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OPTIMAL DESIGN OF A VIBRATING BEAM WITH COUPLED BENDING AND TORSION

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Abstract

The problem of maximizing the fundamental frequency of a thin walled beam with coupled bending and torsional modes has been studied in this paper. An optimality criterion approach has been used to locate stationary values of an appropriate objective function subject to constraints. Optimal designs with and without coupling have been discussed.

1. Introduction

A first investigation of the optimal beam vibration problem is attributed to Niordson[1]. He considered the problem of finding the best taper that yields the highest possible natural frequency. Following the initial work of Niordson, many different investigators have considered different problems in the field of optimal vibrations of beams[2-20]. References 2-8 are concerned with the maximization of fundamental frequencies. Olhoff[9-10] has addressed the problem of maximizing higher order frequencies and rotating beams[4]. The problem of minimizing weight for a specified frequency constraint has been addressed in References 12-18. Multiple frequency constraints have been addressed in References 19-23. An optimality criteria approach has been discussed in References 17 and 1%.

An application to the helicopter blade design problem has been presented by Peters et al.[24] In their work, the problem of optimum distribution of mass and stiffness for a frequency constraint has been discussed. In most cases this is the dual of the problem of maximizing the frequencies[12-16], which is considered as a primal problem. It is possible to solve several primal problems to obtain a solution to a dual probkm. Either of these approaches results in an optimum design and a structural dynamic model corresponding to the optimal design.

The resulting mathematical model can be used as a model for tests and improvements of these models by identification techniques. In an application of this[24] and in all other optimal vibration problems, only uncoupled vibration modes have been considered. In the helicopter design problem[24] and many other practical situations, elastic axes do not coincide with the inertial axes, resulting in a coupling between some of the bending modes and torsional modes. This paper has addressed the problem of maximizing the fundamental frequency of a thin walled beam with coupled bending and torsional modes. This is achieved through an optimality criterion approach to locate stationary values of a proper objective function. The results show that the optimum designs are very different from the design obtained for beams with uncoupled vibration, showing that the coupling must not be ignored in the optimization process.

2. Primal Optimization Problem for a Continuous System

A beam of channel cross section with one axis of section symmetry experiencing vibration in simple harmonic motion of frequency ω is considered. The maximum strain energy determined from the sum of Eqs. (A 8) and (A12) is

$$2U_{max} = \int_L \left[(EI)_y \left(\frac{d^2 w_r}{dx^2}\right)^2 + 2(EI)_{\theta w} \frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2} + (EC)_{wr}\left(\frac{d^2\theta}{dx^2}\right)^2 + \overline{GJ}\left(\frac{d\theta}{dx}\right)^2 + (EI)_z\left(\frac{d^2 v_r}{dx^2}\right)^2 \cdots \right] dx \tag{2.1}$$

The maximum kinetic energy follows from Eqs. (A 9) and (A13), with the addition of non-structural concentrated masses.

$$2T_{max} = \omega^2 (2\overline{T}_{max}) \tag{2.2}$$

with

$$2\overline{T}_{max} = \int_L \left(m w_r^2 + 2\bar{m} w_r \theta + \bar{I}_{pr}\theta^2 + m v_r^2 \right) dx + \sum_i \left(M_i w_{ri}^2 + 2\overline{M}_i w_{ri}\theta_i + J_{pri}\theta_i^2 + M_i v_{ri}^2 \right) \tag{2.3}$$

From the requirement that $2U_{max} = 2T_{max}$, with the constraint that $2\overline{T}_{max} = 1$, it follows that

$$\omega^2 = 2U_{max} \tag{2.4}$$

For the optimization process, $\phi_j(x)$, $j=1,2,\ldots,N_\phi$, denotes the j^{th} design variable, limited in this paper to the flange and web thicknesses. The primal problem is to determine the wall thicknesses which provide the maximum value of the fundamental frequency subject to

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the constraint that the beam mass be equal to some specified value. The formulation of equations is as follows.

maximize

$$\omega^2 = \int_L \left[\begin{array}{l} (EJ)_y \left(\frac{d^2 w_r}{dx^2}\right)^2 + 2(EJ)_{\theta w} \frac{d^2 w_r}{dx^2} \frac{d^2\theta}{dx^2} \\ + (EC)_{wr} \left(\frac{d^2\theta}{dx^2}\right)^2 + \overline{GJ} \left(\frac{d\theta}{dx}\right)^2 \\ + (EJ)_z \left(\frac{d^2 v_r}{dx^2}\right)^2 \end{array} \right] dx \qquad (2.5)$$

subject to the constraints of satisfaction of equilibrium equations

$$\frac{d^2}{dx^2}\left[(EJ)_y \frac{d^2 w_r}{dx^2} + (EJ)_{\theta w} \frac{d^2\theta}{dx^2}\right] - \omega^2 (m w_r + \bar{m}\theta) = 0$$

$$\frac{d^2}{dx^2}\left[(EJ)_{\theta w} \frac{d^2 w_r}{dx^2} + (EC)_{wr} \frac{d^2\theta}{dx^2}\right] - \frac{d}{dx}\left(\overline{GJ} \frac{d\theta}{dx}\right)$$

$$- \omega^2 (\bar{m} w_r + \bar{I}_{pr}\theta) = 0$$

$$\frac{d^2}{dx^2}\left[(EJ)_z \frac{d^2 v_r}{dx^2}\right] - \omega^2 m v_r = 0 \qquad (2.6)$$

with appropriate equilibrium requirements at concentrated masses and appropriate -boundary conditions. There is a normalization constraint

$$\int_L (m w_r^2 + 2\bar{m} w_r \theta + \bar{I}_{pr}\theta^2 + m v_r^2) dx$$

$$+ \sum_i \left(\mathcal{M}_i w_{ri}^2 + 2\bar{\mathcal{M}}_i w_{ri}\theta_i + \mathcal{J}_{pri}\theta_i^2 + \mathcal{M}_i v_{ri}^2\right) - 1 = 0$$

(2.7)

The beam mass is specified

$$\int_L m\,dx - \bar{M} = 0 \qquad (2.8)$$

and there are possible limits on magnitudes of design variables

$$\phi_{j\,min} \le \phi_j \le \phi_{j\,max} \qquad (2.9)$$

This problem will be solved with the optimality criterion *approach,* with the criterion developed by applying the techniques of calculus of variations and Lagrange multipliers, as follows.

A modified frequency functional, F, is defined as follows:

$$F[w_r, \theta, v_r; \phi_j] = \omega^2 - \Omega(2\bar{T}_{max} - 1) - \lambda\left(\int_L m\,dx - \bar{M}\right) \qquad (2.10)$$

That is, the normalization and constant mass constraints are incorporated with Lagrange multipliers Ω and λ, respectively. The problem now is to determine those functions w_r, θ, v_r, and ϕ which give a stationary value to the functional F, subject to equilibrium constraints.

First, the variations of the displacements w_r, θ, and v_r are considered. A typical first variation of F will be

$$\tfrac{1}{2}\delta F_{w_r} = \int_L (EJ)_y \frac{d^2 w_r}{dx^2} \frac{d^2 \delta w_r}{dx^2} + (EJ)_{\theta w} \frac{d^2\theta}{dx^2} \frac{d^2 \delta w_r}{dx^2}$$

$$\tfrac{1}{2}\delta F_{w_r} = \int_L \left[(EJ)_y \frac{d^2 w_r}{dx^2} + (EJ)_{\theta w} \frac{d^2\theta}{dx^2}\right] \frac{d^2 \delta w_r}{dx^2} dx$$

$$- \Omega \left[\begin{array}{l} \int_L (m w_r + \bar{m}\theta)\delta w_r\, dx \\ + \sum_i (\mathcal{M}_i w_{ri} + \bar{\mathcal{M}}_i \theta_i)\delta w_{ri} \end{array} \right]$$

(2.11)

After integration by parts and inclusion of the equilibrium equation constraints, it can be shown that $\delta F_{w_r} = 0$ for every δw_r only if $\Omega = \omega^2$. This same requirement follows from $\delta F_\theta = 0$ and $\delta F_{v_r} = 0$.

Finally, variations of the design variables ϕ_j are considered. It is to be noted that variations of a particular design variable *are* taken only in those regions of the beam domain in which that variable does not have a limiting value set by Eq. (2.9)

$$\delta F_{\phi_j} = \delta\omega^2_{\phi_j} - \omega^2 \delta(2\bar{T}_{max})_{\phi_j} - \lambda \int_L \frac{\partial m}{\partial \phi_j} \delta\phi_j\, dx$$

(2.12)

After evaluating the variations, the requirement that $\delta F_{\phi_j} = 0$ for every $\delta\phi_j$ leads to the optimality criterion for each design variable.

$$H_j[w_r(x), \theta(x), v_r(x), \phi_k(x)] = \lambda, \quad j = 1, 2, \cdots, N_\phi \qquad (2.13)$$

with

$$H_j = \frac{1}{\frac{\partial m}{\partial \phi_j}} \left[\begin{array}{l} \left(\frac{d^2 w_r}{dx^2}\right)^2 \frac{\partial (EJ)_y}{\partial \phi_j} + \left(\frac{d^2 v_r}{dx^2}\right)^2 \frac{\partial (EJ)_z}{\partial \phi_j} \\ + \left(\frac{d^2\theta}{dx^2}\right)^2 \frac{\partial (EC)_{wr}}{\partial \phi_j} + \left(\frac{d\theta}{dx}\right)^2 \frac{\partial (\overline{GJ})}{\partial \phi_j} \\ + 2 \frac{d^2 w_r}{dx^2} \frac{d^2\theta}{dx^2} \frac{\partial (EJ)_{\theta w}}{\partial \phi_j} \\ - \omega^2 \left(\begin{array}{l} w_r^2 \frac{\partial m}{\partial \phi_j} + 2 w_r \theta \frac{\partial \bar{m}}{\partial \phi_j} \\ + \theta^2 \frac{\partial \bar{I}_{pr}}{\partial \phi_j} + v_r^2 \frac{\partial m}{\partial \phi_j} \end{array} \right) \end{array} \right] \qquad (2.14)$$

In words, the optimum design is supposedly achieved when the quantity H_j is constant along the span of the beam for all regions in which the associated ϕ_j does not have a limiting value.

The formulation is summarized as follows. The unknowns *are* three displacement functions (w_r, θ, v_r), N_ϕ design variable functions (ϕ_j), the frequency of vibration (ω^2), and the Lagrange multiplier (λ). Available equations *are* three equilibrium equations with associated boundary conditions and concentrated mass conditions (Eq. (2.6)), No optimality criterion equations (Eqs. (2.13) and (2.14)) or the limiting values (Eq. (2.9)), the normality condition (Eq. (2.7)), and the constant mass constraint equation (Eq. (2.8)). The problem seems to be well-posed; and a simultaneous solution of all equations will lead to possible optimum designs.

Equation (2.6) shows the decoupling between displacement v_r and the displacement pair w_r, θ. There arc two separate eigenvalue problems, leading to eigenvalue ω_v^2 with eigenvector $\hat{v}_r$, $w_r = 0$, $\theta = 0$ and eigenvalue ω_w^2 with eigenvector $\hat{w}_r, \hat{\theta}$, $v_r = 0$. If $\omega_v^2 = \omega_w^2$, then the eigenvector will contain nonzero components for all displacements, with $\hat{w}_r, \hat{\theta}$, and $\hat{v}_r$.

Now, if the physics of the problem is such that one need optimize only for vibration in the plane of symmetry, then it is permissible to set $w_r = 0$, $\theta = 0$. Such singledisplacement optimization problems have been treated many times in the past, most often with cross section area as the design variable. Equations (2.13) and (2.14) will provide the proper optimality criteria for other design variables such as wall thickness.

Likewise, if it is necessary to optimize only for the coupled vibration, then one may set $v = 0$ in Eqs. (2.13) and (2.14) to obtain the correct optimafity criterion. This coupleddisplacement optimization has not been done before, and the reported numerical results in this paper are limited to this problem.

The decoupled optimization problems will lead to valid optimum designs in the following two situations.

If the optimality criteria are satisfied with $v_r \# 0$, $w_r = 0$, $\theta = 0$, and if the optimized ω_v^2 is less than the bending-torsion frequency ω_w^2, then the design is truly optimum. The lowest natural frequency has been raised to the highest value possible.

If Eqs. (2.13) and (2.14) are satisfied with $w_r \neq 0$, $\theta \neq 0$, $v_r = 0$, and if the optimized $\omega_w^2 < \omega_v^2$, the design is truly optiinum. The lowest frequency has been raised.

However, if Eqs. (2.13) and (2.14) *are* satisfied with $v_r \neq 0$, $w_r = 0$, $\theta = 0$ and the optimized $\omega_v^2 > \omega_w^2$ or if $w_r \neq 0$, $\theta \neq 0$, $v_r = 0$ and the optimized $\omega_w^2 > \omega_v^2$, then the designs are not valid. In either case, the design is such that the optimized frequency is not the fundamental frequency, which means that the fundamental frequency has not been optimized.

if decoupled optimization does not provide the optimum design, then the problem must be reformulated. This observation can be explained by beginning an optimization problem with a cross section with specified depth h, width b, mass $\bar{M}$, and uniform wall thickness t such that $\omega_w^2 < \omega_v^2$. In this case, optimization will attempt to raise ω_w^2 by varying the wall thicknesses. This search for the best wall thicknesses can be thought of as a movement through a design space of thicknesses, *seeking* that point which provides the largest ω_w^2. However, because decoupled optimization is presumably inadequate, it follows that at *some* point in the motion through design space, there will be a design for which $\omega_w^2 = \omega_v^2$. That design, while better than the initial uniform thickness design, is not optimum; and if an *even* better design is desired, the movement in design space must satisfy the now active constraint of $\omega_w^2 = \omega_v^2$. This requires mother optimality condition developed as follows. The new modified frequency function is

$$F = \int_L (EI)_z \left(\frac{d^2 v_r}{dx^2}\right)^2 dx - \lambda\left(\int_L m\,dx - \bar{M}\right)$$

$$-\Omega\left[\int_L m v_r^2\,dx + \sum_i M_i v_{ri}^2 - 1\right]$$

$$-\beta\left\{\begin{array}{l}\int_L\left[(EI)_y\left(\frac{d^2 w_r}{dx^2}\right)^2 + 2(EI)_{zw}\frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2} + (EC)_{wr}\left(\frac{d^2\theta}{dx^2}\right)^2 + \overline{GJ}\left(\frac{d\theta}{dx}\right)^2\right]dx \\ -\int_L (EI)_z\left(\frac{d^2 v_r}{dx^2}\right)^2 dx\end{array}\right\}$$

$$-\alpha\left[\begin{array}{l}\int_L\left(m w_r^2 + 2\bar{m} w_r \theta + \bar{I}_{pr}\theta^2\right)dx \\ +\sum_i\left(M_i w_{ri}^2 + 2\bar{M}_i w_{ri}\theta + I_{pri}\theta_i^2\right) - 1\end{array}\right] \qquad (2.15)$$

which is simply the expression for ω_v^2 supplemented by the normality condition for v_r, the constant mass constraint, the constraint drat $\omega_w^2 = \omega_v^2$, and the normality condition for w_r and θ. The variation δF_v leads to $(1+\beta)\omega_v^2 - \Omega = 0$. The variations δF_{w_r} and δF_θ lead to $\beta\omega_w^2 + \alpha = 0$. Finally, variation δF_{ϕ_j} leads to the new optimality criterion for each design variable

$$\left(\frac{d^2 v_r}{dx^2}\right)^2\frac{\partial(EI)_z}{\partial\phi_j} - \omega^2 v_r^2\frac{\partial m}{\partial\phi_j} - \lambda\frac{\partial m}{\partial\phi_j}$$

$$-\beta\left[\begin{array}{l}\left(\frac{d^2 w_r}{dx^2}\right)^2\frac{\partial(EI)_y}{\partial\phi_j} + 2\frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2}\frac{\partial(EI)_{zw}}{\partial\phi_j} \\ +\left(\frac{d^2\theta}{dx^2}\right)^2\frac{\partial(EC)_{wr}}{\partial\phi_j} + \left(\frac{d\theta}{dx}\right)^2\frac{\partial(\overline{GJ})}{\partial\phi_j} \\ -\left(\frac{d^2 v_r}{dx^2}\right)^2\frac{\partial(EI)_z}{\partial\phi_j} \\ -\omega^2\left(w_r^2\frac{\partial m}{\partial\phi_j} + 2w_r\theta\frac{\partial\bar{m}}{\partial\phi_j} + \theta^2\frac{\partial\bar{I}_{pr}}{\partial\phi_j} - v_r^2\frac{\partial m}{\partial\phi_j}\right)\end{array}\right] = 0 \qquad (2.16)$$

The first line of Eq. (2.16) is associated with optimizing ω_v^2 alone. The remaining terms, with the Lagrange multiplier β, appear because of the additional constraint that $\omega_v^2 = \omega_w^2 = \omega^2$.

For this coupled optimization problem, the unknowns have been augmented by the additional Lagrange multiplier, β, and an additional frequency *of* vibration, ω; but the equations have been augmented by an additional normality equation and the constraint equation of equality of frequencies. The problem remains conceptually solvable, but the solution will be more difficult because of the *second* Lagrange multiplier.

3 Development of a Finite Ekrnent Model

A channel cross section with constant specified web depth, h, and constant specified flange width, b is considered. For numerical results to be presented, the beam is modeled as a collection of finite elements; and it is necessary to develop proper stiffness and mass matrices for each element.

If the thicknesses t_f and t_w have some specified variation within each finite element, say, for example, a linear variation, then displacement based finite element stiffness and mass matrices can be developed from the differential equations (Eqs. (A3), (A4), and (A10)) or the virtual work expressions (Eq. (A27)) or the energy definitions (Eqs. (A8), (A9), (A12), and (A13)). However, in this paper the optimization is based on finite elements with uniform thicknesses. Therefore, appropriate matrices have been formulated by taking available matrices based on shear center displacements w_s, θ, v_s and transforming to reference axis displacements w_r, θ, v_r, as follows.

Matrices $[\bar{K}^e_{ws}]$ and $[\bar{M}^e_{ws}]$ denote 8 x 8 element stiffness and mass matrices developed with nodal degrees of freedom w_s, dw_s/dx, θ, $\frac{d\theta}{dx}$. At each node, the transformation from reference point, r, to shear center, s, is

$$\begin{Bmatrix} w_s \\ \frac{dw_s}{dx} \\ \theta \\ \frac{d\theta}{dx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -e & 0 \\ 0 & 1 & 0 & -e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w_r \\ \frac{dw_r}{dx} \\ \theta \\ \frac{d\theta}{dx} \end{Bmatrix} \tag{3.1}$$

where

$$e = -\frac{I_{s\omega}}{I_y} = \frac{3t_f b^2}{6t_f b + t_w h} \tag{3.2}$$

locates the shear center for each finite element cross section (Fig. 1). In condensed notation, Eq. (3.1) is written as

$$\{\bar{w}_s\} = [\bar{T}]\{\bar{w}_r\} \tag{3.3}$$

where $\{\bar{w}_s\}$ and $\{\bar{w}_r\}$ denote 4 x 1 displacement vectors at a single node and $[\bar{T}]$ is the 4 x 4 transformation array. The two nodal displacement vectors are combined to give 8 x 1 total element displacement vectors, $\{w_s\}$ and $\{w_r\}$, which are related by a properly constructed 8 x 8 transformation, $[T]$, as follows

$$\{w_s\} = [T]\{w_r\} \tag{3.4}$$

Finally, the transformed stiffness and mass matrices are

$$[K^e_{wr}] = [T]^T[\bar{K}^e_{wr}][T] \tag{3.5}$$

$$[M^e_{wr}] = [T]^T[\bar{M}^e_{wr}][T] \tag{3.6}$$

The transformed elemental matrices of Eqs. (3.5) and (3.6) can now be merged in the usual manner to form the total structure matrices, $[K_w]$ and $[M_w]$.

The uncoupled beam vibration in the y-direction can be treated with the usual stiffness and mass matrices, $[K_v]$ and $[M_v]$. Note that there will be only two degrees of freedom at each node, v_s and dv_s/dx.

4. Finite Element Formulation of the Primary Problem

A channel cross section beam is considered to be composed of a specified number of finite elements with possibly differing values of web thickness, t^e_w, flange thickness, t^e_f, and length, L^e. (The superscript e denotes element values.) The problem is to determine the set of wall thicknesses and lengths which will provide a maximum value for the fundamental frequency of vibration subject to the constraint of constant total volume (for uniform density material) and the constraint that the summation of element lengths is equal to the total length. In addition, there may be the so-called coupling constraint if the optimum design occurs with $\omega_w = \omega_v$ as discussed earlier.

For the problem of optimizing the coupled bending-torsion frequency, ω^2_θ, without the constraint of $\omega^2_\theta = \omega^2_v$, the modified objective function, which is the finite element form of Eq. (2.10), is given by

$$F(t^e_r, L^e, q_i) = K_{ij}(t^e_r, L^e) q_i q_j - \lambda\Big[\sum_e A^e(t^e_r)L^e - \bar{V}\Big]$$
$$-\Omega\big[M_{ij}(t^e_r, L^e) q_i q_j - 1\big] - \Delta\Big[\sum_e L^e - \bar{L}\Big] \tag{4.1}$$

where K_{ij}, M_{ij} = element in the ith row and jth column of the total beam stiffness and mass matrices, respectively; associated with coupled bending-torsion vibration.

q_i = i^{th} degree of freedom for the system in coupled bending-torsion vibration

t^e_r = r^{th} design variable (t^e_w or t^e_f) in element e

L^e = length of element e

$\bar{V}$, $\bar{L}$ = specified values of volume and length, respectively.

There is the additional constraint that

$$(K_{ij} - \omega^2 M_{ij})\, q_j = 0 \tag{4.2}$$

Note the use of the summation convention in Eqs. (4.1) and (4.2).

The first necessary condition for a differentiable maximum of F is $\frac{\partial F}{\partial q_i} = 0$, from which it follows, after substitution from Eq. (4.2), that

$$\Omega = \omega^2 \tag{4.3}$$

The next requirements are $\frac{\partial F}{\partial t^e_r} = 0$ and $\frac{\partial F}{\partial L^e} = 0$, from which follow the optimality criteria given below in Eqs. (4.6) and (4.8) respectively. In developing those equations, there will be terms of the form $[\partial K_{ij}/\partial t^e_r]\, q_i q_j$. Note, however, that the design variables t^e_r and L^e occur only in element e. Therefore, the derivatives involve only the appropriate stiffness and mass matrices for element e, and the only degrees of freedom which need be considered are those associated with element e. This means that the derivative terms can be written as $(\partial K_{ij}^e/\partial t_r^e) q_i^e q_j^e$, as shown in Eqs. (4.6) and (4.8).

The formulation can be summarized as follows. The unknowns are N_q values of q_i, N_t values of t_r^e, N_e values of L^e, one value of ω^2, one value of λ, and one value of Δ. The equations are

N_q equilibrium $$(K_{ij} - \omega^2 M_{ij}) q_j = 0, \quad i = 1,2,\cdots N_q \tag{4.4}$$

One normalization $$M_{ij} q_i q_j - 1 = 0 \tag{4.5}$$

N_t optimality $$\frac{1}{\frac{\partial A^e}{\partial t_r^e} L^e}\left(\frac{\partial K_{ij}^e}{\partial t_r^e} - \omega^2 \frac{\partial M_{ij}^e}{\partial t_r^e}\right) q_i^e q_j^e - \lambda = 0, \quad r = 1,2,\cdots,N_t \tag{4.6}$$

One constraint $$\sum_e A^e L^e - \bar{V} = 0 \tag{4.7}$$

N_e optimality $$\left(\frac{\partial K_{ij}^e}{\partial L^e} - \omega^2 \frac{\partial M_{ij}^e}{\partial L^e}\right) q_i^e q_j^e - \lambda A^e - \Delta = 0, \quad e = 1,2,\cdots,N_e \tag{4.8}$$

One constraint $$\sum_e L^e - \bar{L} = 0 \tag{4.9}$$

4 simultaneous solution of Eqs. (4.4) - (4.9) will lead to possible optimum designs.

When speaking of N_q equations of equlibrium, as in Eq. (4.4) and subsequently, there are of course only N_q - I independent equations. The remaining needed equation is the characteristic equation established from vanishing of the appropriate determinant.

Some of the design variables might take on specified values, such as a thickness equal to a lower limit value, or an element length might be fixed. If this occurs, simply give those variables the specified values wherever they occur and remove the optimality criteria associated with differentiation with respect to those variables. In particular, if all L^e are specified and fixed, remove Eqs. (4.8) and (4.9) from the formulation. This removes N_e + 1 equations and the N_e + 1 unknowns, L^e and Δ.

The next case to investigate is when the optimum design occurs with $\omega_w^2 = \omega_v^2$; and the modified objective function, which is the finite element form of Eq. (2.15), is

$$\begin{aligned} F(t_r^e, L^e, q_{wi}, q_{vi}) = {} & K_{vij} q_{vi} q_{vj} - \lambda\left(\sum_e A^e L^e - \bar{V}\right) \\ & - \Omega\left(M_{vij} q_{vi} q_{vj} - 1\right) - \Delta\left(\sum_e L^e - \bar{L}\right) \\ & - \beta\left(K_{wij} q_{wi} q_{wj} - K_{vij} q_{vi} q_{vj}\right) \\ & - \alpha\left(M_{wij} q_{wi} q_{wj} - 1\right) \end{aligned} \tag{4.10}$$

It is now necessary to distinguish between coupled bending-torsion vibration, denoted by subscript w, and the uncoupled bending, denoted by subscript v. The derivatives with respect to q_{vi} lead to $(1+\beta)\omega_v^2 - \Omega = 0$, and the derivatives with respect to q_{wi} lead to $\beta\omega_w^2 + \alpha = 0$. The derivatives with respect to design variables t_r^e and L^e lead to the optimality criteria shown below in Eqs. (4.15) and (4.18), respectively.

This coupled optimization problem is summarized as follows. The unknowns are N_w values of q_{wi}, N_v values of q_{vi}, N_t values of t_r^e, N_e values of L^e, two values of ω^2, one value of λ, one value of β, and one value of A. The equations are

N_w equilibrium $$(K_{wij} - \omega^2 M_{wij}) q_{wj} = 0, \quad i = 1,2,\cdots,N_w \tag{4.11}$$

One normalization $$M_{wij} q_{wi} q_{wj} - 1 = 0 \tag{4.12}$$

N_v equilibrium $$(K_{vij} - \omega^2 M_{vij}) q_{vj} = 0, \quad i = 1,2,\cdots,N_v \tag{4.13}$$

One normalization $$M_{vij} q_{vi} q_{vj} - 1 = 0 \tag{4.14}$$

N_t optimality $$\left(\frac{\partial K_{vij}^e}{\partial t_r^e} - \omega^2 \frac{\partial M_{vij}^e}{\partial t_r^e}\right) q_{vi}^e q_{vj}^e - \lambda \frac{\partial A^e}{\partial t_r^e} L^e - \beta \left[\begin{aligned} & \left(\frac{\partial K_{wij}^e}{\partial t_r^e} - \omega^2 \frac{\partial M_{wij}^e}{\partial t_r^e}\right) q_{wi}^e q_{wj}^e \\ & - \left(\frac{\partial K_{vij}^e}{\partial t_r^e} - \omega^2 \frac{\partial M_{vij}^e}{\partial t_r^e}\right) q_{vi}^e q_{vj}^e \end{aligned} \right] = 0, \quad r = 1,2,\cdots,N_t \tag{4.15}$$

One constraint $$\sum_e A^e L^e - \bar{V} = 0 \tag{4.16}$$

One constraint $$K_{wij} q_{wi} q_{wj} - K_{vij} q_{vi} q_{vj} = 0 \tag{4.17}$$

N_e optimality $$\left(\frac{\partial K_{vij}^e}{\partial L^e} - \omega^2 \frac{\partial M_{vij}^e}{\partial L^e}\right) q_{vi}^e q_{vj}^e - \lambda A^e - \Delta - \beta \left[\begin{aligned} & \left(\frac{\partial K_{wij}^e}{\partial L^e} - \omega^2 \frac{\partial M_{wij}^e}{\partial L^e}\right) q_{wi}^e q_{wj}^e \\ & - \left(\frac{\partial K_{vij}^e}{\partial L^e} - \omega^2 \frac{\partial M_{vij}^e}{\partial L^e}\right) q_{vi}^e q_{vj}^e \end{aligned} \right] = 0, \quad e = 1,2,\cdots,N_e \tag{4.18}$$

One constraint $$\sum_e L^e - \bar{L} = 0 \tag{4.19}$$

The optimum design is contained somewhere within Eqs. (4.11)-(4.19), but finding it is surely a difficult problem.

5. Recursion Relationship for the Primal Problem

For the primal problem with uncoupled optimization, the optimization process begins with some known distribution of design variables which satisfy the geometric constraints of Eqs. (4.7) and (4.9). For this initial design, Eqs. (4.4) and (4.5) are solved for the eigenvalue, ω, and the associated normalized eigenvector, q_i. Then it is possible to substitute into the optimality conditions of Eqs. (4.6) and (4.8). Only on rare occasions will these equations provide immediate solutions for the Lagrange multipliers, and so what is required is a procedure for moving through design variable space in such a manner as to eventually locate a design which permits satisfaction of the optimality criteria. This will be done with an iteration scheme developed as follows.

First introduce the definitions

$$U_r^e = \frac{\partial K_{ij}^e}{\partial t_r^e} q_i^e q_j^e \ , \quad T_r^e = \frac{\partial M_{ij}^e}{\partial t_r^e} q_i^e q_j^e \ , \quad A_r^e = \frac{\partial A^e}{\partial t_r^e} \qquad (5.1)$$

$$U_e^e = \frac{\partial K_{ij}^e}{\partial L^e} q_i^e q_j^e \ , \quad T_e^e = \frac{\partial M_{ij}^e}{\partial L^e} q_i^e q_j^e \qquad (5.2)$$

Now the optimality criteria, Eqs. (4.6) and (4.8), can be respectively written as

$$U_r^e - \omega^2 T_r^e - \lambda A_r^e L^e = 0 \qquad (5.3)$$

$$U_e^e - \omega^2 T_e^e - \lambda A^e - \Delta = 0 \qquad (5.4)$$

Next define

$$Z_r^e = U_r^e - \omega^2 T_r^e \ , \quad Z_e^e = U_e^e - \omega^2 T_e^e \qquad (5.5)$$

so that the optimality criteria can be written as

$$Z_r^e - \lambda A_r^e L^e = 0 \ , \quad r = 1,2,\cdots,N_r \qquad (5.6)$$

$$Z_e^e - \lambda A^e - \Delta = 0 \ , \quad e = 1,2,\cdots,N_e \qquad (5.7)$$

At the optimum design there will be a single value for λ which satisfies all N_r equations of Eq. (5.6) and a single value for A which satisfies all N_e equations of Eq. (5.7). However, for a non-optimum design, there is no single value of A and single value of Δ; and what will prove useful is some type of "best" values for λ and Δ, say $\bar{\lambda}$ and $\bar{\Delta}$, which approximately satisfy Eqs. (5.6) and (5.7) according to some criterion of goodness.

There are several ways to determine these best values, including methods which treat Eqs. (5.6) and (5.7) simultaneously. However, the simplest, and perhaps the best, method is to treat the equations separately as follows. From Eq. (5.6), define

$$\lambda_r = \frac{Z_r^e}{A_r^e L^e} \qquad (5.8)$$

Evidently, λ_r is the estimate for Lagrange multiplier λ based on the r^{th} equation. Then Eq. (5.6) can be written as

$$(\lambda_r - \lambda) A_r^e L^e = 0 \quad or \quad \lambda_r - \lambda = 0 \qquad (5.9)$$

The best value of λ is determined by the method of weighted residuals, as follows.

Define

$$R_r = (\lambda_r - \lambda) C_r \qquad (5.10)$$

where C_r is a weighting number the r^{th} difference $(\lambda_r - \lambda)$. Then if the measure of error is given by

$$E = \sum_r R_r^2 \qquad (5.11)$$

it follows that the value of λ, say $\bar{\lambda}$, which minimizes E is given by

$$\bar{\lambda} = \frac{\sum_r \lambda_r C_r^2}{\sum_r C_r^2} \qquad (5.12)$$

Note that if C_r is constant for all design variables t_r^e, then

$$\bar{\lambda} = \frac{1}{N_r} \sum_r \lambda_r \qquad (5.13)$$

so that $\bar{\lambda}$ is simply the arithmetic average of the λ_r. If the weighting number is chosen as $C_r = A_r^e t_r^e$, then $\bar{\lambda}$ from Eq. (5.12) is the same as $\bar{\lambda}$ derived from mean square error considerations of Eq. (5.6).

With $\bar{\lambda}$ now known, the optimality condition of Eq. (5.7) can be written as

$$Z_e^e - \bar{\lambda} A^e - \Delta = 0 \qquad (5.14)$$

Define

$$\Delta_e = Z_e^e - \bar{\lambda} A^e \qquad (5.15)$$

Once again, the optimality criterion requires uniform value for all A_e; and if the Δ_e are not constant, then the best value can be determined from

$$\bar{\Delta} = \frac{\sum \Delta_e D_e^2}{\sum_e D_e^2} \qquad (5.16)$$

with weighting numbers, D_e; or, for uniform D_e,

$$\bar{\Delta} = \frac{1}{N_e} \sum_t \Delta^e \qquad (5.17)$$

The next step is to assume that the $(\nu+1)$ iteration values can be expressed in terms of the ν iteration, as follows.

$$\begin{Bmatrix} U_r^e \\ T_r^e \end{Bmatrix}^{\nu+1} = \left[\frac{(t_r^e)^{\nu}}{(t_r^e)^{\nu+1}} \right]^{\alpha} \begin{Bmatrix} U_r^e \\ T_r^e \end{Bmatrix}^{\nu} \qquad (5.18)$$

$$\begin{Bmatrix} U_e^e \\ T_e^e \end{Bmatrix}^{\nu+1} = \left[\frac{(L^e)^{\nu+1}}{(L^e)^{\nu}} \right]^{\gamma} \begin{Bmatrix} U_e^e \\ T_e^e \end{Bmatrix}^{\nu} \qquad (5.19)$$

where α and γ are positive exponents. No attempt is made to derive these relationships. For some optimization problems in which the optimality criteria can be expressed in terms of potential and kinetic energies, it is possible to make some plausibility arguments relating $(\nu+1)$ and ν energies. These arguments are simply carried without change to this problem for which the optimality criteria can not be expressed in terms of energy, leading to Eqs. (5.18) and (5.19). The only proof of validity is utilitarian - do the assumed relationships lead to procedures which do indeed achieve optimum design?

Substitution of Eqs. (5.18) and (5.19) into Eq. (5.5) gives

$$(Z_r^e)^{\nu+1} = \left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\alpha (Z_r^e)^\nu \quad (5.20)$$

$$(Z_e^e)^{\nu+1} = \left[\frac{(L^e)^{\nu+1}}{(L^e)^\nu}\right]^\eta (Z_e^e)^\nu \quad (5.21)$$

Note that $(\omega^2)^\nu$ is used in the definition of $(Z_r^e)^{\nu+1}$ and $(Z_e^e)^{\nu+1}$. Substitute Eq. (5.20) into Eq. (5.8) and Eq. (5.21) into Eq. (5.15) and get the following approximations for $\lambda_r^{\nu+1}$ and $\Delta_e^{\nu+1}$.

$$\lambda_r^{\nu+1} = \frac{(Z_r^e)^{\nu+1}}{(A_r^e L^e)^\nu} = \left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\alpha \lambda_r^\nu \quad (5.22)$$

$$\Delta_e^{\nu+1} = (Z_e^e)^{\nu+1} - (\bar\lambda A^e)^\nu = \left[\frac{(L^e)^{\nu+1}}{(L^e)^\nu}\right]^\eta (Z_e^e)^\nu - (\bar\lambda A^e)^\nu \quad (5.23)$$

Now, the new design variables are selected so that the $\lambda_r^{\nu+1}$ as defined above are equal to each other for all values of r and the $\Delta_e^{\nu+1}$ are equal for all value of e. This movement toward equality of λ_r and Δ_e is expected to be a movement toward the optimum design. The equal values are chosen to be $\bar\lambda^\nu$ and $\bar\Delta^\nu$, so that

$$\lambda_r^{\nu+1} = \left[\frac{(t_r^e)^\nu}{(t_r^e)^{\nu+1}}\right]^\alpha \lambda_r^\nu = \bar\lambda^\nu \quad (5.24)$$

$$\Delta_e^{\nu+1} = \left[\frac{(L^e)^{\nu+1}}{(L^e)^\nu}\right]^\eta (Z_e^e)^\nu - (\bar\lambda A^e)^\nu = \bar\Delta^\nu \quad (5.25)$$

Therefore, the $(\nu+1)$ values can be written in terms of the ν values, as follows.

$$(t_r^e)^{\nu+1} = a f_r^\nu (t_r^e)^\nu, \quad \text{with} \quad f_r^\nu = \left(\frac{\lambda_r^\nu}{\bar\lambda^\nu}\right)^n \quad (5.26)$$

$$(L^e)^{\nu+1} = b g_e^\nu (L^e)^\nu, \quad \text{with} \quad g_e^\nu = \left[\frac{(\bar\lambda A^e)^\nu + \bar\Delta^\nu}{(\bar\lambda A^e)^\nu + \Delta_e^\nu}\right]^m \quad (5.27)$$

where $n = 1/\alpha$ and $m = 1/\eta$ are positive exponents.

Equations (5.26) and (5.27) include scalar multipliers a and b which are used to force the $(\nu+1)$ design variables to satisfy the length and volume constraints. Because there might be active geometric constraints of the type of Eq. (2.9) acting on some of the design variables, the length and volume constraints can be written as

$$\sum_{e=1}^{N_e - N_{ce}} (L^e)^{\nu+1} = \bar L - L_c \quad (5.28)$$

$$\sum_{e=1}^{N_e - N_{cr}} A^e[(t_r^e)^{\nu+1}](L^e)^{\nu+1} = \bar V - V_c \quad (5.29)$$

where N_{ce} denotes the number of elements with active constraint on L^e, N_{cr} denotes the number of elements with specified cross section area, L_c denotes the total length of elements with constrained length, and V_c denotes the total volume of elements with constrained area. Substitution of Eq. (5.27) into Eq. (5.28) gives

$$b = \frac{\bar L - L_c}{\sum_{e=1}^{N_e-N_{ce}} g_e^\nu (L^e)^\nu} \quad (5.30)$$

and Eqs. (5.26) and (5.29) give

$$a = \frac{\bar V - V_c}{\sum_{e=1}^{N_e-N_{cr}} A^e[f_r^\nu (t_r^e)^\nu](L^e)^{\nu+1}} \quad (5.31)$$

Note that when developing $A^e[(t_r^e)^{\nu+1}]$, it is recognized that the cross section area is a linear function of design variable t_r^e for the channel section with constant h and b (see Eq. (A35)).

Equations (5.26) and (5.27) are useful only when the quantities f_r and g_e are defined, which requires $(\lambda_r^\nu/\bar\lambda^\nu)$ and $[(\bar\lambda A^e)^\nu + \bar\Delta^\nu]/[(\bar\lambda A^e)^\nu + \Delta_e^\nu]$ to be defined and positive. If these requirements are not satisfied, then proceed as follows. Write Eq. (5.3) in the forms

if $\bar\lambda^\nu > 0$, $\lambda_r^\nu < 0$

$$(U_r^e)^{\nu+1} = (\omega^2)^\nu (T_r^e)^\nu + \bar\lambda^\nu (A_r^e L^e)^\nu$$

$$(U_r^e)^{\nu+1} = (U_r^e)^\nu + (\bar\lambda^\nu - \lambda_r^\nu)(A_r^e L^e)^\nu \quad (5.32)$$

if $\bar\lambda^\nu < 0$, $\lambda_r^\nu > 0$

$$(\omega^2)^\nu (T_r^e)^{\nu+1} = (U_r^e)^\nu + (-\bar\lambda)^\nu (A_r^e L^e)^\nu$$

$$(\omega^2)^\nu (T_r^e)^{\nu+1} = (\omega^2)^\nu (T_r^e)^\nu + (\lambda_r^\nu - \bar\lambda^\nu)(A_r^e L^e)^\nu \quad (5.33)$$

Write Eq. (5.4) in the forms

if $(\bar\lambda A^e)^\nu + \bar\Delta^\nu > 0$ and $(\bar\lambda A^e)^\nu + \Delta_e^\nu < 0$ which implies $\bar\Delta^\nu > \Delta_e^\nu$

$$(U_e^e)^{\nu+1} = (\omega^2)^\nu (T_e^e)^\nu + (\bar\lambda A^e)^\nu + \bar\Delta^\nu$$

$$(U_e^e)^{\nu+1} = (U_e^e)^\nu + (\bar\Delta^\nu - \Delta_e^\nu) \quad (5.34)$$

if $(\bar\lambda A^e)^\nu + \bar\Delta^\nu < 0$ and $(\bar\lambda A^e)^\nu + \Delta_e^\nu > 0$ which implies $\Delta_e^\nu > \bar\Delta^\nu$

$$(\omega^2)^\nu (T_e^e)^{\nu+1} = (U_e^e)^\nu + \{-[(\bar\lambda A^e)^\nu + \bar\Delta^\nu]\}$$

$$(\omega^2)^\nu (T_e^e)^{\nu+1} = (\omega^2)^\nu (T_e^e)^\nu + (\Delta_e^\nu - \bar\Delta^\nu) \quad (5.35)$$

From positive definiteness of energies, it follows that U^e_r, T^e_r, U^e_e, and T^e_e are all positive definite. Also $\omega^2 > 0$ always. Clearly, Eqs. (5.32) - (5.35) have been written in such a way as to guarantee positive quantities on each side of each equation. In each case, the intention is for the $(\nu+1)$ design to be such that the left hand side will be increased to the ν value of the right hand side. Substitution from Eqs. (5.18) and (5.19) and introduction of the scalars a and b gives the following results.

$$\text{if } \bar{\lambda}^\nu > 0 \ , \ \lambda^\nu_r < 0$$

$$(t^e_r)^{\nu+1} = a\left[\frac{(U^e_r)^\nu}{(U^e_r)^\nu + (\bar{\lambda}^\nu - \lambda^\nu_r)(A^e_r L^e)^\nu}\right]^n (t^e_r)^\nu \quad (5.36)$$

$$\text{if } \bar{\lambda}^\nu < 0 \ , \ \lambda^\nu_r > 0$$

$$(t^e_r)^{\nu+1} = a\left[\frac{(\omega^2)^\nu (T^e_r)^\nu}{(\omega^2)^\nu (T^e_r)^\nu + (\lambda^\nu_r - \bar{\lambda}^\nu)(A^e_r L^e)^\nu}\right](t^e_r)^\nu \quad (5.37)$$

$$\text{if } (\bar{\lambda} A^e)^\nu + \bar{\Delta}^\nu > 0 \ , \ (\bar{\lambda} A^e)^\nu + \Delta^\nu_e < 0$$

$$(L^e)^{\nu+1} = b\left[\frac{(U^e_e)^\nu + (\bar{\Delta}^\nu - \Delta^\nu_e)}{(U^e_e)^\nu}\right]^m (L^e)^\nu \quad (5.38)$$

$$\text{if } (\bar{\lambda} A^e)^\nu + \bar{\Delta}^\nu < 0 \ , \ (\bar{\lambda} A^e)^\nu + \Delta^\nu_e > 0$$

$$(L^e)^{\nu+1} = b\left[\frac{(\omega^2)^\nu (T^e_e)^\nu + (\Delta^\nu_e - \bar{\Delta}^\nu)}{(\omega^2)^\nu (T^e_e)^\nu}\right]^m (L^e)^\nu \quad (5.39)$$

The scalars b and a are again found form Eqs. (5.30) and (5.31) with proper definitions for the quantities f^ν_r and g^ν_e.

In summary, the recursion relations are as follows. Use Eqs. (5.26) and (5.27) if valid, because these equations account for simultaneous changes in both U^e and T^e. These equations should certainly be valid when the design becomes sufficiently close to an optimum design. If, in the early stages of the iteration process, Eqs. (5.26) and (5.27) are not valid for some design variables, then we Eqs. (5.36) - (5.39) as appropriate to modify those particular variables.

Note that all proposed recursion relationships will automatically stop at an optimum design. This follows because at an optimum design, all $\lambda_r = \bar{\lambda}$ and all $\Lambda_e = \bar{\Delta}$; and all recursion relations give $(t^e_r)^{\nu+1} = (t^e_r)^\nu$ or $(L^e)^{\nu+1} = (L^e)^\nu$.

6. Numerical Results

A channel cross section of the following dimensions has been considered (Figure 1).

h = 0.5 in.

b = 0.975 in.

t_w = 0.025 in

ρ = 0.243 x 10^{-3} lb-sec 2/in4

E = 10 x 106 psi

G = 3.8 x 10^6 psi

The beam length is 40 inches, and that length has been divided into 10 equal length finite elements. Therefore, L^e is fixed; and Eqs. (4.8) and (4.9) are removed from the formulation. There is only one design variable for each finite element, and that is the flange thickness t^e_f. Equation (A35) shows that the cross section area is a linear function of the design variable t_f; and in this case, the volume constraint of Eq. (4.7) reduces to

$$\sum_{e=1}^{N_e - m_c} (c_1 + c_2 t^e_f) = (\bar{V} - V_c)/L^e \quad (6.1)$$

where c_1 = 0.0125 and c_2 = 1.95. The number m_c denotes the number of elements with the active geometric constraint of t_f equal to the specified minimum value; and V_c denotes the total volume of elements with that active constraint. Both simply supported and cantilever boundary conditions have been studied; and for simple support, the minimum thickness is t_f = 0.003 in, while for the cantilever beam, minimum t_f = 0.0004.

For the results to be presented, the optimization process started with a uniform wall thickness, which means $t_f = t_w$ = 0.025 in. The recursion relations are Eqs. (5.26), (5.36), or (5.37) as appropriate, with $t^e_r = t^e_f$ and $A^e_r = c_2$. The scaling factor is given by Eq. (5.31), with $(L^e)^{\nu+1}$ equal to the specified constant L^e.

There are two criteria which might be used to identify the optimum design. The first criterion is satisfaction of the optimality criterion in the form of Eq. (5.9), which requires a constant value for all λ_r. With one design variable per finite element, it follows that there will be one λ_r per element; and the uniformity of those λ_r can be evaluated by the requirement

$$\left| \frac{\lambda_r|_{max}}{\lambda_r|_{min}} - 1 \right| < \epsilon \quad (6.2)$$

where ϵ is a measure of acceptable error.

Another convergence criterion would appear to be of the form

$$\left| \frac{(\omega)^{\nu+1}}{(\omega)^\nu} - 1 \right| < \epsilon \quad (6.3)$$

Equation (6.3) is very simple to implement and will often indicate an optimum design. However, it is possible that Eq. (6.3) will be satisfied but Eq. (6.2) will not be satisfied. Therefore, Eq. (6.2) provides a more rigorous measure of satisfaction of the optimality criterion; and that equation has been used in the present analyses, with ϵ = 0.001.

Convergence to the optimum frequency was smooth and monotonic. The rate of convergence was a function of the initial choice of the exponent n which appears in [illegible] iterations and also a function of how n was [illegible]ed. In this work, n was initially set to a value of 1.5. During the iteration process, if at any stage $(\omega)^2$ is less than $(\omega)^2$, then the value of n is reduced by 75%.

The optimal flange thicknesses are shown in Figures 2-4 and summarized in Table 1-4. Patterns I and 2 in Tables 2 and 4 are explained in Figure 5. For the simply supported beam, a 40.71% increase in the first frequency, ω_1, is realized when compared to the corresponding value for a beam with uniform flange thickness. When geometric constraints are imposed the increase in the value of optimum ω_1, did not change significantly. The increase was 40.65% compared to the beam with uniform flange thickness. In the case of the cantilever beam, the increase in ω_1 in comparison to a cantilever beam of uniform flange thickness is 210.22% when no geometric constraints are imposed. The corresponding value with geometric constraint is 178.9%.

It is interesting to note that the percentage increase in ω_1 with respect to the uniform beam differs very little between the unconstrained and constrained optimization processes for a simply supported beam, whereas this difference is significant in the case of the cantilever. The reason is attributed to the fact that in the case of a simply supported beam the inequality constraint imposed on the design variable becomes active only over very few elements, whereas for the cantilever, the design variables become very small over a large number of elements near the free end and fall below the posed constraint. As a result their values are raised and made equal to $\bar{t}_f$ in the constrained problem. So, this minimum constraint become: critical over a large number of elements; hence, one is left with only a few elements for which the design variables may change during the optimization process.

Some important observations regarding the optimum design variable distributions are made at this point. In case of the solid, simply supported beam undergoing flexural vibration, the optimum area distribution corresponding to the maximum fundamental frequency appeared to follow the pattern of the corresponding mode shape. In other words, the optimum distribution assumed a maximum at the center with minimum at the two ends (Fig. 6). The flange thickness distribution corresponding to the optimum fundamental frequency of the simply supported channel Section, however, assumes a minimum at the center with maximum at the two ends (Fig. 2 and 3). The difference is attributed to the following reason. A beam with a thin-walled open section like a channel is very weak in resistance towards torsion. So, the fundamental mode of coupled vibration is a predominantly torsion dominated mode. Since the twisting moment distribution of a simply supported channel beam has its maximum at the two ends and a minimum at the center, the optimum distribution tends to follow this pattern. Also, for solid sections, beams with second area moments of inertia proportional to the square of the cross-sectional area have been considered; whereas, in the case of the beam with channel cross-section, the design variable yields a linear relation of the type,

$$I(x) = \alpha_0 + \alpha_1 A(x) \qquad (C.4)$$

which may also contribute towards changing the nature of the optimum distribution. This is not true for the cantilever channel beam though. The optimum flange thickness distribution in this case is similar in nature to that of the solid cantilever undergoing only flexural motian. The reason for this is that although the first coupled mode of vibration is still a torsion dominated mode, the twisting moment drrtribution in the case of a cantilever beam has its maximum at the root and minimum at the free end. Although the optimum distribution tends to follow the torsion dominated first natural mode, it is similar in pattern to the optimum distribution of a solid cantilever under bending only (Figs. 4 and 7).

7. Conclusions

In this paper an optimality criterion approach has been developed to maximize the fundamental frequency of a thin walled beam with coupled bending and torsional modes. The results show that the optimum designs, in some cases, are very different from the designs obtained for beams with uncoupled vibrations. This suggests further studies in this field, including the dual problem of minimizing the weight for frequency restraints, beams with closed cross sections and multiple frequency constraints. In practical applications where the coupling of bending and torsional modes can not be avoided, such as in rotorcraft technology, any analysis that ignores the effect of coupling may lead to erroneous results.

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Appendix: Force Vibration of Channel Sections with Nonuniform Wall Thickness

A beam of channel cross section with dimensions and coordinate system as shown in Figure 1 has been considered. The web depth, h, and the flange width, b, are constant along the length; but the thicknesses t_w and t_f are nonuniform along the length of the beam and possibly nonuniform in the cross section subject to a requirement of symmetry about the y-axis. First, the case of uniform thicknesses in the cross section is considered. At each cross section, the shear center s and centroid c are located by their y-coordinate

$$e = 3b^2\left(6b + \frac{h t_w}{t_f}\right)^{-1} \tag{A1}$$

$$c = b^2\left(2b + \frac{h t_w}{t_f}\right)^{-1} \tag{A2}$$

If the thicknesses t_w and t_f are varied in such a way that the ratio t_w/t_f is constant, then the loci of shear centers and centroids will be straight lines; and shear center displacements will provide elastic decoupling of rotation and displacement just as for the uniform channel section. However, for more general axial variations of thickness, the shear centers will be along a curved line which is not so suitable for the beam reference axis. Therefore, the reference axis should be chosen so as to be straight for any variation of thickness; and for the problems considered in this paper, an appropriate reference axis passes through the web center at each cross section. Because there is no taper along the length, the web centers will indeed lie along a straight line; and this choice for reference axis exploits the given cross section symmetry about the y-axis.

Free vibration in the x-y plane occurs without twisting. The usual Bernoulli-Euler equations for nonuniform beams describe this motion. However, free vibration in the x-z plane is coupled with cross section twisting. The double coupling equations of motion are well-known for a uniform beam with straight elastic axis through the shear center. The purpose of this Appendix is to derive the appropriate equations for reference axis at the middle of the web.

The fundamental assumptions are the usual two assumptions for thin walled beams. First, each cross section is assumed to twist without distortion. Second, there is no shear deformation in the middle surface of the beam.

The equations of dynamic equilibrium are derived from a differential beam element and can be written as follows.

$$\frac{\partial^2 M_y}{\partial x^2} + m\frac{\partial^2 w_r}{\partial t^2} + \bar{m}\frac{\partial^2 \theta}{\partial t^2} = 0 \tag{A3}$$

$$\frac{\partial^2 M_{wr}}{\partial x^2} - \frac{\partial M_{xsv}}{\partial x} + \bar{I}_{pr}\frac{\partial^2 \theta}{\partial t^2} + \bar{m}\frac{\partial^2 w_r}{\partial t^2} = 0 \tag{A4}$$

where

$$m(x) = \int_{-\frac{h}{2}}^{h/2} \rho t_w \, dz + 2\int_0^b \rho t_f \, dy \tag{A5}$$

$$\bar{m}(x) = 2\int_0^b \rho t_f y \, dy \tag{A6}$$

$$I_{pr}(x) = \int_{-h/2}^{+h/2} \rho t_w z^2 dz + 2\int_0^b \rho t_f [(\tfrac{h}{2})^2 + y^2] dy \quad (A7)$$

and ρ is the mass density of the material.

The strain energy in the beam is given by

$$U = \frac{1}{2}\int_L \Big[(E\mathcal{I})_y \Big(\frac{\partial^2 w_r}{\partial x^2}\Big)^2 + 2(E\mathcal{I})_{zw} \frac{\partial^2 w_r}{\partial x^2}\frac{\partial^2 \theta}{\partial x^2} + (EC)_{wr}\Big(\frac{\partial^2 \theta}{\partial x^2}\Big)^2 + \overline{GJ}\Big(\frac{\partial \theta}{\partial x}\Big)^2 \Big] dx \quad (A8)$$

and the kinetic energy is

$$T = \frac{1}{2}\int_L \Big[m\Big(\frac{\partial w_r}{\partial t}\Big)^2 + 2\bar{m}\frac{\partial w_r}{\partial t}\frac{\partial \theta}{\partial t} + \bar{I}_{pr}\Big(\frac{\partial \theta}{\partial t}\Big)^2 \Big] dx \quad (A9)$$

The equation of dynamic equilibrium for uncoupled vibration in the y-direction is

$$\frac{\partial^2}{\partial x^2}\Big[(E\mathcal{I})_z \frac{\partial^2 V_r}{\partial x^2} \Big] + m\frac{\partial^2 V_r}{\partial t^2} = 0 \quad (A10)$$

where $(E\mathcal{I})_z$ is the modulus weighted moment of inertia about a line parallel to the z-axis passing through the modulus weighted centroid. The virtual work equation is

$$\int_L \Big[(E\mathcal{I})_z \frac{\partial^2 V_r}{\partial x^2}\frac{\partial^2 \delta V_r}{\partial x^2} + m\frac{\partial^2 V_r}{\partial t^2}\delta V_r \Big] dx = 0, \quad (A11)$$

the strain energy is

$$U = \frac{1}{2}\int_L (E\mathcal{I})_z \Big(\frac{\partial^2 V_r}{\partial x^2}\Big)^2 dx \quad (A12)$$

and the kinetic energy is

$$T = \frac{1}{2}\int_L m\Big(\frac{\partial V_r}{\partial t}\Big)^2 dx \quad (A13)$$

This Appendix closes with consideration of the simplified, but most common, case in which the elastic moduli, E and G, and the mass density, ,have constant values in each cross section. Furthermore, the channel wall thicknesses, t_w and t_f, do not vary in a cross section. For this case, it is possible to calculate cross section geometric properties. Then the beam stiffness and mass per unit length quantities can be written as products of E, G, or ρ multiplied by appropriate geometric properties. Results are as follows.

$$I_y = \frac{t_f b h^2}{2} + \frac{t_w h^3}{12}, \quad (E\mathcal{I})_y = EI_y \quad (A14)$$

$$I_z = \frac{t_f^2 b^4 + 2t_f t_w b^3 h}{3(2t_f b + t_w h)}, \quad (E\mathcal{I})_z = EI_z \quad (A15)$$

$$I_{zw} = -\frac{t_f h^2 b^3}{4}, \quad (E\mathcal{I})_{zw} = EI_{zw} \quad (A16)$$

$$C_{wr} = \frac{t_f h^2 b^3}{6}, \quad (EC)_{wr} = EC_{wr} \quad (A17)$$

$$J = \frac{t_w^3 h}{3} + \frac{2t_f^3 b}{3}, \quad (\overline{GJ}) = GJ \quad (A18)$$

$$A = 2t_f b + t_w h, \quad m = \rho A \quad (A19)$$

$$S_z = t_f b^2, \quad \bar{m} = \rho S_z \quad (A20)$$

$$I_{pr} = \frac{t_w h^3}{12} + 2t_f b\Big(\frac{h^2}{4} + \frac{b^2}{3}\Big), \quad \bar{I}_{pr} = \rho I_{pr} \quad (A21)$$

Table 1 Numerical Results for the Simply Supported Channel Beam Shown in Fig. 3

Element No.	$(t_1)_1$ (inch)
1	0.1011
[illegible]	0.0113
3	0.0066
4	0.0030
5	0.0030
6	0.0030
7	0.0030
8	0.0066
9	0.0113
10	0.1011

Table 2 Optimum Frequency of the Simply Supported Channel Beam with 10 Elements;

$(\omega_1)_a$: Pattern 1 , $(\omega_1)_b$: Pattern 2

	$(\omega_1)_a^+$ rad/sec	$(\omega_1)_b^\bullet$ rad/sec	% Increase§ in $(\omega_1)_a$	% Increase§ in $(\omega_1)_b$
Unconstrained	213.6	931.0	40.71	14.44
Constrained	213.5	934.4	40.65	14.86

$(\omega_1)_{a_{UNI}}$ = 151.8 rad/sec

$(\omega_1)_{b_{UNI}}$ = 813.5 rad/sec

+ Refer Fig. 5 a

• Refer Fig. 5 b

§ Increase with reference to uniform beam

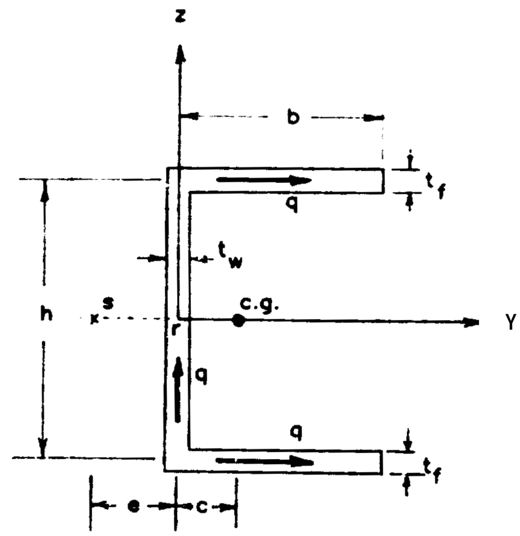

Figure 1 Section Geometry and Shear Flow

Table 3 Numerical Results for the Cantilever Channel Beam Shown in Fig. 4

Element No.	$(t_1)_i$ (inch)
1	0.0771
2	0.0638
3	0.0499
4	0.0352
5	0.0218
6	0.0004
7	0.0004
8	0.0004
9	0.0004
10	0.0004

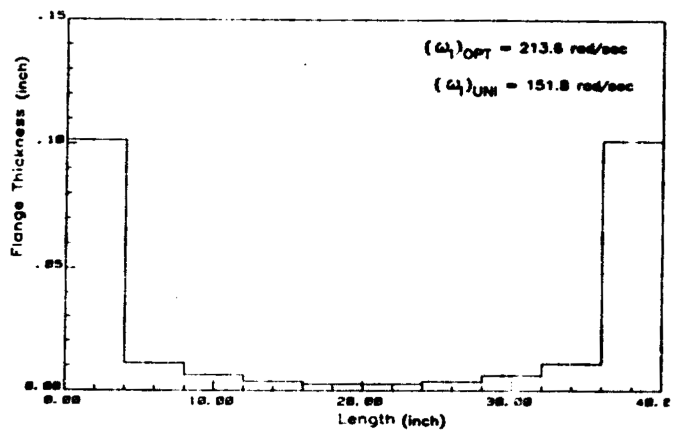

Figure 2 Optimum Flange Thickness Distribution of a Simply Supported Channel Beam; Unconstrained Case

Table 4 Optimum Frequency of the Cantilever Channel Beam with 10 Elements;

$(\omega_1)_a$: Pattern 1, $(\omega_1)_b$: Pattern 2

	$(\omega_1)_a^+$ rad/sec	$(\omega_1)_b^\bullet$ rad/sec	% Increase§ in $(\omega_1)_a$	% Increase§ in $(\omega_1)_b$
Unconstrained	217.9	879.6	210.22	132.58
Constrained	195.9	757.9	178.9	100.4

$(\omega_1)_{a_{UNI}}$ = 70.24 rad/sec

$(\omega_1)_{b_{UNI}}$ = 378.2 rad/sec

+ Refer Fig. 5 a

• Refer Fig. 5 b

§ Increase with reference to uniform beam

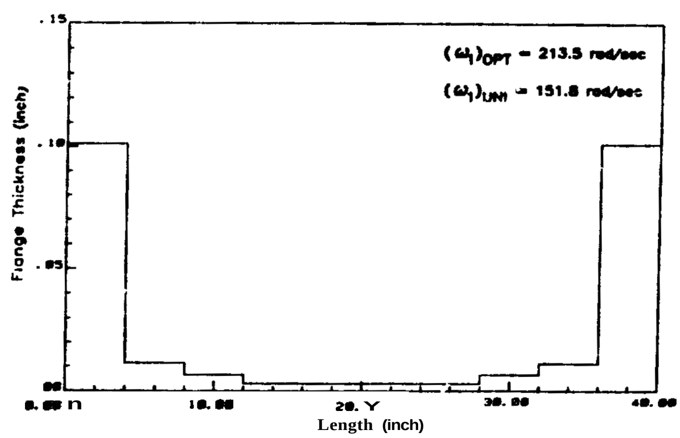

Figure 3 Optimum Flange Thickness Distribution of a Simply Supported Channel Beam; Constrained Case

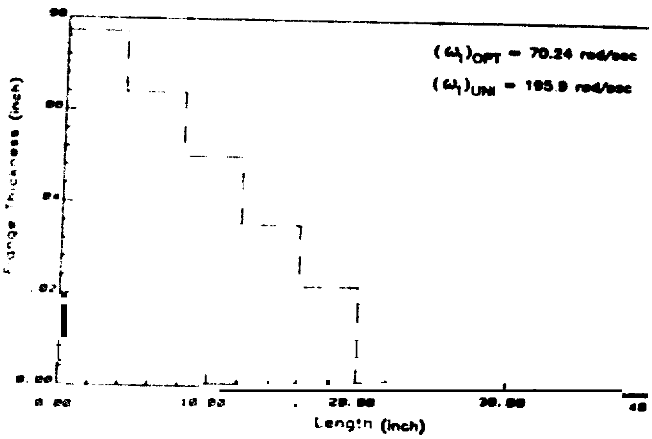

Figure 4 Optimum Flange Thickness Distribution of a Cantilever Channel Beam, Constrained Case

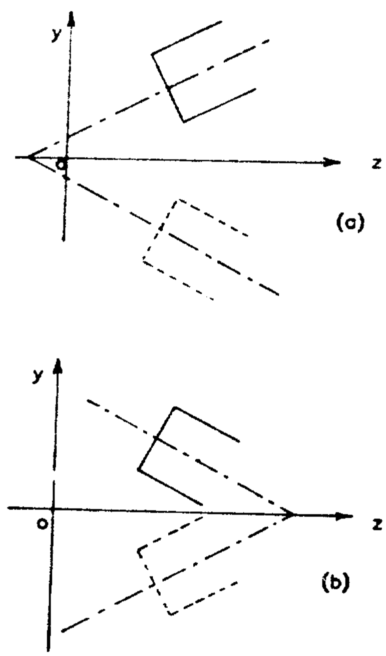

Figure 5 Vibration Patterns for a Channel Section,

(a) Pattern 1

(b) Pattern 2

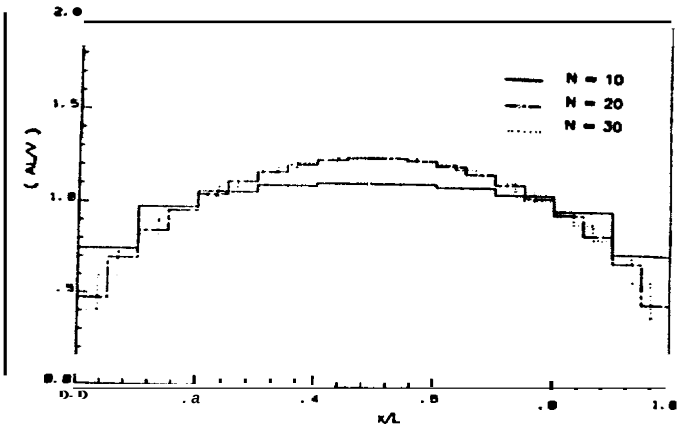

Figure 3 Optimum Area Distribution of a Simply Supported Vibrating Beam in ω_1^2 Maximization Case

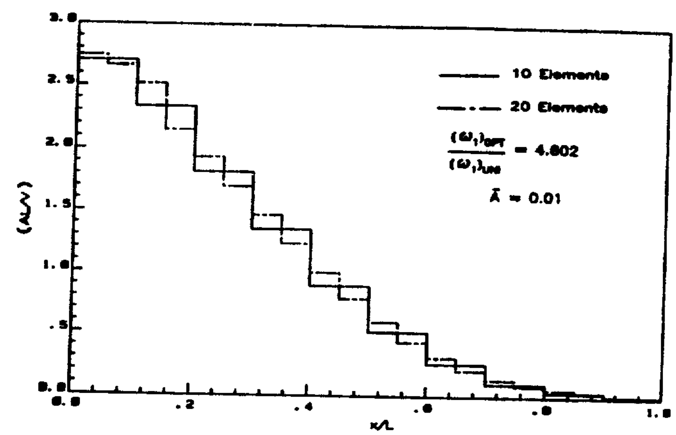

Figure 7 Optimum Area Distribution of a Vibrating Cantilever in ω_1^2 Maximization Case

85-0643

OPTIMAL DESIGN OF A VIBRATING BEAM WITH COUPLED BENDING AND TORSION

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Abstract

The problem of maximizing the fundamental frequency of a thin walled beam with coupled bending and torsional modes has been studied in this paper. An optimality criterion approach has been used to locate stationary values of an appropriate objective function subject to constraints. Optimal designs with and without coupling have been discussed.

1. Introduction

A first investigation of the optimal beam vibration problem is attributed to Niordson[1]. He considered the problem of finding the best taper that yields the highest possible natural frequency. Following the initial work of Niordson, many different investigators have considered different problems in the field of optimal vibrations of beams[2-20]. References 2-8 are concerned with the maximization of fundamental frequencies. Olhoff[9-10] has addressed the problem of maximizing higher order frequencies and rotating beams[4]. The problem of minimizing weight for a specified frequency constraint has been addressed in References 12-18. Multiple frequency constraints have been addressed in References 19-23. An optimality criteria approach has been discussed in References 17 and 1%.

An application to the helicopter blade design problem has been presented by Peters et al.[24] In their work, the problem of optimum distribution of mass and stiffness for a frequency constraint has been discussed. In most cases this is the dual of the problem of maximizing the frequencies[12-16], which is considered as a primal problem. It is possible to solve several primal problems to obtain a solution to a dual probkm. Either of these approaches results in an optimum design and a structural dynamic model corresponding to the optimal design.

The resulting mathematical model can be used as a model for tests and improvements of these models by identification techniques. In an application of this[24] and in all other optimal vibration problems, only uncoupled vibration modes have been considered. In the helicopter design problem[24] and many other practical situations, elastic axes do not coincide with the inertial axes, resulting in a coupling between some of the bending modes and torsional modes. This paper has addressed the problem of maximizing the fundamental frequency of a thin walled beam with coupled bending and torsional modes. This is achieved through an optimality criterion approach to locate stationary values of a proper objective function. The results show that the optimum designs are very different from the design obtained for beams with uncoupled vibration, showing that the coupling must not be ignored in the optimization process.

2. Primal Optimization Problem for a Continuous System

A beam of channel cross section with one axis of section symmetry experiencing vibration in simple harmonic motion of frequency ω is considered. The maximum strain energy determined from the sum of Eqs. (A 8) and (A12) is

$$2U_{max} = \int_L \left[(EI)_y \left(\frac{d^2 w_r}{dx^2}\right)^2 + 2(EI)_{\theta w} \frac{d^2 w_r}{dx^2}\frac{d^2\theta}{dx^2} + (EC)_{wr}\left(\frac{d^2\theta}{dx^2}\right)^2 + \overline{GJ}\left(\frac{d\theta}{dx}\right)^2 + (EI)_z\left(\frac{d^2 v_r}{dx^2}\right)^2 \cdots \right] dx \tag{2.1}$$

The maximum kinetic energy follows from Eqs. (A 9 I and (A13), with the addition of non-structural concentrated masses.

$$2T_{max} = \omega^2 (2\overline{T}_{max}) \tag{2.2}$$

with

$$2\overline{T}_{max} = \int_L \left(m w_r^2 + 2\bar{m} w_r \theta + \bar{I}_{pr}\theta^2 + m v_r^2 \right) dx + \sum_i \left(M_i w_{ri}^2 + 2\overline{M}_i w_{ri}\theta_i + J_{pri}\theta_i^2 + M_i v_{ri}^2 \right) \tag{2.3}$$

From the requirement that $2U_{max} = 2T_{max}$, with the constraint that $2\overline{T}_{max} = 1$, it follows that

$$\omega^2 = 2U_{max} \tag{2.4}$$

For the optimization process, $\phi_j(x)$, j=1,2,..., N_ϕ, denotes the j^{th} design variable, limited in this paper to the flange and web thicknesses. The primal problem is to determine the wall thicknesses which provide the maximum value of the fundamental frequency subject to

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