THERMAL STRESSES AROUND AN INSULATED CRACK IN AN INFINITE PLATE SUBJECTED TO A UNIFORM HEAT FLOW

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In an earlier analysis Sih [1] presented a solution for stresses around the tip of an insulated arbitrary crack in an infinite thin plate subjected to a uniform heat flow using the complex variable method. Solutions obtained by Florence and Goodier [2] to the problem of an insulated elliptic hole in a similar plate subjected to a uniform heat flow provided the basis for the crack solution.

In this paper an alternative formulation is presented for determining the state of thermal stresses in the neighbourhood of an insulated crack in an infinite thin plate subjected to a uniform heat flow. Both the steady state temperature and stress problems are formulated in an elliptic coordinate system. The method developed in respect of the latter involves determining the solution completely in elliptic coordinates using biharmonic functions and then determining the singular stresses by carrying out a transformation to polar coordinates with the crack tip as the origin through a series expansion. The final results obtained for the thermal stresses in the neighbourhood of the crack tip agree completely with those obtained by Sih [1].

The temperature field T satisfying the steady state heat conduction equation, $\nabla^2 T = 0$, may be written as

$$T(\xi,\eta) = qa(\sinh \xi \sin \eta + e^{-\xi} \sin \eta) \tag{1}$$

Eq. (1) also satisfied the following boundary conditions:

$$\begin{aligned} &\text{At large distances from the crack } (\xi \to \infty),\ T = qy \\ &\text{At the insulated crack boundary } (\xi = 0),\ \partial T/\partial \xi = 0 \end{aligned} \tag{2}$$

In (1) the first term does not produce stresses, and only the second term is relevant in our further discussion.

The governing equation for the Airy stress function F in the case of plane thermal stresses is

$$\nabla^4 F + \alpha E \nabla^2 T = 0 \tag{3}$$

The stresses σ_ξ, σ_η, $\sigma_{\xi\eta}$ can be determined from F; the relevant expressions can be found in [3] (in which h should be replaced by a). The solution F should also satisfy the following boundary conditions:

$$\begin{aligned} &\text{At the crack boundary } (\xi = 0),\ \sigma_\xi = \sigma_{\xi\eta} = 0 \\ &\text{At large distances from the crack, } \sigma_\xi = \sigma_{\xi\eta} = \sigma_\eta = 0 \end{aligned} \tag{4}$$

The homogeneous and particular solutions of (3) may be written as

$$F(\xi,\eta) = Ae^{-\xi}\sin\eta + B(e^{-\xi}\sin 3\eta + e^{-\xi}\sin\eta) - (E\alpha qa^3/4)[1/8\ e^{-\xi}\sin 3\eta + 1/8\ e^{-3\xi}\sin\eta + \xi\sinh\xi\sin\eta] \quad (5)$$

The constants A and B determined from boundary conditions (4) are given as

$$A = 0; \quad B = E\alpha qa^3/32 \quad (6)$$

In the vicinity of the crack tip where $\rho << 1$ (see Fig. 1) F can be expanded in dimensionless polar coordinates (ρ,β) as,

$$F(\rho,\beta) = -(E\alpha qa^3/4\sqrt{2})[\rho^{3/2}(\sin\beta/2 + \sin 3\beta/2)] \quad (7)$$

The presence of $\rho^{3/2}$ in (7) gives rise to singular stresses. The stress components σ_ρ, $\sigma_{\rho\beta}$, σ_β and σ_x, σ_{xy}, σ_y can be determined in the usual way [3]. Expressions for the latter are given below:

$$\sigma_x = -\{E\alpha qa/[4\sqrt{(2\rho)}]\}[(7/4)\sin\beta/2 + (1/4)\sin 5\beta/2] + O(\rho^\circ)$$

$$\sigma_y = -\{E\alpha qa/[4\sqrt{(2\rho)}]\}[(1/4)\sin\beta/2 - (1/4)\sin 5\beta/2] + O(\rho^\circ)$$

$$\sigma_{xy} = +\{E\alpha qa/[4\sqrt{(2\rho)}]\}[(3/4)\cos\beta/2 + (1/4)\cos 5\beta/2] + O(\rho^\circ)$$

These equations for the thermal stress distribution in the neighbourhood of the crack tip are identical with those obtained by Sih* (corresponding to the imaginary part of (7) in [1]). The stresses at the crack tip exhibit the inverse square root singularity peculiar to crack solutions [1,3].

REFERENCES

[1] G. C. Sih, *Journal of Applied Mechanics* 29 (1962) 587-589-

[2] A. L. Florence and J. N. Goodier, *Journal of Applied Mechanics* 27 (1960) 635-639.

[3] M. N. Bapu Rao, T. Ariman, and L. H. N. Lee, *International Journal of Solids and Structures* 8 (1972) 945-959.

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APPENDIX

The following relations connecting the elliptic and polar coordinates are valid when ρ 1 (see Fig. 1)

*There is a typographical error in (11) of this paper for the constant A. The denominator of this equation should read (1 + k) and not 2(1 + k).

$$e^{-\xi} = 1 + (1 + \cos\beta)\rho/2 - \sqrt{2\rho}\cos\beta/2 - \frac{1}{2\sqrt{2}}\rho^{3/2}\cos\beta/2 + 0(\rho^2)$$

$$\cos\eta = 1 - \rho\sin^2\beta/2 + 0(\rho^2)$$

The expansions for e^{ξ}, ξ, $\sin\eta$ also required in the analysis, can be expressed in a similar manner.

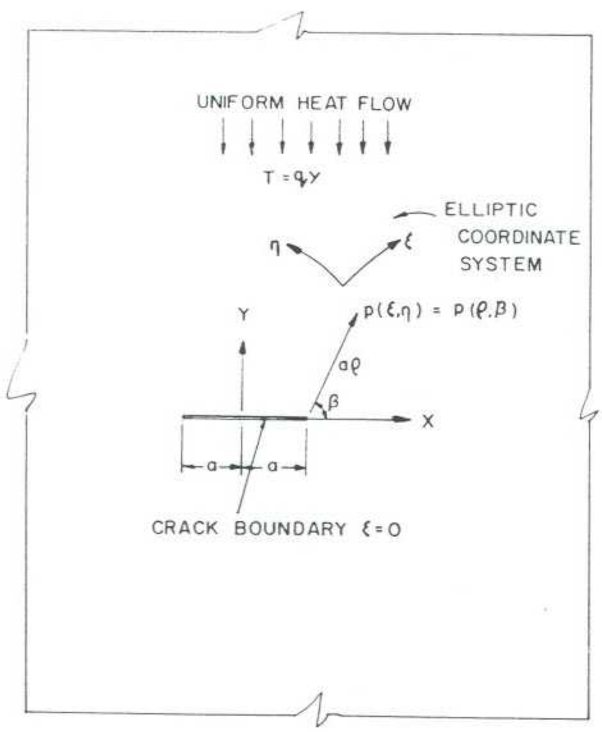

Figure 1. An Infinite Plate with an Insulated Crack Subjected to a Uniform Heat Flow