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# On the oscillation of solutions of stochastic difference equations

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**Abstract**

This paper considers the pathwise oscillatory behaviour of the scalar nonlinear stochastic difference equation

$$X(n+1) = X(n) - F(X(n)) + G(n, X(n))\xi(n+1), \quad n = 0, 1, \dots,$$

with non-random initial value $X_0$. Here $(\xi(n))_{n\geq 0}$ is a sequence of independent random variables with zero mean and unit variance. The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are presumed to be continuous.

**Keywords:** stochastic difference equations; almost sure oscillations

**MSC(2000):** 34F05, 60H10, 65C05, 93E15

## 1 Introduction

Oscillations occur not only in physical systems but also in biological systems and in human society. Neural oscillations refers to rhythmic or repetitive neural activity in the central nervous system. The insulin concentration in blood increases after meals and gradually returns to basal levels during 1-2 hours. So the investigation of the presence of oscillations is very important for the applications. Since the real systems are subjects to the random disturbances, it is also important to investigate the oscillations in the stochastic systems.

The oscillation of the solutions of deterministic difference equation has been discussed in many papers; a comprehensive survey of this literature is contained in [1]. In this paper we concentrate on the oscillation of solutions to scalar stochastic non-linear difference equations. Aside from results concerning the preservation of oscillation and non–oscillation in solutions of discretised linear stochastic delay differential equations in [2, 3] there is little known about the oscillation of the solution of stochastic non-linear difference equations. However, two related questions, positivity and boundedness of some linear and nonlinear logistic stochastic difference equations has been discussed in [6], [7] and [8]. Recently, results on the oscillation of the solutions of the stochastic non-linear and linear difference equations with state-independent noise were proved in [4]. We are going to discuss some of these results in this paper.

We say that the solution is oscillatory if it changes sign infinitely many time. In this paper we consider the oscillatory behaviour of sample paths of the stochastic difference equation

$$X(n+1) = X(n) - F(X(n)) + G(n, X(n))\xi(n+1), \quad n = 0, 1, \dots, \tag{1}$$

with non-random initial value $X_0$, sequence $(\xi(n))_{n\geq 0}$ of independent random variables with zero mean and unit variance, and continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. We are going to distinguish state-dependent and state-independent equations. The state-independent equation, i.e. when $G(n, u)$ does not depend on the second variable, was investigated in the paper [4]. In Section 3 we present some results from [4] and discuss open problems, connected with state-independent case. Section 4 is devoted to the state-dependent equation, when function $G(n, u)$ does depend on the second variable but does not depend on the first one. In Section 2 we give some necessary definitions and assumptions.

## 2 Definitions and Assumptions

Throughout this paper, we say that a sequence $\nu = \{\nu(n) : n \geq 0\} = (\nu(n))_{n\geq 0}$ is in $l_2$ if $\sum_{n=0}^{\infty} \nu^2(n) < \infty$. We say that the equilibrium point 0 of equation $x(n + 1) = x(n) - f(x(n)), \quad n = 0, 1, 2, \dots$ is *hyperbolic* if $f'(0) \neq 0$, and is *non–hyperbolic* if $f'(0) = 0$ (see, e.g., Elaydi [5]).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. We suppose that

**Assumption 2.1.** *$(\xi(n))_{n\in\mathbb{N}}$ is a sequence of independent random variables with distribution functions $F_n$ and with $\mathbb{E}[\xi(n)] = 0$, $\mathbb{E}\left[\xi^2(n)\right] = 1$.*

We suppose that filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is naturally generated, namely that $\mathcal{F}_n = \sigma\{\xi(0), \xi(1), \dots \xi(n)\}$. Among all sequences $(X(n))_{n\in N}$ of random variables we distinguish those for which $X(n)$ are $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. We use the standard abbreviation "*a.s.*" for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. For more details on stochastic concepts and notations, the reader may consult [9].

**Definition 2.2.** *The solution $(X(n))_{n\in\mathbb{N}}$ of equation (1) is said to be a.s. oscillatory if*

$$\mathbb{P}\{X(n) < 0 \quad i.o\} = 1, \quad \mathbb{P}\{X(n) > 0 \quad i.o\} = 1. \tag{2}$$

## 3 State-independent equation

In this section we discuss some results from [4]. We consider the equation

$$X(n+1) = X(n) - f(X(n)) + \sigma(n)\xi(n+1), \quad n = 0, 1, \dots, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$xf(x) > 0, \quad x \neq 0, \quad f(0) = 0. \tag{4}$$

We note that (3) is a partial case of the equation (1), when $G(n, u) = \sigma(n)$.

To analyze the effect of the introduction of the noise term, we often find it instructive to compare the oscillatory behaviour of (3) with its unperturbed deterministic counterpart

$$x(n+1) = x(n) - f(x(n)), \quad n = 0, 1, 2, \ldots. \tag{5}$$

In [4] it was shown that when the noise is persistent or decays slowly in the sense that $\sigma \notin \boldsymbol{\ell_2}$, the solution $X$ of (3) oscillates almost surely regardless of the function $f$. In particular the following result was proved there.

**Theorem 3.1.** *Let $f$ be a continuous function satisfying* (4)*. Let $(\xi(n))_{n\geq 0}$ be identically distributed random variables satisfying Assumption 2.1. If $(X(n))_{n\geq 0}$ is a solution of (3), and $(|\sigma(n)|)_{n\geq 0}$ is a non–increasing sequence such that*

$$\sum_{n=0}^{\infty} \sigma^2(n) = \infty. \tag{6}$$

*Then $(X(n))_{n\geq 0}$ oscillates a.s.*

The proof of Theorem 3.1 was based on the Law of the Iterated Logarithm for the weighted sums of independent and identically distributed random variables. When $\xi(n)$ are just independent but not identically distributed, a different approach, based on the use of Central Limit Theorem and Kolmogorov's Zero-One Law (see e.g. [9]) was applied in [4]. It appears that in this situation to guarantee oscillation we need to demand a more stringent restriction on the noise intensities than (6). This extra restriction depends on the particular distribution of the $\xi$'s. For example, in order to ensure the oscillation of solutions when the probability densities of each $\xi(n)$ decay polynomially with degree $m$, it is sufficient that $\sigma(n)$ decays to zero more slowly than $n^{-\frac{m-3}{2(m-1)}}$. Also, when the probability densities of each $\xi(n)$ decay exponentially, it suffices that $\sigma(n)$ decays to zero more slowly then some power of $\ln n$.

When the intensity of the noise perturbation decays more quickly (i.e., when $\sigma \in \boldsymbol{\ell_2}$) it is possible for the solution of (3) to be oscillatory or non–oscillatory according to the form of the mean–reversion which results from the presence of the function $f$. When the function $f$ grows relatively quickly for large departures from the equilibrium level in the sense that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 2, \tag{7}$$

the following results was proved in [4].

**Theorem 3.2.** *Suppose that $f$ is a continuous function obeying* (4) *and* (7)*. Suppose that $\sigma \in l_2$. Let $(X(n))_{n\geq 0}$ be a solution of (3) with initial condition $X_0 \in \mathbb{R}$. Then for all $\gamma \in (0,1)$ there exist an event $\Omega_\gamma \subseteq \Omega$ with $\mathbb{P}[\Omega_\gamma] > 1-\gamma$, and a number $d(\gamma) > 0$, such that for all $|X_0| > d(\gamma)$ we have*

$$\liminf_{n\to\infty} X(n,\omega) = -\infty, \quad \limsup_{n\to\infty} X(n,\omega) = \infty, \quad \textit{for a.s. } \omega \in \Omega_\gamma.$$

We note that all solutions of in corresponding deterministic equation (5) with sufficiently large initial values oscillate. Thus the presence of the noise perturbation does not change significantly the oscillatory property of the solution and the oscillations of solution $X(n)$ of equation (3) occur independently on the behaviour of $\sigma$ and $\xi$.

Aside from general forms of the function $f$, paper [4] considers the oscillation and non–oscillation of solutions of (3) in the case when the equation is linear (i.e., $f(x) = ax$ for $a > 0$) or when the deterministic equation (5) has a hyperbolic equilibrium at zero (i.e., $f'(0) \neq 0$). Theorem 3.3 presented below furnishes us with a complete picture of the oscillatory behaviour of the linear equation in the important special case when the random process $(\xi(n))_{n\geq 0}$ is Gaussian. More precisely, it is supposed that in the equation

$$X(n+1) = X(n) - aX(n) + \sigma(n)\xi(n+1), \quad n = 0,1,\ldots, \quad X_0 \in \mathbb{R}, \tag{8}$$

$(\xi(n))_{n\in\mathbb{N}}$ are independent normal random variables, $a \geq 0$ and $\sigma \not\equiv 0$.

**Theorem 3.3.** *Suppose that $(\xi(n))_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed standard normal random variables. Let $(X(n))_{n\geq 0}$ be a solution of (8).*

*If $\sigma \notin \boldsymbol{\ell_2}$, then $X$ oscillates a.s.*

*Suppose $\sigma \in \boldsymbol{\ell_2}$.*

*(i) Let $1 \leq a$. Then $X$ is a.s. oscillatory.*

*(ii) Let $0 \leq a < 1$.*

*(a) If $\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j) < +\infty$, then $X$ is a.s. non-oscillatory;*

*(b) If $\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j) = +\infty$, then $X$ is a.s. oscillatory.*

Theorem 3.3 shows that while $\sigma \in \boldsymbol{\ell_2}$ is not sufficient to ensure the non–oscillation of solutions of (3), that the transition between oscillation and non–oscillation occurs at a critical increasing weight function in a weighted space of $\boldsymbol{\ell_2}$ sequences; in other words, it suggests that as far as oscillation of solutions of (3) is concerned, particular attention should be given to noise intensities in $\boldsymbol{\ell_2}$ and related weighted $\boldsymbol{\ell_2}$ spaces.

This suggests a general principle guiding the oscillatory behaviour of solutions of mean–reverting stochastic difference equations of the form (3) where the noise perturbation is independent of the state. The conjectured principle is that the introduction of such a noise perturbation into (5) can preserve oscillation if it is already present, and can induce it if oscillation is absent. However, it will not prevent oscillation if it already present in solutions of (5).

Based on a rather complete picture from the linear case we conjecture that when noise intensity decays quite rapidly ($\sigma \in \boldsymbol{\ell_2}$) the rate of decay of $\sigma$ to zero and the speed of mean reversion to zero given by the behaviour of $f$ local to zero, will interact to produce either oscillation or non-oscillation of the solution $(X(n))_{n\geq 0}$. In does not appear, based on the evidence from the linear case, that the rate of decay of the tails of the distributions of $(\xi(n))_{n\geq 0}$ influence greatly the presence of oscillatory or non–oscillatory solutions. We hope that the connection between the rate of decay of $\sigma$ and the rate of decay of the unperturbed equation will afford us the possibility of determining the rate of decay to zero of the solution $(X(n))_{n\geq 0}$ of (3); specifically we conjecture that a change in the pathwise rate of decay of the solution when the decay of $\sigma$ reaches a critical rate is coincident with a change in the behaviour of solutions from non–oscillatory to oscillatory. Some results in this direction about stochastic equation with non-hyperbolic $f$ are already obtained, but they are not included in this paper.

## 4 State-dependent equation

In this section we consider stochastic nonlinear difference equation

$$X(n+1) = X(n) + f(X(n)) + g(X(n))\xi_{n+1}, \quad n = 1, 2, \dots, \tag{9}$$

where $(\xi_n)_{n\in\mathbf{N}}$ are i.i.d. $\mathcal{N}(0,1)$ random variables, and $X_0 > 0$. We suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, nonrandom functions with

$$f(0) = g(0) = 0. \tag{10}$$

We discuss conditions which guarantee non-positivity and oscillations of solution of (9). In fact, our results are also valid when all $\xi_n$ are i.i.d. random variables with symmetric continuous probability distribution $F$ satisfying

$$\forall n \in \mathbb{N} \ : \ F_{\xi_n} = F_{-\xi_n}, supp(F_{\xi_n}) = \mathbb{R}^1. \tag{11}$$

In the following subsection 4.1 we explore conditions on $f$ and $g$ which guarantee that solution $x_n$ to (9) with positive initial value $X_0 > 0$ becomes negative a.s. We use obtained result in subsection 4.2 when proving oscillations of solutions $X_n$ with probability 1. In subsection 4.3 we present two computer simulations for state-independent and for state-dependent equations whose solutions demonstrate oscillation.

### 4.1 Non-positivity with probability 1

In this subsection we suppose that

$$\inf_{u>0}\left\{-\frac{u+f(u)}{|g(u)|}\right\} = M > -\infty. \tag{12}$$

**Lemma 4.1.** *Let conditions (10), (11) and (12) hold. Assume that $X(n)$ is any solution to (9) with positive initial condition $X_0 > 0$. Then there exists an a.s. finite stopping time $\tau_0 : \Omega \to \mathbb{N}$ such that*

$$X(\tau_0(\omega), \omega) \le 0 \qquad a.s.$$

### 4.2 Oscillation with probability 1

In this section, in addition to condition (12) we need the following condition to be fulfilled

$$\sup_{u\le 0}\left\{-\frac{u+f(u)}{|g(u)|}\right\} = L < +\infty. \tag{13}$$

**Theorem 4.2.** *Let conditions (10), (12) and (13) hold. Then the solution $X(n)$ to (9) with arbitrary nonrandom initial value $X_0 \neq 0$ oscillates a.s. around* $0$.

*Proof.* Suppose that $X_0 > 0$. Lemma 4.1 proves that $X(\tau_0(\omega), \omega) \le 0$ for some a.s. finite stopping time $\tau_0 \in \mathbb{N}$. We define

$$p_L = \mathbb{P}\left\{\omega \in \Omega : \zeta(\omega) \le L\right\}, \tag{14}$$

where $\zeta$ is a $\mathcal{N}(0,1)$–distributed random variable. For each $n \in \mathbb{N}$, define

$$\bar{B}_n = \left\{\omega \in \Omega : X(i,\omega) \le 0, \forall i = \tau_0(\omega), \tau_0(\omega)+1, \dots, \tau_0(\omega)+n\right\}. \tag{15}$$

Since

$$B_n = \left\{\omega \in \Omega : \exists i \text{ among the integers } \tau_0(\omega)+1, \dots, \tau_0(\omega)+n : X(i,\omega) > 0\right\}, \tag{16}$$

we can conclude from Borel-Cantelli lemma that solution $X$ to equation (9) becomes positive with probability 1 on the interval $(\tau_0(\omega), +\infty)$ if

$$\sum_{n=0}^{\infty} \mathbb{P}(\bar{B}_n) < +\infty. \tag{17}$$

To see this, we estimate $\mathbb{P}(\bar{B}_n)$ from above by

$$\begin{aligned}\mathbb{P}(\bar{B}_n) &= \sum_{N=1}^{\infty} \mathbb{P}\left\{X(i,\omega) \le 0, \forall i = \tau_0(\omega)+1, \dots, \tau_0(\omega)+n \middle| \tau_0 = N\right\} \mathbb{P}\{\tau_0 = N\} \\ &= \sum_{N=1}^{\infty} \left( \prod_{j=N+1}^{n+N} \mathbb{P}\left\{X(j-1) + f(X(j-1)) + g(X(j-1))\xi_j \le 0 \middle| \right.\right. \\ &\qquad \left.\left. X(j-1) \le 0, \tau_0 = N\right\} \right) \mathbb{P}\{\tau_0 = N\}.\end{aligned}$$

If $g(X(j-1,\omega)) > 0$, the inequality

$$X(j-1,\omega) + f(X(j-1,\omega)) + g(X(j-1,\omega))\xi_j(\omega) \le 0 \tag{18}$$

is equivalent to $\xi_j(\omega) \le -\dfrac{X(j-1,\omega) + f(X(j-1,\omega))}{g(X(j-1,\omega))}$. If $g(X(j-1,\omega)) < 0$, the inequality (18) is equivalent to $-\xi_j(\omega) \le -\dfrac{X(j-1,\omega) + f(X(j-1,\omega))}{-g(X(j-1,\omega))}$. We note that $-\xi_n$ has also $\mathcal{N}(0,1)$–distribution, and hence its distribution $F_{\xi_n}$ is symmetric and supported on the entire real axis $\mathbb{R}^1$. Thus, in both cases inequality (18) is equivalent to the inequality

$$\zeta \le -\frac{X(j-1,\omega) + f(X(j-1,\omega))}{|g(X(j-1,\omega))|}.$$

with $\mathcal{N}(0,1)$-distributed random variable $\zeta$, which is independent on $x_{j-1}$ (since $\xi_j$ is independent of $\xi_0, \xi_1, \dots, \xi_{j-1}$). We note that $\zeta$ is also independent of events $\{\tau_0 = N\}$ which belong to the $\sigma$-algebra $\mathcal{F}_N$ with $N < j$. Thus

$$\prod_{j=N+1}^{n+N} \mathbb{P}\left\{ X(j-1) + f(X(j-1)) + g(X(j-1)\xi_j(\omega) \le 0 \middle| X(j-1) \le 0, \tau_0 = N \right\}$$
$$\le \prod_{j=N+1}^{n+N} \mathbb{P}\left\{ \omega\in\Omega : \zeta \le L \middle| X(j-1) \le 0, \tau_0 = N \right\} = \prod_{j=N+1}^{n+N} \mathbb{P}\left\{\omega\in\Omega : \zeta \le L\right\}$$
$$= p_L^n.$$

Therefore, we have

$$\mathbb{P}(\bar{B}_n) \le \sum_{N=1}^{+\infty} \left( \prod_{j=N+1}^{n+N} \mathbb{P}\left\{\zeta \le L\right\} \right) \mathbb{P}\{\tau_0 = N\} \le p_L^n \sum_{N=1}^{+\infty} \mathbb{P}\{\tau_0 = N\} = p_L^n.$$

Thus, $\displaystyle\sum_{n=0}^{\infty} \mathbb{P}(\bar{B}_n) \le \frac{1}{1-p_L} < +\infty$, and we conclude that there exists an a.s. finite stopping time $\tau_1 > \tau_0$ such that $X(\tau_1(\omega),\omega) > 0$ a.s. Repeating the same approach and applying (12) and (13), respectively, using mathematical induction we obtain that $X(n)$ changes sign infinitely often and with probability 1. Similarly, we verify the assertion for the case $X(0) < 0$ (just start with the negative event $\{\tau_0 = 0\}$ and proceed as above). Thus, the proof of Theorem 4.2 is complete. □

### 4.3 Computer simulations

In this section we consider computer simulations for two stochastic difference equations. In both equations $\{\xi(n)\}$ is suppose to be a sequence of independent $\mathcal{N}(0,1)$-distributed random variables. The first equation

$$X(n+1) = X(n) - X^3(n) + \sigma(n)\xi(n+1),$$

is non-linear with state independent noise and with the drift coefficient satisfying condition (7). We consider two different coefficients $\sigma_n$: coefficient $\sigma_n = n^{-0.6}$

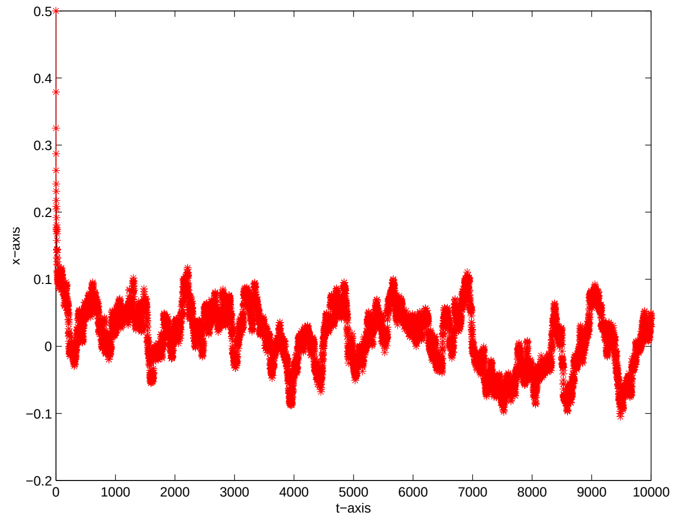


Figure 1: graph of $x(n+1) = x(n) - x^3(n) + n^{(-0.6)} * \xi(n+1)$

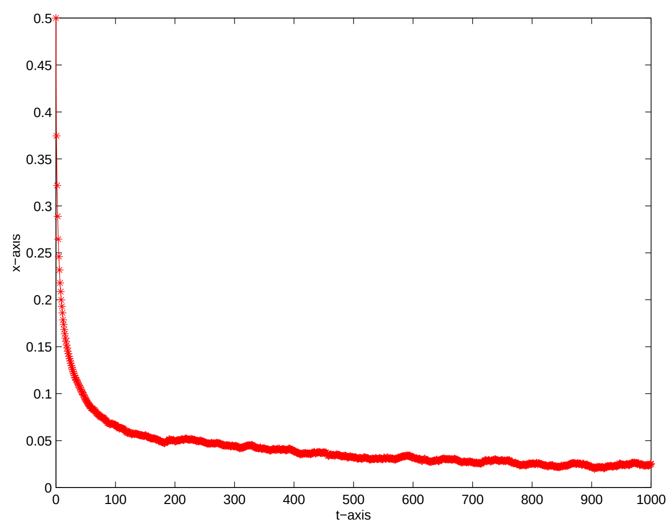


Figure 2: graph of $x(n+1) = x(n) - x^3(n) + n^{(-1.1)} * \xi(n+1)$

in the Figure 1 decays more slowly then coefficient $\sigma_n = n^{-1.1}$ in the Figure 2. However both $\sigma_n$ belong to $\boldsymbol{\ell_2}$. The solution in Figure 1 clearly demonstrates oscillation while the solution in the Figure 2 does not.

The second equation

$$x(n+1) = x(n) - 2x(n) + 0.3x(n)\xi(n+1)$$

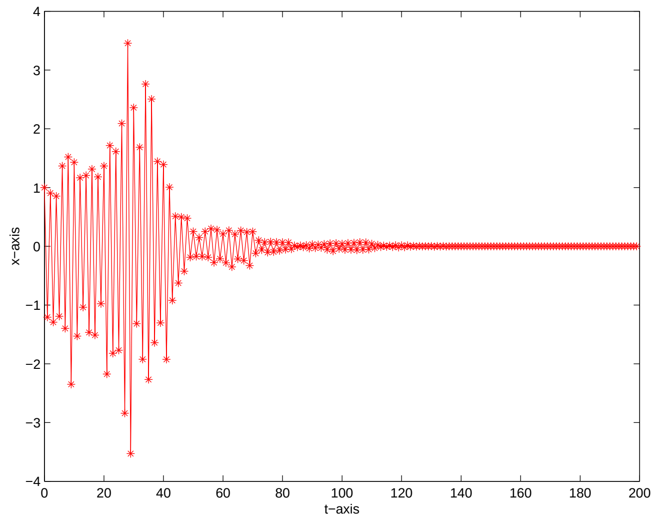


Figure 3: graph of $x(n+1) = x(n)(1 - a + d * \xi(n+1))$

is a linear stochastic difference equation with state-dependent noise. Both conditions (12) and (13) clearly hold. The solution in the Figure 3 demonstrates oscillations.

## References

[1] Agarwal, R. P., Bohner, M., Grace, S.R., and O'Regan, D. Discrete Oscillation Theory. Contemporary Mathematics and Its Applications, Vol. 1, 2006.

[2] Appleby, J. A. D., and Kelly, C. Oscillation of solutions of a nonuniform discretisation of linear stochastic differential equations with vanishing delay, Dynam. Con. Disc. Imp. Sys. Ser A Math Anal., 13B, suppl. (2006), pp. 535–550.

[3] Appleby, J. A. D., and Kelly, C. Spurious oscillation in a uniform Euler discretisation of linear stochastic differential equations with vanishing delay, Journal of Computational and Applied Mathematics, 205(2) (2007), pp.923-935.

[4] Appleby, J. A. D., and Rodkina, A. On the Oscillation of Solutions of Stochastic Difference Equations with State-Independent Perturbations, International Journal of Difference Equations, Vol.2 (2), (2007), pp. 139–164.

[5] Elaydi, S. N. An Introduction to Difference Equations (2nd edition), Springer, Berlin, 1999.

[6] Schurz,H. Numerical regularization for SDEs: Construction of nonnegative solutions, Dynam. Syst. Appl. Vol. 5 (1), (1996), pp. 323-352.

[7] Schurz, H. Modelling and analysis of stochastic innovation diffusion, Z. Angew. Math. Mech. Vol. 76 (Suppl. 3, I-XV), (1996), pp. 366-369.

[8] Schurz, H. Modeling, analysis and discretization of stochastic logistic equations, Int. J. Numer. Anal. Model, Vol. 4 (2), (2007), pp. 178-197.

[9] Shiryaev, A.N. Probability (2nd Ed.), Springer, New York. 1996.

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