# Part I

# Modelling and System Identification

# Determination of Model Order for Dynamical System

R. V. JATEGAONKAR, I. R. RAOL, AND S. BALAKRISHNA

*Abstract*—The problem of model order determination, which is an integral part of system identification for dynamical system, of auto-regressive and least square structures is investigated. Twelve model order testing criteria available in the literature are critically evaluated from simulated data and from two different physical processes, viz., 1) human operator responses in compensatory tracking manual control, and 2) wind runnel mounted model responses to tunnel flow instabilities. The study revealed that of all the criteria tested, only a subset is adequate to establish reliable model order. Based on this observation, a computationally effective working rule is proposed for model order determination for practical dynamical systems.

## I. Introduction

Time series methods are gaining considerable acceptance in system identiiication in new of their inherent simplicity and flexibility [1]–[6]. These techniques provide external descriptions of systems under study and lead to parsimonious representation of the process. The accurate determination of the dynamical order of time series models is a necessary first step in identification and continues to be a subject of extensive contemporary research.

A literature study reveals that many statistical tests are available to find the model order for any given process. Selection of a reliable and efficient test criterion has been elusive since most criteria are sensitive to statistical properties of the process. Validation of most of the available criteria has generally been through simulated data. However, these order determination techniques have to be used with practical systems with unknown structures and finite data length. It is therefore necessary to validate any model order criterion using a wide variety of data base.

The work reported in this paper was motivated by the need to determine dynamic derivatives from free response of a wind tunnel model to tunnel flow [8], and human operator model determination from a fixed base simulator [9]. The data sets are from two distinctly different real processes and are considered to be good candidates for critically evaluating and comparing various criteria

## II. Model Order Determination

In the absence of *a priori* knowledge, any system that is generating time series output can be represented by an auto-regressive (AR) or a least squares (LS) model structure [1], [2]. Both these structures represent a general $n$th order discrete linear time invariant system disturbed randomly. The model order determination problem is one of assigning model dimension so that it adequately represents the unknown system. A review of the literature on order determination relevant to present study is summarized in Tables I and II as subjective and objective types for convenience [7], [10].

The subjective tests have been used in many applications [1], [3], [11], [12] and the main difficulty in using these have been the choice of proper levels of significance. By assuming the same statistical risk level, the subjective tests tend to ignore the increase of variability of estimated parameters for large model orders. It is common to assume a 5 percent risk level as acceptable for $F$-test [3] and whiteness tests [13], [14] arbitrarily. However, the recently suggested whiteness test-2 [15] does consider the cumulative effects of autocorrelations of residuals. The pole-zero cancellations [3], [12], [11] are made visually and are subjective. A systematic exact pole-zero cancellation is possible [16], but is computationally complex. Fit error methods are useful but again subjective. Some of these tests have been compared for their quality in [11] and [12] using simulated second-order plants.

Table II summarizes the objective-type tests, in which a local minimum of a criterion function is usually sought. The final prediction error (FPE) criterion due to Akaike [17] is based on one step ahead prediction and is essentially designed for white noise corrupted processes. The Akaike information criterion (AIC) [17] is a generalized concept based on a mean log likelihood function. Both the FPE and AIC depend only on residual variance and number of estimated parameters and may yield residuals which are correlated. Sometimes these tests yield multiple minima. The criterion autoregressive transfer function (CAT) by Parzen has been proposed as the best finite AR model derived from finite sample data generated by AR model of infinite order [18].

The modified final prediction error (AMFPE) [21] has been developed for autoregressive moving average (ARMA) models in terms of estimated variances of the process correlation function and estimated parameters. A simplified version of AMFPE has been used in this paper to be applicable to AR models. CAT3 is a modification of CAT2 [20] to account for any ambiguity which may arise for "true" first-order AR processes due to omission of $\sigma_0^2$ terms.

The methods discussed so far involve explicit parameter identification before criteria are evaluated. The entropy test [22] is an exception and is based on an information measure which is related to time series of the process. In this study the entropy test has been used for real process data.

## III. Applications

Three applications of model order estimation are now considered through various criteria detailed in Tables I and II. The data chains for the tests were derived from (1) a simulated second order system, (2) human activity from a fixed base simulator and (3) forces on a wind tunnel model exposed to mildly turbulent flows. The AR model identification was carried out as in [2] and the LS model identification as in [1]. The results of the criteria as applied to model order estimation are analyzed.

### *Simulated Second-Order System*

A second-order system of known $\omega_n$ and $\zeta$ patched on an analog computer was excited by a 200-Hz low-pas filtered Gaussian white noise, and the input–output pair were digitized at a sampling rate of 1 ms.

The AR parameters were evaluated in a model order scan using only the output data. Fig. 1 shows a plot of various indices of the criteria as a function of model order. The entropy, FPE, AMFPE, CAT2, and CAT3 explicitly indicate a minimum at a model order of 2. The prediction fit error shows a minimum at model order of 2 and above. The $F$-test almost nearly satisfies a 5 percent risk hypothesis at $F_{2,3}$. Thus, the objective tests clearly indicate the correct model order, and the subjective tests strongly suggest the correct model order.

The LS model parameters were evaluated next, again in an ascending model order scan, and the results are illustrated in Fig. 2. The pole-zero pattern in the unit circles clearly point to a second-order system. The cancellation of real pole-zero for the third-order model and a pair of complex pole-zero pair for the fourth-order model can be seen in Fig. 2. The $F$-test points to a second-order system with $F_{2,3} = 0.389$. The prediction and deterministic fit errors also suggest the same model order.

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R. V. Jategaonkar is with the Institute for Flightmechanics, DFVLR, 3300 Braunschweig-Flughafer, West Germany on leave from the National Aeronautical Laboratory, Bangalore, India.

I. R. Raol and S. Balakrishna are with the Systems Engineering Division, National Aeronautical Laboratory, Post Bag No. 1779, Bangalore 560 017, India.

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TABLE I
SUBJECTIVE TESTS FOR MODEL ORDER DETERMINATION

| Test | Criterion | Features | Comments |
|---|---|---|---|
| 1) *F*-Test | $F_{n_1n_2} = \dfrac{V_{n_1} - V_{n_2}}{V_{n_2}} \cdot \dfrac{N - 2n_2}{2(n_2 - n_1)}$ for $N > 100$, test $F_{n_1n_2} < 3$ for 95 percent confidence level. | Insignificant reduction in loss function by repeated least squares. | Two model orders to be estimated each time. |
| 2) Prediction fit error | $\sum_{k=1}^{N} r^2(k) \Big/ \sum_{k=1}^{N} y^2(k)$. Locate the knee of the curve fit error versus model order. | Insignificant change in fit error is sought. | Gives ability of the estimated parameters to predict the system response. |
| 3) Deterministic fit error | $y(t) - \dfrac{B(q^{-1})}{A(q^{-1})}u(t)$. | Accounts for the effects of nonlinearities, computational errors. | Useful for input-output model. |
| 4) Whiteness test | a) $R_e^2(\tau)/R_e^2(o) \leqslant \pm 1.96/\sqrt{N}$. Check for number of autocorrelations outside specified band for 95 percent confidence level. | The autocorrelations tend to be impulse function if residuals are uncorrelated. | |
| | b) $\sum_{i=1}^{n} R_e^2(i) \leqslant (k + 1.65\sqrt{2k})R_e^2(o)/N$ Find $n$ satisfying above condition with 5 percent risk level. | Cummulative effect of autocorrelations of residuals is evaluated. | Results are identical to whiteness test (a) (detailed discussions in Sections III and IV). |

The frequency spectra were found from a transformation [2], [8] for various models, and the $\omega_n$ and $\zeta$ derived from the AR/LS spectra are plotted in Fig. 3. The system constants $\omega_n$ and $\zeta$ (for $n > 2$) were directly evaluated from discrete parametric models by appropriate transformation [24] and are shown in Fig. 3. The correct value of system damping is predicted only by the chosen model order. Further, the log-decrement method of free oscillation [23] correctly estimates the system constants. Thus both the objective and subjective order test criteria provide sharp and consistent model order since the simulated response data is statistically well-behaved.

*Human Activity in Manual Control Task*

The time series data for human response were derived from a compensatory tracking experiment conducted on a fixed base research simulator. Assuming that the human activity can be represented by AR/LS models [9], the problem of model order determination is addressed here. A record length of 500 data points sampled at 50 ms was used for the analysis. The AR parameters and indices of various order criteria are presented in Figs. 4 and 5.

From Fig. 4 it can be noted that autocorrelation of the residuals suggest a sixth-order model. The whiteness test 2 confirms a sixth-order AR model. The *F*-test nearly satisfies the hypothesis of a sixth-order model with $F_{6,7} = 3$. The prediction fit error flattens nicely after $n > 6$, and thus all the subjective tests of Table I confirm the choice of a sixth-order AR model for human

In Fig. 5, the entropy FPE, AIC, CAT2, and CAT3 explicitly point to a fifth-order model. However, the AMFPE suggests a sixth-order model. It is pertinent to note that entropy test has a very well-defined minimum.

The same data were used to fit LS models in a model order scan from 1 to 8, and the results are shown in Fig. 6. The *F*-test points to a second-order model with $F_{2,3} = 4.2$, which does not improve for higher order hypothesis. From a pole-zero cancellation analysis, as in Fig. 6, a second-order model is suggested. Fit error analysis also indicates the same choice. Further, residuals were found to be correlated by both the whiteness tests for all model order choices.

The effect of the model order on the frequency description of the LS pilot models is shown in Fig. 7. It is clear that these remain essentially the same for second and higher order models. Though the AR pilot model differs from the LS pilot model in model order, the order choice is consistent within each structure.

*Dynamic Stability Derivative Measurement from Wind Tunnel Model Force Data*

Estimation of pitch damping derivatives using random flow fluctuations inherent in the tunnel flow has been recently proposed and validated [8]. This experiment uses an aircraft model mounted on a single degree of freedom flexture having a dominant second-order response. Since the excitation to the model is inaccessible, and AR model is the obvious choice, and an order test has been carried out using a 1000 sample data chain. The

TABLE II
OBJECTIVE TESTS FOR MODEL DETERMINATION

| Test | Criterion | Features | Comments |
|---|---|---|---|
| 1) Final prediction error | $FPE(n,N) = \frac{N+n+1}{N-n-1}\hat{\sigma}^2$ | Minimization of average error for one step prediction considering both the errors due to innovation part of the process and errors due to inaccuracies in estimating parameten. | Developed for univariate processes corrupted by white noise. The penalty for degrees of freedom is significantly reduced for large $N$. |
| 2) Information criterion | $AIC(n) = N \ln \hat{\sigma}_n^2 + 2n$ | Generalized concept of FPE. Estimate of mean log likelihood. | |
| 3) Modified final prediction error | $AMFPE(n)^* = \frac{1+A}{1-A}$ | Minimization of one step ahead prediction error from covariance matrix and estimated variance of the process itself. | Doer not explicitly depend on number of data $N$. Could be used for processes corrupted by color noise. |
| 4) Criteria autoregressive transfer function | a) $CAT1 = 1 - \frac{\sigma_\infty^2}{\sigma_n^2} + \frac{n}{N}$ | Minimization of integrated relative mean square error between $n$th order AR model and theoretical AR($\infty$) model. | Computationally difficult. |
| | b) $CAT2 = \frac{1}{N}\sum_{K=1}^{n}\frac{1}{\sigma_k^2} - \frac{1}{\sigma_n^2}$ minima | Modified CAT1. Computationally superior. | For any arbitrary model order $n$, all $n$ models to be estimated |
| | c) $CAT3 = \frac{1}{N}\sum_{k=0}^{n}\frac{1}{\sigma_k^2} - \frac{1}{\sigma_n^2}$ minima | Includes value in the definition of CAT2 to avoid any possible ambiguity for "true" first-order AR process. | For real physical processes that are seldom true first-order AR type. The advantages are not very significant. |
| 5) Process information | $\text{entropy}^* = \log\frac{N-n}{N-2n-1}$ <br> $\log\lvert S_{i+1}\rvert - \log\lvert S_i\rvert$ minima | Minimization of the difference in adjacent entropies. Not based on the study of residuals of estimating procedure. | The order of the AR model can be judged before estimating the AR parameters. |

*See Appendix

AMFPE criteria point to a tenth-order model. The $F$-test also indicates a tenth-order model.

Since the response is known to be dominantly second order, the natural frequency was determined by evaluating the spectra using a frequency transformation of the discrete AR models. The natural frequency is plotted in Fig. 8 as a function of model order. It can be seen that estimated from AR spectrum stabilizes to a constant value for $n > 10$. The predicted time response and the error time history is shown in Fig. 9.

The effect of record length on AR model fitting is shown in Fig. 10. For a sample length of 4600, the various model order criteria point to a fifteenth-order AR model. The entropy criterion and the natural frequency determined from AR spectrum confirm the same model dimension. This increase in model dimension as a function of record length has also been noted in earlier studies [19].

## IV. DISCUSSION

In the previous section, typical results from AR and LS model order determination studies were presented. Based on these and large number of similar other results, an effort is now made to investigate the relative merits of the various model order determination criteria generally used in the literature. Since the same test procedures have been used on three different data sources, it is hoped to derive some general understanding of the various criteria.

For a simulated second-order system, all the test methods consistently predict the correct model order. In the case of data from physical systems, given a model structure, all the test criteria predict same model orders though they seem to differ for different model structures. Further, in the case of AR modeling, data length affects the model order.

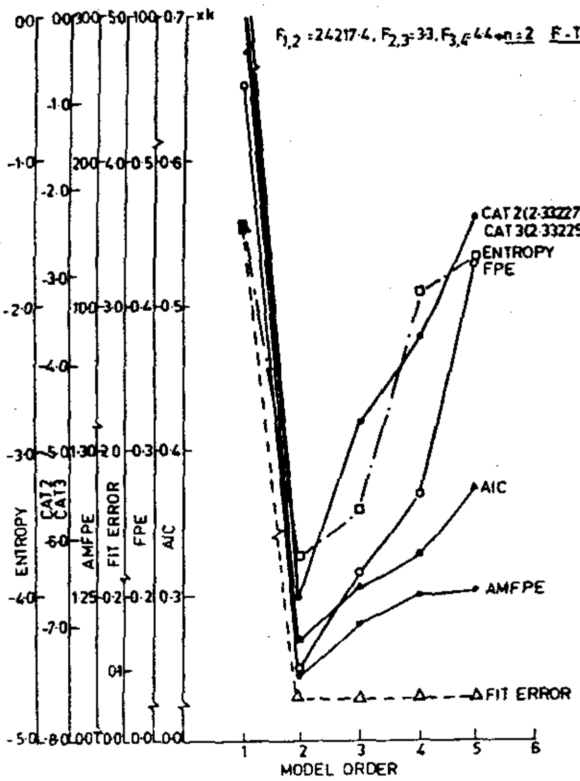
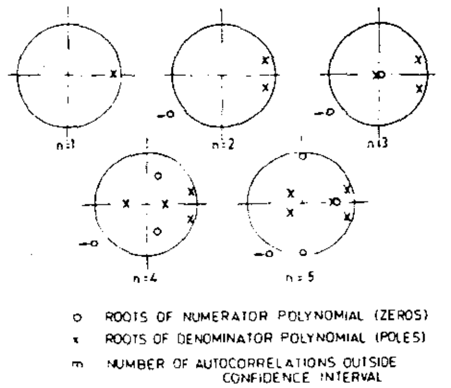



Fig. 1. AR model order for simulated system.




| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| % FIT ERROR | 2·342 | 0·720 | 0·714 | 0·692 | 0·684 |
| $F_{n_1 n_2}$ | 1127·0 | 4·2 | 16·1 | 6·6 | 36·3 |
| WHITENESS TEST- m | 7 | 7 | 6 | 5 | 4 |

Fig. 2. Pole-zero pattern and other model order tests for LS model of simulated system.

Regarding the criteria themselves, the analytical equivalence of FPE and AIC has been known [19], and the present analysis confirm their identical behavior. FPE and CAT2 show good agreement, but evaluation of CAT2 for a given model order $n$ requires estimation of all the $n$ models. In contrast, FPE evaluation can be achieved by a limited order scan. CAT3 is of limited significance since real processes are seldom "true" first-order AR types. The AMFPE meant for colored noise did not show any significant advantage over FPE in the current study. In our view FPE emerges as a reliable test criterion. Further, the entropy test turns out to be an effective pre-estimation test method.

Among the subjective tests investigated, fit error tests tend to be highly biased and are considered to be of limited utility. The

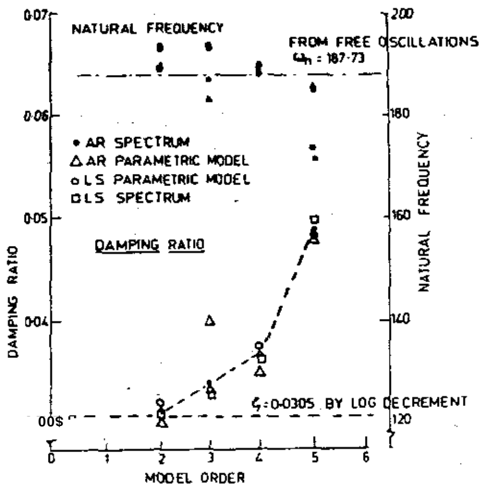



Fig. 3. Comparison of system constants.

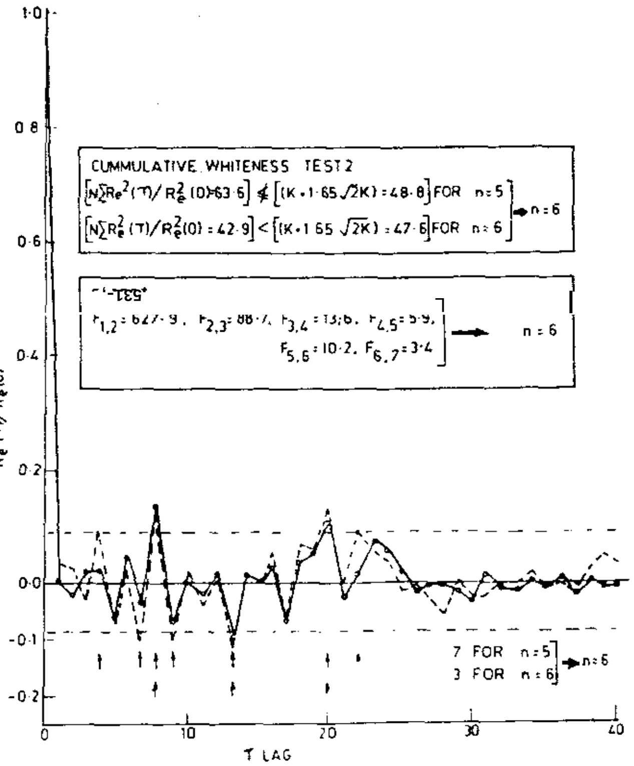



Fig. 4. Subjective tests for AR model of human operator.

$F$-test becomes objective once a level of significance is assigned. Comparison of whiteness tests 1 and 2 reveal no marked difference for the data base used in this study. The study further indicates that pole-zero cancellation in LS models do aid in the selection of lower equivalent model orders when so desired.

Based on the experience gained in the study, the following working rule is considered adequate for selection of the proper model order to fit typical experimental data.

*Order determination:*

evaluate entropy criterion (AR only),
evaluate FPE,
perform $F$-test,
check for pole-zero cancellations.

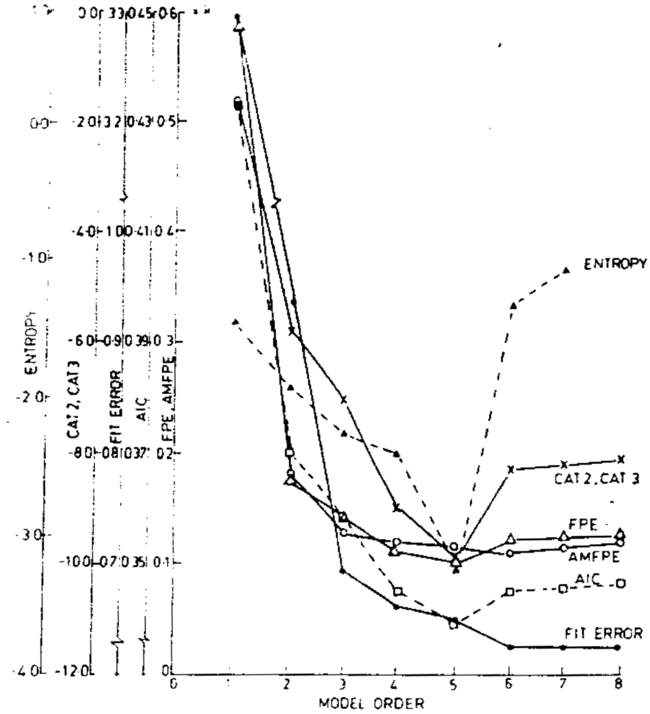
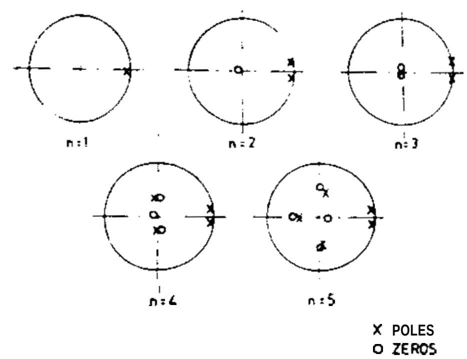



Fig 5. AR model for human performance-order determination




| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $F_{n_1 n_2}$ | 25523.2 | 0.389 | - | - | - |
| % FIT ERROR | 3.128 | 0.030 | 0.030 | 0.046 | 0.081 |
| % DETERMINISTIC FIT ERROR | 94.070 | 17.710 | 19.170 | 72.73 | - |

Fig. 6. LS model for human performance-order determination.

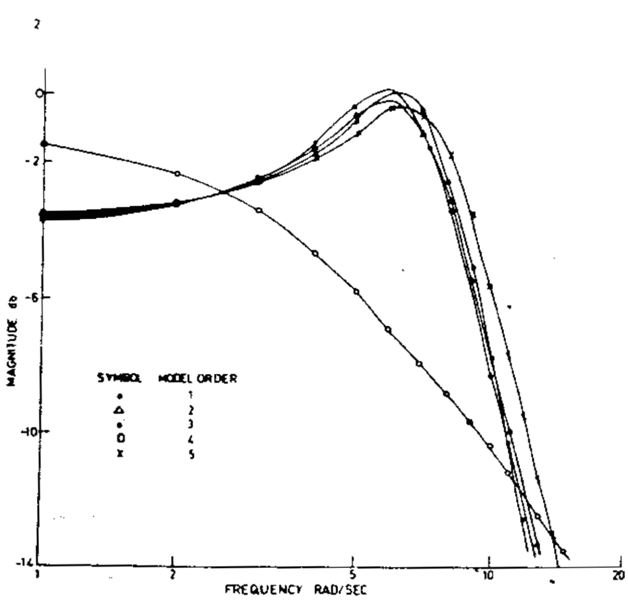



Fig. 7. Comparison of frequency plots for LS models of human operator.

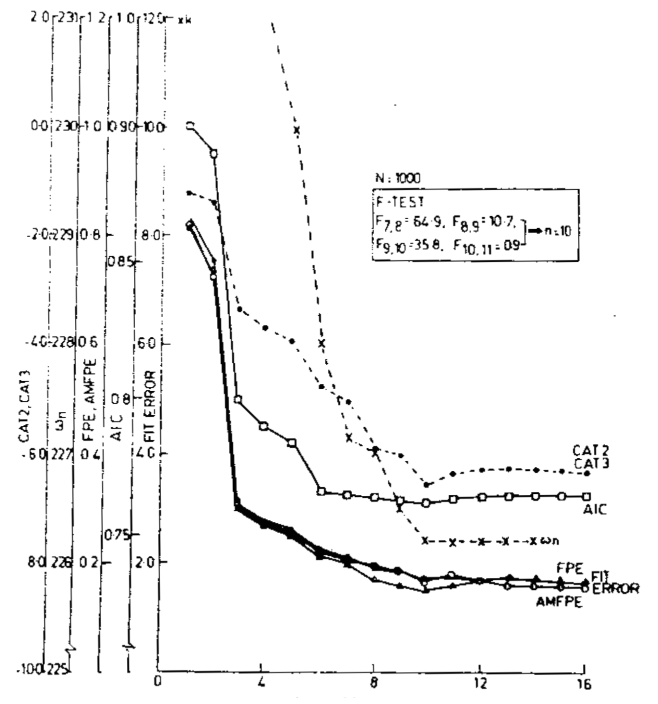



Fig 8 Criteria for AR model order determination of random response. $N =$ two.

*Model validation:*

time **history** prediction,
test residuals for whiteness

## V. Conclusion

The ambiguous problem of the determination of the model order of practical dynamical systems with *a priority* assumed **AR** or LS structure was analyzed using I2 contemporary model order test criteria. The data base was derived from physical situations **with** widely different dynamical characteristics. **This** model determination effort has provided a basis on which the utility of the different test criteria can be compared and evaluated. For the typical dynamical system considered, and similar situations, a minimal set of test criteria has been proposed **as** a working procedure to determine and validate realistic model orders without expending significant computational effort.

## Nomenclature

| | |
|---|---|
| AIC | Akaike's information criterion. |
| AMFPE | Modified final prediction error. |
| **AR** | Autoregressive. |
| **ARMA** | Autoregressive moving average. |
| CAT | Criteric autoregressive transfer function. |
| F | Statistical $F$-test symbol |
| FPE | Final prediction error. |

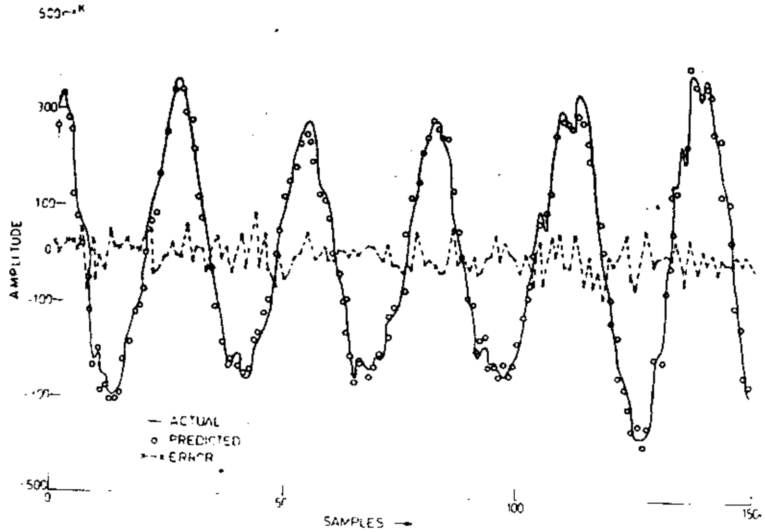



Fig. 9. Actual, predicted, and error time histories of random response.

**LS** Least squares.
$n$ Model order of AR/LS description
$N$ Number of samples.
$q^{-1}$ Back shift operator.
$r$ Residual of estimation procedure.
$R_e$ Autocorrelation of residual.
$V$, Loss function.
$Y$ System output.
$\omega_n$ Natural frequency, radians, second.
$\zeta$ Damping ratio.
$\sigma^2$ Variance.
$\{A(q^{-1}), B(q^{-1})\}$ Polynomials with regression coefficients of the output and input sequences.

## APPENDIX

The modified final prediction error (AMFPE) is evaluated for AR models as [21]

$$\text{AMFPE} = \frac{1 + A}{1 - A}$$

where

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 v}{\partial a_i \partial a_j} R_{yy}(i-j)$$

and

$$R_{yy}(i-j) = \frac{1}{N-q} \sum_{q+1}^{N} Y_{k-i} Y_{k-j}$$

$$q = \max(i, j).$$

The entropy criterion for AR models is evaluated from the correlation matrix $[S_i]$ as [22]

$$\text{entropy} = \log \frac{N-1}{N-2n-1} + \log |S_{i-1}| - \log |S_i|$$

where

$$S_i = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{i-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{i-2} \\ \vdots & & & & \\ \vdots & & & & \\ \rho_{i-1} & \rho_{i-2} & & \cdots & 1 \end{bmatrix}.$$

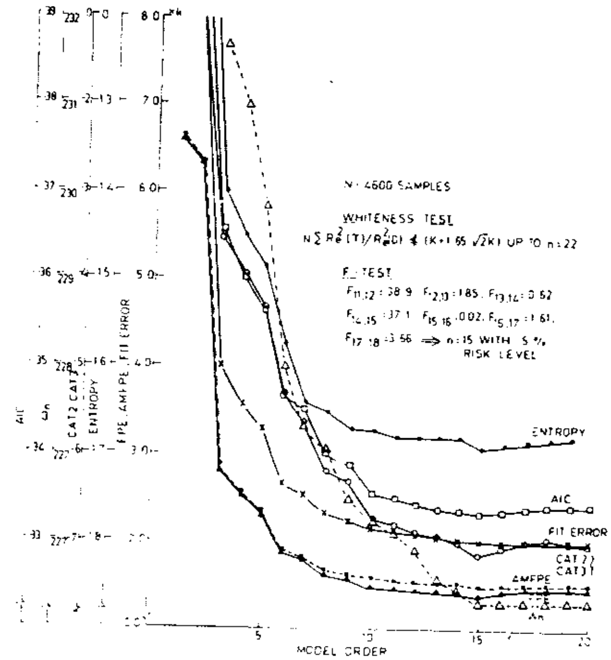



Fig. 10. Order test criteria for AR model of random response, $N = 4600$.

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# Near-Nash Feedback Control of a Composite System with a Time-Scale Hierarchy

UMIT OZGUNER, MEMBER, IEEE

*Abstract*—A large-reale composite system structure encountered in economic systems, manufacturing processes, power systems, etc., which has a two-time-scale hierarchy, is considered. A dynamic game with quadratic cost criteria associated with each subsystem is set up and the Nash feedback strategy analyzed. It is shown that a near-Nash solution can he obtained from low order independent control problem, in the fast time-scale and an independent game, related to a low order subsystem, played in the slow time scale. Furthermore, a decentralized information structure turns out to be sufficient to implement the feedback controls.

## I. INTRODUCTION

An important concept in the area of analysis and control of large-scale interconnected systems is that of *hierarchy*. Although this concept can be viewed in a number of ways, one special class, that of a *time-scale hierarchy*, is especially suited to mathematical formulation. Interconnected systems with an inherent time-scale hierarchy also appear a lot in practice, especially in economics. Consider a model of a number of companies together with the effects of government regulation. One can view the companies' day to day decisions regarding their pricing policy, etc., as constituting a lower level fast time-scale problem, whereas the regulating decision of the government plus the interactions through the market are higher level slow phenomena. Consider also a hierarchy (Fig. 1) formed, say, within a company to carry out a specific project. Each level will be acting on a faster time scale than the one above, and slower than one below. If the tasks in each level are uncoupled, necessary *aggregate* information will be supplied from above and a direct information link will not have to be set up.

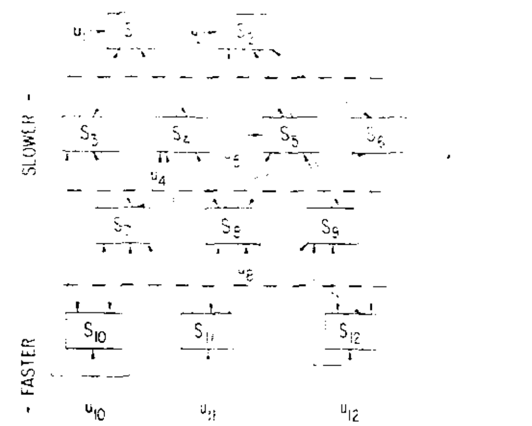


Fig. 1. Four-level hierarchy of interconnected systems.

Recent results in singularly perturbed systems [1], [2] have provided a suitable framework lor a mathematical model for the above. Sandell *et al.* [3] have indicated that two-time-scale filtering might be viewed as a hierarchy. A presentation of the quadratic regulator for multi-time-scale Interconnected systems with due consideration of the hierarchy involved and the necessary information transfer was given by Özgüner [4], [5]. The observer problem was also investigated [6]. The main extension provided to the results of Chow–Kokotovic [1], [2] is based on assuming a structure where uncoupled subsystems exist in any time scale. This results in the analysis/control problem also decoupling in each time scale. Furthermore, the necessary information transfer to implement the near-optimal control is clearly identified.

Especially in view of the applications in economics as indicated, for a number of problems a dynamic game setting seems more natural than the optimal control problem solved by Özgüner in [5]. Some interest has recently been shown in the area of problems lor singularly perturbed systems. Gardner and Cruz and Khalil and Kokotovic [8] investigated the well-posedness Nash solutions and Özgüner [9] considered the simplificat provided by specific cost criteria.

In this correspondence we will consider near-Nash feedb solutions to a quadratic game for a two-time-scale system with multiple subsystems in the lower (laster) level in the hierar The subsystems in the lower level will he shown to have cor (regulator) problems to solve in the fast time scale, whereas actual game is in fact played in the slow time scale. All regu problems and the game will be independent of each other ar much lower order than the original problem. Furthermore feedback solutions will require an inlormation pattern comp with rhe interconnection structure.

## II. THE TWO-TIME-SCALE MULTISUBSYSTEM GAME

We consider a specific structure of $m-1$ lower level, ently "fast" subsystems interconnected through the dynar a higher level "slow" subsystems (Fig. 2). The mather

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The author is with the Department of Electrical Engineering, Ohio State University, Columbus, OH 43210.

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