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CONSISTENT FORCE RESULTANT DISTRIBUTIONS IN DISPLACEMENT ELEMENTS WITH VARYING SECTIONAL PROPERTIES

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SUMMARY

Force fields computed directly from strains calculated in a displacement type finite element description of a structural element of varying sectional rigidities show extraneous oscillations. The origin of these oscillations is traced to the fact that the displacement type finite element procedure determines strains derived from the displacement field in a least squares correct sense and that force resultants computed using these strain fields and the actual sectional rigidities result in unwanted oscillations. It is necessary to introduce the concept of redistributed assumed force resultant fields that maintain a 'consistent' relationship to the strain fields and also are orthogonal to these strain functions. In this paper, the Hu–Washizu theorem is invoked to justify the introduction of an orthogonally correct reconstituted assumed force resultant field which will then be free of extraneous oscillations. The quadratic isoparametric tapered bar element serves to illustrate the underlying principles.

It follows that the extremely general Hu–Washizu principle is the most practical procedure of implementing an assumed force resultant, assumed strain displacement type formulation to introduce consistency and thereby remove problems associated with field-inconsistency (such as cause locking in constrained media elasticity) and force resultant oscillations due to varying sectional properties.

INTRODUCTION

Conventional displacement type formulations using exact integration of all strain energies based on strains derived directly from kinematically admissible displacement fields lead to the phenomenon of locking and stress oscillations in problems where constrained multi-strain fields are present.[1] Many *ad hoc* practices, e.g. reduced integration, addition of bubble modes, assumed strain methods, mode decomposition, etc., alleviate this problem. Recently, a variational basis for such procedures has emerged.[2–5]

Prathap[6] showed that the variationally correct way to formulate the class of constrained media problems so that locking, poor convergence and stress oscillations due to field-inconsistency are removed is to identify the terms of the assumed strain field so that only consistent terms are retained and to determine the constants associated with these terms in the 'assumed' strain field from the constants in the strain field derived directly from the displacement fields using an orthogonality condition that emerges from the Hellinger–Reissner formulation. This derived assumed strain field can be introduced directly into the variational indicator for the minimum total potential principle and an integration of the energies based on these strains will yield stiffness matrices that are free of all field-inconsistencies.

In this paper, we are interested in a related theme pertaining to the evaluation of force or stress resultant fields from a displacement type formulation where the rigidities vary over the element

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volume. We shall use the very simple example of a tapered quadratic bar element to show that the direct use of strains derived from displacement fields leads to extraneous oscillations in the final computed stress or force resultant fields. We show that there is a 'consistent' level up to which force fields can be represented and that the stiffness matrix obtained by the congruent transformations can reflect energies only from this 'consistent' part of the description. Therefore, the direct multiplication of varying sectional rigidities with kinematically derived strain fields retains higher order 'inconsistent' terms which do not contribute to the stiffness matrix but get reflected as extraneous force oscillations if forces are computed from this. The Hu–Washizu theorem provides the variational justification for reconstituting the 'consistent' assumed force resultant field from the 'inconsistent' kinematically derived force field. A simple device of expansion of the latter by Legendre polynomials and suitable truncation yields a 'consistent' representation that satisfies the orthogonality condition imposed by the Hu–Washizu theorem.

The simple example of a tapered quadratic isoparametric bar element allows error estimates to be made which can be verified by numerical exercises. These confirm the arguments presented in this paper relating to the need to represent force fields in a consistent manner so that extraneous force oscillations can be eliminated. These findings will have important extensions to tapered plate and shell formulations.

TAPERED QUADRATIC ISOPARAMETRIC BAR ELEMENT—FORMULATION

Minimum total potential energy principle

In a formulation based on the minimum total potential energy principle, the stiffness matrix is derived from a variational indicator written as

$$\pi = 1/2 \int N^{\mathrm{T}} \varepsilon \, \mathrm{d}x - W \tag{1}$$

where $\varepsilon = \mathrm{d}u/\mathrm{d}x$ is the axial strain, $N = EA(x)\varepsilon$ is the kinematically constituted axial force, $W = \int pu \, \mathrm{d}x$ is the potential of external forces, u the axial displacement, p the distributed axial load, E Young's modulus of elasticity and $A(x)$ is the varying cross-sectional area.

In a quadratic element of length $2l$ with the mid-node exactly at the mid-point of the element, the following interpolations can be made in terms of nodal values of x, u and A:

$$x = x_2 + (x_3 - x_1)/2 \ \xi$$

$$u = u_2 + (u_3 - u_1)/2 \ \xi + (u_1 - 2u_2 + u_3)/2 \ \xi^2$$

$$A = A_2 + (A_3 - A_1)/2 \ \xi + (A_1 - 2A_2 + A_3)/2 \ \xi^2$$

We shall now examine how the congruent transformation implied by $N^{\mathrm{T}}\varepsilon$ and its integration over the element volume takes place. We compute the kinematically derived ε and N as

$$\varepsilon = (u_3 - u_1)/2l + (u_1 - 2u_2 + u_3)/l \ \xi \tag{2}$$

$$N = EA\varepsilon$$

$$= N_1 + N_2\xi + N_3(1 - 3\xi^2) + N_4(3\xi - 5\xi^3) \tag{3}$$

where

$$N_1 = E[A_2(u_3 - u_1)/2l + (A_3 - A_1)/6 \ (u_1 - 2u_2 + u_3)/l$$

$$+ (A_1 - 2A_2 + A_3)/6 \ (u_3 - u_1)/2l] \tag{4a}$$

$$N_2 = E[A_2(u_1-2u_2+u_3)/l+(A_3-A_1)/2\,(u_3-u_1)/2l$$
$$+3(A_1-2A_2+A_3)/10\,(u_1-2u_2+u_3)/l] \quad (4b)$$

$$N_3 = -E[(A_3-A_1)/2\,(u_1-2u_2+u_3)/l$$
$$+(A_1-2A_2+A_3)/2\,(u_3-u_1)/2l]/3 \quad (4c)$$

$$N_4 = -E[(A_1-2A_2+A_3)/2\,(u_1-2u_2+u_3)/l]/5 \quad (4d)$$

and where we have carefully expanded the axial force resultant in terms of the Legendre polynomial forms for reasons which will become transparent soon.

The strain energy of deformation is then expressed as

$$U = 1/2\int N^{\mathrm{T}}\varepsilon\,\mathrm{d}x$$

Owing to the orthogonal nature of the Legendre polynomials it emerges that N_3 and N_4 will not contribute to the energy and therefore not to the stiffness matrix! Thus,

$$U = [N_1(u_3-u_1)/2l + N_2/3(u_1-2u_2+u_3)/l]l$$

Here, we see clearly a pointer to how the 'consistent' representation of the force field denoted by $\bar{N}$ must be made—it should comprise only the terms that will contribute to the stiffness and strain energy and the simplest way to do this was to expand the kinematically determined N in terms of Legendre polynomials, retaining only terms that will meaningfully contribute to the energy in $N^{\mathrm{T}}\varepsilon$. Thus, $\bar{N}$ must be 'consistent' with ε, i.e. in this case, retain only up to linear terms, i.e.

$$\bar{N} = N_1 + N_2\xi \quad (5)$$

If this rule had not been observed, then the force field computed directly using axial displacements obtained in a finite element solution will have extraneous force oscillations due to the N_3 and N_4 terms described in equations (3) and (4c) and (4d)! To see the variational basis for the novel procedure adopted so far, we shall introduce the formulation of this problem according to the Hu–Washizu principle.

Hu–Washizu principle

In forming the Hu–Washizu functional for the total potential, an as yet undetermined assumed force function $\bar{N}$ is introduced, but the assumed strain field $\bar{\varepsilon}$ can be safely retained as ε without any loss of flexibility in this instance (note that, in a constrained media problem,[6] it will be required to introduce a field-consistent $\bar{\varepsilon}$ that will be different from the kinematically derived and therefore usually field-inconsistent ε). The variational indicator now becomes

$$\pi = \int\{1/2(EA\bar{\varepsilon})^{\mathrm{T}}\bar{\varepsilon} + \bar{N}(\varepsilon-\bar{\varepsilon})\}\,\mathrm{d}x - W \quad (6)$$

A variation of the Hu–Washizu energy functional with respect to the kinematically admissible degree of freedom u gives the equilibrium equation

$$\int\delta u\cdot\{\mathrm{d}\bar{N}/\mathrm{d}x - p\}\,\mathrm{d}x = 0 \quad (7)$$

Variation with respect to the assumed strain field ε gives rise to a constitutive relation

$$\int \delta\bar{\varepsilon} \cdot \{ -\bar{N} + EA\bar{\varepsilon} \} \mathrm{d}x = 0 \tag{8}$$

and variation with respect to the assumed force field $\bar{N}$ gives rise to the condition

$$\int \delta\bar{N} \cdot \{ \varepsilon - \bar{\varepsilon} \} \mathrm{d}x = 0 \tag{9}$$

We observe that equations (8) and (9) are curious orthogonality conditions. If $\bar{\varepsilon}$ is taken to be identical to ε, equation (9) is identically satisfied. We now turn our attention to equation (8). We note that this can be rewritten as

$$\int \delta\bar{\varepsilon} \cdot \{ -\bar{N} + N \} \mathrm{d}x = 0 \tag{10}$$

If N is expanded in terms of Legendre polynomials, it can be proved that $\bar{N}$, which is 'consistent' and orthogonally satisfies equation (10), is obtained very simply by retaining all the Legendre polynomial terms that are 'consistent' with $\bar{\varepsilon}$, i.e. as shown in equation (5). Thus the procedure adopted has a variational justification according to the Hu–Washizu principle.

NUMERICAL EXPERIMENTS

We shall perform computational exercises with two versions of the element:

- BAR3.0 is the conventional element using N for stiffness matrix evaluation and recovery of the force resultant;
- BAR3.1 is the 'consistent' element using $\bar{N}$ for stiffness matrix evaluation and recovery of the force resultant.

Note that, in both cases, the stiffness matrices and computed displacements are identical.

The numerical experiments will be based on a single element representation of a quadratic bar. Figure 1 shows a tapered bar clamped at node 1 and subjected to an axial force P at node 3. The taper is defined by the parameters

$$\alpha = (A_3 - A_1)/2A_2 \quad \text{and} \quad \beta = (A_1 - 2A_2 + A_3)/2A_2$$

It is possible now to analytically simulate a single element finite element computation of this problem by identifying generalized displacement parameters

$$a_1 = (u_3 - u_1)/2l$$
$$a_2 = (u_1 - 2u_2 + u_3)/l$$

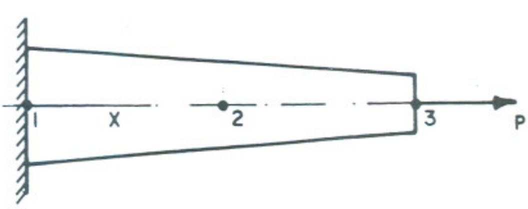

Figure 1. Single element cantilever bar

and, noting that $u_1 = 0$, one can establish that

$$a_1 = (P/EA)\frac{(1 + 3\beta/5)}{(1 + 3\beta/5)(1 + \beta/3) - \alpha^2/3}$$

$$a_2 = (P/EA)\frac{-\alpha}{(1 + 3\beta/5)(1 + \beta/3) - \alpha^2/3}$$

Substituting into equation (3), one obtains

$$N = P - P/3(1 - 3\xi^2)\frac{(\beta(1 + 3\beta/5) - \alpha^2)}{(1 + 3\beta/5)(1 + \beta/3) - \alpha^2/3}$$

$$- P/5(3\xi - 5\xi^3)\frac{(-\alpha\beta)}{(1 + 3\beta/5)(1 + \beta/3) - \alpha^2/3} \qquad (11)$$

The constant and linear components (i.e. $N_1 = P$ and $N_2 = 0$) are correctly recovered. However, there are noticeable quadratic and cubic oscillations for a general quadratic taper (i.e. $\alpha \neq 0$ and $\beta \neq 0$) if force resultants are computed directly from equation (3). However, the 'consistent' force resultant field yields the correct forces.

Case a—Linearly tapered single cantilever bar element

We shall verify the above model by comparison with the actual finite element results from a computational exercise using the two versions described above for a bar with cross section tapering linearly from the root to the tip. Thus α varies from 0 to $-1{\cdot}0$ while $\beta = 0$. Figure 2

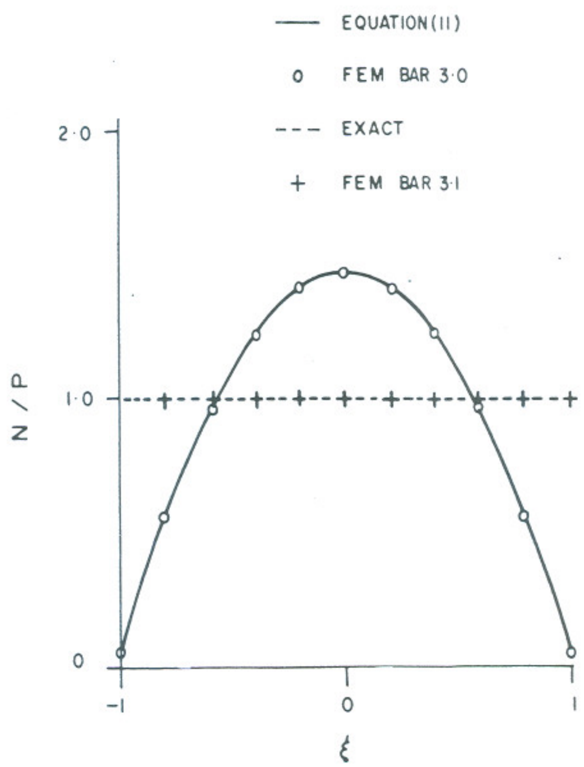

Figure 2. Axial force pattern for linearly tapered bar with $\alpha = 0{\cdot}9802$ ($A_3 = 0{\cdot}01A_1$)

shows the axial force patterns obtained from the finite element digital computation for a case with $\alpha = -0.9802$, and these are seen to coincide with the analytically deduced expression given in equation (11).

Figure 3 illustrates the manner in which the magnitude of the quadratic oscillations in the kinematically derived force resultant fields varies with taper. The taper is now described in the fractional sense as $A^* = (A_1 - A_3)/A_1$. If $N^* = (N' - P)/P$, where N' is the value at $\xi = 0$, then from equation (11), with $\beta = 0$, we get

$$N^* = \alpha^2/(3 - \alpha^2) \tag{12}$$

The figure compares this analytically simulated relationship with that obtained from the finite element digital computations. The agreement is seen to be very good.

Case b—Linear and quadratically varying taper giving rise to cubic force oscillations

We choose a case where $\beta(1 + 3\beta/5) = \alpha^2$ by computing the cross-sectional areas necessary for this as $\alpha = -1.0639411$ and $\beta = 0.7732352$. From equation (11) we see that the quadratic oscillations disappear, leaving behind the cubic oscillations. A finite element computation is performed with a single element cantilever bar with a tip force P and the results are presented in Figure 4. The finite element results agree exactly with those computed from equation (11) based on the analytical simulation of the present problem.

Case c—Linear and quadratically varying taper giving rise to quadratic and cubic force oscillations

We examine a general case of taper with $A_1 = 1.0$, $A_2 = 0.36$ and $A_3 = 0.04$. The area ratios are $\alpha = -4/3$ and $\beta = 4/9$. The analytical simulation predicts that the axial force computed by the BAR3.0 element will be

$$N/P = 1 + 0.4699152(1 - 3\xi^2) - 0.13375358(3\xi - 5\xi^3) \tag{13}$$

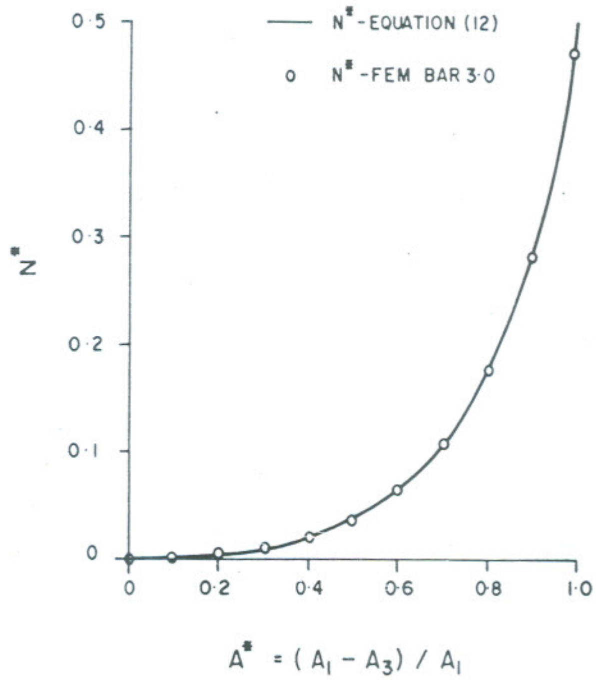

Figure 3. Magnitude of quadratic oscillation in linearly tapered bar

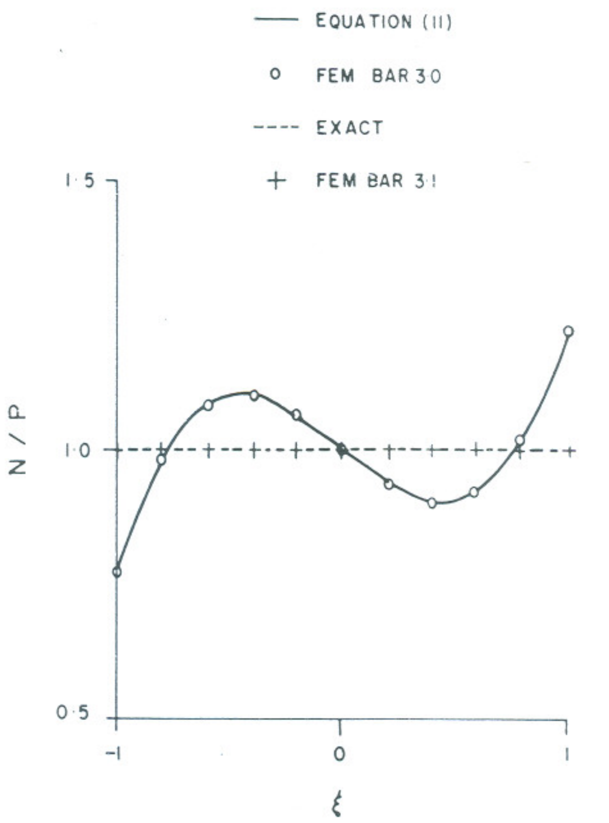

Figure 4. Axial force pattern for bar with combined linear and quadratic taper ($\alpha = -1{\cdot}0639411$, $\beta = 0{\cdot}7732352$)

Figure 5 shows the results from the finite element computations and those predicted by equation (13). Owing to the presence of the cubic oscillations, the Barlow points ($\xi = \pm 1/\sqrt{3}$) are no longer points of accurate force recovery! It is necessary to perform a reconstitution of the force resultant fields on a consistency basis as is done here before reliable force recovery can be made.

CONCLUSIONS

In this paper, we have introduced another variation on the 'field-consistency' theme. Earlier studies, reviewed in Reference 1, showed that in a class of problems recognised as constrained multi-strain-field problems, a certain 'consistency' in the strain-field definitions was required so that only 'true' or physically relevant constraints emerged in the penalty regimes of constraining of the concerned strain fields. It was seen that the Hellinger–Reissner principle[6] offered a means to allow flexibility in designing the assumed strain field such that the desired 'consistency' of the strain to displacement relationship was achieved.

The present studies demonstrate that there is another class of problems, namely where the sectional rigidities vary, where a further 'consistency' requirement, that of the constitutive relationship, comes into play. There is now a determinable stress resultant field that must satisfy a 'consistency' equilibrium with the strain field derived using the gradient operators on the kinematically admissible displacement fields. The more general Hu–Washizu theorem must now be introduced to admit flexibility of design of the 'consistent' stress resultant field.

We have derived a rule to construct variationally consistent interpolations for the force resultant functions for instances where the elastic medium to be modelled by displacement type

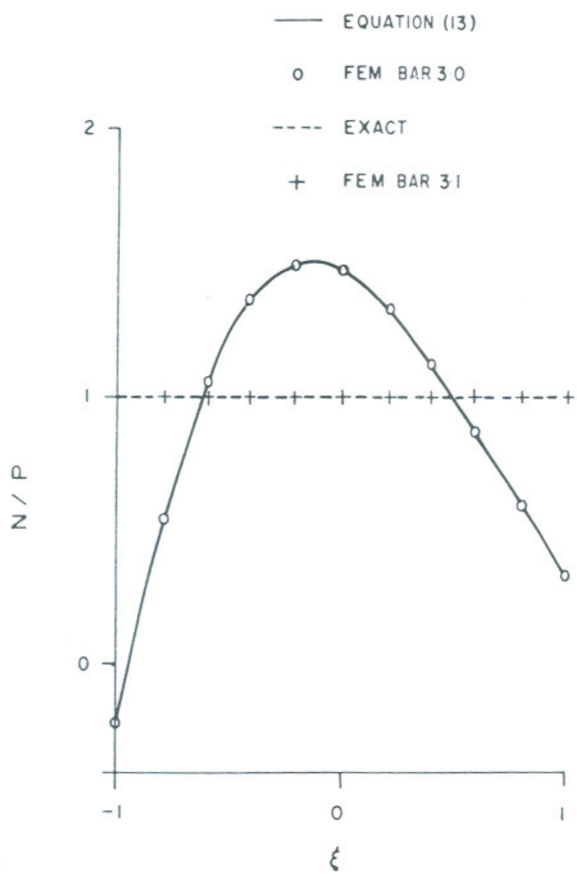

Figure 5. Axial force pattern for bar with combined linear and quadratic taper ($\alpha = -4/3$, $\beta = 4/9$)

finite elements has varying sectional rigidities. In such cases, use of kinematically computed force resultant fields leads to additional spurious oscillations in stress recovery. The Hu–Washizu principle forms the basis for the procedure adopted to reconstitute the 'consistent' force resultant field from that obtained from the kinematically admissible displacement fields. A tapered quadratic isoparametric bar element is used to demonstrate the basic principles involved.

The findings have an important bearing on the performance in the prediction of force and stress resultants of structural elements such as tapered curved beams, plates and shells, and work on these is underway and will be reported separately.

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REFERENCES

1. G. Prathap, 'Field-consistency—Toward a science of constrained multi-strain-field finite element formulation', *SADHANA*, **9**, 345–353 (1986).
2. J. C. Simo and T. J. R. Hughes, 'On the variational foundations of assumed strain methods', *J. Appl. Mech. ASME*, **53**, 51–54 (1986).

3. H. Stolarski and T. Belytschko, 'On the equivalence of mode decomposition and mixed finite elements based on the Hellinger-Reissner principle. Part I—Theory', *Comp. Methods Appl. Mech. Eng.*, **58**, 249–263 (1986).
4. H. Stolarski and T. Belytschko, 'On the equivalence of mode decomposition and mixed finite elements based on the Hellinger-Reissner principle. Part II—Applications', *Comp. Methods Appl. Mech. Eng.*, **58**, 265–284 (1986).
5. D. Karaminlidis and R. Jasti, 'On the variational foundations of the substitute shape function technique', *Comp. Struct.*, **26**, 1041–1042 (1987).
6. G. Prathap, 'The variational basis for least squares field-re-distribution of strain functions in the finite element formulation of constrained media elasticity', *TM ST8801*, National Aeronautical Laboratory, Bangalore, India, 1988.