment distribution obtained from using a one-element idealization using the various forms of the quadratic element. BM3.1 captures the correct linear bending moment variation. BM3.2 shows a line with a slope increased by one-third, confirming the prediction projected by equation (22) which we have derived using the functional re-constitution exercise. BM3.3 is unable to recognize

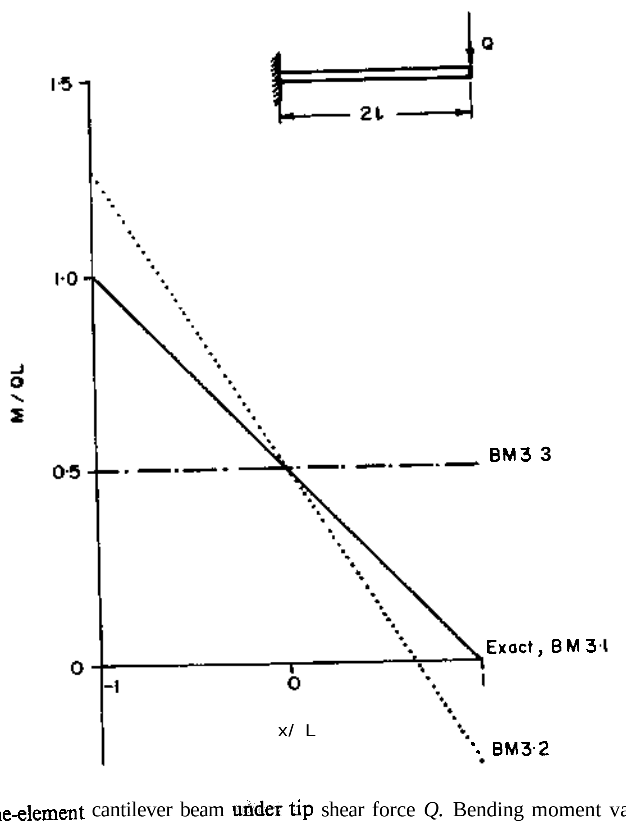

Figure 1. One-element cantilever beam under tip shear force Q. Bending moment variation (BM3)

that the bending moment should be varying linearly and instead gives a constant value over the entire element length corresponding to the correct value at the centroid, thereby confirming the prediction made in equation (25). Figure 2 shows that a two-element idealization produces the same trends. The shear force distributions (not shown here) are correct for all the three field-consistent elements, i.e. BM3.1, BM3.2 and BM3.3, irrespective of whether they are orthogonally correct or not. Of course, the field-inconsistent BM3.0 yields quadratic shear force oscillations (not shown here, but see Reference 5).

Test case 1b. Cantilever beam under uniformly distributed load of intensity q. Figure 3 shows the convergence of the dimensionless tip deflections w and tip rotations θ for 1, 2 and 4 elements in the beam. It is clear that only the orthogonally correct field-consistent element BM3.1 gives exact answers, even with one element. However, all element versions converge rapidly and the ill-effects of lack of consistency (BM3.0) or lack of non-orthogonality of the assumed shear strain interpolations (BM3.2 and BM3.3) are disguised.

Figure 4 shows the shear force variations for the three field-consistent elements. Again, BM3.1 can correctly pick up the linear variation which is the natural result for such a loading. BM3.2 and BM3.3 show changes in the slope of the shear force variation by + 33 1/3 and — 100 per cent respectively. It is seen that these factors are curiously the same as those changing the slopes of the moment variation (equations (22) and (25) and also Figures 1 and 2). The shear forces at *the* centroid of the element are always correct!

It is necessary to account for this in some way, and the only plausible explanation seems to be that this is the price to pay for not conserving orthogonality (or variational equivalence to the

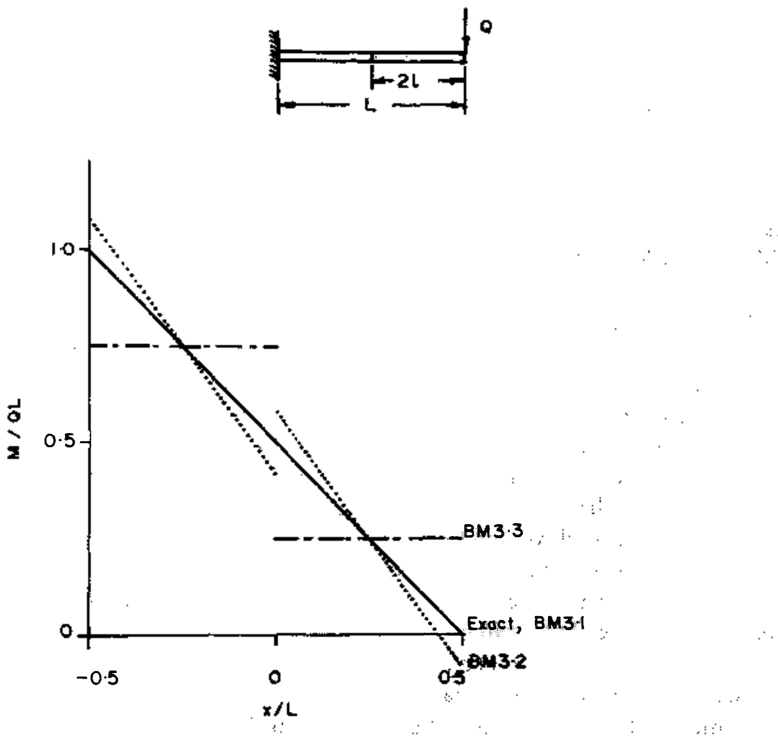

Figure 2. Two-element cantilever under tip shear force Q. Bending moment variation (BM3).

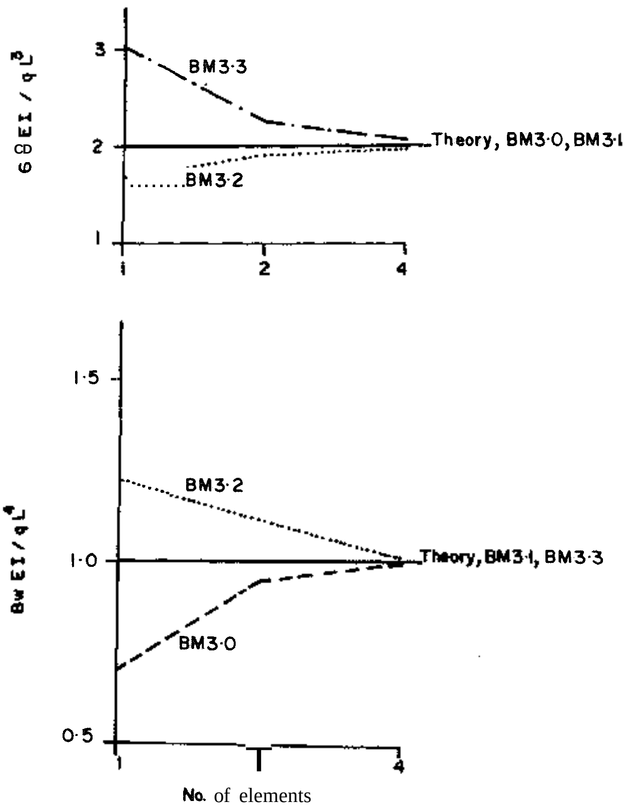

Figure 3. Convergence of tip deflections and rotations for cantilever beam under uniformly distributed load (BM3)

Hu-Washizu or Hellinger-Reissner variational theorems) when the assumed shear strains for the BM3.2 and BM3.3 elements were designed. We argue that a spurious load mechanism has been introduced as a result of the violation of this condition. We can work out the form of this mechanism as follows.

Spurious load mechanism in BM3.2 and BM3.3 elements. The mechanism must be such that the responses seen in test cases la and 1b can be rationalized. From equation (15b), we know that for the orthogonally correct field-consistent element BM3.1, the slope of the shear force predictions reflects the true distributed loading acting on it. However, we know as an *a posteriori* fact from our numerical studies above that the BM3.2 elements actually senses an additional spurious load of intensity *q/3* as well. Within the scope of the engineering level of analysis we have used so far, we are unable to describe this phenomenon in as rigorous an analytical basis as we could do for the way the bending moment slope was changed by using functional re-constitution procedures. We shall, however, proceed by accepting this a *posteriori* determined fact to describe the behaviour of the non-orthogonal assumed strain formulation. We attribute this spurious load mechanism to the fact that variational equivalence has been violated by choosing an assumed shear strain field $\gamma^{(2)}$ that was not orthogonal to the difference $(y \sim \gamma^{(2)})$ as indicated by recent studies on the variational basis for the assumed strain methods.[2,3]

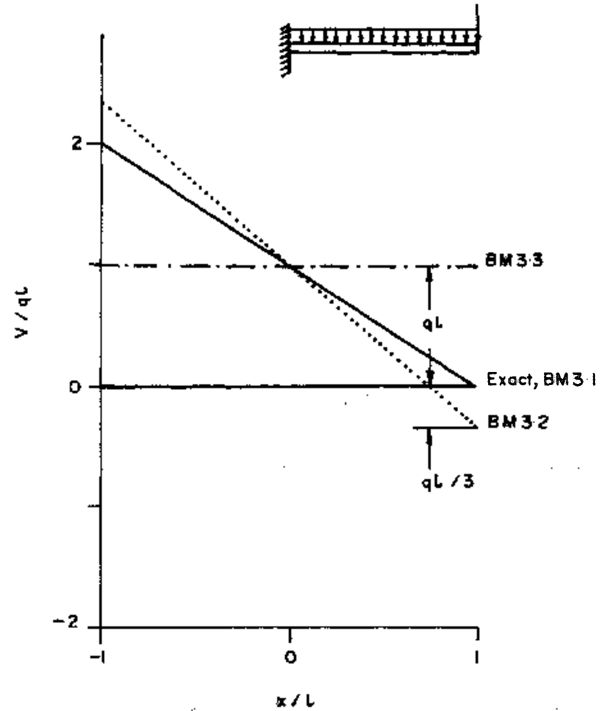

Figure 4. One-element cantilever beam under uniformly distributed loads. Shear force variation (BM3)

Figure 5 shows how the load mechanism operates on a single-element beam. BM3.1 only the true distributed load q acting on it. Correspondingly, it integrates the shear force correctly to give at the centroid, the values

$$V_0^{(1)} = ql$$
$$M_0^{(1)} = 2ql^2/3 \tag{40}$$

The BM3.2 element senses, in addition to this, an artificially activated load system comprising a distributed system $q/3$ and two end forces $ql/3$ so that the entire spurious mechanism is self-equilibrating. On the basis of this projected representation, we can argue that the BM3.2 element will integrate these loads to give the following values of shear force and bending moment at the centroid:

$$V_0^{(2)} = ql$$
$$M_0^{(2)} = M_0^{(1)} - 1/3\ ql^2/3 \tag{41}$$

We shall see later that this is exactly consistent with the actual predictions obtained with a finite element model using the BM3.2 element

Corresponding to the mechanism worked out thus, the slope of the shear force line is no longer q as in equation (15b) but is now

$$V_1^{(2)}/l = q + q/3$$

As before, our numerical studies are made on the action of a uniformly distributed load f on a

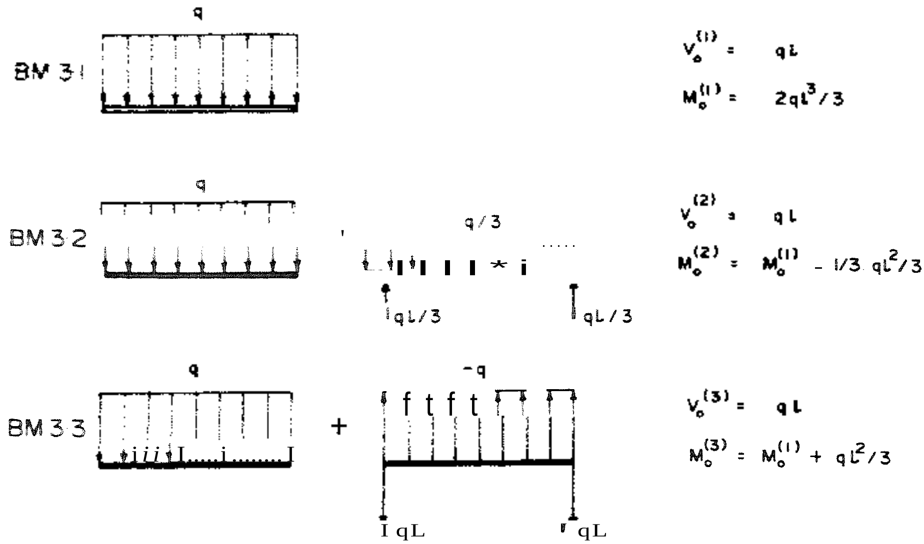

Figure 5. Spurious load mechanisms in BM 3.2 and BM 3.3 elements

cantilever beam modelled with BM3.3 elements and we notice that they actually sense an additional spurious load of intensity $-q$ as well. Figure 5 shows how the load mechanism operates on a single-element beam. The BM3.3 element senses in addition to the true distributed load q, an artifically activated load system comprising a distributed system $-q$ and two end forces $-ql$ so that the entire spurious mechanism is self-equilibrating. On this basis, we can argue that the BM3.3 element will integrate these loads to give the following values of shear force and bending moment at the centroid:

$$V_0^{(3)} = q \qquad M_0^{(3)} = M_0^{(1)} + 1/3\, ql^2/3 \tag{42}$$

We shall see now that this is exactly consistent with the actual predictions obtained with a finite element model using the BM3.2 element. The slopes of both the shear force and bending moment distributions sensed by this element are now identically zero.

Figure 6 shows the bending moment distributions obtained over the length of a single element used to model the cantilever beam with uniformly distributed load q (test case 1b). The results confirm that the load mechanisms are as shown in Figure 5 and that the predictions made by equations (41) and (42) exactly describe this spurious load mechanism.

Experiments with BM4 elements

Test case 2a. Cantilever beam under tip shear force Q. All versions of the BM4 element, i.e. BM4.0 to BM4.3, irrespective of lack of consistency or lack of orthogonality or not, yield exact deflections, rotations, bending moments and shear forces for the case of a cantilever beam with tip shear force Q, even when only one element is used. This can be explained by the fact that the parameter a_3 which is excited when inconsistency or non-orthogonality is present represents the action of the uniformly distributed load q and can only be excited by the presence of such a load.

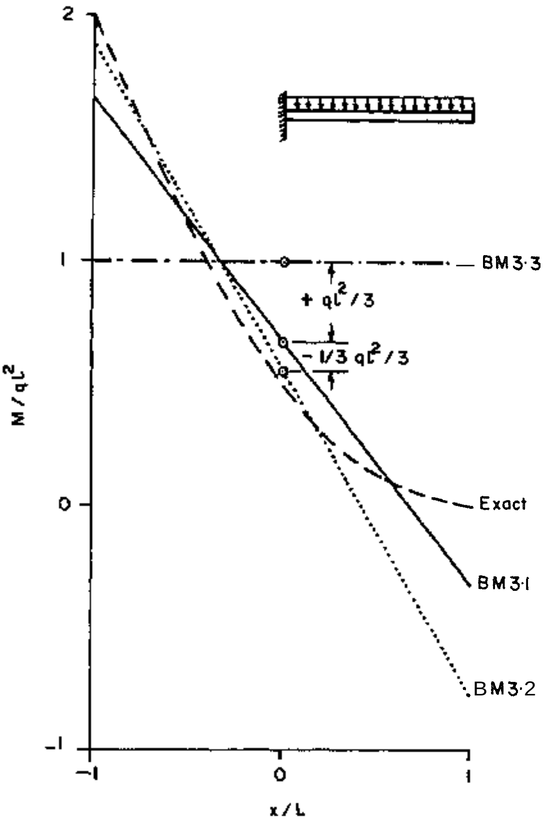

Figure 6. One-element cantilever beam under uniformly distributed load. Bending moment variation (BM3)

Test Case 2b. Cantilever beam under uniformly distributed load of intensity q. It is under the action of uniformly distributed load q that the spurious disturbances are activated. The BM4.0 element reveals cubic shear force and quadratic bending moment oscillations which are superimposed on the correct quadratic variation (not shown here). All three field-consistent elements, i.e. BM4.1 to BM4.3, give correct shear forces—i.e. they can pick up the linear variation without error.

Figure 7 shows the bending moment distribution picked up by a single-element model. While BM4.1 picks up the exact quadratic variation, BM4.2 and BM4.3 are both in error, by precisely the terms reflected in equations (36) and (39). We can again argue that these can be physically equated to the presence of a spurious distributed couple loading of intensities $M(\xi) = 7ql^2\xi/8$ and $-ql^2\zeta$ for BM4.2 and BM4.3 respectively with additional concentrated moments at the end nodes for self-equilibrium, as shown in Figure 8.

CONCLUSIONS

We have used the example of quadratic and cubic Timoshenko beam elements to demonstrate the need to ensure that the field-consistent assumed strain interpolations for the constrained strain fields (in this case, the shear strains must vanish in the Kirchhoff sense) must also satisfy an important orthogonality condition to ensure variational correctness (also described sometimes as

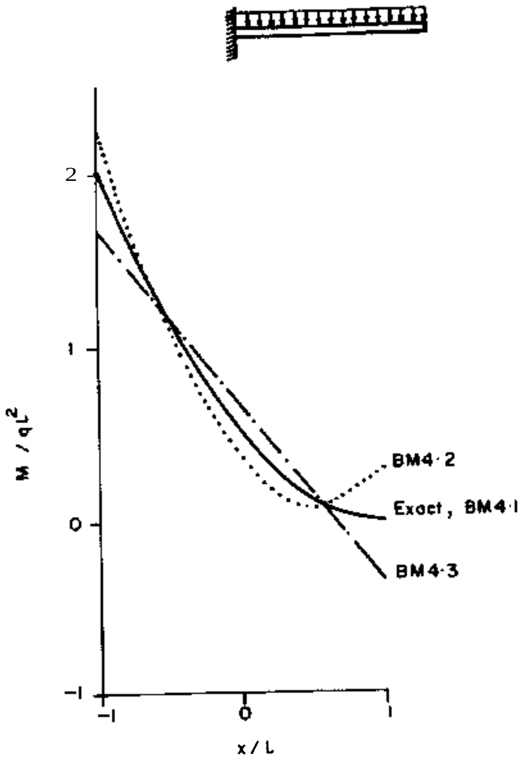

Figure 7. One-element cantilever beam under uniformly distributed load. Bending moment variation (BM4)

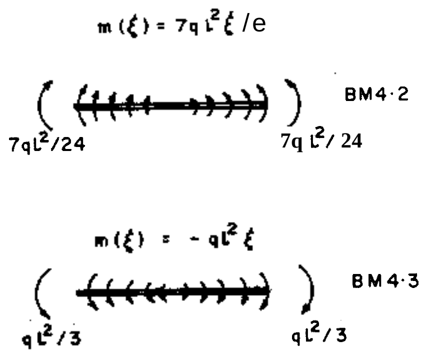

Figure 8. Spurious load mechanisms in BM4.2 and 4.3 element

variational consistency). If this is violated, the variationally inconsistent element will show spurious stress oscillations because of spurious loading mechanisms which originate from the violation of the variational equivalence required of the assumed strain displacement method to the mixed methods. This is true for more general situations as well, as some recent studies of the assumed strain 8-node plate elements show.[4]

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