

CLONES WITH FINITELY MANY RELATIVE \mathcal{R} -CLASSES

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ABSTRACT. For each clone \mathcal{C} on a set A there is an associated equivalence relation analogous to Green's \mathcal{R} -relation, which relates two operations on A if and only if each one is a substitution instance of the other using operations from \mathcal{C} . We study the clones for which there are only finitely many relative \mathcal{R} -classes.

1. INTRODUCTION

Green's relations play a central role in semigroup theory. Two elements a, b of a monoid M are related by Green's \mathcal{R} -relation if and only if they generate the same right ideal $aM = bM$. In particular, if M is a transformation monoid on a set A , then two elements $f = f(x)$ and $g = g(x)$ of M are \mathcal{R} -related exactly when $f(h_1(x)) = g(x)$ and $g(h_2(x)) = f(x)$ for some $h_1, h_2 \in M$, that is, each one of f, g is a substitution instance of the other by transformations from M . For example, if $M = T_A$ is the full transformation monoid on A , then $f \mathcal{R} g$ if and only if f, g have the same range.

Henno [9] generalized Green's relations to Menger algebras (essentially, abstract clones, the multi-variable versions of monoids), and described Green's relations on the clone \mathcal{O}_A of all operations on A for each set A . He proved that two finitary operations on A are \mathcal{R} -related if and only if they have the same range.

Relativized versions of Green's \mathcal{R} -relation on the clone $\mathcal{O}_{\{0,1\}}$ of Boolean functions have been used in computer science to classify Boolean functions. In [21] and [22] a Boolean function g is defined to be a *minor* of another Boolean function f if and only if g can be obtained from f by substituting for each variable of f a variable, a negated variable, or one of the constants 0 or 1. A more restrictive notion of Boolean minor, namely when negated variables are not allowed, is employed in [5] and [23], while in the paper [8] two n -ary Boolean functions are considered equivalent if they are substitution instances of each other with respect to the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ or the affine general linear group $\mathrm{AGL}(n, \mathbb{F}_2)$ where \mathbb{F}_2 is the two-element field.

The notions of 'minor' and ' \mathcal{R} -equivalence' for operations on a set A can be defined relative to any subclone \mathcal{C} of \mathcal{O}_A as follows: for $f, g \in \mathcal{O}_A$, g is a \mathcal{C} -minor of f if g can be obtained from f by substituting operations from \mathcal{C} for the variables of f , and g is \mathcal{C} -equivalent to f if f and g are \mathcal{C} -minors of each other. Thus, for example, Henno's \mathcal{R} -relation on \mathcal{O}_A is nothing else than \mathcal{O}_A -equivalence, and the

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concepts of Boolean minor mentioned in the preceding paragraph are the special cases of the notion of \mathcal{C} -minor where \mathcal{C} is the essentially unary clone of Boolean functions generated by negation and the two constants, or by the two constants only. Further applications of \mathcal{C} -minors and \mathcal{C} -equivalence where \mathcal{C} is a clone of essentially unary operations can be found in [3], [4], and [14].

The question we are interested in is the following:

Question. For which clones \mathcal{C} are there only finitely many relative \mathcal{R} -classes?

That is, we want to know for which clones \mathcal{C} it is the case that the \mathcal{C} -equivalence relation on \mathcal{O}_A has only finitely many equivalence classes. Let \mathfrak{F}_A denote the set of all such clones on A . It is easy to see that \mathcal{C} -equivalent operations have the same range, therefore if A is infinite, then there will be infinitely many \mathcal{C} -equivalence classes for every clone \mathcal{C} , so \mathfrak{F}_A is empty. If A is finite, then the result of Henno [9] mentioned above implies that $\mathcal{O}_A \in \mathfrak{F}_A$. It is not hard to see that \mathfrak{F}_A is an order filter (up-closed set) in the lattice of all clones on A (Proposition 2.1). Moreover, if $|A| > 1$ then the clone \mathcal{P}_A of projections fails to belong to \mathfrak{F}_A , because \mathcal{P}_A -equivalent operations have the same essential arity (i.e., depend on the same number of variables), and on a set with more than one element there exist operations of arbitrarily large essential arity. Thus the order filter \mathfrak{F}_A is proper.

The results of this paper show that the family \mathfrak{F}_A of clones is quite restricted. Every clone \mathcal{C} in \mathfrak{F}_A has to be ‘large’ quantitatively in the sense that it contains a lot of n -ary operations for each n (Proposition 3.3), and it has to be ‘large’ in the sense that there are strong restrictions on the relations that are invariant with respect to the operations in \mathcal{C} (Corollary 3.2).

There is a rich literature of classification results for ‘large’ subclones of \mathcal{O}_A when A is finite (see [11] and the references there) where ‘large’ is usually taken to mean ‘near the top of the lattice of clones on A ’. Our interest in the order filter \mathfrak{F}_A stems from the fact that the property of being in \mathfrak{F}_A is a different kind of ‘largeness’. Since the family \mathfrak{F}_A is quite restricted, the clones in \mathfrak{F}_A may be classifiable. At the same time, \mathfrak{F}_A contains interesting families of clones: e.g., all discriminator clones ([13], see Theorem 2.3) and all clones determined by a chain of equivalence relations on A together with a set of invariant permutations and an arbitrary family of subsets of A (Theorem 4.1).

Using Rosenberg’s description of the maximal clones $\mathcal{M} = \text{Pol } \rho$ on a finite set A (see Theorem 2.2) we determine which maximal clones belong to \mathfrak{F}_A (see Theorem 7.1 and Table 1). Furthermore, for each maximal clone \mathcal{M} that belongs to \mathfrak{F}_A we find families of subclones of \mathcal{M} that also belong to \mathfrak{F}_A . We also investigate which intersections of maximal clones are in \mathfrak{F}_A .

2. PRELIMINARIES

Let A be a fixed nonempty set. If n is a positive integer, then by an n -ary operation on A we mean a function $A^n \rightarrow A$, and we will refer to n as the *arity* of the operation. The set of all n -ary operations on A will be denoted by $\mathcal{O}_A^{(n)}$, and we will write \mathcal{O}_A for the set of all finitary operations on A . For $1 \leq i \leq n$ the i -th n -ary projection is the operation $p_i^{(n)}: A^n \rightarrow A$, $(a_1, \dots, a_n) \mapsto a_i$.

For arbitrary positive integers m and n there is a one-to-one correspondence between the functions $f: A^n \rightarrow A^m$ and the m -tuples $\mathbf{f} = (f_1, \dots, f_m)$ of functions

ρ (h -ary)	$\text{Pol } \rho \stackrel{?}{\in} \mathfrak{F}_A$	Proof
bounded partial order	no	Thm 2.5
prime permutation	yes	Cor 2.4
nontrivial equivalence relation	yes	Thm 4.1
prime affine relation	no	Thm 2.5
central relation		
$h = 1$	yes	Cor 2.4
$2 \leq h \leq A - 2$	no	Thm 5.3
$h = A - 1$	yes	Thm 5.2
h -regular relation		
$h < A $	no	Thm 6.3
$h = A $	yes	Thm 6.1

TABLE 1. The membership of the maximal clones in \mathfrak{F}_A .

$f_i: A^n \rightarrow A$ ($i = 1, \dots, m$) via the correspondence

$$f \mapsto \mathbf{f} = (f_1, \dots, f_m) \quad \text{with} \quad f_i = p_i^{(m)} \circ f \text{ for all } i = 1, \dots, m.$$

In particular, $\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)})$ corresponds to the identity function $A^n \rightarrow A^n$. From now on we will identify each function $f: A^n \rightarrow A^m$ with the corresponding m -tuple $\mathbf{f} = (f_1, \dots, f_m) \in (\mathcal{O}_A^{(n)})^m$ of n -ary operations. Using this convention the *composition* of two functions $\mathbf{f} = (f_1, \dots, f_m): A^n \rightarrow A^m$ and $\mathbf{g} = (g_1, \dots, g_k): A^m \rightarrow A^k$ can be described as follows:

$$\mathbf{g} \circ \mathbf{f} = (g_1 \circ \mathbf{f}, \dots, g_k \circ \mathbf{f}) = (g_1(f_1, \dots, f_m), \dots, g_k(f_1, \dots, f_m))$$

where

$$g_i(f_1, \dots, f_m)(\mathbf{a}) = g_i(f_1(\mathbf{a}), \dots, f_m(\mathbf{a})) \quad \text{for all } \mathbf{a} \in A^n \text{ and for all } i.$$

A *clone* on A is a subset \mathcal{C} of \mathcal{O}_A that contains the projections and is closed under composition; that is, $p_i^{(n)} \in \mathcal{C}$ for all $1 \leq i \leq n$ and $g \circ \mathbf{f} \in \mathcal{C}^{(n)}$ whenever $g \in \mathcal{C}^{(m)}$ and $\mathbf{f} \in (\mathcal{C}^{(n)})^m$ ($m, n \geq 1$). The clones on A form a complete lattice under inclusion. Therefore for each set $F \subseteq \mathcal{O}_A$ of operations there exists a smallest clone that contains F , which will be denoted by $\langle F \rangle$ and will be referred to as the *clone generated by F* .

Clones can also be described via invariant relations. For an n -ary operation $f \in \mathcal{O}_A^{(n)}$ and an r -ary relation ρ on A we say that f *preserves* ρ (or ρ is *invariant* under f , or f is a *polymorphism* of ρ), if whenever f is applied coordinatewise to r -tuples from ρ , the resulting r -tuple belongs to ρ . If ρ is an r -ary relation on A and n is a positive integer, ρ^n will denote the r -ary relation ‘‘coordinatewise ρ -related’’ on A^n ; more formally, for arbitrary n -tuples $\mathbf{a}_i = (a_{i1}, \dots, a_{in}) \in A^n$ ($1 \leq i \leq r$)

$$(\mathbf{a}_1, \dots, \mathbf{a}_r) \in \rho^n \iff (a_{1j}, \dots, a_{rj}) \in \rho \text{ for all } j \text{ (} 1 \leq j \leq n \text{)}.$$

We will say that $\mathbf{f} = (f_1, \dots, f_m) \in (\mathcal{O}_A^{(n)})^m$ *preserves* an r -ary relation ρ on A if each f_i ($1 \leq i \leq m$) does; that is,

$$(\mathbf{a}_1, \dots, \mathbf{a}_r) \in \rho^n \implies (\mathbf{f}(\mathbf{a}_1), \dots, \mathbf{f}(\mathbf{a}_r)) \in \rho^m \text{ for all } \mathbf{a}_1, \dots, \mathbf{a}_r \in A^n.$$

For any family R of (finitary) relations on A , the set $\text{Pol } R$ of all operations $f \in \mathcal{O}_A$ that preserve every relation in R is easily seen to be a clone on A . Moreover, if A

is finite, then it is a well-known fact that every clone on A is of the form $\text{Pol } R$ for some family of relations on A (see, e.g., [1, 6, 11, 15, 20]). If $R = \{\rho\}$, we will write $\text{Pol } \rho$ for $\text{Pol } \{\rho\}$.

Throughout the paper we will use the following additional notation concerning operations and relations. The constant tuple (a, \dots, a) of any length is denoted by \bar{a} (the length will be clear from the context). If θ is an equivalence relation on A , then the equivalence class containing $a \in A$ is denoted by a/θ . For any operation f on A that preserves θ , f^θ denotes the natural action of f on the set A/θ of θ -classes. Furthermore, for any set F of operations contained in $\text{Pol } \theta$ we write F^θ for the set $\{f^\theta : f \in F\}$. The range of an arbitrary function φ will be denoted by $\text{Im } \varphi$.

Now let \mathcal{C} be a fixed clone on a set A of any cardinality. For arbitrary operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that

- f is a \mathcal{C} -minor of g , in symbols $f \leq_{\mathcal{C}} g$, if $f = g \circ \mathbf{h}$ for some $\mathbf{h} \in (\mathcal{C}^{(n)})^m$;
- f and g are \mathcal{C} -equivalent, in symbols $f \equiv_{\mathcal{C}} g$, if $f \leq_{\mathcal{C}} g$ and $g \leq_{\mathcal{C}} f$.

It is easy to verify (see [13]) that $\leq_{\mathcal{C}}$ is a quasiorder on \mathcal{O}_A , and hence $\equiv_{\mathcal{C}}$, the intersection of $\leq_{\mathcal{C}}$ with its converse, is an equivalence relation on \mathcal{O}_A .

\mathfrak{F}_A will denote the collection of all clones \mathcal{C} on A such that the equivalence relation $\equiv_{\mathcal{C}}$ has only finitely many equivalence classes. As we discussed in the Introduction, if A is infinite, then $\mathfrak{F}_A = \emptyset$, while if A is finite and $|A| > 1$, then the clone \mathcal{O}_A of all operations is in \mathfrak{F}_A , and the clone \mathcal{P}_A of projections is not.

From now on we will assume that A is finite. The next proposition contains some useful basic facts about \mathfrak{F}_A .

Proposition 2.1 ([13]). *Let \mathcal{C} be a clone on a finite set A .*

- (i) $\mathcal{C} \in \mathfrak{F}_A$ if and only if there exists an integer $d > 0$ such that every operation on A is \mathcal{C} -equivalent to a d -ary operation on A .
- (ii) \mathfrak{F}_A is an order filter in the lattice of all clones on A ; that is, if $\mathcal{C} \in \mathfrak{F}_A$, then $\mathcal{C}' \in \mathfrak{F}_A$ for every clone \mathcal{C}' that contains \mathcal{C} .

It is well known that every clone on A other than \mathcal{O}_A is contained in a maximal clone. Since $\mathcal{O}_A \in \mathfrak{F}_A$ and \mathfrak{F}_A is an order filter of clones on A , it is natural to ask which maximal clones belong to \mathfrak{F}_A . To answer this question we will use Rosenberg's description of the maximal clones.

Theorem 2.2 (Rosenberg [17]). *For each finite set A with $|A| \geq 2$ the maximal clones on A are the clones of the form $\text{Pol } \rho$ where ρ is a relation of one of the following six types:*

- (1) a bounded partial order on A ,
- (2) a prime permutation on A ,
- (3) a prime affine relation on A ,
- (4) a nontrivial equivalence relation on A ,
- (5) a central relation on A ,
- (6) an h -regular relation on A .

Here a partial order on A is called *bounded* if it has both a least and a greatest element. A *prime permutation* on A is (the graph of) a fixed point free permutation on A in which all cycles are of the same prime length, and a *prime affine relation* on A is the graph of the ternary operation $x - y + z$ for some elementary abelian p -group $(A; +, -, 0)$ on A (p prime). An equivalence relation on A is called *nontrivial* if it is neither the equality relation $\mathbf{0}_A$ on A nor the full relation $\mathbf{1}_A$ on A .

To describe central relations and h -regular relations we call an h -ary relation ρ on A *totally reflexive* if ρ contains all h -tuples from A^h whose coordinates are not pairwise distinct, and *totally symmetric* if ρ is invariant under any permutation of its coordinates. We say that ρ is a *central relation* on A if $\emptyset \neq \rho \neq A^h$, ρ is totally reflexive and totally symmetric, and there exists an element $c \in A$ such that $\{c\} \times A^{h-1} \subseteq \rho$. The elements c with this property are called the *central elements* of ρ . Note that the arity h of a central relation on A has to satisfy $1 \leq h \leq |A| - 1$, and the unary central relations are just the nonempty proper subsets of A .

For an integer $h \geq 3$ a family $T = \{\theta_1, \dots, \theta_r\}$ ($r \geq 1$) of equivalence relations on A is called *h -regular* if each θ_i ($1 \leq i \leq r$) has exactly h blocks, and for arbitrary blocks B_i of θ_i ($1 \leq i \leq r$) the intersection $\bigcap_{i=1}^r B_i$ is nonempty. To each h -regular family $T = \{\theta_1, \dots, \theta_r\}$ of equivalence relations on A we associate an h -ary relation λ_T on A as follows:

$$\lambda_T = \{(a_1, \dots, a_h) \in A^h : \text{for each } i, a_1, \dots, a_h \text{ is not a transversal} \\ \text{for the blocks of } \theta_i\}.$$

Relations of the form λ_T are called *h -regular* (or *h -regularly generated*) *relations*. It is clear from the definition that h -regular relations are totally reflexive and totally symmetric, their arity h satisfies $3 \leq h \leq |A|$, and $h = |A|$ holds if and only if T is the one-element family consisting of the equality relation.

We conclude this section by summarizing earlier known results proving some of the maximal clones from Theorem 2.2 to belong or not to belong to \mathfrak{F}_A .

Theorem 2.3 ([13]). *Let A be a finite set with $|A| \geq 2$.*

- (i) *The clone on A generated by the ternary discriminator function*

$$t_A(x, y, z) = \begin{cases} z, & \text{if } x = y, \\ x, & \text{otherwise} \end{cases} \quad (x, y, z \in A)$$

is a minimal member of \mathfrak{F}_A . Hence every clone containing t_A belongs to \mathfrak{F}_A .

- (ii) *If $|A| = 2$, then a clone is in \mathfrak{F}_A if and only if it contains t_A .*

It is well known and easy to check that every maximal clone determined by a prime permutation on A or by a proper subset of A contains t_A . Therefore we get the following corollary.

Corollary 2.4. *Every maximal clone determined by a prime permutation on A or by a proper subset of A (i.e., a unary central relation on A) belongs to \mathfrak{F}_A .*

Theorem 2.5 ([12]). *If A is a finite set with $|A| \geq 2$, then the maximal clones determined by bounded partial orders or by prime affine relations do not belong to \mathfrak{F}_A .*

3. TWO NECESSARY CONDITIONS

In this section we establish some necessary conditions for a clone \mathcal{C} on a finite set A to belong to \mathfrak{F}_A . The first condition shows that for $\mathcal{C} \in \mathfrak{F}_A$ it is necessary that for each subset B of A , the operations from \mathcal{C} restrict to B so that the restrictions that are operations on B form a clone belonging to \mathfrak{F}_B .

Proposition 3.1. *Let \mathcal{C} be a clone on a finite set A , let B be a nonempty subset of A , and let \mathcal{C}_B be the clone on B defined as follows:*

$$\mathcal{C}_B = \{f|_B : f \in \mathcal{C} \cap \text{Pol } B\}.$$

If $\mathcal{C} \in \mathfrak{F}_A$, then $\mathcal{C}_B \in \mathfrak{F}_B$.

Proof. We will prove the contrapositive, so suppose that $\mathcal{C}_B \notin \mathfrak{F}_B$. Our goal is to show that $\mathcal{C} \notin \mathfrak{F}_A$. Since $\mathcal{C}_A = \mathcal{C}$, there is nothing to prove if $B = A$. Therefore let us assume that B is a proper subset of A , and let $0 \in A \setminus B$. Using the assumption $\mathcal{C}_B \notin \mathfrak{F}_B$, select representatives g_i ($i = 1, 2, \dots$) of infinitely many different $\equiv_{\mathcal{C}_B}$ -classes. Define f_i on A such that $f_i(\mathbf{x}) = g_i(\mathbf{x})$ if all coordinates of the tuple \mathbf{x} are in B and $f_i(\mathbf{x}) = 0$ otherwise. We will prove $\mathcal{C} \notin \mathfrak{F}_A$ by showing that the operations f_i ($i = 1, 2, \dots$) belong to pairwise different $\equiv_{\mathcal{C}}$ -classes.

Suppose that there exist operations f_i, f_j ($i \neq j$) such that $f_i \equiv_{\mathcal{C}} f_j$, that is, $f_i = f_j \circ \mathbf{h}$ and $f_j = f_i \circ \mathbf{h}'$ for some $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ and $\mathbf{h}' \in (\mathcal{C}^{(n)})^m$ where m is the arity of f_i and n is the arity of f_j . Let $\mathbf{x} \in B^m$. Then $f_i(\mathbf{x}) = g_i(\mathbf{x}) \in B$, so $f_j(\mathbf{h}(\mathbf{x})) = (f_j \circ \mathbf{h})(\mathbf{x}) = f_i(\mathbf{x}) \in B$. Since $f_j(\mathbf{y}) = 0 \notin B$ if $\mathbf{y} \notin B^n$, we get that $\mathbf{h}(\mathbf{x}) \in B^n$. This shows that \mathbf{h} preserves B , hence $\mathbf{h}|_B \in (\mathcal{C}_B^{(m)})^n$. Similarly, by interchanging the roles of f_i and f_j we conclude that \mathbf{h}' also preserves B , and $\mathbf{h}'|_B \in (\mathcal{C}_B^{(n)})^m$. By construction, f_i, f_j preserve B as well, therefore $f_i|_B = f_j|_B \circ \mathbf{h}|_B$ and $f_j|_B = f_i|_B \circ \mathbf{h}'|_B$. This implies that $g_i = g_j \circ \mathbf{h}|_B$ and $g_j = g_i \circ \mathbf{h}'|_B$. Hence $g_i \equiv_{\mathcal{C}_B} g_j$, which contradicts the choice of the operations g_i, g_j . \square

Corollary 3.2. *Let ρ be a relation on a finite set A . If A has a nonempty subset B such that for the clone determined by the restriction $\rho|_B$ of ρ to B we have that $\text{Pol } \rho|_B \notin \mathfrak{F}_B$, then $\text{Pol } \rho \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \rho$, and let \mathcal{C}_B be the clone defined in Proposition 3.1. First we will show that $\mathcal{C}_B \subseteq \text{Pol } \rho|_B$. Indeed, every operation in \mathcal{C}_B is of the form $f|_B$ for some $f \in \mathcal{C} \cap \text{Pol } B = \text{Pol}\{\rho, B\}$. Since f preserves ρ and B , it also preserves $\rho \cap B^2 = \rho|_B$. Thus $f|_B$ also preserves $\rho|_B$, that is, $f|_B \in \text{Pol } \rho|_B$.

If $\text{Pol } \rho|_B \notin \mathfrak{F}_B$, then the fact that \mathcal{C}_B is a subclone of $\text{Pol } \rho|_B$ implies by Proposition 2.1 (ii) that $\mathcal{C}_B \notin \mathfrak{F}_B$. Therefore it follows from Proposition 3.1 that $\text{Pol } \rho = \mathcal{C} \notin \mathfrak{F}_A$, as claimed. \square

The second necessary condition for $\mathcal{C} \in \mathfrak{F}_A$ is a quantitative condition indicating that the clones in \mathfrak{F}_A are large in the sense that they must have a lot of n -ary operations for each n .

Proposition 3.3. *Let A be a k -element set. If $\mathcal{C} \in \mathfrak{F}_A$, then there exists a positive constant c such that $|\mathcal{C}^{(n)}| \geq ck^{k^n/n}$ for all $n \geq 1$.*

Proof. Denote the number of $\equiv_{\mathcal{C}}$ -classes by μ . For every $n \geq 1$ the number of n -ary operations on A is k^{k^n} , therefore there must be a $\equiv_{\mathcal{C}}$ -class B such that $|B^{(n)}| \geq k^{k^n}/\mu$. Any $f \in B^{(n)}$ has at most $|\mathcal{C}^{(n)}|^n$ n -ary \mathcal{C} -minors, so we have that

$$|\mathcal{C}^{(n)}|^n \geq |\{f \circ \mathbf{g} : \mathbf{g} \in (\mathcal{C}^{(n)})^n\}| \geq |B^{(n)}| \geq k^{k^n}/\mu.$$

It follows that

$$|\mathcal{C}^{(n)}| \geq (k^{k^n}/\mu)^{1/n} = k^{k^n/n}/\mu^{1/n} \geq k^{k^n/n}/\mu.$$

The claim now follows by letting $c = 1/\mu$. \square

Corollary 3.4. *Let \mathcal{C} be a clone on a k -element set A . If $|\mathcal{C}^{(n)}| \leq k^{p(n)}$ for all n , where $p: \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $\lim_{n \rightarrow \infty} \frac{p(n)}{k^n/n} = 0$, then $\mathcal{C} \notin \mathfrak{F}_A$.*

Proof. Suppose that the assumptions of the corollary hold, but $\mathcal{C} \in \mathfrak{F}_A$. Proposition 3.3 implies then that for some positive constant c we have $ck^{k^n/n} \leq |\mathcal{C}^{(n)}|$ for all n . Hence $ck^{k^n/n} \leq k^{p(n)}$ for all n . Since $k > 1$, we get that $\log_k c + k^n/n \leq p(n)$ for all n , or equivalently, $\frac{\log_k c}{k^n/n} + 1 \leq \frac{p(n)}{k^n/n}$ for all n . Taking the limit of both sides as $n \rightarrow \infty$ we get that $1 \leq 0$, a contradiction. \square

Every polynomial function p satisfies the condition $\lim_{n \rightarrow \infty} \frac{p(n)}{k^n/n} = 0$, hence the following statement is a special case of Corollary 3.4.

Corollary 3.5. *If \mathcal{C} is a clone on a k -element set A such that for some polynomial function p we have $|\mathcal{C}^{(n)}| \leq k^{p(n)}$ for all n , then $\mathcal{C} \notin \mathfrak{F}_A$.*

Remark 3.6. The converse of Proposition 3.3 is not true, that is, there exist clones $\mathcal{C} \notin \mathfrak{F}_A$ that satisfy the conclusion of Proposition 3.3. For example, if $A = \{0, 1\}$ is a 2-element set and \leq is the natural order $0 \leq 1$ on A , then it follows from part (ii) of Theorem 2.3 that the clone $\mathcal{M} := \text{Pol } \leq$ of all monotone Boolean functions is not in \mathfrak{F}_A . However, Gilbert [7] proved that

$$|\mathcal{M}^{(n)}| \geq 2^{\binom{n}{\lfloor n/2 \rfloor}} \quad \text{for all } n \geq 1.$$

If $n \geq 2$, then $\binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{k}$ for all $0 < k < n$ and $\binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{0} + \binom{n}{n}$, therefore $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$. Hence $|\mathcal{M}^{(n)}| \geq 2^{2^n/n}$ holds for all $n \geq 2$. For $n = 1$ we have $|\mathcal{M}^{(1)}| = 3 \geq \frac{1}{2} \cdot 2^{2^1/1}$. Thus $|\mathcal{M}^{(n)}| \geq \frac{1}{2} \cdot 2^{2^n/n}$ for all $n \geq 1$, which shows that the clone $\mathcal{C} = \mathcal{M}$ satisfies the conclusion of Proposition 3.3.

Another example is the clone \mathcal{B}_{k-2} on a k -element set A with $k \geq 4$ that consists of all essentially at most unary operations and all operations whose range has at most $k-2$ elements (see Section 6). We will show in Theorem 6.1 that $\mathcal{B}_{k-2} \notin \mathfrak{F}_A$. On the other hand,

$$|\mathcal{B}_{k-2}^{(n)}| \geq (k-2)^{k^n} = k^{k^n \log_k(k-2)} = k^{dk^n},$$

where $1/2 \leq d = \log_k(k-2) < 1$. Now if we choose $c = k^{-k/2}$ ($0 < c < 1$) then for $n = 1$ we have that

$$k^{dk^1} \geq k^{k/2} = k^{-k/2} k^k = ck^{k^1/1},$$

and for $n \geq 2$ we have that $dn \geq 1$ and so

$$k^{dk^n} \geq k^{dk^n/dn} = k^{k^n/n} \geq ck^{k^n/n}.$$

Thus, $|\mathcal{B}_{k-2}^{(n)}| \geq ck^{k^n/n}$ for all $n \geq 1$, proving that the conclusion of Proposition 3.3 holds for \mathcal{B}_{k-2} .

Remark 3.7. Theorem 2.5 can be derived from Corollaries 3.2, 3.5, and the fact that $\mathcal{M} \notin \mathfrak{F}_{\{0,1\}}$ holds for the clone \mathcal{M} of monotone Boolean functions (see Remark 3.6). Indeed, let first $\text{Pol } \leq$ be a maximal clone on A determined by a bounded partial order \leq . We may assume without loss of generality that $A = \{0, 1, 2, \dots, k-1\}$ where $k \geq 2$ and the least and greatest elements of \leq are 0 and 1. Thus $\text{Pol } \leq|_{\{0,1\}} = \mathcal{M} \notin \mathfrak{F}_{\{0,1\}}$, so Corollary 3.2 implies that $\text{Pol } \leq \notin \mathfrak{F}_A$.

Next let \mathcal{C} be a maximal clone on A determined by a prime affine relation. In this case $|A| = q^r$ for some prime q and some positive integer r . Moreover, there exists an elementary abelian q -group $(A; +)$ such that the n -ary operations in \mathcal{C} are exactly the operations $\sum_{i=1}^n M_i x_i + a$ where $a \in A$ and each M_i is an $r \times r$ matrix over the q -element field. Thus, using the notation $k := |A| = q^r$ we get that $|\mathcal{C}^{(n)}| \leq (q^{r^2})^n q^r = k^{rn+1}$. Hence Corollary 3.5 implies that $\mathcal{C} \notin \mathfrak{F}_A$.

We conclude this section by two further applications of Propositions 3.1, 3.3 and their corollaries. Recall that *Burle's clone* on a finite set A is the subclone of \mathcal{O}_A that consists of all essentially at most unary operations and all *quasilinear operations*, i.e., all operations of the form $g(h_1(x_1) \oplus \cdots \oplus h_n(x_n))$ where $h_1, \dots, h_n: A \rightarrow \{0, 1\}$, $g: \{0, 1\} \rightarrow A$ are arbitrary mappings and \oplus denotes addition modulo 2. We will denote Burle's clone by \mathcal{B}_1 (see Section 6).

Corollary 3.8. *If A is a finite set with at least two elements, then $\mathcal{B}_1 \notin \mathfrak{F}_A$.*

Proof. If $|A| = 2$, then Burle's clone is the unique maximal clone determined by a prime affine relation. As discussed in Remark 3.7, in this case $\mathcal{B}_1 \notin \mathfrak{F}_A$ can be proved using Corollary 3.5. From now on let $|A| = k \geq 3$, and assume without loss of generality that $A = \{0, 1, 2, \dots, k-1\}$. In this case we can employ either one of Corollaries 3.2 and 3.5 to prove that $\mathcal{B}_1 \notin \mathfrak{F}_A$.

First we will discuss the proof that relies on Corollary 3.2. It is well known that $\mathcal{B}_1 = \text{Pol } \beta$ where β is the 4-ary relation on A that consists of all tuples of the form (x, x, y, y) , (x, y, x, y) , and (x, y, y, x) with $x, y \in A$. Since $\beta|_{\{0,1\}}$ is the unique prime affine relation on $\{0, 1\}$, our argument in Remark 3.7 shows that $\text{Pol } \beta|_{\{0,1\}} \notin \mathfrak{F}_{\{0,1\}}$. Thus Corollary 3.2 yields that $\mathcal{B}_1 = \text{Pol } \beta \notin \mathfrak{F}_A$.

To get the same conclusion using Corollary 3.5 we have to estimate the number of n -ary operations in \mathcal{B}_1 . The number of functions $A \rightarrow \{0, 1\}$ is 2^k , and the number of functions $\{0, 1\} \rightarrow A$ is k^2 , so the number of n -ary quasilinear operations on A is at most $k^2(2^k)^n$. The number of functions $A \rightarrow A$ is k^k , so the number of n -ary, essentially at most unary operations on A is at most nk^k . Thus,

$$|\mathcal{B}_1^{(n)}| \leq k^2(2^k)^n + nk^k \leq k^k(k^k)^n + (k^k)^n k^k \leq k k^k (k^k)^n = k^{kn+k+1},$$

where the second inequality holds because $k > 2$ and hence $n \leq (k^k)^n$ for all $n \geq 1$. It follows from Corollary 3.5 that $\mathcal{B}_1 \notin \mathfrak{F}_A$. \square

In the proof of Theorem 4.1 we will see an application of Corollary 3.4 where the function p is not a polynomial.

Our last application answers a question on minimal clones raised by P. Mayr. Recall that a clone \mathcal{C} on A is called *minimal* if \mathcal{C} is not the clone \mathcal{P}_A of projections, and \mathcal{P}_A is the only proper subclone of \mathcal{C} . Equivalently, \mathcal{C} is a minimal clone on A if and only if $\mathcal{C} \setminus \mathcal{P}_A \neq \emptyset$ and $\langle f \rangle = \mathcal{C}$ for all $f \in \mathcal{C} \setminus \mathcal{P}_A$.

Corollary 3.9. *If A is a finite set with at least two elements, then no minimal clone on A belongs to \mathfrak{F}_A .*

Proof. Assume that the statement is false, and let A be a finite set of minimum size $|A| \geq 2$ such that \mathfrak{F}_A contains a minimal clone \mathcal{C} . Let B be any 2-element subset of A . Since \mathcal{C} is a minimal clone, the clone $\mathcal{C} \cap \text{Pol } B$ is either \mathcal{P}_A or \mathcal{C} . Hence the clone $\mathcal{C}_B = \{f|_B : f \in \mathcal{C} \cap \text{Pol } B\}$ defined in Proposition 3.1 is either \mathcal{P}_B or a minimal clone on B . By Proposition 3.1, the assumption $\mathcal{C} \in \mathfrak{F}_A$ implies that $\mathcal{C}_B \in \mathfrak{F}_B$. However, as we discussed in the introduction, $\mathcal{P}_B \notin \mathfrak{F}_B$. Therefore \mathcal{C}_B is

a minimal clone on B that is a member of \mathfrak{F}_B . The minimality of A implies that $B = A$ and hence $|A| = 2$. It is well known from [16] that there are seven minimal clones on a 2-element set, and each one of them is either a subclone of the maximal clone \mathcal{M} of all monotone Boolean functions, or a subclone of the maximal clone \mathcal{B}_1 of all linear Boolean functions. Therefore Theorem 2.3 or Theorem 2.5 (see also Remark 3.7) implies that $\mathcal{C} \notin \mathfrak{F}_A$. This contradicts our assumption on \mathcal{C} , and hence proves Corollary 3.9. \square

4. EQUIVALENCE RELATIONS

Let E be a set of equivalence relations on a finite set A . Our aim in this section is to show that $\text{Pol } E \in \mathfrak{F}_A$ if and only if E is a chain (with respect to inclusion). We will in fact prove the following stronger theorem.

Theorem 4.1. *Let A be a finite set, and let E be a set of equivalence relations on A , Γ a set of permutations of A , and Σ a set of nonempty subsets of A . The clone $\text{Pol}(E, \Gamma, \Sigma)$ is a member of \mathfrak{F}_A if and only if*

- (a) E is a chain (i.e., any two members of E are comparable), and
- (b) $\Gamma \subseteq \text{Pol } E$.

For any set E of equivalence relations on A we call a permutation γ of A E -invariant if $\gamma \in \text{Pol } E$, that is, if γ is an automorphism of the relational structure $(A; E)$. Therefore we denote the group of E -invariant permutations of A by $\text{Aut } E$. Furthermore, we denote the set of all nonempty subsets of A by $\mathcal{P}^+(A)$. Thus, in Theorem 4.1, Σ is an arbitrary subset of $\mathcal{P}^+(A)$ and (b) requires that $\Gamma \subseteq \text{Aut } E$.

Proof of Theorem 4.1. Necessity. Let $\mathcal{C} = \text{Pol}(E, \Gamma, \Sigma)$ and $k = |A|$. We want to show that if (a) or (b) fails, then $\mathcal{C} \notin \mathfrak{F}_A$. Assume first that (a) fails, that is, E contains equivalence relations α and β such that $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$. Clearly, $\mathcal{C} \subseteq \text{Pol}(\alpha, \beta)$, therefore in view of Proposition 2.1 (ii) it suffices to prove that the clone $\mathcal{E} = \text{Pol}(\alpha, \beta)$ fails to belong to \mathfrak{F}_A . Let \mathbf{A} denote the algebra $(A; \mathcal{E})$. Since $\mathcal{E} = \text{Pol}(\alpha, \beta)$, it follows that α and β are congruences of \mathbf{A} , and the clones of the corresponding quotient algebras are $\text{Clo}(\mathbf{A}/\alpha) = \mathcal{E}^\alpha$ and $\text{Clo}(\mathbf{A}/\beta) = \mathcal{E}^\beta$, the natural actions of \mathcal{E} on A/α and A/β .

First we will consider the case when $\alpha \wedge \beta = \mathbf{0}_A$. Then the embedding $\mathbf{A} \rightarrow \mathbf{A}/\alpha \times \mathbf{A}/\beta$, $a \mapsto (a/\alpha, a/\beta)$ represents \mathbf{A} as a subdirect product of \mathbf{A}/α and \mathbf{A}/β . Hence $\mathcal{E} \rightarrow \mathcal{E}^\alpha \times \mathcal{E}^\beta$, $h \mapsto (h^\alpha, h^\beta)$ is a clone embedding. This implies that for each n ,

$$|\mathcal{E}^{(n)}| \leq |(\mathcal{E}^\alpha)^{(n)}| \cdot |(\mathcal{E}^\beta)^{(n)}|.$$

The assumption that α and β are incomparable ensures that $|\mathbf{A}/\alpha| \leq k - 1$ and $|\mathbf{A}/\beta| \leq k - 1$. Thus

$$|\mathcal{E}^{(n)}| \leq (k - 1)^{(k-1)^n} \cdot (k - 1)^{(k-1)^n} = (k - 1)^{2(k-1)^n} < k^{2(k-1)^n}.$$

Since $\lim_{n \rightarrow \infty} \frac{2(k-1)^n}{k^{2(k-1)^n}} = 0$, Corollary 3.4 implies that $\mathcal{E} \notin \mathfrak{F}_A$.

To prove the statement in the general case let $\theta = \alpha \wedge \beta$, and consider the algebra \mathbf{A}/θ and its congruences α/θ and β/θ . Clearly, the clone of \mathbf{A}/θ is $\text{Clo}(\mathbf{A}/\theta) = \mathcal{E}^\theta$, and the assumptions ensure that $\alpha/\theta \not\subseteq \beta/\theta$ and $\beta/\theta \not\subseteq \alpha/\theta$. Since $\alpha/\theta \wedge \beta/\theta = \mathbf{0}_{\mathbf{A}/\theta}$, the special case established in the preceding paragraph shows that $\mathcal{E}^\theta \notin \mathfrak{F}_{\mathbf{A}/\theta}$. Hence there exists an infinite sequence of operations g_n ($n \geq 1$) on \mathbf{A}/θ such that

$g_i \not\equiv_{\mathcal{E}^\theta} g_j$ for all $i \neq j$. Now choose and fix operations f_n ($n \geq 1$) on A such that $g_n = f_n^\theta$ for each n . Then $f_n \in \text{Pol } \theta$ ($n \geq 1$) and $f_i^\theta \not\equiv_{\mathcal{E}^\theta} f_j^\theta$ whenever $i \neq j$. We claim that $f_i \not\equiv_{\mathcal{E}} f_j$ whenever $i \neq j$. Suppose otherwise, and let $i \neq j$ be such that $f_i \equiv_{\mathcal{E}} f_j$. Then there exist tuples of operations \mathbf{h} and \mathbf{h}' in \mathcal{E} such that $f_i = f_j \circ \mathbf{h}$ and $f_j = f_i \circ \mathbf{h}'$. Since all operations in \mathbf{h} and \mathbf{h}' belong to $\mathcal{E} = \text{Pol}(\alpha, \beta)$, they preserve $\theta = \alpha \wedge \beta$. Hence we get that $f_i^\theta = f_j^\theta \circ \mathbf{h}^\theta$ and $f_j^\theta = f_i^\theta \circ (\mathbf{h}')^\theta$, which contradicts the choice of the operations $g_n = f_n^\theta$. Thus there are infinitely many $\equiv_{\mathcal{E}}$ -classes, and hence $\mathcal{E} \notin \mathfrak{F}_A$. This proves the necessity of condition (a).

Now assume that condition (b) fails, and let $\gamma \in \Gamma$ be such that $\gamma \notin \text{Pol } E$, that is, $\gamma \notin \text{Pol } \rho$ for some $\rho \in E$. Let $\gamma(\rho) = \{(\gamma(a), \gamma(b)) : (a, b) \in \rho\}$, and let $E' = E \cup \{\gamma(\rho)\}$. Clearly, $\gamma(\rho)$ is an equivalence relation on A , and $\gamma(\rho) \neq \rho$, since $\gamma \notin \text{Pol } \rho$. As A is finite, and ρ and $\gamma(\rho)$ have the same system of block sizes, it follows that ρ and $\gamma(\rho)$ are incomparable. Hence E' is a set of equivalence relations that is not a chain. It is easy to verify that every operation that preserves both γ and ρ also preserves $\gamma(\rho)$. Therefore $\mathcal{C} \subseteq \text{Pol } E'$, and the failure of condition (a) shows that $\text{Pol } E' \notin \mathfrak{F}_A$. Thus Proposition 2.1 (ii) implies that $\mathcal{C} \notin \mathfrak{F}_A$, establishing the necessity of condition (b).

Sufficiency. Given a chain E of equivalence relations, there is a smallest clone of the form $\text{Pol}(E, \Gamma, \Sigma)$ satisfying the assumptions of the theorem and also condition (b), namely the clone $\text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$. Therefore, by Proposition 2.1 (ii), it suffices to prove that this clone belongs to \mathfrak{F}_A . This claim, which is the hardest part of Theorem 4.1, is stated below as Theorem 4.2, and will be proved separately. \square

Theorem 4.2. *If E is a chain of equivalence relations on a finite set A , then $\text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A)) \in \mathfrak{F}_A$.*

Remark 4.3. For every chain E of equivalence relations on A , the clone $\text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$ contains a 2/3-minority operation, i.e., a ternary operation m such that

$$(4.1) \quad m(x, x, y) = y, \quad m(x, y, x) = x, \quad \text{and} \quad m(x, y, y) = x$$

for all $x, y \in A$. To define such an operation let $\theta(a, b)$ denote the least equivalence relation $\varepsilon \in E \cup \{\mathbf{0}_A, \mathbf{1}_A\}$ such that $a \varepsilon b$ ($a, b \in A$). It is clear that if $a, b, c \in A$ and $\theta(a, b) \leq \theta(a, c), \theta(b, c)$, then $\theta(a, c) = \theta(b, c)$. We will write $a \sim b \sim c$ to denote that $\theta(a, b) = \theta(a, c) = \theta(b, c)$, and $a \sim b \not\sim c$ to denote that $\theta(a, b) < \theta(a, c) = \theta(b, c)$. Since $E \cup \{\mathbf{0}_A, \mathbf{1}_A\}$ is a chain, it follows that exactly one of the following conditions holds for any triple $(a, b, c) \in A^3$:

$$(i) \ a \sim b \sim c, \quad (ii) \ a \sim b \not\sim c, \quad (iii) \ a \sim c \not\sim b, \quad (iv) \ b \sim c \not\sim a.$$

We define a ternary operation m on A as follows:

$$m(x, y, z) = \begin{cases} z & \text{if } x \sim y \sim z \text{ or } x \sim y \not\sim z, \\ x & \text{if } x \sim z \not\sim y \text{ or } y \sim z \not\sim x \end{cases} \quad (x, y, z \in A).$$

For any $x, y \in A$ we have $x \sim x \sim y$ if $x = y$ and $x \sim x \not\sim y$ if $x \neq y$. Hence, in either case, the definition of m shows that the equalities in (4.1) hold, which proves that m is a 2/3-minority operation. Since on any input triple the value of m equals one of the inputs, it follows that m preserves all nonempty subsets of A . If $\gamma \in \text{Aut } E$, then $\theta(a, b) = \theta(\gamma(a), \gamma(b))$ holds for all $a, b \in A$. Consequently, for each one of conditions (i)–(iv), a triple $(a, b, c) \in A^3$ satisfies this condition if and only if the triple $(\gamma(a), \gamma(b), \gamma(c))$ does. This implies that m preserves all permutations $\gamma \in \text{Aut } E$.

Finally, to see that m preserves all equivalence relations in E let $\rho \in E$, and let $(a, b, c) \rho^3 (a', b', c')$. As we will now show, the latter assumption implies that

$$(4.2) \quad \begin{aligned} \theta(a, b) \vee \rho &= \theta(a', b') \vee \rho, \\ \theta(a, c) \vee \rho &= \theta(a', c') \vee \rho, \\ \theta(b, c) \vee \rho &= \theta(b', c') \vee \rho. \end{aligned}$$

Indeed, by our assumption we have that $a \rho a'$ and $b \rho b'$, therefore

$$(a', b') \in \rho \circ \theta(a, b) \circ \rho \subseteq \theta(a, b) \vee \rho.$$

Here $\theta(a, b) \vee \rho$ is the larger one of $\theta(a, b)$ and ρ in the chain E , so $(a', b') \in \theta(a, b) \vee \rho$ implies that the least equivalence relation $\theta(a', b')$ in E containing the pair (a', b') satisfies $\theta(a', b') \leq \theta(a, b) \vee \rho$. Hence $\theta(a', b') \vee \rho \leq \theta(a, b) \vee \rho$. By interchanging the roles of a, b and a', b' we get the reverse inclusion $\theta(a, b) \vee \rho \leq \theta(a', b') \vee \rho$, which proves the first equality in (4.2). The second and third equalities can be proved similarly.

Our goal is to verify that the assumption $(a, b, c) \rho^3 (a', b', c')$ implies that $m(a, b, c) \rho m(a', b', c')$. If $a \rho b \rho c$ or $a' \rho b' \rho c'$, then by the assumption $(a, b, c) \rho^3 (a', b', c')$ all six elements a, b, c, a', b', c' lie in the same ρ -class, so $m(a, b, c)$ and $m(a', b', c')$, too, lie in that ρ -class, because $m(a, b, c) \in \{a, c\}$ and $m(a', b', c') \in \{a', c'\}$. Thus $m(a, b, c) \rho m(a', b', c')$ holds in this case.

Now assume for the rest of the proof that

$$(4.3) \quad a, b, c \text{ are not all } \rho\text{-related, and } a', b', c' \text{ are not all } \rho\text{-related.}$$

We want to prove that

$$(*) \text{ for each one of conditions (i)–(iv), } (a, b, c) \text{ satisfies this condition if and only if } (a', b', c') \text{ does.}$$

By the definition of m , this will imply that $(m(a, b, c), m(a', b', c')) = (a, a')$ or (c, c') , hence $m(a, b, c) \rho m(a', b', c')$. Since statement $(*)$ is invariant under performing the same permutation on the coordinates of the two triples, and since the roles of the two triples are symmetric, $(*)$ will follow if we show that $a \sim b \sim c$ implies $a' \sim b' \sim c'$, and $a \sim b \not\sim c$ implies $a' \sim b' \not\sim c'$. So, let us assume first that $a \sim b \sim c$, that is, $\theta(a, b) = \theta(a, c) = \theta(b, c)$. Since $E \cup \{\mathbf{0}_A, \mathbf{1}_A\}$ is a chain, our assumption (4.3) forces that $\theta(a, b) = \theta(a, c) = \theta(b, c) > \rho$. Therefore (4.2) implies that

$$\rho < \theta(a, b) = \theta(a, c) = \theta(b, c) = \theta(a', b') \vee \rho = \theta(a', c') \vee \rho = \theta(b', c') \vee \rho.$$

The inequality $\rho < \theta(a', b') \vee \rho$ shows that $\rho < \theta(a', b')$. Similarly, $\rho < \theta(a', c')$ and $\rho < \theta(b', c')$. Now the displayed equalities imply that $\theta(a', b') = \theta(a', c') = \theta(b', c') (= \theta(a, b))$, and hence $a' \sim b' \sim c'$. Next let us assume that $a \sim b \not\sim c$. Thus, $\theta(a, c) = \theta(b, c) > \theta(a, b)$, and since $E \cup \{\mathbf{0}_A, \mathbf{1}_A\}$ is a chain, we get from our assumption (4.3) that $\theta(a, c) = \theta(b, c) > \rho$. This inequality, combined with the second and third equalities in (4.2) yields, as before, that

$$\theta(a', c') = \theta(b', c') = \theta(a, c) = \theta(b, c) > \rho.$$

The same holds with ρ replaced by $\theta(a, b)$, since $\theta(a, c) > \theta(a, b)$. Therefore ρ can also be replaced by $\alpha := \theta(a, b) \vee \rho$, the larger one of $\theta(a, b)$ and ρ . Hence

$$\theta(a', c') = \theta(b', c') > \alpha.$$

Making use of (4.2) again we also get that $\alpha \geq \theta(a', b')$, because

$$\alpha = \theta(a, b) \vee \rho = \theta(a', b') \vee \rho \geq \theta(a', b').$$

Thus $a' \sim b' \not\sim c'$, which completes the proof of (*), and thereby establishes the existence of a 2/3-minority operation in the clone $\text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$ for every chain E of equivalence relations on A .

Remark 4.4. If $E = \emptyset$ (or $E \subseteq \{\mathbf{0}_A, \mathbf{1}_A\}$), then $\text{Aut } E$ is the full symmetric group on A , the 2/3-minority operation m defined in Remark 4.3 is the ternary discriminator t_A on A , and $\text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$ is the clone generated by t_A . Therefore Theorem 4.2 includes the statement $\langle t_A \rangle \in \mathfrak{F}_A$ from Theorem 2.3 (i) as a special case.

Let E be a chain of equivalence relations on A , let $\Gamma = \text{Aut } E$, and let $\mathcal{C} = \text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$. We will prove Theorem 4.2 by associating to each operation on A a finite structure of bounded size in such a way that if two operations have isomorphic structures associated to them, then they are in the same $\equiv_{\mathcal{C}}$ -class. This finite structure, to be defined in detail below, will be a Γ -set with a tree structure on it, and the leaves of the tree will have a labeling that is compatible with the action of Γ .

Let G be an arbitrary group. A G -set is a unary algebra $(U; G)$ such that each $g \in G$ acts on U by a permutation $U \rightarrow U$, $u \mapsto g \cdot u$, and for any $g, g' \in G$ and $u \in U$, we have $gg' \cdot u = g \cdot (g' \cdot u)$. Since each $g \in G$ acts by a permutation of U , it follows that the neutral element 1 of G acts by the identity permutation, that is, $1 \cdot u = u$ holds for all $u \in U$. Consequently, for any $g \in G$, the actions of g and g^{-1} are inverses of each other. If there is no danger of confusion, we will write gu instead of $g \cdot u$. For any element $u \in U$, the stabilizer of u in G is the subgroup $G_u := \{g \in G : gu = u\}$ of G . For $u \in U$ the subalgebra $G_u := \{gu : g \in G\}$ of $(U; G)$ generated by $u \in U$ is called the G -orbit of u . It is well known and easy to check that the G -orbits of $(U; G)$ are minimal subalgebras, and therefore they partition U . If $(U; G)$ and $(V; G)$ are G -sets, then a mapping $\varphi: U \rightarrow V$ is a homomorphism $(U; G) \rightarrow (V; G)$ of G -sets, if $\varphi(gu) = g \cdot \varphi(u)$ holds for all $u \in U$ and $g \in G$. By a pointed G -set $(U; u, G)$ we mean a G -set $(U; G)$ with a distinguished element $u \in U$. If $U = Gu$ is a G -orbit, we will call the pointed G -set $(U; u, G)$ as well as the pointed set $(U; u)$ (if the G -set structure is irrelevant) a pointed G -orbit. A homomorphism $(U; u, G) \rightarrow (V; v, G)$ between pointed G -sets is a homomorphism $\varphi: (U; G) \rightarrow (V; G)$ between the underlying G -sets such that $\varphi(u) = v$. If $(U; u, G)$ and $(V; v, G)$ are pointed G -orbits, that is, $U = Gu$ and $V = Gv$, then a homomorphism $\varphi: (U; u, G) \rightarrow (V; v, G)$ exists between them if and only if $G_u \subseteq G_v$; moreover, φ is uniquely determined: $\varphi: U = Gu \rightarrow Gv = V$, $gu \mapsto gv$ for all $g \in G$. We will denote this homomorphism (if it exists) by $\chi_{u,v}$. Clearly, $\chi_{u,v} = \chi_{gu,gv}$ for all $g \in G$, and $\chi_{u,v}$ is an isomorphism if and only if $G_u = G_v$.

By a tree we mean a finite partial algebra $\mathbf{P} = (P; *, 1_P)$ where $*$: $P \setminus \{1_P\} \rightarrow P$ is a function, called the successor function, such that the distinguished element 1_P can be obtained from any other element $a \in P \setminus \{1_P\}$ by repeated application of $*$. Denoting the i -th power of $*$ by $*^i$ we get that for each $a \in P$ there is a unique integer $d \geq 0$ such that $a^{*^d} = 1_P$, which will be called the depth of a . The only element of depth 0 is 1_P . An element a of \mathbf{P} will be called a leaf if it is not in the range of the successor function. We will denote the set of leaves of \mathbf{P} by \mathbf{P}_{\min} . If

every leaf of \mathbf{P} has the same depth d , we will say that *the tree \mathbf{P} has uniform depth d* .

If $\mathbf{P} = (P; *, 1_P)$ and $\mathbf{Q} = (Q; *, 1_Q)$ are trees, we will call a function $\varphi: P \rightarrow Q$ a *homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$* of trees if

- (H0) $\varphi(1_P) = 1_Q$,
- (H1) φ maps leaves to leaves, that is, $\varphi(\mathbf{P}_{\min}) \subseteq \mathbf{Q}_{\min}$, and
- (H2) $\varphi(a^*) = \varphi(a)^*$ for all $a \in P \setminus \{1_P\}$.

An *automorphism* of \mathbf{P} is a bijective homomorphism $\mathbf{P} \rightarrow \mathbf{P}$.

A tree $\mathbf{Q} = (Q; *, 1_Q)$ is a *subtree* of another tree $\mathbf{P} = (P; *, 1_P)$ if $Q \subseteq P$ and the identity function $Q \rightarrow P$, $q \mapsto q$ is a homomorphism $\mathbf{Q} \rightarrow \mathbf{P}$. Thus \mathbf{Q} is a subtree of \mathbf{P} if and only if $Q \subseteq P$, $1_Q = 1_P$, $\mathbf{Q}_{\min} \subseteq \mathbf{P}_{\min}$, and the successor function of \mathbf{Q} is the restriction to $Q \setminus \{1_Q\}$ of the successor function of \mathbf{P} .

Let G be a group. We define a G -tree to be a tree on which G acts by automorphisms; more precisely, a G -tree is a structure $\mathbf{P} = (P; *, 1_P, G)$ such that $(P; *, 1_P)$ is a tree, $(P; G)$ is a G -set, and for each $g \in G$ the permutation $a \mapsto ga$ of P is an automorphism of the tree $(P; *, 1_P)$. The assumption that G acts by tree automorphisms implies that in every G -tree $\mathbf{P} = (P; *, 1_P, G)$,

$$g \cdot 1_P = 1_P \quad \text{for all } g \in G,$$

and

$$a^{*^d} = 1_P \iff (ga)^{*^d} (= ga^{*^d}) = 1_P \quad \text{for all } a \in P \setminus \{1_P\} \text{ and } g \in G.$$

Therefore each G -orbit Ga of \mathbf{P} consists of elements of the same depth. Similarly, if a is a leaf, then so are all elements in the G -orbit Ga of a . Thus the leaves of \mathbf{P} form a G -set $(\mathbf{P}_{\min}; G)$.

For arbitrary G -trees $\mathbf{P} = (P; *, 1_P, G)$ and $\mathbf{Q} = (Q; *, 1_Q, G)$ a G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$ is a mapping $\varphi: P \rightarrow Q$ that is a homomorphism $(P; *, 1_P) \rightarrow (Q; *, 1_Q)$ of trees and also a homomorphism $(P; G) \rightarrow (Q; G)$ of G -sets; that is, in addition to (H0), (H1), and (H2), φ also satisfies

- (H3) $\varphi(ga) = g \cdot \varphi(a)$ for all $a \in P$ and $g \in G$.

A G -tree $\mathbf{Q} = (Q; *, 1_Q, G)$ is a G -subtree of $\mathbf{P} = (P; *, 1_P, G)$ if $Q \subseteq P$ and the identity function $Q \rightarrow P$, $q \mapsto q$ is a G -homomorphism $\mathbf{Q} \rightarrow \mathbf{P}$. Thus \mathbf{Q} is a G -subtree of \mathbf{P} if and only if $(Q; *, 1_Q)$ is a subtree of $(P; *, 1_P)$ and the action of each $g \in G$ on Q is the restriction to Q of the action of g on P . Hence, if $\mathbf{P} = (P; *, 1_P, G)$ is a G -tree, then a subtree $(Q; *, 1_Q)$ of $(P; *, 1_P)$ is (the underlying tree of) a G -subtree of \mathbf{P} if and only if Q is a union of G -orbits of \mathbf{P} .

Next we will introduce the concept of a labeled G -tree. The labels will come from a structure $(S; \leq, G)$ where $(S; \leq)$ is a partially ordered set on which G acts by automorphisms; more precisely, $(S; \leq, G)$ is a structure such that $(S; \leq)$ is a partially ordered set, $(S; G)$ is a G -set, and for each $g \in G$, the permutation $s \mapsto gs$ of S is an automorphism of $(S; \leq)$. If $\mathbf{P} = (P; *, 1_P, G)$ is a G -tree, then an S -labeling of the leaves of \mathbf{P} is a homomorphism $\ell: (\mathbf{P}_{\min}; G) \rightarrow (S; G)$ of G -sets. An S -labeled G -tree is a structure $(\mathbf{P}; \ell) = (P; *, 1_P, G; \ell)$ where $\mathbf{P} = (P; *, 1_P, G)$ is a G -tree and ℓ is an S -labeling of the leaves of \mathbf{P} . If the labeling ℓ is understood, we will write \mathbf{P} instead of $(\mathbf{P}; \ell)$.

For arbitrary S -labeled G -trees $\mathbf{P} = (P; *, 1_P, G; \ell_P)$ and $\mathbf{Q} = (Q; *, 1_Q, G; \ell_Q)$ a *label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$* is a G -homomorphism $\varphi: (P; *, 1_P, G) \rightarrow (Q; *, 1_Q, G)$ with the additional property that

$$(H4) \quad \ell_P(a) = \ell_Q(\varphi(a)) \text{ for all } a \in \mathbf{P}_{\min},$$

and a *label-increasing G -homomorphism* $\mathbf{P} \nearrow \mathbf{Q}$ is a G -homomorphism $\varphi: (P; *, 1_P, G) \rightarrow (Q; *, 1_Q, G)$ with the additional property that

$$(H5) \quad \ell_P(a) \leq \ell_Q(\varphi(a)) \text{ for all } a \in \mathbf{P}_{\min}.$$

Clearly, every label-preserving G -homomorphism is a label-increasing G -homomorphism. Moreover, the composition of label-preserving G -homomorphisms is a label-preserving G -homomorphism, and the same holds for label-increasing G -homomorphisms. An *isomorphism between S -labeled G -trees* is a bijective, label-preserving G -homomorphism. As usual, if there exists an isomorphism $\mathbf{P} \rightarrow \mathbf{Q}$ between two S -labeled G -trees \mathbf{P} and \mathbf{Q} , then \mathbf{P} and \mathbf{Q} are said to be *isomorphic*; in symbols: $\mathbf{P} \cong \mathbf{Q}$.

An S -labeled G -tree $\mathbf{Q} = (Q; *, 1_Q, G; \ell_Q)$ is an *S -labeled G -subtree* of $\mathbf{P} = (P; *, 1_P, G; \ell_P)$ if $Q \subseteq P$ and the identity function $Q \rightarrow P$, $q \mapsto q$ is a label-preserving G -homomorphism $\mathbf{Q} \rightarrow \mathbf{P}$; or equivalently, if $(Q; *, 1_Q, G)$ is a G -subtree of $(P; *, 1_P, G)$ and ℓ_Q is the restriction of ℓ_P to \mathbf{Q}_{\min} .

The main examples of labeled trees we will be concerned with are obtained from chains E of equivalence relations as follows. Let $E = \{\rho_i : 1 \leq i \leq r\}$, say, $\rho_0 := \mathbf{0}_A < \rho_1 < \dots < \rho_{r-1} < \rho_r < \mathbf{1}_A =: \rho_{r+1}$, and let $\Gamma := \text{Aut } E$. Since Γ is a group of permutations on A , $(A; \Gamma)$ becomes a Γ -set with the natural action defined by $\gamma a = \gamma(a)$ for all $a \in A$ and $\gamma \in \Gamma$. For each integer $n \geq 1$, the n -th power of $(A; \Gamma)$ is the Γ -set $(A^n; \Gamma)$ where Γ acts coordinatewise on n -tuples in A^n ; that is, $\gamma \mathbf{a} = (\gamma(a_1), \dots, \gamma(a_n))$ for all $\mathbf{a} = (a_1, \dots, a_n) \in A^n$. Since each permutation $\gamma \in \Gamma$ is ρ_i -invariant for all i ($0 \leq i \leq r+1$), these equivalence relations are congruences of $(A; \Gamma)$, and for each $n \geq 1$, the equivalence relations $(\rho_i)^n$ are congruences of $(A^n; \Gamma)$. Hence we get quotient Γ -sets $(A^n; \Gamma)/(\rho_i)^n = (A^n/(\rho_i)^n; \Gamma)$ whose elements are the blocks of $(\rho_i)^n$, and Γ acts on them the natural way: if B is a block of $(\rho_i)^n$ and $\gamma \in \Gamma$, then γB is the block $\{\gamma \mathbf{x} : \mathbf{x} \in B\}$ of $(\rho_i)^n$. Thus the Γ -orbit of any block B of $(\rho_i)^n$ is the set $\Gamma B = \{\gamma B : \gamma \in \Gamma\}$. For $i = 0$ we will identify $A^n/(\rho_0)^n = A^n/\mathbf{0}_{A^n}$ with A^n , and accordingly, if $B = \{\mathbf{x}\}$, then we will write $\Gamma \mathbf{x}$ for $\Gamma\{\mathbf{x}\}$.

For each integer $n \geq 1$ we define a Γ -tree $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ of uniform depth $r+1$ associated to E as follows:

- $P_n(E) := \{(i, B) : 0 \leq i \leq r+1, B \text{ is a block of } (\rho_i)^n \text{ on } A^n\}$,
- $1_{P_n(E)} := (r+1, A^n)$,
- the successor of each element (i, B) ($0 \leq i \leq r$) is defined by $(i, B)^* := (i+1, C)$ where C is the unique block of ρ_{i+1} with $B \subseteq C$, and
- $\gamma \cdot (i, B) := (i, \gamma B)$ for all $(i, B) \in P_n(E)$.

It is clear that $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ is indeed a Γ -tree of uniform depth $r+1$.

Example 4.5. Figure 1 depicts the Γ -tree $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ for the case when $n = 1$, $A = \{1, 2, 3, 4\}$, $E = \{\rho_1, \rho_2\}$, and ρ_1 has blocks $\{1\}$, $\{2\}$, $\{3, 4\}$, while ρ_2 has blocks $\{1, 2\}$, $\{3, 4\}$. It is easy to see that $\Gamma = \text{Aut } E$ is the 4-element group generated by the transpositions $(1\ 2)$ and $(3\ 4)$. The transposition $(1\ 2)$ acts by switching $(0, 1)$ with $(0, 2)$, $(1, \{1\})$ with $(1, \{2\})$, and fixing all other vertices of the tree, while the transposition $(3\ 4)$ acts by switching $(0, 3)$ with $(0, 4)$ and fixing all other vertices.

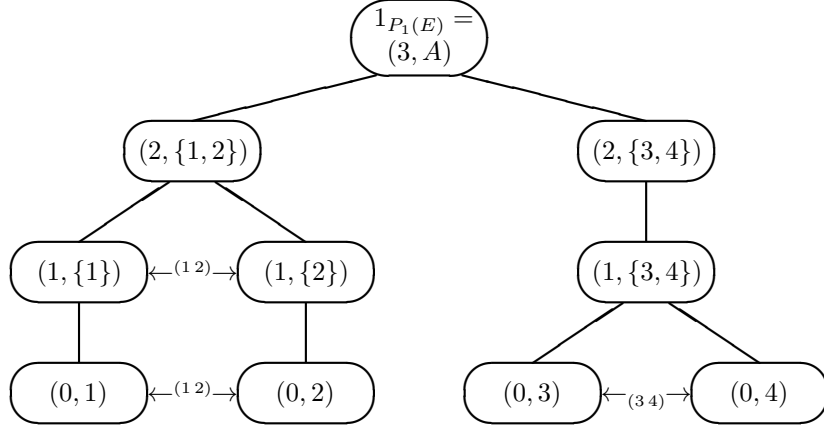


FIGURE 1

We return to the discussion of the Γ -trees $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ introduced before the example, where E is an arbitrary chain of equivalence relations on a finite set A , $\Gamma = \text{Aut } E$, and $n \geq 1$. To describe the labelings of the leaves of $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ that we will need later on, we have to first define the appropriate partially ordered Γ -set of labels. To this end let \mathbb{S} denote the set of all functions $(\Gamma \mathbf{y}, \mathbf{y}) \rightarrow A$ whose domains are pointed Γ -orbits in A^m for some $m \geq 1$. We define an action of Γ on \mathbb{S} as follows:

- for arbitrary element $\mu: (U, \mathbf{y}) \rightarrow A$ of \mathbb{S} with $U = \Gamma \mathbf{y}$ and for any $\gamma \in \Gamma$, the function $\gamma \mu$ is μ considered as a function $(U, \gamma \mathbf{y}) \rightarrow A$.

That is, the only difference between μ and $\gamma \mu$ is in the distinguished element of the orbit U . Clearly, $\gamma \mu \in \mathbb{S}$ and $(\gamma \gamma') \mu = \gamma(\gamma' \mu)$ hold for all $\gamma, \gamma' \in \Gamma$ and $\mu \in \mathbb{S}$, so we have obtained a Γ -set $(\mathbb{S}; \Gamma)$.

Now we define a quasiorder \preceq on \mathbb{S} . Let $\mu: (U, \mathbf{y}) \rightarrow A$ and $\nu: (V, \mathbf{z}) \rightarrow A$ be arbitrary elements of \mathbb{S} where $U = \Gamma \mathbf{y}$, $V = \Gamma \mathbf{z}$, and $\mathbf{y} \in A^m$, $\mathbf{z} \in A^n$. For any tuple $\mathbf{x} \in A^m$ let \mathbf{x}^b denote the set of coordinates of \mathbf{x} . We define $\mu \preceq \nu$ by the following condition:

- $\mu \preceq \nu$ if and only if $\Gamma_{\mathbf{y}} \subseteq \Gamma_{\mathbf{z}}$, $\mathbf{y}^b \supseteq \mathbf{z}^b$, and $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$ where $\chi_{\mathbf{y}, \mathbf{z}}$ is the unique homomorphism $(\Gamma \mathbf{y}; \mathbf{y}, \Gamma) \rightarrow (\Gamma \mathbf{z}; \mathbf{z}, \Gamma)$, $\gamma \mathbf{y} \mapsto \gamma \mathbf{z}$ of pointed Γ -sets.

\sim will denote the intersection of \preceq with its converse. It follows from the definitions of \sim and \preceq that $\mu \sim \nu$ if and only if $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{z}}$, $\mathbf{y}^b = \mathbf{z}^b$, and $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$, $\nu = \mu \circ \chi_{\mathbf{z}, \mathbf{y}}$. The equality $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{z}}$ implies that $\chi_{\mathbf{y}, \mathbf{z}}$ and $\chi_{\mathbf{z}, \mathbf{y}}$ are mutually inverse isomorphisms between the pointed Γ -sets $(\Gamma \mathbf{y}; \mathbf{y}, \Gamma)$ and $(\Gamma \mathbf{z}; \mathbf{z}, \Gamma)$. Therefore

- $\mu \sim \nu$ if and only if $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{z}}$, $\mathbf{y}^b = \mathbf{z}^b$, and $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$ where $\chi_{\mathbf{y}, \mathbf{z}}$ is the unique isomorphism $(\Gamma \mathbf{y}; \mathbf{y}, \Gamma) \rightarrow (\Gamma \mathbf{z}; \mathbf{z}, \Gamma)$, $\gamma \mathbf{y} \mapsto \gamma \mathbf{z}$ of pointed Γ -sets.

The next lemma summarizes some elementary consequences of these definitions that we will need later on.

Lemma 4.6. *Let $(\mathbb{S}; \Gamma)$ be the Γ -set, and let \preceq and \sim be the relations on \mathbb{S} defined above.*

- (1) \preceq is a quasiorder, i.e., it is reflexive and transitive.

- (2) \sim is an equivalence relation, and \preceq induces a partial order \leq on the quotient set \mathbb{S}/\sim by

$$\mu/\sim \leq \nu/\sim \iff \mu \preceq \nu \quad \text{for all } \mu, \nu \in \mathbb{S}.$$

- (3) Γ acts on \mathbb{S} by automorphisms of the relational structure $(\mathbb{S}; \preceq, \sim)$.
(4) The quotient structure $(\mathbb{S}/\sim; \leq, \Gamma)$ is a partially ordered set on which Γ acts by automorphisms of $(\mathbb{S}/\sim; \leq)$.
(5) The number of \sim -classes of \mathbb{S} is at most $|A|^{|A|+2|\Gamma|}$, hence \mathbb{S}/\sim is finite.

Proof. Let $\lambda: (T, \mathbf{x}) \rightarrow A$, $\mu: (U, \mathbf{y}) \rightarrow A$, and $\nu: (V, \mathbf{z}) \rightarrow A$ be arbitrary elements of \mathbb{S} where $T = \Gamma\mathbf{x}$, $U = \Gamma\mathbf{y}$, $V = \Gamma\mathbf{z}$, and $\mathbf{x} \in A^l$, $\mathbf{y} \in A^m$, $\mathbf{z} \in A^n$.

(1) $\mu \preceq \mu$, since $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{y}}$, $\mathbf{y}^b = \mathbf{y}^b$, and $\chi_{\mathbf{y}, \mathbf{y}}$ is the identity function $U \rightarrow U$, so $\mu = \mu \circ \chi_{\mathbf{y}, \mathbf{y}}$. Thus \preceq is reflexive. To verify that \preceq is transitive, assume that $\lambda \preceq \mu \preceq \nu$, that is, $\Gamma_{\mathbf{x}} \subseteq \Gamma_{\mathbf{y}} \subseteq \Gamma_{\mathbf{z}}$, $\mathbf{x}^b \supseteq \mathbf{y}^b \supseteq \mathbf{z}^b$, and $\lambda = \mu \circ \chi_{\mathbf{x}, \mathbf{y}}$, $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$. Then $\Gamma_{\mathbf{x}} \subseteq \Gamma_{\mathbf{z}}$, $\mathbf{x}^b \supseteq \mathbf{z}^b$, and $\lambda = \nu \circ (\chi_{\mathbf{y}, \mathbf{z}} \circ \chi_{\mathbf{x}, \mathbf{y}})$. Since $\chi_{\mathbf{y}, \mathbf{z}} \circ \chi_{\mathbf{x}, \mathbf{y}} = \chi_{\mathbf{x}, \mathbf{z}}$, we get that $\lambda = \nu \circ \chi_{\mathbf{x}, \mathbf{z}}$, proving that $\lambda \preceq \nu$.

(2) is an immediate consequence of (1).

(3) Since \sim is the intersection of \preceq and its converse, it is enough to prove that Γ acts by automorphisms of $(\mathbb{S}; \preceq)$. To this end we need to show that $\mu \preceq \nu$ implies $\gamma\mu \preceq \gamma\nu$ for all $\gamma \in \Gamma$. Let $\mu \preceq \nu$, that is, $\Gamma_{\mathbf{y}} \subseteq \Gamma_{\mathbf{z}}$, $\mathbf{y}^b \supseteq \mathbf{z}^b$, and $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$. Then

$$\begin{aligned} \Gamma_{\gamma\mathbf{y}} &= \gamma\Gamma_{\mathbf{y}}\gamma^{-1} \subseteq \gamma\Gamma_{\mathbf{z}}\gamma^{-1} = \Gamma_{\gamma\mathbf{z}}, \\ (\gamma\mathbf{y})^b &= \gamma(\mathbf{y}^b) \supseteq \gamma(\mathbf{z}^b) = (\gamma\mathbf{z})^b, \end{aligned}$$

and $\gamma\mu = \gamma\nu \circ \chi_{\gamma\mathbf{y}, \gamma\mathbf{z}}$, because $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$, $\chi_{\mathbf{y}, \mathbf{z}} = \chi_{\gamma\mathbf{y}, \gamma\mathbf{z}}$, and $\mu, \gamma\mu$ are the same function $U \rightarrow A$ and $\nu, \gamma\nu$ are the same function $V \rightarrow A$. This proves that $\gamma\mu \preceq \gamma\nu$.

(4) is an immediate consequence of (2) and (3).

(5) We saw earlier that $\mu \sim \nu$ if and only if $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{z}}$, $\mathbf{y}^b = \mathbf{z}^b$, and $\mu = \nu \circ \chi_{\mathbf{y}, \mathbf{z}}$ for the unique isomorphism $\chi_{\mathbf{y}, \mathbf{z}}$ between the pointed Γ -sets $(U; \mathbf{y}, \Gamma)$ and $(V; \mathbf{z}, \Gamma)$. The equality $\Gamma_{\mathbf{y}} = \Gamma_{\mathbf{z}}$ also implies that $(U; \mathbf{y}, \Gamma)$ and $(V; \mathbf{z}, \Gamma)$ are isomorphic to the pointed Γ -set $(\Gamma/\Gamma_{\mathbf{y}}; \Gamma_{\mathbf{y}}, \Gamma)$ of the left cosets of $\Gamma_{\mathbf{y}}$ under the natural action of Γ by left multiplication. Therefore the number of \sim -classes in \mathbb{S} is at most the number of triples $(\mathbf{y}^b, \Gamma_{\mathbf{y}}, \sigma)$ where \mathbf{y}^b is a subset of A , $\Gamma_{\mathbf{y}}$ is a subgroup of Γ , and σ is a function $(\Gamma/\Gamma_{\mathbf{y}}; \Gamma_{\mathbf{y}}) \rightarrow A$. Hence the number of \sim -classes is at most $2^{|A|} 2^{|\Gamma|} |A|^{|\Gamma|} \leq |A|^{|A|+2|\Gamma|}$, as claimed. \square

If g is an n -ary operation on A , we define an \mathbb{S}/\sim -labeling ℓ_g of the leaves of the Γ -tree $(P_n(E); *, 1_{P_n(E)}, \Gamma)$ by

$$\bullet \ell_g((0, \mathbf{x})) = g|_{(\Gamma\mathbf{x}, \mathbf{x})}/\sim \text{ for all } \mathbf{x} \in A^n$$

where $g|_{(\Gamma\mathbf{x}, \mathbf{x})}$ denotes the restriction of g to the pointed Γ -orbit $(\Gamma\mathbf{x}, \mathbf{x})$; thus $g|_{(\Gamma\mathbf{x}, \mathbf{x})}$ is an element of \mathbb{S} . This labeling yields an \mathbb{S}/\sim -labeled Γ -tree $(P_n(E); *, 1_{P_n(E)}, \Gamma; \ell_g)$, which we will denote by $\mathbf{P}_g(E)$, and will call *the \mathbb{S}/\sim -labeled Γ -tree associated to f* .

Example 4.7. Let A , E , and Γ be as in Example 4.5, and let g be the unary operation on A defined by $g(1) = 2$, $g(2) = 4$, $g(3) = 4$, and $g(4) = 3$. The \mathbb{S}/\sim -labeled Γ -tree $\mathbf{P}_g(E)$ is obtained from the Γ -tree $(P_1(E); *, 1_{P_1(E)}, \Gamma)$ in Example 4.5 by labeling the leaves via ℓ_g . For each leaf $(0, x)$ ($x \in A = \{1, 2, 3, 4\}$), the label of

$(0, x)$ is the equivalence class μ_x/\sim where $\mu_x: (\Gamma x, x) \rightarrow A$ is the restriction of g to the pointed Γ -orbit of x ; i.e.,

$$\begin{aligned}\mu_1: (\{1, 2\}, 1) &\rightarrow A, & 1 \mapsto 2, 2 \mapsto 4; \\ \mu_2: (\{1, 2\}, 2) &\rightarrow A, & 1 \mapsto 2, 2 \mapsto 4; \\ \mu_3: (\{3, 4\}, 3) &\rightarrow A, & 3 \mapsto 4, 4 \mapsto 3; \\ \mu_4: (\{3, 4\}, 4) &\rightarrow A, & 3 \mapsto 4, 4 \mapsto 3.\end{aligned}$$

The functions μ_x ($x \in A$) belong to pairwise different \sim -classes, because $x^b = \{x\} \neq \{y\} = y^b$ for distinct elements $x, y \in A$. Therefore the labeling ℓ_g assigns four distinct labels to the four leaves.

The next lemma shows the relevance of the \mathbb{S}/\sim -labeled Γ -trees $\mathbf{P}_f(E)$ and $\mathbf{P}_g(E)$ to the problem of determining whether $f \leq_C g$ holds for two operations f, g on A .

Lemma 4.8. *Let E be a chain of equivalence relations on a finite set A , and let $\mathcal{C} = \text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$. For arbitrary operations f, g on A , $f \leq_C g$ if and only if there exists a label-increasing Γ -homomorphism $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$ between the \mathbb{S}/\sim -labeled Γ -trees associated to f and g .*

Proof. Let f be m -ary and g be n -ary. To prove the forward implication assume that $f \leq_C g$, and let $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ be such that $f = g \circ \mathbf{h}$. Since \mathbf{h} preserves the equivalence relations in E , \mathbf{h} maps each block B of $(\rho_i)^m$ into a block of $(\rho_i)^n$. Thus \mathbf{h} induces a map

$$\psi: P_m(E) \rightarrow P_n(E), \quad (i, B) \mapsto (i, \overline{\mathbf{h}(B)})$$

where $\overline{\mathbf{h}(B)}$ denotes the block of $(\rho_i)^n$ containing $\mathbf{h}(B)$. We claim that ψ is a label-increasing Γ -homomorphism $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$. Clearly, ψ maps $1_{P_m(E)} = (r+1, A^m)$ to $1_{P_n(E)} = (r+1, A^n)$, and it maps leaves to leaves. Furthermore, if $(i, B) \in P_m(E)$ with $0 \leq i \leq r$, then $(i, B)^* = (i+1, C)$ for the unique block C of $(\rho_{i+1})^m$ satisfying $B \subseteq C$. Therefore $\mathbf{h}(B) \subseteq \mathbf{h}(C)$, so $\overline{\mathbf{h}(B)} \subseteq \overline{\mathbf{h}(C)}$, which shows that

$$\psi((i, B)^*) = \psi((i+1, C)) = (i+1, \overline{\mathbf{h}(C)}) = (i, \overline{\mathbf{h}(B)})^* = \psi((i, B))^*.$$

Thus ψ is a homomorphism of trees. Next, if $(i, B) \in P_m(E)$ and $\gamma \in \Gamma$, then

$$\psi(\gamma((i, B))) = \psi((i, \gamma(B))) = (i, \overline{\mathbf{h}(\gamma(B))})$$

and

$$\gamma(\psi((i, B))) = \gamma((i, \overline{\mathbf{h}(B)})) = (i, \gamma(\overline{\mathbf{h}(B)})) = (i, \overline{\gamma(\mathbf{h}(B))}).$$

Since \mathbf{h} preserves γ , we have $\mathbf{h}(\gamma(B)) = \gamma(\mathbf{h}(B))$, proving $\psi(\gamma((i, B))) = \gamma(\psi((i, B)))$. Hence ψ is a Γ -homomorphism $(P_m(E); *, 1_{P_m(E)}, \Gamma) \rightarrow (P_n(E); *, 1_{P_n(E)}, \Gamma)$.

Finally, if $(0, \mathbf{x})$ is a leaf of $P_m(E)$, then using the definition of the labelings ℓ_f and ℓ_g and the relationship $f = g \circ \mathbf{h}$ we get that

$$\begin{aligned}\ell_f((0, \mathbf{x})) &= f|_{(\Gamma \mathbf{x}, \mathbf{x})/\sim}, \\ \ell_g(\psi((0, \mathbf{x}))) &= \ell_g((0, \mathbf{h}(\mathbf{x}))) = g|_{(\Gamma \mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{x}))/\sim},\end{aligned}$$

and

$$f|_{(\Gamma \mathbf{x}, \mathbf{x})} = (g \circ \mathbf{h})|_{(\Gamma \mathbf{x}, \mathbf{x})} = g|_{(\Gamma \mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{x}))} \circ \mathbf{h}|_{(\Gamma \mathbf{x}, \mathbf{x})}.$$

Here $\mathbf{h}|_{(\Gamma\mathbf{x}, \mathbf{x})}: (\Gamma\mathbf{x}, \mathbf{x}) \rightarrow (\Gamma\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{x}))$ is a homomorphism of pointed Γ -orbits, since \mathbf{h} preserves all permutations $\gamma \in \Gamma$. Thus $\Gamma_{\mathbf{x}} \subseteq \Gamma_{\mathbf{h}(\mathbf{x})}$ and $\mathbf{h}|_{(\Gamma\mathbf{x}, \mathbf{x})} = \chi_{\mathbf{x}, \mathbf{h}(\mathbf{x})}$. In addition, we have $\mathbf{x}^b \supseteq \mathbf{h}(\mathbf{x})^b$, since \mathbf{h} preserves all subsets of A . Thus

$$f|_{(\Gamma\mathbf{x}, \mathbf{x})} = g|_{(\Gamma\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{x}))} \circ \chi_{\mathbf{x}, \mathbf{h}(\mathbf{x})} \preceq g|_{(\Gamma\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{x}))},$$

implying that $\ell_f((0, \mathbf{x})) \leq \ell_g(\psi((0, \mathbf{x})))$. This proves that ψ is a label-increasing Γ -homomorphism $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$, and hence concludes the proof of the forward implication.

For the converse, assume that there exists a label-increasing Γ -homomorphism $\psi: \mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$. Our goal is to show that $f \leq_C g$. Since ψ is a homomorphism of trees, therefore it maps each leaf of $\mathbf{P}_f(E)$ into a leaf of $\mathbf{P}_g(E)$. Hence ψ yields a function $\mathbf{h}: A^m \rightarrow A^n$ such that $\psi((0, \mathbf{x})) = (0, \mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \in A^m$. We will establish $f \leq_C g$ by proving that $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ and $f = g \circ \mathbf{h}$.

First we will show that \mathbf{h} preserves all equivalence relations ρ_i ($1 \leq i \leq r$). Let $\mathbf{x}, \mathbf{y} \in A^m$ be such that $\mathbf{x} (\rho_i)^m \mathbf{y}$. Then \mathbf{x}, \mathbf{y} are in the same block B of $(\rho_i)^m$, i.e., $(0, \mathbf{x})^{*i} = (i, B) = (0, \mathbf{y})^{*i}$. Since ψ is a homomorphism of trees, we get that

$$\begin{aligned} (0, \mathbf{h}(\mathbf{x}))^{*i} &= \psi((0, \mathbf{x})^{*i}) = \psi((0, \mathbf{x})^{*i}) \\ &= \psi((0, \mathbf{y})^{*i}) = \psi((0, \mathbf{y})^{*i}) = (0, \mathbf{h}(\mathbf{y}))^{*i}. \end{aligned}$$

Hence $\mathbf{h}(\mathbf{x})$ and $\mathbf{h}(\mathbf{y})$ are in the same block of $(\rho_i)^n$, that is, $\mathbf{h}(\mathbf{x}) (\rho_i)^n \mathbf{h}(\mathbf{y})$.

Next we show that \mathbf{h} preserves all permutations $\gamma \in \Gamma$. Since ψ is a Γ -homomorphism and Γ acts on the leaves of $\mathbf{P}_f(E)$ and $\mathbf{P}_g(E)$ by $\gamma \cdot (0, \mathbf{u}) = (0, \gamma\mathbf{u})$ for all \mathbf{u} and γ , we get that

$$\begin{aligned} (0, \mathbf{h}(\gamma\mathbf{x})) &= \psi((0, \gamma\mathbf{x})) = \psi(\gamma \cdot (0, \mathbf{x})) \\ &= \gamma \cdot (\psi(0, \mathbf{x})) = \gamma \cdot (0, \mathbf{h}(\mathbf{x})) = (0, \gamma\mathbf{h}(\mathbf{x})) \end{aligned}$$

for all $\mathbf{x} \in A^m$ and $\gamma \in \Gamma$. Hence $\mathbf{h}(\gamma\mathbf{x}) = \gamma(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \in A^m$ and $\gamma \in \Gamma$, as claimed.

This also proves that \mathbf{h} restricts to every pointed Γ -orbit $(\Gamma\mathbf{u}, \mathbf{u})$ in A^m as a homomorphism $\mathbf{h}|_{(\Gamma\mathbf{u}, \mathbf{u})}: (\Gamma\mathbf{u}, \mathbf{u}) \rightarrow (\Gamma\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{u}))$ between two pointed Γ -orbits. Since such a homomorphism exists only if $\Gamma_{\mathbf{u}} \subseteq \Gamma_{\mathbf{h}(\mathbf{u})}$, and when it exists, it is uniquely determined, we get that $\mathbf{h}|_{(\Gamma\mathbf{u}, \mathbf{u})} = \chi_{\mathbf{u}, \mathbf{h}(\mathbf{u})}$.

Since

$$\begin{aligned} \ell_f((0, \mathbf{u})) &= f|_{(\Gamma\mathbf{u}, \mathbf{u})}/\sim, \\ \ell_g(\psi((0, \mathbf{u}))) &= \ell_g((0, \mathbf{h}(\mathbf{u}))) = g|_{(\Gamma\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{u}))}/\sim, \end{aligned}$$

and ψ is label-increasing, we get that $f|_{(\Gamma\mathbf{u}, \mathbf{u})} \preceq g|_{(\Gamma\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{u}))}$. By the definition of \preceq this means that $\Gamma_{\mathbf{u}} \subseteq \Gamma_{\mathbf{h}(\mathbf{u})}$, $\mathbf{u}^b \supseteq \mathbf{h}(\mathbf{u})^b$, and $f|_{(\Gamma\mathbf{u}, \mathbf{u})} = g|_{(\Gamma\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{u}))} \circ \chi_{\mathbf{u}, \mathbf{h}(\mathbf{u})}$. Combining this with the equality $\mathbf{h}|_{(\Gamma\mathbf{u}, \mathbf{u})} = \chi_{\mathbf{u}, \mathbf{h}(\mathbf{u})}$ we get that

$$f|_{(\Gamma\mathbf{u}, \mathbf{u})} = g|_{(\Gamma\mathbf{h}(\mathbf{u}), \mathbf{h}(\mathbf{u}))} \circ \mathbf{h}|_{(\Gamma\mathbf{u}, \mathbf{u})}.$$

Since A^m is the union of all Γ -orbits $\Gamma\mathbf{u}$, we obtain from the last displayed equality that $f = g \circ \mathbf{h}$. The property that $\mathbf{u}^b \supseteq \mathbf{h}(\mathbf{u})^b$ for all $\mathbf{u} \in A^m$ shows that \mathbf{h} preserves all subsets of A . Thus $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ and $f = g \circ \mathbf{h}$, which proves that $f \leq_C g$. \square

It follows from Lemma 4.8 that $f \equiv_C g$ holds for two operations f, g on A if and only if there exist label-increasing Γ -homomorphisms $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$ and

$\mathbf{P}_g(E) \nearrow \mathbf{P}_f(E)$ between the \mathbb{S}/\sim -labeled Γ -trees associated to f and g . Since the size of $\mathbf{P}_f(E)$ increases with the arity of f , this lemma alone is not enough to conclude that the number of $\equiv_{\mathcal{C}}$ -classes is finite. We want to replace each $\mathbf{P}_f(E)$ by an \mathbb{S}/\sim -labeled Γ -tree $\widehat{\mathbf{P}}_f$ that is

- homomorphically equivalent to $\mathbf{P}_f(E)$, that is, there exist label-preserving Γ -homomorphisms $\mathbf{P}_f(E) \rightarrow \widehat{\mathbf{P}}_f$ and $\widehat{\mathbf{P}}_f \rightarrow \mathbf{P}_f(E)$, and
- as small as possible with this property.

The first condition is to ensure that the analog of Lemma 4.8 remains true if, instead of $\mathbf{P}_f(E)$, we associate $\widehat{\mathbf{P}}_f$ to each operation f . The second condition will allow us to prove that, up to isomorphism, there are only finitely many $\widehat{\mathbf{P}}_f$'s, and hence it will follow that the number of $\equiv_{\mathcal{C}}$ -classes is finite.

The intended relationship between $\widehat{\mathbf{P}}_f$ and $\mathbf{P}_f(E)$ is captured by the concept of a *core*, which applies to arbitrary finite structures. For our purposes it will be enough to discuss cores of S -labeled G -trees.

Let $\mathbf{P} = (P; *, 1_P, G; \ell_P)$ and $\mathbf{Q} = (Q; *, 1_Q, G; \ell_Q)$ be S -labeled G -trees. We say that

- (1) \mathbf{Q} is a *core* if every label-preserving G -homomorphism $\mathbf{Q} \rightarrow \mathbf{Q}$ is onto;
- (2) \mathbf{Q} is a *core of \mathbf{P}* if
 - \mathbf{Q} is *homomorphically equivalent* to \mathbf{P} , that is, there exist label-preserving G -homomorphisms $\mathbf{P} \rightarrow \mathbf{Q}$ and $\mathbf{Q} \rightarrow \mathbf{P}$, and
 - \mathbf{Q} is minimal with this property (i.e., no proper labeled G -subtree of \mathbf{Q} is homomorphically equivalent to \mathbf{P}).

For the reader's convenience we will state and prove the basic properties of cores for S -labeled G -trees. The first one of these properties is that the two uses of the word 'core' in the definitions above are compatible: every core of an S -labeled G -tree [in the sense of (2)] is actually a core [in the sense of (1)]. We will use this property later on without further reference. The second and third properties show that every S -labeled G -tree has a core (in fact, it has one among its S -labeled G -subtrees), and the core is uniquely determined, up to isomorphism.

Lemma 4.9. *Let \mathbf{P} be an S -labeled G -tree.*

- (1) *Every core of \mathbf{P} is a core.*
- (2) *If $\widehat{\mathbf{P}}$ is minimal, with respect to inclusion, among all S -labeled G -subtrees \mathbf{P}' of \mathbf{P} for which there exists a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{P}'$, then $\widehat{\mathbf{P}}$ is a core of \mathbf{P} .*
- (3) *Any two cores of \mathbf{P} are isomorphic.*

Proof. (1) Let \mathbf{Q} be a core of \mathbf{P} . It follows that there exist label-preserving G -homomorphisms $\varphi: \mathbf{P} \rightarrow \mathbf{Q}$ and $\psi: \mathbf{Q} \rightarrow \mathbf{P}$. To prove that \mathbf{Q} is a core, we need to show that every label-preserving G -homomorphism $\tau: \mathbf{Q} \rightarrow \mathbf{Q}$ is onto. The range \mathbf{R} of τ is an S -labeled G -subtree of \mathbf{Q} , therefore the identity embedding $\iota: \mathbf{R} \rightarrow \mathbf{Q}$ is a label-preserving G -homomorphism. Thus $\tau = \iota \circ \tilde{\tau}$ for some label-preserving G -homomorphism $\tilde{\tau}: \mathbf{Q} \rightarrow \mathbf{R}$. Hence we have label-preserving G -homomorphisms

$$\mathbf{P} \begin{array}{c} \xrightarrow{\varphi} \\ \xrightarrow{\psi} \end{array} \mathbf{Q} \begin{array}{c} \xrightarrow{\tilde{\tau}} \\ \xrightarrow{\iota} \end{array} \mathbf{R},$$

which implies that \mathbf{R} is homomorphically equivalent to \mathbf{P} , as witnessed by $\tilde{\tau} \circ \varphi: \mathbf{P} \rightarrow \mathbf{R}$ and $\psi \circ \iota: \mathbf{R} \rightarrow \mathbf{P}$. Since \mathbf{Q} is a core of \mathbf{P} , the S -labeled G -subtree \mathbf{R} of \mathbf{Q} cannot be proper. Thus $\mathbf{R} = \mathbf{Q}$ and τ is onto.

(2) Let $\widehat{\mathbf{P}}$ be minimal, with respect to inclusion, among all S -labeled G -subtrees \mathbf{P}' of \mathbf{P} for which there exists a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{P}'$. Such a $\widehat{\mathbf{P}}$ exists, since \mathbf{P} is finite. Moreover, the identity embedding $\widehat{\mathbf{P}} \rightarrow \mathbf{P}$ is a label-preserving G -homomorphism, because $\widehat{\mathbf{P}}$ is an S -labeled G -subtree of \mathbf{P} . Thus $\widehat{\mathbf{P}}$ is homomorphically equivalent to \mathbf{P} . The choice that $\widehat{\mathbf{P}}$ is minimal among the S -labeled G -subtrees \mathbf{P}' of \mathbf{P} for which there exists a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{P}'$ ensures that $\widehat{\mathbf{P}}$ is also minimal among the S -labeled G -subtrees of \mathbf{P} that are homomorphically equivalent to \mathbf{P} . This proves that $\widehat{\mathbf{P}}$ is a core of \mathbf{P} , as claimed.

(3) Let \mathbf{Q} and \mathbf{Q}' be cores of \mathbf{P} . Then \mathbf{Q} and \mathbf{Q}' are homomorphically equivalent to \mathbf{P} , so we can choose label-preserving G -homomorphisms

$$\mathbf{Q} \begin{array}{c} \xrightarrow{\varphi} \\ \xleftarrow{\psi} \end{array} \mathbf{P} \begin{array}{c} \xrightarrow{\varphi'} \\ \xleftarrow{\psi'} \end{array} \mathbf{Q}'$$

witnessing this fact. Thus we have label-preserving G -homomorphisms

$$\mathbf{Q} \xrightarrow{\sigma} \mathbf{Q}', \quad \mathbf{Q} \xleftarrow{\sigma'} \mathbf{Q}', \quad \mathbf{Q} \xrightarrow{\sigma' \circ \sigma} \mathbf{Q}, \quad \text{and} \quad \mathbf{Q}' \xleftarrow{\sigma \circ \sigma'} \mathbf{Q}'$$

where $\sigma = \varphi' \circ \psi$ and $\sigma' := \varphi \circ \psi'$. Since \mathbf{Q} and \mathbf{Q}' are cores by part (1), the latter two label-preserving G -homomorphisms are onto. Since \mathbf{Q} and \mathbf{Q}' are finite, they are also one-to-one. This implies that σ and σ' are both onto and one-to-one, hence they are isomorphisms. \square

To prove that for each d there are, up to isomorphism, only finitely many S -labeled trees of uniform depth d that are cores (Lemma 4.13), we need some necessary conditions for an S -labeled G -tree to be a core (Corollary 4.12). These necessary conditions will be derived from a general lemma on label-preserving G -homomorphisms between S -labeled G -trees (Lemma 4.11).

We start with some preparation. Let $\mathbf{P} = (P; *, 1_P, G; \ell)$ be an S -labeled G -tree. The set of elements of depth 1 in \mathbf{P} , that is, the set of elements $a \in P$ such that $a^* = 1_P$, will be denoted by \mathbf{P}_{\max} . For any $a \in \mathbf{P}_{\max}$ the set of all elements $b \in P$ such that $b^{*i} = a$ for some integer $i \geq 0$ will be denoted by $[a]$. The next lemma summarizes some basic facts on \mathbf{P}_{\max} and $[a]$ ($a \in \mathbf{P}_{\max}$) that we will need later on.

Lemma 4.10. *Let $\mathbf{P} = (P; *, 1_P, G; \ell)$ be an S -labeled G -tree.*

- (1) $[a] \cap [b] = \emptyset$ if a, b are distinct elements of \mathbf{P}_{\max} , and

$$P = \{1_P\} \cup \bigcup ([a] : a \in \mathbf{P}_{\max}).$$

- (2) \mathbf{P}_{\max} and \mathbf{P}_{\min} are unions of G -orbits.
(3) For each $a \in \mathbf{P}_{\max}$,
(i) if $c \in [a]$, $c \neq a$, then $c^* \in [a]$;
(ii) if $g \in G$, then $g \cdot [a] = [ga]$; hence $g \cdot [a] = [a]$ if and only if $g \in G_a$.
(4) For each $a \in \mathbf{P}_{\max}$, $[a]$ is the underlying set of an S -labeled G_a -tree

$$[a]_{\mathbf{P}} := ([a]; *, 1_{[a]}, G_a; \ell)$$

where $1_{[a]} = a$, $*$ is the restriction of the successor function of \mathbf{P} to the set $(a] \setminus \{a\}$, the action of each $g \in G_a$ on $(a]$ is obtained by restricting the action of g to $(a]$, and ℓ is the restriction of the labeling of the leaves of \mathbf{P} to the leaves of $(a]$.

- (5) $((a]_{\mathbf{P}})_{\min} = \mathbf{P}_{\min} \cap (a]$ for each $a \in \mathbf{P}_{\max}$, so if $|P| > 1$, then

$$\mathbf{P}_{\min} = \bigcup \{((a]_{\mathbf{P}})_{\min} : a \in \mathbf{P}_{\max}\}.$$

Proof. Recall that for each element $u \in P \setminus \{1_P\}$ there exists a unique positive integer d , the depth of u , such that $u^{*^d} = 1_P$. Thus $u^{*^{d-1}} \in \mathbf{P}_{\max}$ and $u \in (u^{*^{d-1}}]$, which proves the displayed equality in (1). Moreover, if $u \in (a]$ for some $a \in \mathbf{P}_{\max}$, then the definitions of \mathbf{P}_{\max} and $(a]$ yield that $a^* = 1_P$ and $u^{*^i} = a$ for some integer $i \geq 0$. Thus $u^{*^{i+1}} = 1_P$, and the uniqueness of the depth of u implies that $d = i + 1$. Hence $a = u^{*^{d-1}}$, showing that $u \in (a]$ for a unique $a \in \mathbf{P}_{\max}$. This completes the proof of (1).

(2) and (3) are immediate consequences of the definitions, using also the fact that each $g \in G$ acts by automorphisms of the tree $(P; 1_P, *)$. (3) ensures that $*$ and $g \in G_a$ restrict to $(a]$ as claimed. The properties of the operations of $((a]; *, 1_{[a]}, G_a)$ that make it a G_a -tree are inherited from \mathbf{P} . Furthermore, it follows from the definition of $(a]$ that the leaves of the tree $((a]; 1_{[a]}, *)$ are exactly the leaves of \mathbf{P} that are in $(a]$. This establishes the first equality in (5), and also implies that the restriction of ℓ to $(a]$ (also denoted by ℓ) yields an S -labeling of the leaves of the G_a -tree $((a]; *, 1_{[a]}, G_a)$. This proves (4). Finally, the displayed equality in (5) follows from the equality $((a]_{\mathbf{P}})_{\min} = \mathbf{P}_{\min} \cap (a]$ proved earlier and the displayed equality in (1). \square

It follows from the preceding lemma that every S -labeled G -tree is the disjoint union of the S -labeled G_a -trees $(a]_{\mathbf{P}}$ ($a \in \mathbf{P}_{\max}$) with a new top element 1_P added. In the next lemma we will use this structure of S -labeled G -trees to analyze the label-preserving G -homomorphisms between them.

Lemma 4.11. *Let $\mathbf{P} = (P; *, 1_P, G; \ell_P)$ and $\mathbf{Q} = (Q; *, 1_Q, G; \ell_Q)$ be S -labeled G -trees, and let $\{a_i : 1 \leq i \leq t\}$ be a transversal for the G -orbits of \mathbf{P}_{\max} . If b_i ($1 \leq i \leq t$) are elements of \mathbf{Q}_{\max} such that $G_{a_i} = G_{b_i}$ for each i , then*

- (1) every family $\{\psi_i : 1 \leq i \leq t\}$ of label-preserving G_{a_i} -homomorphisms $\psi_i : (a_i]_{\mathbf{P}} \rightarrow (b_i]_{\mathbf{Q}}$ has a unique extension to a label-preserving G -homomorphism $\varphi : \mathbf{P} \rightarrow \mathbf{Q}$.
- (2) φ is onto if and only if every G -orbit of \mathbf{Q}_{\max} contains at least one b_i , and

$$(b_i] = \bigcup \{h \cdot \text{Im } \psi_j : 1 \leq j \leq t, h \in G, hb_j = b_i\}$$

for each i ($1 \leq i \leq t$).

- (2)' In particular, if $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} , then φ is onto if and only if each ψ_i is onto.
- (3) φ is bijective if and only if $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} and each ψ_i is bijective.

Proof. Let a_i, b_i ($1 \leq i \leq t$) satisfy the assumptions of the lemma. Fix an i ($1 \leq i \leq t$), and consider the G -orbit $Ga_i = \{ha_i : h \in G\}$ of a_i . As we noticed in Lemma 4.10 (2), $Ga_i \subseteq \mathbf{P}_{\max}$. We claim that the subset $P_i = \{1_P\} \cup \bigcup_{h \in G} (ha_i]$ of P is the underlying set of an S -labeled G -subtree \mathbf{P}_i of \mathbf{P} . Indeed, the definition of

\mathbf{P}_{\max} and Lemma 4.10 (3) shows that the successor of every element of $P_i \setminus \{1_P\}$ is in P_i , and that P_i is closed under the action of G . Furthermore, it follows from the first equality in Lemma 4.10 (5) that $(\mathbf{P}_i)_{\min} = \mathbf{P}_{\min} \cap \mathbf{P}_i$. This proves that P_i is the underlying set of an S -labeled G -subtree \mathbf{P}_i of \mathbf{P} . In fact, \mathbf{P}_i is the smallest S -labeled G -subtree of \mathbf{P} that contains $(a_i]$. For, if \mathbf{P}'_i is an S -labeled G -subtree of \mathbf{P} such that $(a_i] \subseteq P'_i$, then $1_P \in P'_i$ by the definition of a subtree, and $(ha_i] = h \cdot (a_i] \subseteq P'_i$, since \mathbf{P}'_i is closed under the action of G . Thus $P_i \subseteq P'_i$.

Similarly, for each i ($1 \leq i \leq t$), the subset $Q_i = \{1_Q\} \cup \bigcup_{h \in G} (hb_i]_{\mathbf{Q}}$ of Q is the underlying set of an S -labeled G -subtree \mathbf{Q}_i of \mathbf{Q} , and \mathbf{Q}_i is the smallest S -labeled G -subtree of \mathbf{P} that contains $(b_i]$.

(1) Now assume that $\{\psi_i : 1 \leq i \leq t\}$ is a family of label-preserving G_{a_i} -homomorphisms $\psi_i : (a_i]_{\mathbf{P}} \rightarrow (b_i]_{\mathbf{Q}}$. First we prove the uniqueness of the extension φ claimed in (1). Assume $\varphi : \mathbf{P} \rightarrow \mathbf{Q}$ is a label-preserving G -homomorphism that extends all ψ_i . Then $\varphi(1_P) = 1_Q$ and, by Lemma 4.10 (3)(ii), for each $h \in G$ and $c \in (ha_i]$ we have $h^{-1}c \in (a_i]$, so

$$\varphi(c) = \varphi(h(h^{-1}c)) = h\varphi(h^{-1}c) = h\psi_i(h^{-1}c).$$

This proves that φ is uniquely determined by the ψ_i 's.

To prove the existence of φ we will verify that under the assumptions of the lemma, for each i ($1 \leq i \leq t$),

(I) _{i} the rule

$$\varphi_i(c) = \begin{cases} 1_Q & \text{if } c = 1_P \\ h\psi_i(h^{-1}c) & \text{if } c \in (ha_i] \text{ (} h \in G \text{)} \end{cases}$$

defines a label-preserving G -homomorphism $\varphi_i : \mathbf{P}_i \rightarrow \mathbf{Q}_i$ that extends ψ_i ,

and

(II) for any family $\{\varphi_i : 1 \leq i \leq t\}$ of label-preserving G -homomorphisms $\varphi_i : \mathbf{P}_i \rightarrow \mathbf{Q}_i$, the union φ of the φ_i 's is a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$.

We will start with (II). By Lemma 4.10 (1) every element c of \mathbf{P} other than 1_P belongs to a subset of the form $(a]$ for a unique $a \in \mathbf{P}_{\max}$. Since $\{a_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{P}_{\max} , the G -orbits Ga_i partition \mathbf{P}_{\max} . Moreover, $(\mathbf{P}_i)_{\max} = Ga_i$ for each i , therefore it follows that every element c of \mathbf{P} other than 1_P belongs to exactly one of the G -subtrees \mathbf{P}_i of \mathbf{P} . As for 1_P , we have $\varphi_i(1_P) = 1_Q$ for each i , since φ_i is a homomorphism of trees. Thus we get that $\varphi := \bigcup_{i=1}^t \varphi_i$ is a well-defined function $\mathbf{P} \rightarrow \mathbf{Q}$.

To prove that φ is a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$ we have to verify that it satisfies conditions (H1)–(H4) ((H0) is established already). Since all φ_i are label-preserving G -homomorphisms, they satisfy conditions (H1)–(H4). In particular, φ_i maps $(\mathbf{P}_i)_{\min}$ into $(\mathbf{Q}_i)_{\min}$. But the displayed equality in Lemma 4.10 (5) applied to \mathbf{P} and each \mathbf{P}_i shows that $\mathbf{P}_{\min} = \bigcup_{i=1}^t (\mathbf{P}_i)_{\min}$, so (H1) follows for φ . Since each \mathbf{P}_i is an S -labeled G -subtree of \mathbf{P} , conditions (H2)–(H4) immediately follow from the corresponding conditions for the φ_i 's. This completes the proof of (II).

For each i , statement (I) _{i} is a special case of the general statement about the existence of φ , namely the special case when \mathbf{P}_{\max} is a single G -orbit Ga . Therefore all (I) _{i} will be proved if we show the existence of φ for the case when $t = 1$ holds for \mathbf{P} . To simplify notation, we will omit subscripts; that is, we let a be an element

of \mathbf{P}_{\max} , and assume that $\mathbf{P} = \{1_P\} \cup \bigcup_{h \in G} (ha]_{\mathbf{P}}$. Furthermore, we let b be an element of \mathbf{Q}_{\max} with $G_a = G_b$, and let $\psi: (a]_{\mathbf{P}} \rightarrow (b]_{\mathbf{Q}}$ be a label-preserving G_a -homomorphism. Our goal is to show that

$$\varphi(c) = \begin{cases} 1_Q & \text{if } c = 1_P \\ h\psi(h^{-1}c) & \text{if } c \in (ha] \ (h \in G) \end{cases}$$

defines a label-preserving G -homomorphism $\varphi: \mathbf{P} \rightarrow \mathbf{Q}$ that extends ψ .

First we show that φ is a well-defined function $P \rightarrow Q$. If $c \in (ha]$, then $h^{-1}c \in (a]$, therefore $\psi(h^{-1}c)$ is defined, and hence so is $h\psi(h^{-1}c)$. Suppose now that $c \in (ha]$ and $c \in (ga]$. Since $ha, ga \in \mathbf{P}_{\max}$, Lemma 4.10 (1) shows that $ha = ga$. Thus $g^{-1}ha = a$, that is, $g^{-1}h \in G_a$. Hence

$$g\psi(h^{-1}c) = g\psi((g^{-1}h)(h^{-1}c)) = g(g^{-1}h)\psi(h^{-1}c) = h\psi(h^{-1}c),$$

where the middle equality holds, because ψ is a G_a -homomorphism. This shows that φ is well-defined. Clearly, φ is an extension of ψ , for if $c \in (a]$, then an application of the definition of φ to $h = 1$, the neutral element of G , yields that $\varphi(c) = \psi(c)$.

To prove that φ is a label-preserving G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$, we need to check that conditions (H0)–(H4) hold for φ . (H0) is obvious from the definition of φ , and (H1) holds, because ψ as well as the actions of $h \in G$ map leaves to leaves. To show that (H2) holds let $c \in P \setminus \{1_P\}$. As $c \neq 1_P$, we have that $c \in (ha]$ for some $h \in G$. Assume first that $c = ha$. Since ψ is a G_a -homomorphism $(a]_{\mathbf{P}} \rightarrow (b]_{\mathbf{Q}}$ and $a = 1_{(a]}$, $b = 1_{(b]}$, therefore we get that $\psi(a) = b$. Hence if $c = ha$, then $\varphi(c) = h\psi(h^{-1}c) = h\psi(a) = hb \in \mathbf{Q}_{\max}$, so $\varphi(c)^* = 1_Q = \varphi(1_P) = \varphi(c^*)$. Now assume that $c \in (ha]$ but $c \neq ha$. Then $h^{-1}c \in (a]$ and $h^{-1}c \neq a$. Hence $(h^{-1}c)^*$ in $(a]_{\mathbf{P}}$ is the same as $(h^{-1}c)^*$ in \mathbf{P} , which is equal to $h^{-1}c^*$. Using this (in the fourth equality below) we get that

$$\varphi(c)^* = (h\psi(h^{-1}c))^* = h(\psi(h^{-1}c))^* = h\psi((h^{-1}c)^*) = h\psi(h^{-1}c^*) = \varphi(c^*),$$

which completes the proof of (H2). Next we prove (H3). Every $g \in G$ acts by tree automorphisms, therefore $g \cdot 1_P = 1_P$ and $g \cdot 1_Q = 1_Q$, whence $\varphi(g \cdot 1_P) = \varphi(1_P) = 1_Q = g \cdot 1_Q = g\varphi(1_P)$. To prove (H3) for elements $c \neq 1_P$ let $c \in (ha]$ and $g \in G$. Then $gc \in (gha]$, hence $\varphi(gc) = gh\psi((gh)^{-1}gc) = gh\psi(h^{-1}c) = g\varphi(c)$. Thus (H3) holds for φ . Finally, we verify (H4). Let $c \in \mathbf{P}_{\min}$. Then c is a leaf in $(ha]_{\mathbf{P}}$ for some $h \in G$, and hence $h^{-1}c$ is a leaf in $(a]_{\mathbf{P}}$. Since $\psi: (a]_{\mathbf{P}} \rightarrow (b]_{\mathbf{Q}}$ is a label-preserving G_a -homomorphism, $\psi(h^{-1}c)$ is a leaf in $(b]_{\mathbf{Q}}$. Using the facts that ℓ_P, ℓ_Q are labelings of \mathbf{P} and \mathbf{Q} , and their restrictions are the labelings of $(a]_{\mathbf{P}}$ and $(b]_{\mathbf{Q}}$, we get that

$$\begin{aligned} \ell_Q(\varphi(c)) &= \ell_Q(h\psi(h^{-1}c)) = h\ell_Q(\psi(h^{-1}c)) \\ &= h\ell_P(h^{-1}c) = hh^{-1}\ell_P(c) = \ell_P(c), \end{aligned}$$

proving (H4). This finishes the proof of statement (1) of the lemma.

(2) We return to the general case; that is, $\{a_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{P}_{\max} , $\{b_i : 1 \leq i \leq t\}$ is a subset of \mathbf{Q}_{\max} such that $G_{a_i} = G_{b_i}$ for each i , $\{\psi_i : 1 \leq i \leq t\}$ is a family of label-preserving G_{a_i} -homomorphisms $\psi_i: (a_i]_{\mathbf{P}} \rightarrow (b_i]_{\mathbf{Q}}$, and φ is the unique extension of all ψ_i 's to a label-preserving homomorphism $\varphi: \mathbf{P} \rightarrow \mathbf{Q}$ constructed in part (1). The two-step construction of φ

described in (I)_i and (II) above shows that

$$\varphi(c) = \begin{cases} 1_Q & \text{if } c = 1_P, \\ h\psi_i(h^{-1}c) & \text{if } c \in (ha_i] \text{ for some } 1 \leq i \leq t \text{ and some } h \in G. \end{cases}$$

We claim that an element u of \mathbf{Q} is in the range of φ if and only if either $u = 1_Q$ or $u \in h \cdot \text{Im } \psi_i$ for some i and some $h \in G$. The necessity of this condition is clear from the description of φ above. For the sufficiency, let $u = 1_Q$ or $u \in h \cdot \text{Im } \psi_i$. In the first case, clearly, u is in the range of φ . In the second case $u = h\psi_i(v)$ for some $v \in (a_i]$, so for $c = hv$ we have $c \in (ha_i]$ and $u = h\psi_i(v) = h\psi_i(h^{-1}c) = \varphi(c)$. Thus u is in the range of φ , as claimed. This proves that

$$\text{Im } \varphi = \{1_Q\} \cup \bigcup (G \cdot \text{Im } \psi_j : 1 \leq j \leq t).$$

Since ψ_j maps $a_j = 1_{(a_j]}$ to $1_{(b_j]} = b_j$, we have $b_j \in \text{Im } \psi_j \subseteq (b_j]$. It follows that $\mathbf{Q}_{\max} \cap G \cdot \text{Im } \psi_j = Gb_j$ holds for all j . Thus

$$\mathbf{Q}_{\max} \cap \text{Im } \varphi = \bigcup (\mathbf{Q}_{\max} \cap G \cdot \text{Im } \psi_j : 1 \leq j \leq t) = \bigcup (Gb_j : 1 \leq j \leq t).$$

For any two distinct elements $b, b' \in \mathbf{Q}_{\max}$, the subsets $(b]$ and $(b']$ are disjoint by Lemma 4.10 (1). Therefore $(b_i]$ is disjoint from $h \cdot \text{Im } \psi_j (\subseteq (hb_j])$ unless $b_i = hb_j$, and hence $h \cdot \text{Im } \psi_j \subseteq (b_i]$. Thus, for each i ($1 \leq i \leq t$),

$$\begin{aligned} (b_i] \cap \text{Im } \varphi &= (b_i] \cap \bigcup (G \cdot \text{Im } \psi_j : 1 \leq j \leq t) \\ &= \bigcup (h \cdot \text{Im } \psi_j : 1 \leq j \leq t, h \in G, hb_j = b_i). \end{aligned}$$

Hence, for φ to map onto Q , it is necessary that $\mathbf{Q}_{\max} = \bigcup (Gb_j : 1 \leq j \leq t)$ and $(b_i] = \bigcup (h \cdot \text{Im } \psi_j : 1 \leq j \leq t, h \in G, hb_j = b_i)$ for each i ($1 \leq i \leq t$). This shows that the conditions in (2) are necessary. Conversely, assume that φ satisfies these conditions. The second one of these conditions implies that $(b_i] \subseteq \text{Im } \varphi$ for all i ($1 \leq i \leq t$). Since φ is a G -homomorphism $\mathbf{P} \rightarrow \mathbf{Q}$, its range is closed under the actions of all $g \in G$. Combining this with the condition $\mathbf{Q}_{\max} = \bigcup (Gb_j : 1 \leq j \leq t)$ we obtain that

$$\begin{aligned} Q \setminus \{1_Q\} &= \bigcup ((b] : b \in \mathbf{Q}_{\max}) \\ &= \bigcup ((gb_i] : 1 \leq i \leq t, g \in G) \\ &= \bigcup (G \cdot (b_i] : 1 \leq i \leq t) \subseteq \text{Im } \varphi. \end{aligned}$$

Since $1_Q \in \text{Im } \varphi$, we get that φ is surjective. This proves statement (2).

(2)' Now assume that $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} . Then $hb_j = b_i$ holds for some $1 \leq i, j \leq t$ and $h \in G$ only if $i = j$ and $h \in G_{b_i}$. Therefore

$$\bigcup (h \cdot \text{Im } \psi_j : 1 \leq j \leq t, h \in G, hb_j = b_i) = \bigcup (h \cdot \text{Im } \psi_i : h \in G_{b_i}) = \text{Im } \psi_i.$$

Hence the criterion in (2) implies that in this special case φ is onto if and only if all ψ_i are onto.

(3) To prove the necessity of the conditions in (3) suppose that φ is bijective. First we show that $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} . Since φ is onto, we get from part (2) of the lemma that every G -orbit of \mathbf{Q}_{\max} is represented by at least one element in $\{b_i : 1 \leq i \leq t\}$. If $\{b_i : 1 \leq i \leq t\}$ was not a transversal, there would exist $1 \leq j < l \leq t$ such that $b_j = hb_l$ for some $h \in G$.

Hence $\varphi(a_j) = b_j = hb_l = \varphi(ha_l)$, but $a_j \neq ha_l$ as $\{a_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{P}_{\max} . This shows that $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} . Since φ is onto, it follows from part (2)' of the lemma that each ψ_i is onto; ψ_i is also one-to-one, since φ extends ψ_i and φ is one-to-one. This proves that the conditions in (3) are indeed necessary for φ to be bijective.

Conversely, if φ satisfies the conditions in (3), then it is clearly onto by the criterion in part (2)'. To verify that φ is one-to-one, let c, c' be elements in \mathbf{P} such that $\varphi(c) = \varphi(c')$. It is clear from the description of φ that the only element whose φ -image is 1_Q is 1_P . Therefore if $1_P \in \{c, c'\}$, say $c = 1_P$, then $\varphi(c') = \varphi(c) = 1_Q$, so $c' = 1_P$ and hence $c = c'$. Assume from now on that $c, c' \neq 1_P$. Then $c \in (ha_i]$ and $c' \in (h'a_{i'})]$ for some i, i' ($1 \leq i, i' \leq t$) and $h, h' \in G$. Since $\varphi(c) = h\psi_i(h^{-1}c) \in h \cdot (b_i] = (hb_i]$ and similarly $\varphi(c') \in (h'b_{i'})]$, the assumption that $\varphi(c) = \varphi(c')$, combined with Lemma 4.10 (1), implies that $hb_i = h'b_{i'}$. Our assumption that $\{b_i : 1 \leq i \leq t\}$ is a transversal for the G -orbits of \mathbf{Q}_{\max} forces that $i = i'$ and $hG_{b_i} = h'G_{b_{i'}}$. Throughout the lemma we assume $G_{b_i} = G_{a_i}$ for each i , therefore $ha_i = h'a_i$ and both of c, c' belong to $(ha_i]$. Thus the equality $\varphi(c) = \varphi(c')$ can be rewritten as $h\psi_i(h^{-1}c) = h\psi_i(h^{-1}c')$. Hence $\psi_i(h^{-1}c) = \psi_i(h^{-1}c')$, and since ψ_i is bijective, $h^{-1}c = h^{-1}c'$, which implies that $c = c'$. This proves the sufficiency of the conditions in (3), and completes the proof of the lemma. \square

Corollary 4.12. *If $\mathbf{Q} = (Q; *, 1_Q, G; \ell)$ is an S -labeled G -tree that is a core, then the following hold for arbitrary elements a, b of \mathbf{Q}_{\max} :*

- (i) $(a]_{\mathbf{Q}}$, as an S -labeled G_a -tree, is a core.
- (ii) If $G_a = G_b$ and $(a]_{\mathbf{Q}} \cong (b]_{\mathbf{Q}}$ as S -labeled G_a -trees, then $Ga = Gb$.

Proof. (i) Suppose that $(a]_{\mathbf{Q}}$, as an S -labeled G_a -tree, is not a core. Then there exists a label-preserving G_a -homomorphism $\psi: (a]_{\mathbf{Q}} \rightarrow (a]_{\mathbf{Q}}$ that is not surjective. Let $a_1 = a, a_2, \dots, a_t$ be a transversal for the G -orbits of \mathbf{Q}_{\max} , let $\psi_1 = \psi$, and for $2 \leq i \leq t$ let ψ_i be the identity isomorphism $(a_i]_{\mathbf{Q}} \rightarrow (a_i]_{\mathbf{Q}}$. Applying Lemma 4.11 we get that the family $\{\psi_i : 1 \leq i \leq t\}$ can be extended to a unique label-preserving G -homomorphism $\varphi: \mathbf{Q} \rightarrow \mathbf{Q}$. Moreover, since $\psi_i(a_i) = a_i$ for all i , part (2)' of the lemma applies and yields that φ is not surjective. Therefore \mathbf{Q} is not a core.

(ii) Assume that $a, b \in \mathbf{Q}_{\max}$ are in different G -orbits such that $G_a = G_b$ and $(a]_{\mathbf{Q}} \cong (b]_{\mathbf{Q}}$ as S -labeled G_a -trees. Let ψ be a label-preserving G_a -isomorphism $(a]_{\mathbf{Q}} \rightarrow (b]_{\mathbf{Q}}$. We want to show that \mathbf{Q} is not a core. Let $a_1 = a, a_2 = b, a_3, \dots, a_t$ be a transversal for the G -orbits of \mathbf{Q}_{\max} , let $\psi_1 = \psi$, and for $2 \leq i \leq t$ let ψ_i be the identity isomorphism $(a_i]_{\mathbf{Q}} \rightarrow (a_i]_{\mathbf{Q}}$. Applying Lemma 4.11 we get that the family $\{\psi_i : 1 \leq i \leq t\}$ can be extended to a unique label-preserving G -homomorphism $\varphi: \mathbf{Q} \rightarrow \mathbf{Q}$. Since $\{\psi_i(a_i) : 1 \leq i \leq t\} = \{b, a_3, \dots, a_t\}$ does not represent all G -orbits of \mathbf{Q}_{\max} , it follows from part (2) of the lemma that φ is not surjective. Therefore \mathbf{Q} is not a core, as claimed. \square

Lemma 4.13. *For every group G and G -set $(S; G)$ of labels, and for each natural number k there exists an integer $n_k = n_k(G, S)$ depending only on k, G , and $(S; G)$ such that there are at most n_k nonisomorphic S -labeled G -trees of uniform depth k that are cores.*

Proof. Let $\mathbf{Q} = (Q; *, 1_Q, G; \ell)$ be an S -labeled G -tree of uniform depth k that is a core. We want to find an upper bound on the number of possibilities for \mathbf{Q} , up to isomorphism.

If $k = 0$, then $Q = \{1_Q\}$, and the unique element (which is a leaf) can be labeled in $|S|$ different ways. Therefore in this case there are $n_0 = |S|$ possibilities for \mathbf{Q} , up to isomorphism.

Now let $k \geq 1$, and assume that $n_{k-1} = n_{k-1}(G, S)$ has been found for all G and $(S; G)$. Choose a transversal $\{a_i : 1 \leq i \leq t\}$ for the G -orbits of \mathbf{Q}_{\max} , and for each transversal element a_i consider the pair $(G_{a_i}, ((a_i]_{\mathbf{Q}})^{\text{iso}})$ where $((a_i]_{\mathbf{Q}})^{\text{iso}}$ denotes the isomorphism type of $(a_i]_{\mathbf{Q}}$, as an S -labeled G_{a_i} -tree. Each $(a_i]_{\mathbf{Q}}$ has uniform depth $k - 1$, since \mathbf{Q} has uniform depth k . Since \mathbf{Q} is a core, we get from Corollary 4.12 that the S -labeled G_{a_i} -tree $(a_i]_{\mathbf{Q}}$ is a core for every i ($1 \leq i \leq t$). Moreover, if $1 \leq i < j \leq t$, then $G_{a_i} \neq G_{a_j}$ or $(a_i]_{\mathbf{Q}} \not\cong (a_j]_{\mathbf{Q}}$. Thus the pairs $(G_{a_i}, ((a_i]_{\mathbf{Q}})^{\text{iso}})$ ($1 \leq i \leq t$) are pairwise distinct. By part (3) of Lemma 4.11 the set

$$\{(G_{a_i}, ((a_i]_{\mathbf{Q}})^{\text{iso}}) : 1 \leq i \leq t\}$$

determines \mathbf{Q} , up to isomorphism. Therefore the number of possible isomorphism types for \mathbf{Q} is at most

$$n_k(G, S) = 2^s \quad \text{where} \quad s = \sum (n_{k-1}(H, S) : H \text{ is a subgroup of } G).$$

This completes the proof. \square

We return to the proof of Theorem 4.2. As before, let A be a finite set, and let $E = \{\rho_i : 1 \leq i \leq r\}$ be a chain of equivalence relations, say, $\rho_0 := \mathbf{0}_A < \rho_1 < \dots < \rho_{r-1} < \rho_r < \mathbf{1}_A =: \rho_{r+1}$, and let $\Gamma := \text{Aut } E$. Earlier in this section we defined for each operation f on A an \mathbb{S}/\sim -labeled Γ -tree $\mathbf{P}_f(E)$ of uniform depth $r + 1$. By Lemma 4.9 $\mathbf{P}_f(E)$ has a core $\widehat{\mathbf{P}}_f = (\widehat{P}_f, \widehat{\leq}, \widehat{\ell}_f)$ that is an \mathbb{S}/\sim -labeled Γ -subtree of $\mathbf{P}_f(E)$. Thus $\widehat{\mathbf{P}}_f$ is of uniform depth $r + 1$, and there exists a label-preserving Γ -homomorphism $\varphi_f : \mathbf{P}_f(E) \rightarrow \widehat{\mathbf{P}}_f$. Moreover, $\widehat{\mathbf{P}}_f$ is uniquely determined up to isomorphism. We will refer to $\widehat{\mathbf{P}}_f$ as *the core of the \mathbb{S}/\sim -labeled Γ -tree associated to f* .

The following statement is an easy consequence of Lemma 4.8.

Corollary 4.14. *Let E be a chain of equivalence relations on a finite set A , let $\Gamma = \text{Aut } E$, and let $\mathcal{C} = \text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$. For arbitrary operations f, g on A ,*

- (1) $f \leq_{\mathcal{C}} g$ if and only if there exists a label-increasing homomorphism $\widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$ between the cores of the \mathbb{S}/\sim -labeled Γ -trees associated to f and g .
- (2) $f \equiv_{\mathcal{C}} g$ if and only if there exist label-increasing homomorphisms $\widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$ and $\widehat{\mathbf{P}}_g \nearrow \widehat{\mathbf{P}}_f$ between the cores of the \mathbb{S}/\sim -labeled Γ -trees associated to f and g .

Proof. Let f, g be arbitrary operations on A . By construction, there exist label-preserving Γ -homomorphisms $\varphi_f : \mathbf{P}_f(E) \rightarrow \widehat{\mathbf{P}}_f$ and $\varphi_g : \mathbf{P}_g(E) \rightarrow \widehat{\mathbf{P}}_g$. Since $\widehat{\mathbf{P}}_f$ is an \mathbb{S}/\sim -labeled Γ -subtree of $\mathbf{P}_f(E)$, and $\widehat{\mathbf{P}}_g$ is an \mathbb{S}/\sim -labeled Γ -subtree of $\mathbf{P}_g(E)$, the identity mappings $\iota_f : \widehat{\mathbf{P}}_f \rightarrow \mathbf{P}_f(E)$ and $\iota_g : \widehat{\mathbf{P}}_g \rightarrow \mathbf{P}_g(E)$ are also label-preserving Γ -homomorphisms.

By Lemma 4.8, $f \leq_{\mathcal{C}} g$ if and only if there exists a label-increasing Γ -homomorphism $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$. We claim that there exists a label-increasing Γ -homomorphism $\mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$ if and only if there exists a label-increasing Γ -homomorphism $\widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$. Indeed, if $\psi : \mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$, then $\varphi_g \circ \psi \circ \iota_f : \widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$, and conversely, if $\psi' : \widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$, then $\iota_g \circ \psi' \circ \varphi_f : \mathbf{P}_f(E) \nearrow \mathbf{P}_g(E)$, since the

composition of label-increasing (or label-preserving) Γ -homomorphisms is a label-increasing Γ -homomorphism. This proves (1).

The relation \equiv_C is the intersection of \leq_C with its converse, therefore (2) is an immediate consequence of (1). \square

Proof of Theorem 4.2. Let E be a chain of equivalence relations on a finite set A , let $\Gamma = \text{Aut } E$, and let $\mathcal{C} = \text{Pol}(E, \text{Aut } E, \mathcal{P}^+(A))$. Corollary 4.14 implies that $f \equiv_C g$ holds for two operations f and g on A if and only if for the cores $\widehat{\mathbf{P}}_f$ and $\widehat{\mathbf{P}}_g$ of the associated \mathbb{S}/\sim -labeled Γ -trees there exist label-increasing Γ -homomorphisms $\widehat{\mathbf{P}}_f \nearrow \widehat{\mathbf{P}}_g$ and $\widehat{\mathbf{P}}_g \nearrow \widehat{\mathbf{P}}_f$. In particular, it follows that $f \equiv_C g$ if $\widehat{\mathbf{P}}_f \cong \widehat{\mathbf{P}}_g$. By Lemma 4.13 there exist only finitely many isomorphism classes of trees $\widehat{\mathbf{P}}_f$ as f runs over all operations on A . Therefore there exist only finitely many \equiv_C -classes. \square

5. CENTRAL RELATIONS

Let A be a k -element finite set, $k \geq 3$. In this section, our aim is to find all maximal clones $\text{Pol } \rho$ on A that are determined by central relations and are members of \mathfrak{F}_A . Note that the arity r of a central relation on A satisfies $1 \leq r \leq k-1$, and the case of unary central relations is settled in Corollary 2.4. Therefore in this section we consider only central relations of arity $r \geq 2$.

We will show that if ρ has arity $r \leq k-2$, then $\text{Pol } \rho \notin \mathfrak{F}_A$ (Theorem 5.3), while if ρ has arity $r = k-1$, then $\text{Pol } \rho \in \mathfrak{F}_A$ (Theorem 5.2). Note that for each element $c \in A$ there is a unique central relation σ_c of arity $k-1$ with central element c , namely

$$\sigma_c = \{(a_1, \dots, a_{k-1}) \in A^{k-1} : a_i = a_j \text{ for some } 1 \leq i < j \leq k-1, \text{ or} \\ a_i = c \text{ for some } 1 \leq i \leq k-1\}.$$

Therefore all central relations of arity $k-1$ are of the form σ_c for some $c \in A$. In Theorem 5.2 we will, in fact, prove that $\text{Pol}(\sigma_c, \{c\}) \in \mathfrak{F}_A$ for all $c \in A$, which implies by Proposition 2.1(ii) that all maximal clones $\text{Pol } \sigma_c$ ($c \in A$) also belong to \mathfrak{F}_A .

We start by stating Jablonskiĭ's Lemma which we will need in the proof of Theorem 5.2.

Lemma 5.1. (Jablonskiĭ [10]) *Let f be an n -ary operation on a finite set A such that f depends on at least two of its variables. If the range $\text{Im } f$ of f has $r \geq 3$ elements, then there exist $D_1, \dots, D_n \subseteq A$ such that $|D_i| < r$ for all $1 \leq i \leq n$ and $f[D_1 \times \dots \times D_n] = \text{Im } f$.*

Theorem 5.2. *If σ_c is the $(k-1)$ -ary central relation with central element c on a k -element set A ($k \geq 3$), then $\text{Pol}(\sigma_c, \{c\}) \in \mathfrak{F}_A$.*

Proof. We may assume without loss of generality that $c = 0$, and we will write σ for σ_0 . So, let $\mathcal{C} = \text{Pol}(\sigma, \{0\})$. Since σ contains all $(k-1)$ -tuples whose coordinates are not pairwise distinct or include 0, it follows that σ is preserved by

- all operations $f: A^n \rightarrow A$ with $|\text{Im } f| \leq k-2$, and also
- all operations $f: A^n \rightarrow A$ with $|\text{Im } f| = k-1$ and $0 \in \text{Im } f$.

To prove that $\mathcal{C} \in \mathfrak{F}_A$ we partition \mathcal{O}_A into two subsets, O_0 and $O_1 = \mathcal{O}_A \setminus O_0$, as follows: an operation f belongs to O_0 if and only if its domain A^n where n is the arity of f contains a subset $C_1 \times \dots \times C_n$ such that $0 \in C_i \neq A$ for all i ($1 \leq i \leq n$) and $f[C_1 \times \dots \times C_n] = \text{Im } f$. First we will show that all nonsurjective

operations f on A belong to O_0 . Indeed, assume first that f is nonsurjective and essentially unary, say it depends on its first variable only. Then there exists a nonsurjective unary operation f_1 such that $f(\mathbf{x}) = f_1(x_1)$ for all $\mathbf{x} = (x_1, \dots, x_n) \in A^n$. Therefore $f_1(a) = f_1(b)$ for some distinct $a, b \in A$ such that $a \neq 0$. Hence the choice $C_1 = A \setminus \{a\}$, $C_2 = \dots = C_n = \{0\}$ shows that $f \in O_0$. Now assume that f is nonsurjective and depends on at least two of its variables. If $|\text{Im } f| = 2$, then there exists $\mathbf{a} = (a_1, \dots, a_n)$ distinct from $\bar{0}$ such that $f(\mathbf{a}) \neq f(\bar{0})$. Hence the choice $C_i = \{a_i, 0\}$ ($1 \leq i \leq n$) shows that $f \in O_0$ ($C_i \neq A$, since $k = |A| \geq 3$). Finally, if $|\text{Im } f| > 2$ but f is nonsurjective, then by Jablonskiĭ's Lemma (Lemma 5.1) there exist $(k-2)$ -element subsets D_1, \dots, D_n of A such that $f[D_1 \times \dots \times D_n] = \text{Im } f$. Hence we can choose $C_i = D_i \cup \{0\}$ ($1 \leq i \leq n$) to show that $f \in O_0$.

Claim 1. If $\text{Im } f = \text{Im } g$, $f(\bar{0}) = g(\bar{0})$, and $f, g \in O_0$, then $f \equiv_c g$.

Proof of Claim 1. Let f be n -ary and g be m -ary. Using the assumption $f \in O_0$, we fix sets $C_i \subset A$ ($1 \leq i \leq n$) such that $0 \in C_i$ and $f[C_1 \times \dots \times C_n] = \text{Im } f$. Furthermore, we choose a transversal $\{\mathbf{b}_1, \dots, \mathbf{b}_r\} \subseteq C_1 \times \dots \times C_n$ of $\ker f$ where $\mathbf{b}_1 = \bar{0}$. Now we define a function $\mathbf{h}: A^m \rightarrow A^n$ as follows: for each $\mathbf{a} \in A^m$ we have $g(\mathbf{a}) \in \text{Im } g = \text{Im } f$, therefore $g(\mathbf{a}) = f(\mathbf{b}_j)$ for a unique j ; we let $\mathbf{h}(\mathbf{a}) = \mathbf{b}_j$. It is clear from this definition that $g = f \circ \mathbf{h}$. By assumption, $g(\bar{0}) = f(\bar{0})$ and $\mathbf{b}_1 = \bar{0}$, therefore $g(\bar{0}) = f(\mathbf{b}_1)$. Hence $\mathbf{h}(\bar{0}) = \mathbf{b}_1 = \bar{0}$, which implies that \mathbf{h} preserves $\{0\}$. Since $\text{Im } \mathbf{h} = \{\bar{0}, \mathbf{b}_2, \dots, \mathbf{b}_r\} \subseteq C_1 \times \dots \times C_n$, the range of each component h_i of \mathbf{h} satisfies $0 \in \text{Im } h_i \subseteq C_i \neq A$. As was observed at the beginning of the proof, this implies that each h_i preserves σ . Thus \mathbf{h} preserves σ . This proves that $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ and hence $g \leq_c f$. A similar argument shows that $f \leq_c g$. \diamond

Claim 2. If $f(\bar{0}) = g(\bar{0})$ and $f, g \notin O_0$, then $f \equiv_c g$.

Proof of Claim 2. Again, let f be n -ary and g be m -ary. We proved earlier that all nonsurjective operations belong to O_0 , therefore f and g are necessarily surjective. Let $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ be a transversal of $\ker f$ where $\mathbf{b}_1 = \bar{0}$. As before, we define $\mathbf{h}: A^m \rightarrow A^n$ such that for each $\mathbf{a} \in A^m$, $\mathbf{h}(\mathbf{a}) = \mathbf{b}_j$ for the unique j such that $g(\mathbf{a}) = f(\mathbf{b}_j)$. We get, as before, that $g = f \circ \mathbf{h}$ and that \mathbf{h} preserves $\{0\}$. It remains to show that \mathbf{h} preserves σ . Let $\mathbf{a}_1, \dots, \mathbf{a}_{k-1} \in A^m$ be m -tuples such that $(\mathbf{h}(\mathbf{a}_1), \dots, \mathbf{h}(\mathbf{a}_{k-1})) \notin \sigma^n$. Since the range of \mathbf{h} is $\{\bar{0}, \mathbf{b}_2, \dots, \mathbf{b}_k\}$ and σ^n contains every $(k-1)$ -tuple which has repeated coordinates or has $\bar{0}$ as one of its coordinates, we get that $\{\mathbf{h}(\mathbf{a}_1), \dots, \mathbf{h}(\mathbf{a}_{k-1})\} = \{\mathbf{b}_2, \dots, \mathbf{b}_k\}$. Hence

$$\begin{aligned} \{g(\bar{0}), g(\mathbf{a}_1), \dots, g(\mathbf{a}_{k-1})\} &= \{(f \circ \mathbf{h})(\bar{0}), (f \circ \mathbf{h})(\mathbf{a}_1), \dots, (f \circ \mathbf{h})(\mathbf{a}_{k-1})\} \\ &= \{f(\bar{0}), f(\mathbf{b}_2), \dots, f(\mathbf{b}_k)\} = A, \end{aligned}$$

implying that $g[D_1 \times \dots \times D_m] = A = \text{Im } g$ where D_i is the set of i -th coordinates of $\bar{0}, \mathbf{a}_1, \dots, \mathbf{a}_{k-1}$ for each i ($1 \leq i \leq m$). Since by assumption $g \notin O_0$, we have $D_i = A$ for at least one i . Thus $(\mathbf{a}_1, \dots, \mathbf{a}_{k-1}) \notin \sigma^m$, proving that \mathbf{h} preserves σ . This completes the proof that $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ and hence $g \leq_c f$. A similar argument shows that $f \leq_c g$. \diamond

Now consider the mapping

$$\Phi: \mathcal{O}_A \rightarrow \mathcal{P}^+(A) \times A \times \{0, 1\}, \quad f \mapsto (\text{Im } f, f(\bar{0}), i_f)$$

where $i_f = 0$ if $f \in O_0$ and $i_f = 1$ if $f \in O_1$. Claims 1 and 2 show that we have $f \equiv_c g$ whenever $\Phi(f) = \Phi(g)$. Therefore the number of \equiv_c -classes does not exceed the number of kernel classes of Φ . The number of kernel classes of Φ is finite, since

the codomain $\mathcal{P}^+(A) \times A \times \{0, 1\}$ of Φ is finite. Hence the number of $\equiv_{\mathcal{C}}$ -classes is also finite, which proves that $\mathcal{C} \in \mathfrak{F}_A$. \square

Theorem 5.3. *If ρ is an r -ary central relation on a k -element set A such that $2 \leq r \leq k - 2$ ($k \geq 4$), then $\text{Pol } \rho \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \rho$. We may assume without loss of generality that A is the set $\{0, \dots, k - 1\}$, 0 is a central element of ρ , and $(1, 2, \dots, r) \notin \rho$. For each $n \geq 2$ and $1 \leq i \leq n$ we define $2n$ -tuples \mathbf{a}_i^n , \mathbf{b}_i^n , and \mathbf{c}_i^n as follows: $\mathbf{a}_i^n = (0, 0, \dots, 0, 0, 1, 1, 0, 0, \dots, 0, 0)$ with the two 1's occurring in the $(2i - 1)$ -th and $2i$ -th coordinates; $\mathbf{b}_i^n = (1, 2, \dots, 1, 2, 0, 0, 1, 2, \dots, 1, 2)$ with the two 0's occurring in the $(2i - 1)$ -th and $2i$ -th coordinates; $\mathbf{c}_i^n = (0, 0, \dots, 0, 0, 2, 1, 0, 0, \dots, 0, 0)$ with the 2 and 1 occurring in the $(2i - 1)$ -th and the $2i$ -th coordinates. Next we define a $2n$ -ary operation $f_n: A^{2n} \rightarrow A$ for each $n \geq 2$ as follows:

$$f_n(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{a}_i^n \ (1 \leq i \leq n), \\ 1 & \text{if } \mathbf{a} = \mathbf{b}_i^n \ (1 \leq i \leq n), \\ 2 & \text{if } \mathbf{a} = \mathbf{c}_i^n \ (1 \leq i \leq n), \\ u & \text{if } \mathbf{a} = \bar{u} \ (3 \leq u \leq r), \\ r + 1 & \text{otherwise.} \end{cases}$$

We claim that $f_n \not\equiv_{\mathcal{C}} f_m$ whenever $n \neq m$. Suppose on the contrary that $f_n \equiv_{\mathcal{C}} f_m$ for some $m < n$. Then there exists $\mathbf{h} \in (\mathcal{C}^{(2n)})^{2m}$ such that $f_n = f_m \circ \mathbf{h}$. For each element v in the common range $\{0, 1, \dots, r, r + 1\}$ of f_m and f_n , \mathbf{h} maps the inverse image $f_n^{-1}(v)$ of v under f_n into the inverse image $f_m^{-1}(v)$ of v under f_m ; for, if $\mathbf{x} \in f_n^{-1}(v)$, then $v = f_n(\mathbf{x}) = f_m(\mathbf{h}(\mathbf{x}))$, implying that $\mathbf{h}(\mathbf{x}) \in f_m^{-1}(v)$. Thus, in particular, $\mathbf{h}(\bar{u}) = \bar{u}$ for all $3 \leq u \leq r$, and

$$\mathbf{h}(\mathbf{a}_i^n) \in \{\mathbf{a}_1^m, \dots, \mathbf{a}_m^m\}, \quad \mathbf{h}(\mathbf{b}_i^n) \in \{\mathbf{b}_1^m, \dots, \mathbf{b}_m^m\}, \quad \mathbf{h}(\mathbf{c}_i^n) \in \{\mathbf{c}_1^m, \dots, \mathbf{c}_m^m\}$$

for all i ($1 \leq i \leq n$). Since $m < n$, there exist $1 \leq p < q \leq n$ and $1 \leq s \leq m$ such that $\mathbf{h}(\mathbf{a}_p^n) = \mathbf{h}(\mathbf{a}_q^n) = \mathbf{a}_s^m$. We have $(\mathbf{a}_p^n, \mathbf{b}_p^n, \bar{3}, \dots, \bar{r}) \in \rho^{2n}$, as in each coordinate the first or second component of the tuple is 0. Therefore, since \mathbf{h} preserves ρ , we get that

$$(\mathbf{a}_s^m, \mathbf{h}(\mathbf{b}_p^n), \bar{3}, \dots, \bar{r}) = (\mathbf{h}(\mathbf{a}_p^n), \mathbf{h}(\mathbf{b}_p^n), \mathbf{h}(\bar{3}), \dots, \mathbf{h}(\bar{r})) \in \rho^{2m}.$$

If $j \neq s$ then $(\mathbf{a}_s^m, \mathbf{b}_j^m, \bar{3}, \dots, \bar{r}) \notin \rho^{2m}$, because the $2s$ -th coordinate of the tuple is $(1, 2, 3, \dots, r) \notin \rho$. This forces $\mathbf{h}(\mathbf{b}_p^n) = \mathbf{b}_s^m$. The same argument with \mathbf{a}_q^n in place of \mathbf{a}_p^n shows that $\mathbf{h}(\mathbf{b}_q^n) = \mathbf{b}_s^m$. Similarly, since $(\mathbf{c}_p^n, \mathbf{b}_p^n, \bar{3}, \dots, \bar{r}) \in \rho^{2n}$ and \mathbf{h} preserves ρ , we get that

$$(\mathbf{h}(\mathbf{c}_p^n), \mathbf{b}_s^m, \bar{3}, \dots, \bar{r}) = (\mathbf{h}(\mathbf{c}_p^n), \mathbf{h}(\mathbf{b}_p^n), \mathbf{h}(\bar{3}), \dots, \mathbf{h}(\bar{r})) \in \rho^{2m}.$$

Again, if $j \neq s$ then $(\mathbf{c}_j^m, \mathbf{b}_s^m, \bar{3}, \dots, \bar{r}) \notin \rho^{2m}$, because the $2j$ -th coordinate of the tuple is $(1, 2, 3, \dots, r) \notin \rho$. Thus $\mathbf{h}(\mathbf{c}_p^n) = \mathbf{c}_s^m$. The same argument with \mathbf{c}_q^n in place of \mathbf{c}_p^n yields that $\mathbf{h}(\mathbf{c}_q^n) = \mathbf{c}_s^m$. Now we see that $(\mathbf{a}_p^n, \mathbf{c}_q^n, \bar{3}, \dots, \bar{r}) \in \rho^{2n}$, since in each coordinate the first or second component is 0, but

$$(\mathbf{h}(\mathbf{a}_p^n), \mathbf{h}(\mathbf{c}_q^n), \mathbf{h}(\bar{3}), \dots, \mathbf{h}(\bar{r})) = (\mathbf{a}_s^m, \mathbf{c}_s^m, \bar{3}, \dots, \bar{r}) \notin \rho^{2m},$$

because the $(2s - 1)$ -th coordinate is $(1, 2, 3, \dots, r) \notin \rho$. This contradiction shows that $f_n \not\equiv_{\mathcal{C}} f_m$ if $m < n$, and hence proves that $\mathcal{C} \notin \mathfrak{F}_A$. \square

6. h -REGULAR RELATIONS

Let A be a finite set with k elements ($k \geq 3$). In this section our goal is to find all maximal clones $\text{Pol } \lambda_T$ on A determined by h -regular relations λ_T (see Section 2 for the definition) that are members of \mathfrak{F}_A . Recall that the arity of an h -regular relation λ_T is h with $3 \leq h \leq k$. The only h -regular relation with $h = k$ is λ_T where T is the singleton consisting of the equality relation on A , and then $\text{Pol } \lambda_T$ is Słupecki's clone on A .

We will show that $\text{Pol } \lambda_T \notin \mathfrak{F}_A$ unless $\text{Pol } \lambda_T$ is Słupecki's clone (Theorem 6.3). Moreover, we will find an interval in the clone lattice that includes Słupecki's clone and is contained in \mathfrak{F}_A (Theorem 6.1).

As a preparation for stating the latter result we introduce some notation. For $2 \leq i \leq k$, \mathcal{B}_i will denote the subclone of \mathcal{O}_A that consists of all essentially at most unary operations and all operations whose range contains at most i elements. Thus, \mathcal{B}_{k-1} is Słupecki's clone and \mathcal{B}_{k-2} is the clone introduced in Remark 3.6. For $i = 0$, \mathcal{B}_0 will stand for the clone of all essentially at most unary operations, and for $i = 1$, \mathcal{B}_1 denotes Burle's clone defined preceding Corollary 3.8. Furthermore, T_A will denote the full transformation monoid $\mathcal{O}_A^{(1)}$ on A , and T_A^- its submonoid consisting of the identity function and all nonpermutations. For any submonoid M of T_A containing T_A^- and for any i ($1 \leq i < k$) we will use $\mathcal{B}_i(M)$ to denote the clone that arises from \mathcal{B}_i by omitting all operations depending on at most one variable which are outside the clone $\langle M \rangle$.

It is well known (see [19] and [2]) that the subclones of \mathcal{O}_A containing T_A are exactly the clones in the $(k+1)$ -element chain

$$\langle T_A \rangle = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_{k-1} \subset \mathcal{B}_k = \mathcal{O}_A,$$

which is often referred to as the Słupecki–Burle chain. Szabó (unpublished, [20]) extended this result and showed that the proper subclones of \mathcal{O}_A containing T_A^- are exactly the clones $\mathcal{B}_i(M)$ where $0 \leq i < k$ and M is a submonoid of T_A containing T_A^- .

Theorem 6.1. *If \mathcal{C} is a clone on a k -element set A ($k \geq 3$) such that $T_A^- \subseteq \mathcal{C}$, then $\mathcal{C} \in \mathfrak{F}_A$ if and only if $\mathcal{B}_{k-1}(T_A^-) \subseteq \mathcal{C}$.*

Proof. Let $\mathcal{N} = \mathcal{B}_{k-1}(T_A^-)$, which is the subclone of \mathcal{O}_A that consists of all projections and all nonsurjective operations. Assume first that $\mathcal{N} \subseteq \mathcal{C}$. We want to show that $\mathcal{C} \in \mathfrak{F}_A$. By Proposition 2.1 (ii) it suffices to prove that $\mathcal{N} \in \mathfrak{F}_A$. We will start with the following claim.

Claim. If f and g are operations on A that are not essentially unary and satisfy $\text{Im } f = \text{Im } g$, then $f \equiv_{\mathcal{N}} g$.

Proof of Claim. Let $\text{Im } f = \text{Im } g = S$, and let f be n -ary and g be m -ary. Since f and g are not essentially unary, $|S| \geq 2$. If $|S| \geq 3$, then it follows from Jablonskii's Lemma (Lemma 5.1) that there is a transversal $B = \{\mathbf{b}_1, \dots, \mathbf{b}_{|S|}\}$ for $\ker f$ such that $B \subseteq C_1 \times C_2 \times \cdots \times C_n$ for some proper subsets $C_i \subset A$. This condition clearly holds also in the case $|S| = 2$. The assumption $\text{Im } f = \text{Im } g$ combined with the choice of B ensures that for each $\mathbf{a} \in A^m$ there exists a unique $\mathbf{b}_j \in B$ such that $g(\mathbf{a}) = f(\mathbf{b}_j)$. Therefore we get a well-defined function $\mathbf{h}: A^m \rightarrow A^n$ by setting $\mathbf{h}(\mathbf{a}) = \mathbf{b}_j$ whenever $g(\mathbf{a}) = f(\mathbf{b}_j)$. It is clear from this definition that $g = f \circ \mathbf{h}$. Since $B \subseteq C_1 \times C_2 \times \cdots \times C_n$, we see that the components h_i of $\mathbf{h} = (h_1, \dots, h_n)$

are non-surjective, and hence they are members of \mathcal{N} . Thus, $g \leq_{\mathcal{N}} f$. The same argument with the roles of f, g switched shows also that $f \leq_{\mathcal{N}} g$. \diamond

It follows from the Claim above that every operation f on A that is not essentially unary, is \mathcal{N} -equivalent to a binary operation. It is easy to see that for any clone \mathcal{K} , every essentially unary operation is \mathcal{K} -equivalent to a unary operation. Therefore we get from Proposition 2.1 (i) that $\mathcal{N} \in \mathfrak{F}_A$. Proposition 2.1 (ii) thus implies that $\mathcal{C} \in \mathfrak{F}_A$ whenever $\mathcal{N} \subseteq \mathcal{C}$.

For the converse assume that $\mathcal{N} \not\subseteq \mathcal{C}$. Since $T_A^- \subseteq \mathcal{C}$, Szabó's theorem implies that \mathcal{C} is a subclone of \mathcal{B}_{k-2} . Therefore, by Proposition 2.1 (ii), $\mathcal{C} \notin \mathfrak{F}_A$ will follow if we show that $\mathcal{B}_{k-2} \notin \mathfrak{F}_A$. For $k = 3$ the clone \mathcal{B}_{k-2} is Burle's clone, so in this case $\mathcal{B}_{k-2} \notin \mathfrak{F}_A$ follows from Corollary 3.8.

From now on let $k > 3$, and assume without loss of generality that $A = \{0, 1, \dots, k-1\}$. For each $n \geq 2$ and $1 \leq i \leq n$ let \mathbf{u}_i^n denote the n -tuple whose i -th coordinate is 1 and all other coordinates are $k-1$. Now we define an n -ary operation f_n on A by

$$f_n(\mathbf{a}) = \begin{cases} l & \text{if } \mathbf{a} = \bar{l} \text{ with } 1 \leq l \leq k-2, \\ k-1 & \text{if } \mathbf{a} = \mathbf{u}_i^n \text{ (} 1 \leq i \leq n \text{),} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that f_n depends on all of its variables, because it is invariant under all permutations of its variables, and is not constant. Our claim $\mathcal{B}_{k-2} \notin \mathfrak{F}_A$ will follow, if we show that $f_m \not\equiv_{\mathcal{B}_{k-2}} f_n$ whenever $m \neq n$.

Assume that, on the contrary, $f_m \leq_{\mathcal{B}_{k-2}} f_n$ for some $n < m$. Then there exists $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{B}_{k-2}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. This implies that \mathbf{h} maps each kernel class $f_m^{-1}(l)$ ($l \in A$) of f_m to the corresponding kernel class $f_n^{-1}(l)$ of f_n . Applying this for $l \in \{1, \dots, k-2\}$ we obtain that $\mathbf{h}(\bar{l}) = \bar{l}$, so the range of each h_i contains the elements $1, 2, \dots, k-2$. Applying the same property of \mathbf{h} for $l = k-1$ we get that the range of \mathbf{h} must also contain an n -tuple of the form \mathbf{u}_s^n for some $1 \leq s \leq n$. For each $i \neq s$ the i -th coordinate of \mathbf{u}_s^n is $k-1$, therefore for all such i all elements $1, 2, \dots, k-2, k-1$ must be in the range of h_i . This implies that each h_i ($i \neq s$) is essentially unary, because the only members of \mathcal{B}_{k-2} with ranges containing at least $k-1$ elements are essentially unary. On the other hand, it is not the case that h_s , too, is essentially unary, because $n < m$ and f_m depends on all of its variables. Thus h_s has essential arity ≥ 2 . The facts established so far about the ranges of the h_i 's imply that the range of h_s is $\{1, \dots, k-2\}$. Furthermore, the other h_i 's ($1 \leq i \leq n$, $i \neq s$) are of the form $h_i(\mathbf{x}) = h'_i(x_{\sigma(i)})$ for some $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ and some unary operations h'_i that fix the elements $1, \dots, k-2$. Choose and fix $t \neq s$ ($1 \leq t \leq n$) arbitrarily, and let $p = \sigma(t)$; hence $h_t(\mathbf{x}) = h'_t(x_p)$. Since \mathbf{h} maps $f_m^{-1}(k-1)$ to $f_n^{-1}(k-1)$, we get that $\mathbf{h}(\mathbf{u}_p^m) = \mathbf{u}_j^n$ for some j ($1 \leq j \leq n$). Thus $h_i(\mathbf{u}_p^m) = 1$ if $i = j$ and $h_i(\mathbf{u}_p^m) = k-1$ if $i \neq j$. As the range of h_s does not contain $k-1$, it must be the case that $j = s$. Hence $h_t(\mathbf{u}_p^m) = k-1$. Since the p -th coordinate of \mathbf{u}_p^m is 1, we get that $h_t(\mathbf{u}_p^m) = h'_t(1)$, and hence $h'_t(1) = k-1$. This contradicts the fact established earlier that h'_t fixes 1. The proof of Theorem 6.1 is complete. \square

Now we turn to the second main result of this section which shows that if λ_T is an h -regular relation of arity $h < k$, then the maximal clone $\text{Pol } \lambda_T$ is not a member of \mathfrak{F}_A . We will use the notation $\mathbf{h} = \{1, \dots, h\}$ throughout the rest of the section.

The following property of the operations in $\text{Pol } \lambda_T$ will be useful (see, e.g., [18, Lemma 7.3]).

Lemma 6.2. *Let $T = \{\theta_1, \dots, \theta_r\}$ be an h -regular family of equivalence relations on A , let $\theta = \bigcap_{i=1}^r \theta_i$, and let g be an m -ary operation in $\text{Pol } \lambda_T$. If the range of g contains a transversal for the blocks of each θ_i ($1 \leq i \leq r$), then*

- (1) *for each i ($1 \leq i \leq r$) there exist p ($1 \leq p \leq m$) and q ($1 \leq q \leq r$) such that for all $\mathbf{a}, \mathbf{b} \in A^m$,*

$$g(\mathbf{a})\theta_i g(\mathbf{b}) \quad \text{whenever} \quad a_p \theta_q b_p;$$

consequently,

- (2) *g preserves θ , and*
(3) *the operation g^θ on A/θ depends on at most r variables.*

Theorem 6.3. *If λ_T is an h -regular relation such that $h < k$, then $\text{Pol } \lambda_T \notin \mathfrak{F}_A$.*

Proof. Let $T = \{\theta_1, \dots, \theta_r\}$ be an h -regular family of equivalence relations on A , let $\theta = \bigcap_{i=1}^r \theta_i$, and let $\mathcal{C} = \text{Pol } \lambda_T$. First we will consider the case when $r \geq 2$. Since T is h -regular, there exists a surjective function $\varphi: A \rightarrow \mathbf{h}^r$ such that each θ_i is the inverse image under φ of the kernel of the i -th projection map $\pi_i: \mathbf{h}^r \rightarrow \mathbf{h}$. The diagonal $\Delta = \{\bar{u} : u \in \mathbf{h}\}$ of \mathbf{h}^r is a common transversal for the kernel classes of π_i for each i . Therefore by choosing $t_u \in A$ for each $u \in \mathbf{h}$ such that $\varphi(t_u) = \bar{u}$ we get an h -element subset $\{t_u : u \in \mathbf{h}\}$ of A that is a common transversal for the blocks of each $\theta_i \in T$. In particular, t_1, \dots, t_h are pairwise non-equivalent modulo θ . The number of blocks of θ is $h^r > h + 2$ (since $r \geq 2$ and $h \geq 3$). Hence we can extend t_1, \dots, t_h to a transversal $o, e, t_1, \dots, t_h, t_{h+1}, \dots, t_s$ of θ ($s = h^r - 2$).

For $n \geq 2$ define an n -ary operation f_n on A as follows:

$$f_n(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } a_1 \theta a_2 \theta \dots \theta a_n \text{ but } (a_1, e) \notin \theta, \\ e & \text{if } |\{i : a_i \theta e\}| = n - 1, \\ o & \text{otherwise.} \end{cases}$$

We will show that if $f_m \leq_{\mathcal{C}} f_n$ then $m \leq nr$. Hence, if $f_m \equiv_{\mathcal{C}} f_n$ then $n/r \leq m \leq nr$. This will imply that no two operations in the infinite sequence f_{n_ℓ} , $\ell = 1, 2, \dots$, with $n_\ell = r^\ell + r^{\ell-1} + \dots + r + 1$ are in the same $\equiv_{\mathcal{C}}$ -class, and therefore $\mathcal{C} \notin \mathfrak{F}_A$.

Assume that $f_m \leq_{\mathcal{C}} f_n$. Hence there exists $\mathbf{g} = (g_1, \dots, g_n) \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{g}$. Substituting $\bar{t}_u = (t_u, \dots, t_u) \in A^m$ ($1 \leq u \leq h$) into this equality we get that $t_u = f_m(\bar{t}_u) = f_n(\mathbf{g}(\bar{t}_u))$. Since $t_u \neq o$ and $t_u \neq e$, the definition of f_n implies that

$$t_u = g_1(\bar{t}_u) \theta g_2(\bar{t}_u) \theta \dots \theta g_n(\bar{t}_u).$$

Thus, it follows from the choice of t_1, \dots, t_h that the range of each g_j ($1 \leq j \leq n$) contains a transversal for the blocks of all equivalence relations θ_i . Therefore by Lemma 6.2 each g_j preserves θ and the operation g_j^θ on A/θ depends on at most r variables. It is easy to see from their definitions that the operations f_m and f_n also preserve θ . Hence for the operations f_m^θ and f_n^θ on A/θ we get that $f_m^\theta = f_n^\theta \circ \mathbf{g}^\theta$. Since each g_j^θ depends on at most r variables, we conclude that f_m^θ depends on at most nr variables. But the definition of f_m shows that f_m^θ depends on all of its m variables, because it is symmetric in all of its variables (that is, every operation obtained from f_m^θ by permuting variables is f_m^θ itself) and is not constant. This implies that $m \leq nr$, completing the proof of the theorem in the case when $r \geq 2$.

Now let $r = 1$, that is, $T = \{\theta\}$ where θ has $h \geq 3$ blocks, but θ is not the equality relation. We may assume without loss of generality that $A = \{0, 1, \dots, k-1\}$, $h = \{1, 2, \dots, h\}$ is a transversal for the blocks of θ , and $0\theta 1$. For $n \geq 2$ define an n -ary operation f_n on A as follows:

$$f_n(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } a_1\theta a_2\theta \cdots \theta a_n \text{ but } (a_1, 1) \notin \theta, \\ 1 & \text{if } |\{i : a_i\theta 1\}| = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that if $f_m \leq_C f_n$ then $m \leq n$. Hence, if $f_m \equiv_C f_n$ then $m = n$. This will imply that no two operations in the infinite sequence f_n , $n = 2, 3, \dots$, are in the same \equiv_C -class, and hence $\mathcal{C} \notin \mathfrak{F}_A$.

Assume that $f_m \leq_C f_n$. Hence there exists $\mathbf{g} = (g_1, \dots, g_n) \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{g}$. Substituting $\bar{u} = (u, \dots, u) \in A^m$ ($2 \leq u \leq h$) into this equality we get that $u = f_m(\bar{u}) = f_n(\mathbf{g}(\bar{u}))$. Since $u \neq 0$ and $u \neq 1$, the definition of f_n implies that

$$u = g_1(\bar{u})\theta g_2(\bar{u})\theta \cdots \theta g_n(\bar{u}).$$

Thus, the range of each g_j ($1 \leq j \leq n$) contains an element from every θ -block $2/\theta, \dots, h/\theta$ (i.e., from every θ -block other than $1/\theta$).

Now let \mathbf{v}_i denote the m -tuple whose i -th coordinate is 2 and all other coordinates are 1. Substituting the tuple \mathbf{v}_i into $f_m = f_n \circ \mathbf{g}$ we get that $1 = f_m(\mathbf{v}_i) = f_n(\mathbf{g}(\mathbf{v}_i))$. The definition of f_n yields that

(*) for each i ($1 \leq i \leq m$), exactly $n-1$ of the n elements

$$g_1(\mathbf{v}_i), g_2(\mathbf{v}_i), \dots, g_n(\mathbf{v}_i)$$

are in the θ -block $1/\theta$.

This implies that at least $n-1$ of the operations g_1, \dots, g_n have the property that their ranges contain transversals for the blocks of θ . We want to argue that all operations g_1, \dots, g_n have this property.

Assume not, and let, say, g_1 be the unique operation among g_1, \dots, g_n whose range fails to contain a transversal for the blocks of θ . Since the range of g_1 contains an element from each one of the θ -blocks other than $1/\theta$, the range of g_1 must be disjoint from $1/\theta$. Now (*) implies that $g_j(\mathbf{v}_i)\theta 1$ for all $j > 1$ and all i ($2 \leq j \leq n$, $1 \leq i \leq m$). In particular, for $j = 2$, this shows that the range of g_2 contains a transversal for the blocks of θ , so by Lemma 6.2 (case $r = 1$) there must exist a p ($1 \leq p \leq m$) such that for arbitrary arguments $\mathbf{a}, \mathbf{b} \in A^m$

$$g_2(\mathbf{a})\theta g_2(\mathbf{b}) \quad \text{whenever} \quad a_p\theta b_p.$$

However, this fails for $\mathbf{a} = \mathbf{v}_p$ and $\mathbf{b} = \bar{2}$; indeed, the p -th coordinates of \mathbf{v}_p and $\bar{2}$ are both 2, but as we established earlier, $g_2(\mathbf{v}_p)\theta 1$, $g_2(\bar{2})\theta 2$, and $(1, 2) \notin \theta$. This contradiction proves that all operations g_1, \dots, g_n have the property that their ranges contain transversals for the blocks of θ .

Now we can finish the proof the same way as before. It follows from Lemma 6.2 that each g_j preserves θ and the operation g_j^θ on A/θ depends on at most one variable. It is easy to see from their definitions that the operations f_m and f_n also preserve θ . Hence for the operations f_m^θ and f_n^θ on A/θ we get that $f_m^\theta = f_n^\theta \circ \mathbf{g}^\theta$. This implies that f_m^θ depends on at most n variables. But the definition of f_m shows that f_m^θ depends on all of its m variables. Thus $m \leq n$. This completes the proof of Theorem 6.3 \square

7. INTERSECTIONS OF MAXIMAL CLONES

Theorems 4.1, 5.2, 5.3, 6.1, 6.3 from previous sections of this paper, combined with earlier results stated in Theorem 2.5 and Corollary 2.4 completely describe which maximal clones on a finite set A belong to the filter \mathfrak{F}_A . This description is summarized below (see also Table 1).

Theorem 7.1. *Let A be a finite set with k elements ($k \geq 3$). For a maximal clone \mathcal{M} on A we have $\mathcal{M} \in \mathfrak{F}_A$ if and only if \mathcal{M} is one of the following clones:*

- $\mathcal{M} = \text{Pol } \gamma$ for a prime permutation γ on A ,
- $\mathcal{M} = \text{Pol } \varepsilon$ for a nontrivial equivalence relation ε on A ,
- $\mathcal{M} = \text{Pol } B$ for a nonempty proper subset B of A ,
- $\mathcal{M} = \text{Pol } \sigma_c$ for some $c \in A$ where σ_c is the $(k-1)$ -ary central relation with central element c ,
- \mathcal{M} is Słupecki's clone.

In this section we determine for each pair of maximal clones in \mathfrak{F}_A whether or not their intersection is in \mathfrak{F}_A . The results can be summarized as follows.

Theorem 7.2. *Let A be a finite set with k elements ($k \geq 3$), and let \mathcal{M} and \mathcal{N} be distinct maximal clones in \mathfrak{F}_A .*

- (1) *If \mathcal{N} is Słupecki's clone, then $\mathcal{M} \cap \mathcal{N} \notin \mathfrak{F}_A$.*
- (2) *If $\mathcal{N} = \text{Pol } \sigma_c$ for some $c \in A$, then $\mathcal{M} \cap \mathcal{N} \in \mathfrak{F}_A$ if and only if $\mathcal{M} = \text{Pol } \{c\}$.*
- (3) *If $\mathcal{N} = \text{Pol } \varepsilon$ for a nontrivial equivalence relation ε on A and $\mathcal{M} = \text{Pol } \rho$ where ρ is a prime permutation, a nonempty proper subset, or a nontrivial equivalence relation on A , then $\mathcal{M} \cap \mathcal{N} \in \mathfrak{F}_A$ unless*
 - $\rho = \gamma$ is a prime permutation such that $\gamma \notin \mathcal{N}$, or
 - ρ is an equivalence relation incomparable to ε .
- (4) *If $\mathcal{M} = \text{Pol } \rho$ and $\mathcal{N} = \text{Pol } \tau$ where ρ, τ are prime permutations or nonempty proper subsets of A , then $\mathcal{M} \cap \mathcal{N} \in \mathfrak{F}_A$.*

Since every clone in \mathfrak{F}_A other than \mathcal{O}_A is below a maximal clone in \mathfrak{F}_A , the ordered set $\mathfrak{F}_A \setminus \{\mathcal{O}_A\}$ can be decomposed into a union of up-closed sets of the form

$$\mathfrak{F}_A(\mathcal{M}) := \{\mathcal{C} : \mathcal{C} \subseteq \mathcal{M}\}$$

for each maximal clone \mathcal{M} in \mathfrak{F}_A . Statement (1) of Theorem 7.2 shows that for Słupecki's clone \mathcal{N} , the set $\mathfrak{F}_A(\mathcal{N})$ is disjoint from all other $\mathfrak{F}_A(\mathcal{M})$'s. Similarly, statement (2) shows that for each $\mathcal{N} = \text{Pol } \sigma_c$, the set $\mathfrak{F}_A(\mathcal{N})$ is almost disjoint from all other $\mathfrak{F}_A(\mathcal{M})$'s. In contrast, by statements (3) and (4) (or by the more general Theorem 4.1) there are large overlaps between the sets $\mathfrak{F}_A(\mathcal{M})$ for the remaining three types of maximal clones. Thus, Theorem 7.2 can be viewed as a structure theorem for the order filter \mathfrak{F}_A , stating that \mathfrak{F}_A consists of three almost independent parts: (i) the clones contained in Słupecki's clone, (ii) the clones contained in $\text{Pol } \sigma_c$ for some $c \in A$, and (iii) the clones that lie below at least one maximal clone of one of the remaining three types (i.e., a maximal clone determined by a prime permutation, a subset, or an equivalence relation); see Figure 2.

For the proof of Theorem 7.2 we have to verify that almost all intersections $\mathcal{M} \cap \mathcal{N}$ of maximal clones $\mathcal{M}, \mathcal{N} \in \mathfrak{F}_A$ fail to be in \mathfrak{F}_A if \mathcal{N} is Słupecki's clone or a maximal clone determined by a $(k-1)$ -ary central relation. This will be done in Lemmas 7.3–7.6 and Lemmas 7.9–7.12 below.

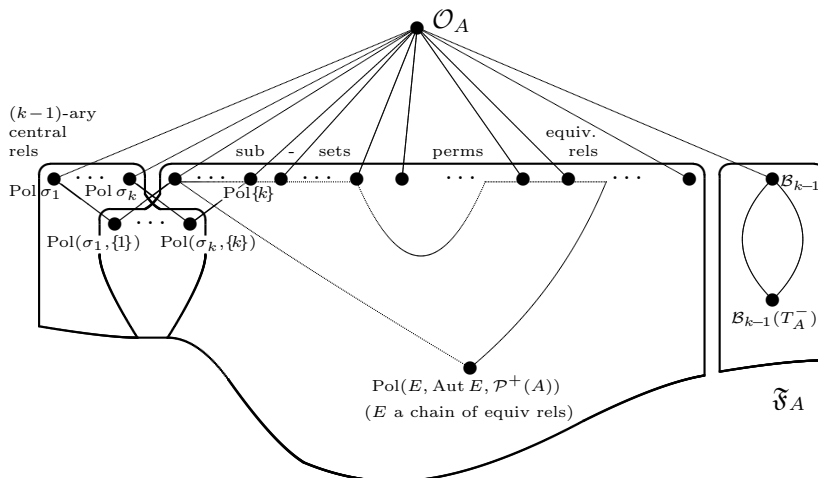


FIGURE 2. The structure of \mathfrak{F}_A for $A = \{1, 2, \dots, k\}$ ($k \geq 3$)

We will assume throughout that A is a finite set with k elements, and will use the notation \mathcal{B}_{k-1} and \mathcal{B}_{k-2} from Section 6 for Shupecki’s clone and its lower cover in the Shupecki–Burle chain.

Lemma 7.3. *If $c \in A$, then $\text{Pol } \sigma_c \cap \mathcal{B}_{k-1} \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \sigma_c \cap \mathcal{B}_{k-1}$, and assume without loss of generality that $A = \{0, 1, \dots, k-1\}$ and $c = 0$. To simplify notation we will write σ for σ_0 . For each $n \geq 1$ define an n -ary operation f_n on A as follows:

$$f_n(\mathbf{a}) = \begin{cases} u & \text{if } \mathbf{a} = \bar{u}, 1 \leq u \leq k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f_n depends on all of its variables, because it is invariant under all permutations of its variables, and is not constant.

We claim that $f_n \not\equiv_{\mathcal{C}} f_m$ whenever $n \neq m$, and hence $\mathcal{C} \notin \mathfrak{F}_A$. For, suppose on the contrary that $f_n \equiv_{\mathcal{C}} f_m$ for some $n < m$. Then there exists $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. Thus \mathbf{h} maps $f_m^{-1}(u)$ into $f_n^{-1}(u)$ for each $u \in A$. Since for $1 \leq u \leq k-1$ the set $f_m^{-1}(u)$ (resp. $f_n^{-1}(u)$) contains the m -tuple (n -tuple) \bar{u} only, we get that $\mathbf{h}(\bar{u}) = \bar{u}$ holds for all u with $1 \leq u \leq k-1$. We will distinguish two cases according to whether or not $\mathbf{h}(\bar{0}) = \bar{0}$.

Assume first that $\mathbf{h}(\bar{0}) = \bar{0}$. Then each h_i ($1 \leq i \leq n$) is surjective and, being a member of \mathcal{B}_{k-1} , h_i is thus essentially unary. Therefore the equality $f_m = f_n \circ \mathbf{h}$ implies that f_m depends on at most n ($< m$) variables. This is impossible, since we established earlier that f_m depends on all m of its variables.

Assume now that $\mathbf{h}(\bar{0}) = \mathbf{b} = (b_1, \dots, b_n) \neq \bar{0}$. Then $b_i = b \neq 0$ for some i ($1 \leq i \leq n$). Then $(\bar{1}, \dots, \overline{b-1}, \bar{0}, \overline{b+1}, \dots, \overline{k-1}) \in \sigma^m$, but

$$\begin{aligned} &(\mathbf{h}(\bar{1}), \dots, \mathbf{h}(\overline{b-1}), \mathbf{h}(\bar{0}), \mathbf{h}(\overline{b+1}), \dots, \mathbf{h}(\overline{k-1})) \\ &= (\bar{1}, \dots, \overline{b-1}, \mathbf{b}, \overline{b+1}, \dots, \overline{k-1}) \notin \sigma^n, \end{aligned}$$

since the i -th coordinate of the tuple is $(1, \dots, b-1, b, b+1, \dots, k-1) \notin \sigma$. This is again impossible, since our assumption that $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ requires that \mathbf{h} preserve σ . \square

Lemma 7.4. *If γ is a prime permutation on A , then $\text{Pol } \gamma \cap \mathcal{B}_{k-1} \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \gamma \cap \mathcal{B}_{k-1}$ and $k = |A|$. Our goal is to prove that $\mathcal{C} \subseteq \mathcal{B}_{k-2}$. Since $\mathcal{B}_{k-2} \notin \mathfrak{F}_A$ by Theorem 6.1, this will imply our claim that $\mathcal{C} \notin \mathfrak{F}_A$.

By assumption, γ is a prime permutation. Therefore γ has no fixed points, and every cycle of γ has the same prime length p . So $k = mp$ for some integer $m \geq 1$. First we will show that the range of every operation in $\text{Pol } \gamma$ is closed under γ . Indeed, let f be an n -ary operation in $\text{Pol } \gamma$, and let $a \in \text{Im } f$, i.e., $a = f(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in A$. Then $\gamma(a) = \gamma(f(a_1, \dots, a_n)) = f(\gamma(a_1), \dots, \gamma(a_n))$ holds because $f \in \text{Pol } \gamma$, hence $\gamma(a) \in \text{Im } f$. This proves that if $f \in \text{Pol } \gamma$, then $\text{Im } f$ is closed under γ . It follows that $\text{Im } f$ is closed under all powers of γ , including $\gamma^{-1} = \gamma^{p-1}$. Hence $A \setminus \text{Im } f$ is also closed under γ . This implies that if $\text{Im } f \neq A$, then $|\text{Im } f| \leq |A| - p = k - p$.

Now we are ready to prove that $\mathcal{C} \subseteq \mathcal{B}_{k-2}$. Let $f \in \mathcal{C}$. If f is essentially at most unary, then $f \in \mathcal{B}_0 \subseteq \mathcal{B}_{k-2}$. So, suppose that f is not essentially at most unary. Then $f \in \mathcal{B}_{k-1}$ implies that $\text{Im } f \neq A$, and hence $f \in \text{Pol } \gamma$ implies, by our discussion in the preceding paragraph, that $|\text{Im } f| \leq k - p$. For $k = p$ this shows that such an f cannot exist, while for $k = mp \geq 2p$ it shows that $f \in \mathcal{B}_{k-p} \subseteq \mathcal{B}_{k-2}$. In either case, this completes the proof that $\mathcal{C} \subseteq \mathcal{B}_{k-2}$, and finishes the proof of the lemma. \square

Lemma 7.5. *If B is a nonempty proper subset of A , then $\text{Pol } B \cap \mathcal{B}_{k-1} \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } B \cap \mathcal{B}_{k-1}$. We may assume without loss of generality that $A = \{0, 1, \dots, k-1\}$ and $B = \{0, 1, \dots, r-1\}$ ($1 \leq r < k$). For the proof of $\mathcal{C} \notin \mathfrak{F}_A$ we will use a description of Słupecki's clone \mathcal{B}_{k-1} via relations. As we mentioned at the beginning of Section 6, $\mathcal{B}_{k-1} = \text{Pol } \lambda_T$ where λ_T is the h -regular relation associated to the singleton $T = \{\mathbf{0}_A\}$ consisting of the equality relation on A . It is clear from the definition of h -regular relations in Section 2 that the relation $\lambda_{\{\mathbf{0}_A\}}$ is nothing else than the k -ary relation

$$\iota_k = \{(a_1, \dots, a_k) \in A^k : a_1, \dots, a_k \text{ are not pairwise distinct}\}.$$

Hence $\mathcal{B}_{k-1} = \text{Pol } \iota_k$.

Now we turn to the proof of $\mathcal{C} \notin \mathfrak{F}_A$. First we will consider the case when $r = 1$, and hence $B = \{0\}$. It is straightforward to verify that

$$\begin{aligned} \sigma_0 &= \{(a_1, \dots, a_{k-1}) \in A^{k-1} : (a_1, \dots, a_{k-1}, 0) \in \iota_k\} \\ &= \{(a_1, \dots, a_{k-1}) \in A^{k-1} : \\ &\quad \text{there exists } a_k \in B \text{ such that } (a_1, \dots, a_{k-1}, a_k) \in \iota_k\}. \end{aligned}$$

This shows that every operation on A that preserves ι_k and B , also preserves σ_0 . Therefore we get that $\mathcal{C} \subseteq \text{Pol } \sigma_0 \cap \mathcal{B}_{k-1}$. By Lemma 7.3, $\text{Pol } \sigma_0 \cap \mathcal{B}_{k-1} \notin \mathfrak{F}_A$, so it follows that $\mathcal{C} \notin \mathfrak{F}_A$.

From now on we will assume that $r \geq 2$. For each a ($0 \leq a \leq k-1$) let \mathbf{e}_a^k denote the k -tuple whose j -th coordinate is $(a+j-1) \bmod k$ for each j . Furthermore, for each a ($0 \leq a \leq k-1$) and $n > k$ let \mathbf{e}_a^n denote the n -tuple that is the concatenation of \mathbf{e}_a^k with the constant $(n-k)$ -tuple repeating the last coordinate of \mathbf{e}_a^k . Thus,

$$\begin{aligned} \mathbf{e}_0^n &= (0, 1, \dots, k-1, k-1, \dots, k-1), \quad \text{and} \\ \mathbf{e}_a^n &= (a, a+1, \dots, k-1, 0, \dots, a-1, a-1, \dots, a-1) \quad \text{for } 1 \leq a \leq k-1. \end{aligned}$$

Note that none of the n -tuples \mathbf{e}_a^n is a member of B^n , since all elements of A occur among the coordinates of \mathbf{e}_a^n .

For each $n \geq k$ we define an n -ary operation f_n on A as follows:

$$f_n(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } (a_1, \dots, a_n) \in B^n, \\ a_1 & \text{if } a_1 = a_2 = \dots = a_n, r \leq a_1 \leq k-1, \\ 1 & \text{if } a_1 = 1 \text{ and } (a_1, \dots, a_n) \notin B \cup \{\mathbf{e}_1^n\}, \\ 1 & \text{if } (a_1, \dots, a_n) = \mathbf{e}_a^n \text{ for some } a \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that f_n depends on all of its variables, which can be seen as follows: for any i with $1 \leq i \leq n$ the n -tuple $\overline{k-1}$ and the n -tuple \mathbf{u}_i obtained from $\overline{k-1}$ by changing the i -th coordinate to 0 satisfy $f_n(\overline{k-1}) = k-1 \neq 0 = f_n(\mathbf{u}_i)$.

We claim that $f_n \not\equiv_{\mathcal{C}} f_m$ whenever $n \neq m$, and hence $\mathcal{C} \notin \mathfrak{F}_A$. For, suppose on the contrary that $f_n \equiv_{\mathcal{C}} f_m$ for some $n < m$. Then there exists $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. Thus \mathbf{h} maps each set $f_m^{-1}(u)$ ($u \in A$) into the set $f_n^{-1}(u)$. Since for $r \leq a \leq k-1$ the set $f_m^{-1}(a)$ (resp. $f_n^{-1}(a)$) contains the m -tuple (n -tuple) \bar{a} only, we get that $\mathbf{h}(\bar{a}) = \bar{a}$ holds for all $r \leq a \leq k-1$. In particular, $h_1(\bar{a}) = a$ for all $r \leq a \leq k-1$.

Now let $\mathbf{a} = (a_1, \dots, a_m) \in B^m$. Since \mathbf{h} preserves B , we have that $\mathbf{h}(\mathbf{a}) = (h_1(\mathbf{a}), \dots, h_n(\mathbf{a})) \in B^n$. So, by applying the definitions of f_m and f_n for tuples in B we get that

$$a_1 = f_m(\mathbf{a}) = f_n(\mathbf{h}(\mathbf{a})) = f_n((h_1(\mathbf{a}), \dots, h_n(\mathbf{a}))) = h_1(\mathbf{a}),$$

that is, h_1 restricted to B is projection onto the first variable. Combining this with the property of h_1 established in the preceding paragraph we get that h_1 is surjective, and hence, being a member of \mathcal{B}_{k-1} , it is essentially unary. The fact that h_1 restricted to the set B of size ≥ 2 depends on its first variable forces that it is the first variable that h_1 depends on. Since $h_1(\bar{a}) = a$ for all a (whether $a \in B$ or $r \leq a \leq k-1$), we conclude that h_1 is projection onto the first variable.

Next we want to determine $\mathbf{h}(\mathbf{e}_a^m)$ for each $a \in A$. Since h_1 is projection onto the first variable and \mathbf{e}_a^m has first coordinate a , we get that $\mathbf{h}(\mathbf{e}_a^m)$ also has first coordinate a . On the other hand,

$$f_n(\mathbf{h}(\mathbf{e}_a^m)) = f_m(\mathbf{e}_a^m) = \begin{cases} 1 & \text{if } a \neq 1, \\ 0 & \text{if } a = 1. \end{cases}$$

If $a \neq 1$, then $\mathbf{h}(\mathbf{e}_a^m)$ is an n -tuple with first coordinate $a \neq 1$ whose f_n -image is 1. It follows from the definition of f_n that the only such n -tuple is \mathbf{e}_a^n , so $\mathbf{h}(\mathbf{e}_a^m) = \mathbf{e}_a^n$. If $a = 1$, then $\mathbf{h}(\mathbf{e}_1^m)$ is an n -tuple with first coordinate 1 whose f_n -image is 0. Again, the definition of f_n shows that the only such n -tuple is \mathbf{e}_1^n , so $\mathbf{h}(\mathbf{e}_1^m) = \mathbf{e}_1^n$. This proves that $\mathbf{h}(\mathbf{e}_a^m) = \mathbf{e}_a^n$ for all $a \in A$.

Since for each i ($1 \leq i \leq n$) the i -th coordinates of the tuples \mathbf{e}_a^n ($a \in A$) exhaust A , we obtain that each h_i is surjective. But, $h_i \in \mathcal{B}_{k-1}$ for each i , therefore each h_i is essentially unary. Hence $f_m = f_n \circ \mathbf{h}$ yields that f_m depends on at most n ($< m$) variables, contradicting the fact that f_m depends on all m of its variables. \square

Lemma 7.6. *If ε is a nontrivial equivalence relation on A , then $\text{Pol } \varepsilon \cap \mathcal{B}_{k-1} \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \varepsilon \cap \mathcal{B}_{k-1}$. We may assume without loss of generality that $A = \{0, 1, \dots, k-1\}$, and the equivalence classes of ε are $\{0, 1, \dots, n_1\}$, $\{n_1+1, \dots, n_2\}$,

$\dots, \{n_{r-1} + 1, \dots, k - 1\}$ for some $r \geq 2$ and some $0 = n_0 + 1 < 1 \leq n_1 < n_2 < \dots < n_{r-1} < n_r = k - 1$.

For $0 \leq a \leq k - 1$ and $n \geq 1$ let \mathbf{e}_a^{kn} denote the kn -tuple that is a concatenation of k constant n -tuples such that the $(jn + 1)$ -th coordinate of \mathbf{e}_a^{kn} is $(a + j) \bmod k$ for each j ($0 \leq j \leq k - 1$); equivalently,

$$\mathbf{e}_a^{kn} = (\bar{a}, \overline{a+1}, \dots, \overline{k-1}, \bar{0}, \dots, \overline{a-1}) \quad (0 \leq a \leq k - 1)$$

where each constant tuple \bar{b} ($0 \leq b \leq k - 1$) has length n . Two properties of these tuples will be important:

- $(\mathbf{e}_a^{kn}, \mathbf{e}_b^{kn}) \notin \varepsilon^{kn}$ if $a \neq b$, and
- $\mathbf{e}_a^{kn} / \varepsilon^{kn}$ has an element other than \mathbf{e}_a^{kn} for each a .

The first property can be verified by observing that if $(a, b) \notin \varepsilon$, then the first coordinates of \mathbf{e}_a^{kn} and \mathbf{e}_b^{kn} are not ε -related, while if $(a, b) \in \varepsilon$, say $n_i + 1 \leq a < b \leq n_{i+1}$, then $(a + (n_{i+1} - b + 1), b + (n_{i+1} - b + 1)) \notin \varepsilon$, so that for $j = n_{i+1} - b + 1$ the $(nj + 1)$ -th coordinates of \mathbf{e}_a^{kn} and \mathbf{e}_b^{kn} are not ε -related. The second property is true because the assumption $(0, 1) \in \varepsilon$ ensures that if for any ℓ ($1 \leq \ell \leq n$) we replace the ℓ -th occurrence of 0 in \mathbf{e}_a^{kn} by 1, we get a kn -tuple $(\mathbf{e}_a^{kn})^{[\ell]}$ which is ε^{kn} -related, but not equal to \mathbf{e}_a^{kn} .

For $n \geq 1$ we now define a kn -ary operation f_n on A as follows:

$$f_n(\mathbf{a}) = \begin{cases} a & \text{if } \mathbf{a} = \mathbf{e}_a^{kn} \text{ for some } a, \\ (a + 1) \bmod k & \text{if } (\mathbf{a}, \mathbf{e}_a^{kn}) \in \varepsilon^{kn}, \mathbf{a} \neq \mathbf{e}_a^{kn} \text{ for some } a, \\ 0 & \text{otherwise.} \end{cases}$$

The properties of \mathbf{e}_a^{kn} established in the preceding paragraph make sure that f_n is well-defined, and that $f_n[\mathbf{e}_a^{kn} / \varepsilon^{kn}]$ is a 2-element set for each a . Moreover, it follows also that f_n depends on all of its variables, because for each j ($1 \leq j \leq kn$) there exist a and ℓ such that the kn -tuples \mathbf{e}_a^{kn} and $(\mathbf{e}_a^{kn})^{[\ell]}$ differ in their j -th coordinates only, and $f_n(\mathbf{e}_a^{kn}) = a \neq (a + 1) \bmod k = f_n((\mathbf{e}_a^{kn})^{[\ell]})$.

We claim that $f_n \not\equiv_{\mathcal{C}} f_m$ whenever $n \neq m$, and hence $\mathcal{C} \notin \mathfrak{F}_A$. For, let $n < m$, and suppose on the contrary that there exists $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. It follows from the definition of f_n that for each block B of ε^{kn} , $f_n[B]$ is the 2-element set consisting of a and $(a + 1) \bmod k$, if $B = \mathbf{e}_a^{kn} / \varepsilon^{kn}$ ($0 \leq a \leq k - 1$), and $f_n[B]$ is the singleton $\{0\}$ otherwise. We want to use this fact to prove that $\mathbf{h}(\mathbf{e}_a^{km}) = \mathbf{e}_a^{kn}$ holds for each a ($0 \leq a \leq k - 1$). Indeed, since \mathbf{h} preserves ε , therefore \mathbf{h} maps $\mathbf{e}_a^{km} / \varepsilon^{km}$ into a single ε^{kn} -block B in A^{kn} . As

$$\{a, (a + 1) \bmod k\} = f_m[\mathbf{e}_a^{km} / \varepsilon^{km}] = f_n[\mathbf{h}[\mathbf{e}_a^{km} / \varepsilon^{km}]] \subseteq f_n[B],$$

we get that $B = \mathbf{e}_a^{kn} / \varepsilon^{kn}$. Since $a = f_m(\mathbf{e}_a^{km}) = f_n(\mathbf{h}(\mathbf{e}_a^{km}))$, and $\mathbf{a} = \mathbf{e}_a^{kn}$ is the only element $\mathbf{a} \in B$ for which $f_n(\mathbf{a}) = a$, we conclude that $\mathbf{h}(\mathbf{e}_a^{km}) = \mathbf{e}_a^{kn}$, as claimed.

Since for each i ($1 \leq i \leq n$), all elements of A occur in the i -th coordinate of some \mathbf{e}_a^{kn} , the equalities $\mathbf{h}(\mathbf{e}_a^{km}) = \mathbf{e}_a^{kn}$ ($a \in A$) imply that each h_i is surjective. As each h_i is a member of \mathcal{B}_{k-1} , we get that each h_i is essentially unary. Hence $f_m = f_n \circ \mathbf{h}$ yields that f_m depends on at most n ($< m$) variables, contradicting the fact established earlier that f_m depends on all m of its variables. \square

Next we will consider intersections of $\text{Pol } \sigma_c$ with other maximal clones in \mathfrak{F}_A . We will start with two auxiliary lemmas.

Lemma 7.7. *Let $A = \{0, 1, \dots, k-1\}$, and let σ_0 be the $(k-1)$ -ary central relation on A with central element 0. A subclone \mathcal{C} of $\text{Pol } \sigma_0$ fails to belong to \mathfrak{F}_A if for some integers $n_0 \geq 3$ and $l \in \{0, 1\}$, there exist n -tuples \mathbf{c}_i^n ($1 \leq i \leq n-l$) for each $n \geq n_0$ such that the following two conditions are satisfied:*

(1) *For all $n \geq n_0$ and $1 \leq i \leq j \leq n-l$ we have*

$$(\mathbf{c}_i^n, \mathbf{c}_j^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n \iff j - i \leq 1.$$

(2) *For all $m, n \geq n_0$, we have $\mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n$ whenever $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ is such that*

(i) $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n-l\}$,

(ii) $\mathbf{h}(\mathbf{c}_{m-l}^m) = \mathbf{c}_{n-l}^n$, and

(iii) $\mathbf{h}(\bar{b}) = \bar{b}$ for all $3 \leq b \leq k-1$.

Proof. Let \mathcal{C} be a subclone of $\text{Pol } \sigma_0$, and assume that conditions (1) and (2) are satisfied for some n -tuples \mathbf{c}_i^n ($n \geq n_0$, $1 \leq i \leq n-l$). For each $n \geq n_0$ we define an n -ary operation f_n on A as follows:

$$f_n(\mathbf{a}) = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{c}_i^n \ (1 \leq i < n-l), \\ 2 & \text{if } \mathbf{a} = \mathbf{c}_{n-l}^n, \\ b & \text{if } \mathbf{a} = \bar{b} \ (3 \leq b \leq k-1), \\ 0 & \text{otherwise.} \end{cases}$$

We will prove $\mathcal{C} \notin \mathfrak{F}_A$ by showing that $f_n \not\equiv_{\mathcal{C}} f_m$ whenever $n \neq m$.

Suppose that, on the contrary, there exist $m < n$ such that $f_n \equiv_{\mathcal{C}} f_m$. Hence there exists $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. Thus \mathbf{h} maps each set $f_m^{-1}(b)$ ($b \in A$) into the set $f_n^{-1}(b)$. Applying this to $3 \leq b \leq k-1$ and to $b = 2$ we get that $\mathbf{h}(\bar{b}) = \bar{b}$ for all $3 \leq b \leq k-1$ and $\mathbf{h}(\mathbf{c}_{m-l}^m) = \mathbf{c}_{n-l}^n$. The same property for $b = 1$ shows that

$$(7.1) \quad \mathbf{h}(\mathbf{c}_j^m) \in \{\mathbf{c}_i^n : 1 \leq i < n-l\} \quad \text{for all } j \ (1 \leq j < m-l).$$

In particular, $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n-l\}$. Thus $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ satisfies all three requirements (i)–(iii) in (2). Therefore we can apply condition (2) to conclude that

$$(7.2) \quad \mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n.$$

By condition (1) we have $(\mathbf{c}_j^m, \mathbf{c}_{j+1}^m, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^m$ for all j ($1 \leq j < m-l$). Since $\mathbf{h} \in (\mathcal{C}^{(m)})^n$, and therefore \mathbf{h} preserves σ_0 , the \mathbf{h} -images of these tuples are in $(\sigma_0)^n$. Since \mathbf{h} satisfies (iii), this means that

$$(7.3) \quad (\mathbf{h}(\mathbf{c}_j^m), \mathbf{h}(\mathbf{c}_{j+1}^m), \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n \quad \text{for all } j \ (1 \leq j < m-l).$$

Using (7.1) we obtain that the $(m-l)$ -element sequence of \mathbf{h} -images

$$\mathbf{c}_1^n = \mathbf{h}(\mathbf{c}_1^m), \mathbf{h}(\mathbf{c}_2^m), \dots, \mathbf{h}(\mathbf{c}_j^m), \mathbf{h}(\mathbf{c}_{j+1}^m), \dots, \mathbf{h}(\mathbf{c}_{m-l-1}^m), \mathbf{h}(\mathbf{c}_{m-l}^m) = \mathbf{c}_{n-l}^n$$

has its first $m-l-1$ members in the set $\{\mathbf{c}_i^n : 1 \leq i < n-l\}$. (The equalities for the first and last members follow from (7.2) and the fact that \mathbf{h} satisfies (ii).) Thus $\mathbf{h}(\mathbf{c}_j^m) = \mathbf{c}_{s_j}^n$ for each j ($1 \leq j \leq m-l$) so that $s_1 = 1$, $1 \leq s_2, \dots, s_{m-l-1} < n-l$, and $s_{m-l} = n-l$. Combining this with (7.3) we get that

$$(\mathbf{c}_{s_j}^n, \mathbf{c}_{s_{j+1}}^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n \quad \text{for all } j \ (1 \leq j \leq m-l-1).$$

Since σ_0 is totally symmetric, we also have that

$$(\mathbf{c}_{s_{j+1}}^n, \mathbf{c}_{s_j}^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n \quad \text{for all } j \ (1 \leq j \leq m-l-1).$$

Thus it follows from condition (1) that $|s_{j+1} - s_j| \leq 1$ for all j ($1 \leq j \leq m-l-1$). Therefore $n-l-1 = |s_{m-l} - s_1| \leq \sum_{j=1}^{m-l-1} |s_{j+1} - s_j| \leq m-l-1$, which contradicts our assumption that $m < n$. This completes the proof of the lemma. \square

Lemma 7.8. *Let $A = \{0, 1, \dots, k-1\}$, let σ_0 be the $(k-1)$ -ary central relation on A with central element 0, and for $n \geq 4$ and $2 \leq i \leq n-1$ let*

$$\mathbf{e}_i^n = (1, \dots, 1, 0, 2, 0, 1, \dots, 1) \in A^n$$

be the n -tuple where the sole 2 is in the i -th coordinate. For all $2 \leq i \leq j \leq n-1$ we have

$$(\mathbf{e}_i^n, \mathbf{e}_j^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n \iff j - i \leq 1.$$

Proof. Let $j \geq i$. If $j = i$ or $j = i+1$, then in each coordinate, the $(k-1)$ -tuple $(\mathbf{e}_i^n, \mathbf{e}_j^n, \bar{3}, \dots, \overline{k-1})$ is of the form $(0, 0, \dots)$, $(1, 1, \dots)$, $(2, 2, \dots)$, $(0, 1, \dots)$, $(2, 0, \dots)$, $(0, 2, \dots)$, or $(1, 0, \dots)$. Thus $(\mathbf{e}_i^n, \mathbf{e}_j^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n$. If $j > i+1$, then in the i -th coordinate of the $(k-1)$ -tuple $(\mathbf{e}_i^n, \mathbf{e}_j^n, \bar{3}, \dots, \overline{k-1})$ we have $(2, 1, 3, \dots, k-1) \notin \sigma_0$, hence $(\mathbf{e}_i^n, \mathbf{e}_j^n, \bar{3}, \dots, \overline{k-1}) \notin (\sigma_0)^n$. \square

Lemma 7.9. *If $c \in A$ and B is a nonempty proper subset of A such that $B \neq \{c\}$, then $\text{Pol } B \cap \text{Pol } \sigma_c \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } B \cap \text{Pol } \sigma_c$. We may assume without loss of generality that $A = \{0, 1, \dots, k-1\}$ and $c = 0$. In view of Lemma 7.7, our claim that $\mathcal{C} \notin \mathfrak{F}_A$ will follow if we exhibit tuples \mathbf{c}_i^n that satisfy conditions (1) and (2). We will distinguish two cases according to whether $c = 0$ is a member of B or not.

Case 1: $0 \in B$. In this case $|B| \geq 2$. Assume without loss of generality that $0, 1 \in B$ and $2 \notin B$. For $n \geq 4$, let $\mathbf{c}_1^n = (0, 0, 1, 1, \dots, 1) \in A^n$, and let $\mathbf{c}_i^n = \mathbf{e}_i^n$ ($2 \leq i \leq n-1$) be the tuples from Lemma 7.8. We claim that (1)–(2) of Lemma 7.7 hold true (with $n_0 = 4$ and $l = 1$). For $j \geq i \geq 2$, condition (1) follows from Lemma 7.8. So, let $i = 1$. Then it is straightforward to check that $(\mathbf{c}_1^n, \mathbf{c}_2^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n$, while if $j > 2$, then $(\mathbf{c}_1^n, \mathbf{c}_j^n, \bar{3}, \dots, \overline{k-1}) \notin (\sigma_0)^n$, because in the j -th coordinate we have $(1, 2, 3, \dots, k-1) \notin \sigma_0$. This proves that condition (1) holds. To establish condition (2) assume that $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ ($m, n \geq 4$) satisfies requirements (i)–(iii) in condition (2); in fact, in this case it will be enough to assume that \mathbf{h} satisfies (i), that is, $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n-1\}$. Here $\mathbf{c}_1^m \in B^m$, $\mathbf{c}_1^n \in B^n$, and $\mathbf{c}_2^n, \dots, \mathbf{c}_{n-2}^n \notin B^n$, because B contains 0, 1 and does not contain 2. Since \mathbf{h} preserves B , we must have that $\mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n$.

Case 2: $0 \notin B$. Assume without loss of generality that $2 \in B$. For $n \geq 4$, consider the following n -tuples: $\mathbf{c}_1^n = \bar{2}$, $\mathbf{c}_2^n = (2, 0, 0, \dots, 0)$, and $\mathbf{c}_i^n = \mathbf{e}_{i-1}^n$ from Lemma 7.8 for $3 \leq i \leq n$. We want to show that conditions (1) and (2) of Lemma 7.7 are satisfied (with $n_0 = 4$ and $l = 0$). For $j \geq i \geq 3$ condition (1) follows from Lemma 7.8. For $i = 2$ we have $(\mathbf{c}_2^n, \mathbf{c}_3^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n$, since in each coordinate one of $\mathbf{c}_2^n, \mathbf{c}_3^n$ is 0. On the other hand, if $j > 3$, then $(\mathbf{c}_2^n, \mathbf{c}_j^n, \bar{3}, \dots, \overline{k-1}) \notin (\sigma_0)^n$, because in the first coordinate we have $(2, 1, 3, \dots, k-1) \notin \sigma_0$. For $i = 1$, $(\mathbf{c}_1^n, \mathbf{c}_2^n, \bar{3}, \dots, \overline{k-1}) \in (\sigma_0)^n$, since in each coordinate this $(k-1)$ -tuple has the form $(2, 2, \dots)$ or $(2, 0, \dots)$. However, for $j > 2$ we have $(\mathbf{c}_1^n, \mathbf{c}_j^n, \bar{3}, \dots, \overline{k-1}) \notin (\sigma_0)^n$, because in every coordinate where \mathbf{c}_j^n is 1 we have $(2, 1, 3, \dots, k-1) \notin \sigma_0$. This proves condition (1). As before, to verify condition (2) let $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ ($m, n \geq 4$) be such that $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n\}$. Here $\mathbf{c}_1^m \in B^m$,

$\mathbf{c}_1^n \in B^n$, and $\mathbf{c}_2^n, \dots, \mathbf{c}_{n-1}^n \notin B^n$, because $2 \in B$ and $0 \notin B$. Since \mathbf{h} preserves B , it must be the case that $\mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n$. \square

Lemma 7.10. *If c and d are distinct elements of A , then $\text{Pol } \sigma_c \cap \text{Pol } \sigma_d \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \sigma_c \cap \text{Pol } \sigma_d$. We may assume without loss of generality that $A = \{0, 1, \dots, k-1\}$, $c = 0$, and $d = 2$. We will again use Lemma 7.7 to show that $\mathcal{C} \notin \mathfrak{F}_A$. In fact, we will show that the tuples \mathbf{c}_i^n ($1 \leq i \leq n$) exhibited for Case 2 of the proof of Lemma 7.9 satisfy conditions (1) and (2) of Lemma 7.7 (with $n_0 = 4$ and $l = 0$) for this clone \mathcal{C} as well. Since (1) is independent of the choice of the subclone \mathcal{C} of $\text{Pol } \sigma_0$, there is nothing more to do to prove (1). It remains to show that condition (2) is satisfied. Let $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ ($m, n \geq 4$), and assume that \mathbf{h} satisfies requirements (i)–(iii) in condition (2), that is, $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n\}$, $\mathbf{h}(\mathbf{c}_m^m) = \mathbf{c}_n^n$, and $\mathbf{h}(b) = \bar{b}$ for all $3 \leq b \leq k-1$. Since \mathcal{C} is a subclone of $\text{Pol } \sigma_2$, \mathbf{h} preserves σ_2 . In particular, the \mathbf{h} -image of the $(k-1)$ -tuple $(\mathbf{c}_1^m, \mathbf{c}_m^m, \bar{3}, \dots, \bar{k-1}) \in (\sigma_2)^m$ is a tuple in $(\sigma_2)^n$; that is,

$$(\mathbf{h}(\mathbf{c}_1^m), \mathbf{c}_n^n, \bar{3}, \dots, \bar{k-1}) \in (\sigma_2)^n.$$

By assumption, $\mathbf{h}(\mathbf{c}_1^m) \in \{\mathbf{c}_i^n : 1 \leq i < n\}$; on the other hand, for $i = 2$ we have $(\mathbf{c}_2^n, \mathbf{c}_n^n, \bar{3}, \dots, \bar{k-1}) \notin (\sigma_2)^n$, because the second coordinate is $(0, 1, 3, \dots, k-1) \notin \sigma_2$, while for $3 \leq i < n$ we have $(\mathbf{c}_i^n, \mathbf{c}_n^n, \bar{3}, \dots, \bar{k-1}) \notin (\sigma_2)^n$, because the last coordinate is $(1, 0, 3, \dots, k-1) \notin \sigma_2$. Thus it must be that $\mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n$, as required. \square

Lemma 7.11. *If γ is a nonidentity permutation of A and $c \in A$, then $\text{Pol } \gamma \cap \text{Pol } \sigma_c \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol } \gamma \cap \text{Pol } \sigma_c$, and assume without loss of generality that $A = \{0, 1, \dots, k-1\}$ and $c = 0$. Let $d = \gamma(0)$, and let $B = \{a \in A : \gamma(a) = a\}$ be the set of fixed points of γ . It is easy to verify that

$$\sigma_d = \{(\gamma(a_1), \dots, \gamma(a_{k-1})) : (a_1, \dots, a_{k-1}) \in \sigma_0\}.$$

Thus it follows that every operation that preserves σ_0 and γ also preserves B and σ_d . Hence $\mathcal{C} \subseteq \text{Pol } \sigma_0 \cap \text{Pol } B \cap \text{Pol } \sigma_d$. In case $d \neq 0$ we get from Lemma 7.10 and Proposition 2.1 (ii) that $\mathcal{C} \notin \mathfrak{F}_A$. If $d = 0$, then $0 \in B$. Moreover, since γ is not the identity permutation, B is a proper subset of A . Therefore Lemma 7.9, combined again with Proposition 2.1 (ii), yields that $\mathcal{C} \notin \mathfrak{F}_A$ unless $B = \{0\}$.

So, it remains to consider the case when $B = \{0\}$, that is, 0 is the unique fixed point of γ . Assume from now on that γ satisfies this condition. To prove that $\mathcal{C} \notin \mathfrak{F}_A$ holds in this case as well, we will use Lemma 7.7, that is, we will exhibit tuples \mathbf{c}_i^n that satisfy conditions (1) and (2) of Lemma 7.7.

First let $k \geq 4$. Since 0 is the only fixed point of γ , we may assume without loss of generality that $\gamma(2) = 3$. Now let \mathbf{c}_i^n ($n \geq 4, 1 \leq i \leq n$) be the tuples defined in Case 2 of the proof of Lemma 7.9. We know from that proof that condition (1) is satisfied. To prove that (2) is satisfied with our current choice of clone \mathcal{C} , consider any $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ ($m, n \geq 4$) that satisfies $\mathbf{h}(\bar{3}) = \bar{3}$, a fragment of requirement (iii) in (2). Since $(\mathbf{c}_1^m, \bar{3}) = (\bar{2}, \bar{3}) \in \gamma^m$ and \mathbf{h} preserves γ , we get that $(\mathbf{h}(\mathbf{c}_1^m), \bar{3}) \in \gamma^n$. As γ is a permutation, it follows that $\mathbf{h}(\mathbf{c}_1^m) = \bar{2} = \mathbf{c}_1^n$.

Finally, let $k = 3$. Since 0 is the only fixed point of γ , γ is the transposition $(1\ 2)$. For each $n \geq 7$ define n -tuples \mathbf{c}_i^n ($1 \leq i \leq n$) as follows: $\mathbf{c}_1^n = (2, 2, \dots, 2, 0, 1, 0)$, $\mathbf{c}_2^n = (2, 0, \dots, 0, 0, 0, 0)$, and for $3 \leq i \leq n$, $\mathbf{c}_i^n = \mathbf{e}_{i-1}^n$ are the tuples from

Lemma 7.8. Since these tuples, with the exception of \mathbf{c}_1^n , are the same as those in Case 2 of the proof of Lemma 7.9, we know from that proof that condition (1) holds whenever $j \geq i \geq 2$. For $i = 1$, clearly, $(\mathbf{c}_1^n, \mathbf{c}_2^n) \in (\sigma_0)^n$, because in each coordinate the pair is $(2, 2)$ or contains a 0. However, if $j > 2$, then $(\mathbf{c}_1^n, \mathbf{c}_j^n) \notin (\sigma_0)^n$, because we have $(2, 1) \notin \sigma_0$ either in the fourth coordinate (if $j = 3$), or in the first coordinate (if $j > 3$). This proves that (1) holds. To prove that (2) also holds, let $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ ($m, n \geq 7$) satisfy requirement (ii) from condition (2), that is, $\mathbf{h}(\mathbf{c}_m^n) = \mathbf{c}_n^n$. Since $(\mathbf{c}_1^m, \mathbf{c}_m^m) \in \gamma^m$ and \mathbf{h} preserves γ , we get that $(\mathbf{h}(\mathbf{c}_1^m), \mathbf{c}_n^n) \in \gamma^n$. As γ is a permutation, it follows that $\mathbf{h}(\mathbf{c}_1^m) = \mathbf{c}_1^n$, completing the proof. \square

Lemma 7.12. *If ε is a nontrivial equivalence relation on A and $c \in A$, then $\text{Pol } \varepsilon \cap \text{Pol } \sigma_c \notin \mathfrak{F}_A$.*

Proof. Let $\mathcal{C} = \text{Pol}(\varepsilon) \cap \text{Pol } \sigma_c$, and assume without loss of generality that $A = \{0, 1, \dots, k\}$, $c = 0$, and 2 is an element of A such that $(0, 2) \notin \varepsilon$, but at least one of the ε -classes $0/\varepsilon$, $2/\varepsilon$ is not a singleton. For $n \geq 4$ let \mathbf{e}_i^n ($2 \leq i \leq n-1$) be the n -tuples from Lemma 7.8. The following two properties of these tuples will be important:

- $(\mathbf{e}_i^n, \mathbf{e}_j^n) \notin \varepsilon^n$ if $i \neq j$;
- $\mathbf{e}_i^n/\varepsilon$ has an element other than \mathbf{e}_i^n for each i .

To verify the first property we may assume that $i < j$, since ε^n is a symmetric relation. If $j = i + 1$, then $(\mathbf{e}_i^n, \mathbf{e}_j^n) \notin \varepsilon^n$, because in the i -th coordinate $(2, 0) \notin \varepsilon$; if $j > i + 1$, then $(\mathbf{e}_i^n, \mathbf{e}_j^n) \notin \varepsilon^n$, because in the j -th and $(j + 1)$ -th coordinates we have the pairs $(1, 2)$, $(1, 0)$, which cannot simultaneously be in ε , or else we would get $(2, 0) \in \varepsilon$. The second property follows from the assumption that at least one of the ε -classes $0/\varepsilon$, $2/\varepsilon$ is not a singleton.

For $n \geq 4$ we now define an n -ary operation f_n on A as follows:

$$f_n(\mathbf{a}) = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{e}_2^n, \\ 2 & \text{if } \mathbf{a} \in \mathbf{e}_2^n/\varepsilon^n \text{ but } \mathbf{a} \neq \mathbf{e}_2^n, \\ 1 & \text{if } \mathbf{a} = \mathbf{e}_i^n \text{ (} 2 < i < n-1 \text{),} \\ 0 & \text{if } \mathbf{a} \in \mathbf{e}_i^n/\varepsilon^n \text{ but } \mathbf{a} \neq \mathbf{e}_i^n \text{ (} 2 < i < n-1 \text{),} \\ 2 & \text{if } \mathbf{a} = \mathbf{e}_{n-1}^n, \\ 0 & \text{if } \mathbf{a} \in \mathbf{e}_{n-1}^n/\varepsilon^n \text{ but } \mathbf{a} \neq \mathbf{e}_{n-1}^n, \\ b & \text{if } \mathbf{a} = \bar{b} \text{ (} 3 \leq b \leq k-1 \text{),} \\ 0 & \text{otherwise.} \end{cases}$$

The properties of \mathbf{e}_i^n established in the preceding paragraph make sure that f_n is well-defined, and that $f_n[\mathbf{e}_i^n/\varepsilon^n]$ is a 2-element set for each $2 \leq i \leq n-1$. Our aim is to prove that $f_m \not\equiv_{\mathcal{C}} f_n$ whenever $m \neq n$, which will show that $\mathcal{C} \notin \mathfrak{F}_A$.

Suppose that, on the contrary, there exist $m < n$ such that $f_m \equiv_{\mathcal{C}} f_n$. Hence there exists $\mathbf{h} \in (\mathcal{C}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$, that is, \mathbf{h} maps each set $f_m^{-1}(b)$ ($b \in A$) into the set $f_n^{-1}(b)$. Applying this to $3 \leq b \leq k-1$ we get that $\mathbf{h}(\bar{b}) = \bar{b}$ for all $3 \leq b \leq k-1$. Since \mathbf{h} preserves ε , it maps each ε^m -class into an ε^n -class. In

particular, let B_i denote the ε^n -class containing $\mathbf{h}[\mathbf{e}_i^m/\varepsilon^m]$ ($2 \leq i \leq m-1$). Then

$$\begin{aligned} \{1, 2\} &= f_m[\mathbf{e}_2^m/\varepsilon^m] = f_n[\mathbf{h}[\mathbf{e}_2^m/\varepsilon^m]] \subseteq f_n[B_2], \\ \{1, 0\} &= f_m[\mathbf{e}_i^m/\varepsilon^m] = f_n[\mathbf{h}[\mathbf{e}_i^m/\varepsilon^m]] \subseteq f_n[B_i] \quad \text{for } 2 < i < m-1, \\ \{2, 0\} &= f_m[\mathbf{e}_{m-1}^m/\varepsilon^m] = f_n[\mathbf{h}[\mathbf{e}_{m-1}^m/\varepsilon^m]] \subseteq f_n[B_{m-1}]. \end{aligned}$$

However, it follows from the definition of f_n that for each ε^n -class B ,

$$f_n[B] = \begin{cases} \{1, 2\} & \text{if } B = \mathbf{e}_2^n/\varepsilon^n, \\ \{1, 0\} & \text{if } B = \mathbf{e}_i^n/\varepsilon^n \ (2 < i < n-1), \\ \{2, 0\} & \text{if } B = \mathbf{e}_{n-1}^n/\varepsilon^n, \\ C \subseteq \{0, 3, \dots, k-1\} & \text{otherwise.} \end{cases}$$

Therefore $B_2 = \mathbf{e}_2^n/\varepsilon^n$, $B_{m-1} = \mathbf{e}_{n-1}^n/\varepsilon^n$, and for each $2 < i < m-1$, $B_i = \mathbf{e}_{s_i}^n/\varepsilon^n$ for some s_i with $2 < s_i < n-1$. Since $1 = f_m(\mathbf{e}_2^m) = f_n(\mathbf{h}(\mathbf{e}_2^m))$, $\mathbf{h}(\mathbf{e}_2^m) \in B_2$, and the only element $\mathbf{a} \in B_2$ with $f_n(\mathbf{a}) = 1$ is $\mathbf{a} = \mathbf{e}_2^n$, we get that $\mathbf{h}(\mathbf{e}_2^m) = \mathbf{e}_2^n$. We conclude similarly that $\mathbf{h}(\mathbf{e}_i^m) = \mathbf{e}_{s_i}^n$ for all $2 < i < m-1$, and $\mathbf{h}(\mathbf{e}_{m-1}^m) = \mathbf{e}_{n-1}^n$. By introducing the notation $s_2 = 2$ and $s_{m-1} = n-1$ we can write these results more compactly as follows:

$$\mathbf{h}(\mathbf{e}_i^m) = \mathbf{e}_{s_i}^n \ (2 \leq i \leq m-1) \text{ where } 2 = s_2 < s_3, \dots, s_{m-2} < s_{m-1} = n-1.$$

Now we can finish the proof the same way as in Lemma 7.7. We know from Lemma 7.8 that $(\mathbf{e}_i^m, \mathbf{e}_{i+1}^m, \bar{3}, \dots, \bar{k-1}) \in (\sigma_0)^m$ for all i ($2 \leq i \leq m-2$). Since \mathbf{h} preserves σ_0 , the \mathbf{h} -images of these tuples are in $(\sigma_0)^n$; that is,

$$(\mathbf{e}_{s_i}^n, \mathbf{e}_{s_{i+1}}^n, \bar{3}, \dots, \bar{k-1}) \in (\sigma_0)^n \quad \text{for all } i \ (2 \leq i \leq m-2).$$

Since σ_0 is totally symmetric, the tuples obtained by interchanging $\mathbf{e}_{s_i}^n$ and $\mathbf{e}_{s_{i+1}}^n$ are also members of $(\sigma_0)^n$. Thus we get from Lemma 7.8 that $|s_i - s_{i+1}| \leq 1$ for all $2 \leq i \leq m-2$. This implies that $(n-1)-2 = |s_{n-1} - s_2| \leq \sum_{i=2}^{m-2} |s_{j+1} - s_j| \leq m-3$, which contradicts our assumption that $m < n$. This completes the proof of the lemma. \square

Proof of Theorem 7.2. Statement (1) follows from Lemmas 7.3–7.6. In Statement (2) the necessity is a consequence of Lemmas 7.3 and 7.9–7.12, while the sufficiency was established in Theorem 5.2. Statements (3) and (4) are special cases of Theorem 4.1 and Theorem 2.3. \square

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