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CLOSED CLASSES OF FUNCTIONS, GENERALIZED CONSTRAINTS AND CLUSTERS

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Abstract. Classes of functions of several variables on arbitrary nonempty domains that are closed under permutation of variables and addition of dummy variables are characterized by generalized constraints, and hereby Hellerstein's Galois theory of functions and generalized constraints is extended to infinite domains. Furthermore, classes of operations on arbitrary nonempty domains that are closed under permutation of variables, addition of dummy variables and composition are characterized by clusters, and a Galois connection is established between operations and clusters.

1. Introduction

Iterative algebras, as introduced by Mal'cev [8], are classes of operations on a fixed base set A that are closed under permutation of variables, addition of dummy variables, identification of variables, and composition. Clones are iterative algebras that contain all projections. The "preservation" relation between operations and relations on A induces a well-known Galois connection, known as the Pol–Inv theory, whose closed subsets of operations are precisely the clones on A. This theory was first established for finite domains by Geiger [4] and independently by Bodnarčuk, Kalužnin, Kotov and Romov [1]. These authors also defined certain operations on the set of all relations on A and showed that the Galois closed subsets of relations are exactly the sets that are closed under these operations. These results were extended to infinite domains by Szabó [12] and independently by Pöschel [10]. More generally, iterative algebras (with or without projections) were described by Harnau [5] in terms of a "preservation" relation between operations and relation pairs (R, R') where $R' \subseteq R$. For general background on function and relation algebras, see the monographs by Pöschel and Kalužnin [11] and Lau [7]. For Galois theories etc., see [3, 7, 11]; in particular, see the survey articles of Erné and Pöschel in [3]. For further information on clones, see the monograph by Szendrei [13].

Classes of functions that are closed under only some of the iterative algebra operations have been studied by several authors, and analogous Galois theories have been developed for these variants to describe the closed classes in terms of a "preservation" relation between functions and some dual objects. While the primal objects are still functions, the dual objects are no longer relations but something more general. We will present a brief survey on what has been done previously in this line of research.

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Pippenger [9] showed that the classes of finite functions that are closed under *identification minors* (permutation of variables, identification of variables, and addition of dummy variables) are precisely the closed classes of the Galois connection induced by the "preservation" relation between functions and so-called constraints. He also described the Galois closed classes of constraints as classes that are closed under certain operations on the set of constraints. This Galois theory was extended to functions and constraints on arbitrary, possibly infinite domains by Couceiro and Foldes [2].

Hellerstein [6] showed that the classes of finite functions that are closed under *special minors* (permutation of variables and addition of dummy variables) are precisely the closed classes of the Galois connection induced by the "preservation" relation between functions and so-called generalized constraints. She also described the Galois closed classes of generalized constraints as classes that are closed under certain operations on the set of generalized constraints. The first objective of the current paper is to extend Hellerstein's Galois theory of functions and generalized constraints to arbitrary, possibly infinite domains.

The second objective of this paper is to describe the classes of operations on arbitrary nonempty sets A that are closed under the iterative algebra operations except for identification of variables, i.e., permutation of variables, addition of dummy variables, and composition. We show that the classes of operations on A that contain all projections and are closed under the operations in question are precisely the closed classes of the Galois connection induced by the "preservation" relation between operations and so-called clusters, which we define as downward closed sets of multisets of m-tuples on A. We also describe the Galois closed classes of clusters as classes that are closed under certain operations on the set of clusters.

2. Preliminaries

2.1. **General notation.** We denote the set of natural numbers by $\omega := \{0, 1, 2, \dots\}$, and we regard its elements as ordinals, i.e., $n \in \omega$ is the set of lesser ordinals $\{0, 1, \dots, n-1\}$. Thus, an n-tuple $\mathbf{a} \in A^n$ is formally a map $\mathbf{a} \colon \{0, 1, \dots, n-1\} \to A$. The notation $(a_i \mid i \in n)$ means the n-tuple mapping i to a_i for each $i \in n$. The notation $(a_1, \dots, a_n)$ means the n-tuple mapping i to a_{i+1} for each $i \in n$.

We view an $m \times n$ matrix $\mathbf{M} \in A^{m \times n}$ with entries in A as an n-tuple of m-tuples $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n)$. The m-tuples $\mathbf{a}^1, \dots, \mathbf{a}^n$ are called the *columns* of $\mathbf{M}$. For $i \in m$, the n-tuple $\bigl(\mathbf{a}^1(i), \dots, \mathbf{a}^n(i)\bigr)$ is called *row* i of $\mathbf{M}$. If for $1 \leq i \leq p$, $\mathbf{M}_i := (\mathbf{a}^i_1, \dots, \mathbf{a}^i_{n_i})$ is an $m \times n_i$ matrix, then we denote by $[\mathbf{M}_1|\mathbf{M}_2|\cdots|\mathbf{M}_p]$ the $m \times \sum_{i=1}^p n_i$ matrix $(\mathbf{a}^1_1, \dots, \mathbf{a}^1_{n_1}, \mathbf{a}^2_1, \dots, \mathbf{a}^2_{n_2}, \dots, \mathbf{a}^p_1, \dots, \mathbf{a}^p_{n_p})$. An *empty matrix* has no columns and is denoted by $()$.

For a function $f \colon A^n \to B$ and a matrix $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n) \in A^{m \times n}$, we denote by $f\mathbf{M}$ the m-tuple $\bigl(f(\mathbf{a}^1(i), \dots, \mathbf{a}^n(i)) \bigm| i \in m\bigr)$ in B^m, in other words, $f\mathbf{M}$ is the m-tuple obtained by applying f to the rows of $\mathbf{M}$.

2.2. **Iterative algebras and a Galois connection between operations and relations.** Let A and B be arbitrary nonempty sets. A *function of several variables from A to B* is a map $f \colon A^n \to B$ for some integer $n \geq 1$, called the *arity* of f. For $n \geq 1$, we denote

$$\mathcal{F}^{(n)}_{AB} := B^{A^n} = \{f \mid f \colon A^n \to B\} \quad \text{and} \quad \mathcal{F}_{AB} := \bigcup_{n \geq 1} \mathcal{F}^{(n)}_{AB}.$$

For a subset $\mathcal{F} \subseteq \mathcal{F}_{AB}$, the *$n$-ary part* of $\mathcal{F}$ is $\mathcal{F}^{(n)} := \mathcal{F} \cap \mathcal{F}^{(n)}_{AB}$. In the case that $A = B$, we call maps $f\colon A^n \to A$ *operations on A*. The set of all operations on A is denoted by $\mathcal{O}_A$. Thus, $\mathcal{O}_A = \mathcal{F}_{AA}$ and $\mathcal{O}^{(n)}_A = \mathcal{F}^{(n)}_{AA}$, for $n \geq 1$.

Mal'cev [8] introduced the operations ζ, τ, Δ, ∇, $*$ on the set $\mathcal{O}_A$ of all operations on A, defined as follows for arbitrary $f \in \mathcal{O}^{(n)}_A$, $g \in \mathcal{O}^{(m)}_A$:

$$\begin{aligned}(\zeta f)(x_1, x_2, \dots, x_n) &:= f(x_2, x_3, \dots, x_n, x_1),\\ (\tau f)(x_1, x_2, \dots, x_n) &:= f(x_2, x_1, x_3, \dots, x_n),\\ (\Delta f)(x_1, x_2, \dots, x_{n-1}) &:= f(x_1, x_1, x_2, \dots, x_{n-1})\end{aligned}$$

for $n > 1$, $\zeta f = \tau f = \Delta f := f$ for $n = 1$, and

$$(\nabla f)(x_1, x_2, \dots, x_{n+1}) := f(x_2, \dots, x_{n+1}),$$

$$(f * g)(x_1, x_2, \dots, x_{m+n-1}) := f\bigl(g(x_1, x_2, \dots, x_m), x_{m+1}, \dots, x_{m+n-1}\bigr).$$

The operations ζ and τ are collectively referred to as *permutation of variables,* Δ is called *identification of variables* (also known as *diagonalization*), ∇ is called *addition of a dummy variable* (or *cylindrification*), and $*$ is called *composition.* The algebra $(\mathcal{O}_A; \zeta, \tau, \Delta, \nabla, *)$ of type $(1, 1, 1, 1, 2)$ is called the *full iterative algebra* on A, and its subalgebras are called *iterative algebras* on A. A subset $\mathcal{F} \subseteq \mathcal{O}_A$ is called a *clone* on A, if it is the universe of an iterative algebra on A that contains all *projections* $(x_1, \dots, x_n) \mapsto x_i$, $1 \leq i \leq n$. Note that the operations ζ, τ, Δ and ∇ can be defined in an analogous way on the set $\mathcal{F}_{AB}$ of functions of several variables from A to B, and we will call the algebra $(\mathcal{F}_{AB}; \zeta, \tau, \Delta, \nabla)$ of type $(1, 1, 1, 1)$ a *full function algebra.*

A *Galois connection* between sets A and B is a pair (σ, τ) of mappings $\sigma\colon \mathcal{P}(A) \to \mathcal{P}(B)$ and $\tau\colon \mathcal{P}(B) \to \mathcal{P}(A)$ between the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$ such that for all $X, X' \subseteq A$ and all $Y, Y' \subseteq B$ the following conditions are satisfied:

$$\begin{aligned}X \subseteq X' &\Longrightarrow \sigma(X) \supseteq \sigma(X'),\\ Y \subseteq Y' &\Longrightarrow \tau(Y) \supseteq \tau(Y'),\end{aligned} \quad \text{and} \quad \begin{aligned}X &\subseteq \tau(\sigma(X)),\\ Y &\subseteq \sigma(\tau(Y)),\end{aligned}$$

or, equivalently,

$$X \subseteq \tau(Y) \Longleftrightarrow \sigma(X) \supseteq Y.$$

The most popular Galois connections are derived from binary relations, as the following well-known theorem shows (for a proof, see, e.g., [3, 7]):

Theorem 2.1. *Let A and B be nonempty sets and let $R \subseteq A \times B$. Define the mappings $\sigma\colon \mathcal{P}(A) \to \mathcal{P}(B)$, $\tau\colon \mathcal{P}(B) \to \mathcal{P}(A)$ by*

$$\begin{aligned}\sigma(X) &:= \{y \in B \mid \forall x \in X\colon (x, y) \in R\},\\ \tau(Y) &:= \{x \in A \mid \forall y \in Y\colon (x, y) \in R\}.\end{aligned}$$

Then the pair (σ, τ) is a Galois connection between A and B.

A prototypical example of a Galois connection is given by the Pol–Inv theory of functions and relations. For $m \geq 1$, we denote

$$\mathcal{R}^{(m)}_A := \{R \mid R \subseteq A^m\} = \mathcal{P}(A^m) \quad \text{and} \quad \mathcal{R}_A := \bigcup_{m \geq 1} \mathcal{R}^{(m)}_A.$$

Let $R \in \mathcal{R}^{(m)}_A$. For a matrix $\mathbf{M} \in A^{m \times n}$, we write $\mathbf{M} \prec R$ to mean that the columns of $\mathbf{M}$ are m-tuples from the relation R. An operation $f\colon A^n \to A$ is said to *preserve* R (or f is a *polymorphism* of R, or R is an *invariant* of f), denoted $f \rhd R$, if for all $m \times n$ matrices $\mathbf{M} \in A^{m \times n}$, $\mathbf{M} \prec R$ implies $f\mathbf{M} \in R$.

For a relation $R \in \mathcal{R}_A$, we denote by $\operatorname{Pol} R$ the set of all operations $f \in \mathcal{O}_A$ that preserve the relation R. For a set $\mathcal{Q} \subseteq \mathcal{R}_A$ of relations, we let $\operatorname{Pol} \mathcal{Q} := \bigcap_{R \in \mathcal{Q}} \operatorname{Pol} R$. The sets $\operatorname{Pol} R$ and $\operatorname{Pol} \mathcal{Q}$ are called the sets of all *polymorphisms* of R and $\mathcal{Q}$, respectively. Similarly, for an operation $f \in \mathcal{O}_A$, we denote by $\operatorname{Inv} f$ the set of all relations $R \in \mathcal{R}_A$ that are preserved by f. For a set $\mathcal{F} \subseteq \mathcal{O}_A$ of functions, we let $\operatorname{Inv} \mathcal{F} := \bigcap_{f \in \mathcal{F}} \operatorname{Inv} f$. The sets $\operatorname{Inv} f$ and $\operatorname{Inv} \mathcal{F}$ are called the sets of all *invariants* of f and $\mathcal{F}$, respectively.

By Theorem 2.1, $(\operatorname{Inv}, \operatorname{Pol})$ is the Galois connection induced by the relation $\rhd$ between the set $\mathcal{O}_A$ of all operations on A and the set $\mathcal{R}_A$ of all relations on A. It was shown by Geiger [4] and independently by Bodnarčuk, Kalužnin, Kotov and Romov [1] that for finite sets A, the closed subsets of $\mathcal{O}_A$ under this Galois connection are exactly the clones on A. These authors also described the closed subsets of $\mathcal{R}_A$ by defining an algebra on $\mathcal{R}_A$ and showing that the closed sets of relations are exactly the subuniverses of this algebra. This can be done as follows, following Lau [7].

We define operations ζ, τ, pr, $\wedge$, $\times$ on $\mathcal{R}_A$ as follows. For $R \in \mathcal{R}_A^{(m)}$, $R' \in \mathcal{R}_A^{(m')}$,

$$\zeta R := \{(a_2, a_3, \dots, a_m, a_1) \mid (a_1, a_2, \dots, a_m) \in R\},$$
$$\tau R := \{(a_2, a_1, a_3, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in R\},$$
$$pr R := \{(a_2, \dots, a_m) \mid \exists a_1 \in A \colon (a_1, a_2, \dots, a_m) \in R\}$$

for $m > 1$ and $\zeta R = \tau R = pr R := R$ for $m = 1$, and

$$R \wedge R' := \{(a_1, \dots, a_m) \mid (a_1, \dots, a_m) \in R \cap R'\}$$

for $m = m'$ and $R \wedge R' := R$ for $m \neq m'$, and

$$R \times R' := \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \mid (a_1, \dots, a_m) \in R \wedge (b_1, \dots, b_{m'}) \in R'\}.$$

The operations ζ and τ are collectively referred to as *permutation of rows,* pr is called *deletion of the first row,* $\wedge$ is called *intersection of relations,* and $\times$ is called *Cartesian product.* Denote $\delta := \{(x, x, y) \in A^3 \mid x, y \in A\}$. The algebra $(\mathcal{R}_A; \delta, \zeta, \tau, pr, \wedge, \times)$ of type $(0, 1, 1, 1, 2, 2)$ is called the *full relation algebra* on A. The subuniverses of $(\mathcal{R}_A; \delta, \zeta, \tau, pr, \wedge, \times)$ are called *relational clones* (or *coclones*) on A.

Theorem 2.2 (Geiger [4]; Bodnarčuk, Kalužnin, Kotov and Romov [1]). *Let A be a finite nonempty set.*

(i) A set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations is the set of polymorphisms of some set $\mathcal{Q} \subseteq \mathcal{R}_A$ of relations if and only if $\mathcal{F}$ is a clone on A.
(ii) A set $\mathcal{Q} \subseteq \mathcal{R}_A$ of relations is the set of invariants of some set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations if and only if $\mathcal{Q}$ is a relational clone on A.

On arbitrary, possibly infinite sets A, the Galois closed sets of operations are the locally closed clones, as shown by Szabó [12] and independently by Pöschel [10]. A set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions is said to be *locally closed,* if it holds that for all $f \in \mathcal{F}_{AB}$, say of arity n, $f \in \mathcal{F}$ whenever for all finite subsets $F \subseteq A^n$, there exists a function $g \in \mathcal{F}^{(n)}$ such that $f|_F = g|_F$. A more general closure condition was defined for sets of relations as well.

ALGEBRA	DUAL OBJECTS	REFERENCE
$(\mathcal{O}_A; \zeta, \tau, \Delta, \nabla, *)$ – with projections (clones)	relations R	Geiger [4]; Bodnarčuk, Kalužnin, Kotov, Romov [1] (finite domains), Szabó [12]; Pöschel [10] (general)
– all iterative algebras	relation pairs (R, R') with $R' \subseteq R$	Harnau [5] (finite domains)
$(\mathcal{F}_{AB}; \zeta, \tau, \Delta, \nabla)$	constraints (R, S)	Pippenger [9] (finite domains), Couceiro, Foldes [2] (general)
$(\mathcal{F}_{AB}; \zeta, \tau, \nabla)$	generalized constraints (ϕ, S)	Hellerstein [6] (finite domains), Theorems 3.3, 4.6 (general)
$(\mathcal{O}_A; \zeta, \tau, \nabla, *)$ with projections	clusters Φ	Theorems 5.11, 6.7 (general)

Table 1. Galois theories for function algebras.

These results were generalized to iterative algebras (with or without projections) by Harnau [5] who defined a preservation relation between operations and relation pairs. An m-ary *relation pair* on A is a pair (R, R') where $R, R' \in \mathcal{R}_A^{(m)}$ for some $m \geq 1$ and $R' \subseteq R$. For $m \geq 1$, denote

$$\mathcal{H}_A^{(m)} := \{(R, R') \mid R' \subseteq R \subseteq A^m\} \quad \text{and} \quad \mathcal{H}_A = \bigcup_{m \geq 1} \mathcal{H}_A^{(m)}.$$

An operation $f \in \mathcal{O}_A$ is said to *preserve* a relation pair $(R, R') \in \mathcal{H}_A^{(m)}$, denoted $f \rhd (R, R')$, if for all matrices $\mathbf{M} \in A^{m \times n}$, $\mathbf{M} \prec R$ implies $f\mathbf{M} \in R'$. In light of Theorem 2.1, the preservation relation $\rhd$ induces a Galois connection between the sets $\mathcal{O}_A$ and $\mathcal{H}_A$. Harnau showed that the closed sets of operations are exactly the universes of iterative algebras. He defined certain operations on the set $\mathcal{H}_A$ of relation pairs and showed that the Galois closed subsets of relation pairs are precisely the subsets that are closed under these operations.

2.3. **Reducts of full iterative algebras and of full function algebras.** Let $(A; F)$ and $(A; F')$ be algebras on the same universe A. $(A; F')$ is called a *reduct* of $(A; F)$ if $F' \subseteq F$, i.e., $(A; F')$ is obtained by omitting some operations from $(A; F)$.

Reducts of full iterative algebras $(\mathcal{O}_A; \zeta, \tau, \Delta, \nabla, *)$ or of full function algebras $(\mathcal{F}_{AB}; \zeta, \tau, \Delta, \nabla)$ have been studied by various authors, and the subuniverses thereof have been described in terms of Galois connections induced by "preservation" relations between functions and certain dual objects. In the remainder of this section, we will present a brief survey of the work previously done in this line of research. For easy reference, this overview is summarized in Table 1.

Pippenger [9] described subuniverses of $(\mathcal{F}_{AB}; \zeta, \tau, \Delta, \nabla)$ in terms of a preservation relation between functions and constraints. An m-ary *constraint* from A to B is a pair (R, S), where $R \in \mathcal{R}_A^{(m)}$ and $S \in \mathcal{R}_B^{(m)}$. For $m \geq 1$, denote

$$\mathcal{C}_{AB}^{(m)} := \{(R, S) \mid R \subseteq A^m \wedge S \subseteq B^m\} \quad \text{and} \quad \mathcal{C}_{AB} := \bigcup_{m \geq 1} \mathcal{C}_{AB}^{(m)}.$$

A function $f \in \mathcal{F}_{AB}^{(n)}$ is said to *preserve* a constraint $(R, S) \in \mathcal{C}_{AB}^{(m)}$, denoted $f \rhd (R, S)$, if for all matrices $\mathbf{M} \in A^{m \times n}$, $\mathbf{M} \prec R$ implies $f\mathbf{M} \in S$. As in Theorem 2.1, the relation $\rhd$ induces a Galois connection between the sets $\mathcal{F}_{AB}$ and $\mathcal{C}_{AB}$. We say that a set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions is *characterized* by a set $\mathcal{C} \subseteq \mathcal{C}_{AB}$ of constraints if $\mathcal{F} = \{f \in \mathcal{F}_{AB} \mid \forall (R, S) \in \mathcal{C} \colon f \rhd (R, S)\}$, i.e., $\mathcal{F}$ is precisely the set of functions that preserve all constraints in $\mathcal{C}$. Similarly, $\mathcal{C}$ is said to be *characterized* by $\mathcal{F}$ if $\mathcal{C} = \{(R, S) \in \mathcal{C}_{AB} \mid \forall f \in \mathcal{F} \colon f \rhd (R, S)\}$, i.e., $\mathcal{C}$ is precisely the set of constraints that are preserved by all functions in $\mathcal{F}$.

In order to describe the closed sets of constraints, we define a few operations on $\mathcal{C}_{AB}$. A constraint $(R, S) \in \mathcal{C}_{AB}^{(m)}$ is a *simple minor* of a constraint $(R', S') \in \mathcal{C}_{AB}^{(n)}$ if there is a natural number p $(0 \leq p \leq n)$ and a map $h \colon \{1, \dots, n\} \to \{1, \dots, m + p\}$ such that

$$(x_1, \dots, x_m) \in R \iff \exists x_{m+1}, \dots, x_{m+p} \colon (x_{h(1)}, \dots, x_{h(n)}) \in R'$$

and

$$(x_1, \dots, x_m) \in S \iff \exists x_{m+1}, \dots, x_{m+p} \colon (x_{h(1)}, \dots, x_{h(n)}) \in S'.$$

We say that a constraint (R, S) is obtained from (R', S) by *restricting the antecedent* if $R \subseteq R'$. We say that a constraint (R, S) is obtained from (R, S') by *extending the consequent* if $S \supseteq S'$. We say that the constraint $(R, S \cap S')$ is obtained from (R, S) and (R, S') by *intersecting consequents.*

A constraint $(R, S) \in \mathcal{C}_{AB}^{(m)}$ where $R = \{(a, \dots, a) \in A^m \mid a \in A\}$ and $S = \{(b, \dots, b) \in B^m \mid b \in B\}$ is called an *equality constraint.* The constraint $(\emptyset, \emptyset)$ is called the *empty constraint* (of any arity).

A set $\mathcal{C} \subseteq \mathcal{C}_{AB}$ of constraints is *minor-closed* if it contains the binary equality constraint and the unary empty constraint, and it is closed under taking simple minors, restricting antecedents, extending consequents and intersecting consequents.

Theorem 2.3 (Pippenger [9])**.** *Let A and B be finite nonempty sets.*

- *(i) A set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions is characterized by some set $\mathcal{C} \subseteq \mathcal{C}_{AB}$ of constraints if and only if $\mathcal{F}$ is a subuniverse of $(\mathcal{F}_{AB}; \zeta, \tau, \Delta, \nabla)$.*
- *(ii) A set $\mathcal{C} \subseteq \mathcal{C}_{AB}$ of constraints is characterized by some set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions if and only if $\mathcal{C}$ is minor-closed.*

Couceiro and Foldes [2] extended Pippenger's results to functions and constraints on arbitrary, possibly infinite domains. In this case, one has to stipulate that the closed sets of functions are locally closed. In order to describe the Galois closed sets of constraints, Couceiro and Foldes defined *conjunctive minors* of constraints and presented a more general closure condition in terms of conjunctive minors.

Subuniverses of $(\mathcal{F}_{AB}; \zeta, \tau, \nabla)$ were described by Hellerstein [6] in terms of generalized constraints. In the following Sections 3, 4, we will present Hellerstein's Galois theory of functions and generalized constraints for finite domains, and we will extend it to arbitrary, possibly infinite domains.

Finally, in Sections 5, 6, we will describe the subuniverses of $(\mathcal{O}_A; \zeta, \tau, \nabla, *)$ by developing an analogous Galois theory for operations and clusters.

3. Classes of functions closed under permutation of variables and addition of dummy variables

Hellerstein [6] showed that for finite domains A and B, the subuniverses of $(\mathcal{F}_{AB}; \zeta, \tau, \nabla)$, i.e., the classes of functions that are closed under permutation of variables and addition of dummy variables, are characterized by generalized constraints. We extend Hellerstein's Galois theory of functions and generalized constraints to arbitrary, possibly infinite domains. Our results and proofs closely follow Hellerstein's analogous statements for functions and generalized constraints on finite domains, which in turn are adaptations of those by Pippenger [9], Geiger [4] and Bodnarčuk, Kalužnin, Kotov, Romov [1].

We will consider the set $\omega \cup \{\omega\}$ with the usual ordering of natural numbers and a new largest element ω adjoined. For $m \geq 1$, an m-ary *repetition function* on A is a map $\phi\colon A^m \to \omega \cup \{\omega\}$. Hellerstein [6] defined an m-ary *generalized constraint* from A to B to be a pair (ϕ, S) where ϕ is an m-ary repetition function on A called the *antecedent*, and $S \subseteq B^m$ is called the *consequent*. The number m is called the *arity* of the generalized constraint. We denote

$$\mathcal{G}_{AB}^{(m)} := \{(\phi, S) \mid \phi\colon A^m \to \omega \cup \{\omega\},\, S \subseteq B^m\} \quad \text{and} \quad \mathcal{G}_{AB} := \bigcup_{m \geq 1} \mathcal{G}_{AB}^{(m)}.$$

For a matrix $\mathbf{M} \in A^{m \times n}$ and a repetition function $\phi\colon A^m \to \omega \cup \{\omega\}$, we write $\mathbf{M} \prec \phi$ to mean that each m-tuple $\mathbf{a} \in A^m$ occurs as a column of $\mathbf{M}$ at most $\phi(\mathbf{a})$ times. If $f \in \mathcal{F}_{AB}^{(n)}$ and $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$, we say that f *preserves* (ϕ, S), denoted $f \rhd (\phi, S)$, if for every matrix $\mathbf{M} \in A^{m \times n}$, $\mathbf{M} \prec \phi$ implies $f\mathbf{M} \in S$.

It is clear from Theorem 2.1 that the relation $\rhd$ establishes a Galois connection between the sets $\mathcal{F}_{AB}$ and $\mathcal{G}_{AB}$. We say that a set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions is *characterized* by a set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints if $\mathcal{F} = \{f \in \mathcal{F}_{AB} \mid \forall (\phi, S) \in \mathcal{G}\colon f \rhd (\phi, S)\}$, i.e., $\mathcal{F}$ is precisely the set of functions that preserve all generalized constraints in $\mathcal{G}$. Similarly, $\mathcal{G}$ is said to be *characterized* by $\mathcal{F}$ if $\mathcal{G} = \{(\phi, S) \in \mathcal{G}_{AB} \mid \forall f \in \mathcal{F}\colon f \rhd (\phi, S)\}$, i.e., $\mathcal{G}$ is precisely the set of generalized constraints preserved by all functions in $\mathcal{F}$. Thus, the Galois closed sets of functions (generalized constraints) are exactly those that are characterized by generalized constraints (functions, respectively).

Theorem 3.1 (Hellerstein [6])**.** *Let A and B be finite nonempty sets. For any set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions, the following two conditions are equivalent:*

(i) $\mathcal{F}$ is closed under permutation of variables and addition of dummy variables.
(ii) $\mathcal{F}$ is characterized by some set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints.

We will extend Theorem 3.1 to functions on arbitrary, possibly infinite domains.

For a matrix $\mathbf{M} \in A^{m \times n}$, the *characteristic function* of $\mathbf{M}$ is defined as the function $\chi_{\mathbf{M}}\colon A^m \to \omega$ given by the rule that for every $\mathbf{a} \in A^m$, $\chi_{\mathbf{M}}(\mathbf{a})$ equals the number of times the m-tuple $\mathbf{a}$ occurs as a column of $\mathbf{M}$. For any $\mathcal{F} \subseteq \mathcal{F}_{AB}$ and for a matrix $\mathbf{M} \in A^{m \times n}$, we denote $\mathcal{F}\mathbf{M} := \{f\mathbf{M} \mid f \in \mathcal{F}^{(n)}\}$.

Lemma 3.2. *If $\mathcal{F} \subseteq \mathcal{F}_{AB}$ is closed under permutation of variables and addition of dummy variables, then for every matrix $\mathbf{M} \in A^{m \times n}$, the generalized constraint $(\chi_{\mathbf{M}}, \mathcal{F}\mathbf{M})$ is preserved by all functions in $\mathcal{F}$.*

Proof. Let $f' \in \mathcal{F}$ be n'-ary, and let $\mathbf{M}'$ be an $m \times n'$ matrix such that $\mathbf{M}' \prec \chi_{\mathbf{M}}$. Then there exists an injection $\sigma\colon \{1, \dots, n'\} \to \{1, \dots, n\}$ such that for each $i \in \{1, \dots, n'\}$, column i of $\mathbf{M}'$ equals column $\sigma(i)$ of $\mathbf{M}$. Let f be the n-ary function defined by $f(x_1, \dots, x_n) = f'(x_{\sigma(1)}, \dots, x_{\sigma(n')})$. We have that $f'\mathbf{M}' = f\mathbf{M}$. Since f is obtained from f' by permutation of variables and addition of dummy variables, it is a member of $\mathcal{F}$, and hence $f\mathbf{M} \in \mathcal{F}\mathbf{M}$. Thus $f'\mathbf{M}' \in \mathcal{F}\mathbf{M}$ and so $f' \rhd (\chi_{\mathbf{M}}, \mathcal{F}\mathbf{M})$. □

We are now ready to describe the classes of functions that are characterized by generalized constraints. Recall that a set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ is locally closed if and only if for all $f \in \mathcal{F}_{AB}$ it holds that $f \in \mathcal{F}$ whenever for all finite subsets $F \subseteq A^n$ (where n is the arity of f) there is a $g \in \mathcal{F}^{(n)}$ such that $f|_F = g|_F$.

Theorem 3.3. *Let A and B be arbitrary, possibly infinite nonempty sets. For any set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions, the following two conditions are equivalent:*

(i) $\mathcal{F}$ is locally closed and it is closed under permutation of variables and addition of dummy variables.
(ii) $\mathcal{F}$ is characterized by some set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints.

Proof. (ii) $\Rightarrow$ (i): As observed in the case of finite domains by Hellerstein [6], it is easy to see, also in general, that the set of functions preserving a generalized constraint (ϕ, S) is closed under permutation of variables and addition of dummy variables. Thus any class of functions characterized by a set $\mathcal{G}$ of generalized constraints is closed under the operations considered.

It remains to show that $\mathcal{F}$ is locally closed. It is clear that $\emptyset$ and $\mathcal{F}_{AB}$ are locally closed, so we may assume that $\emptyset \neq \mathcal{F} \neq \mathcal{F}_{AB}$. Suppose on the contrary that there is a $g \in \mathcal{F}_{AB} \setminus \mathcal{F}$, say of arity n, such that for every finite subset $F \subseteq A^n$, there is an $f \in \mathcal{F}^{(n)}$ for which $g|_F = f|_F$ holds. Since $\mathcal{F}$ is characterized by $\mathcal{G}$ and $g \notin \mathcal{F}$, there is a $(\phi, S) \in \mathcal{G}$, say m-ary, such that $g \not\triangleright (\phi, S)$, and hence for some $m \times n$ matrix $\mathbf{M}$, we have $\mathbf{M} \prec \phi$ but $g\mathbf{M} \notin S$. Let F be the finite set of rows of $\mathbf{M}$. By our assumption, there is an $f \in \mathcal{F}^{(n)}$ such that $g|_F = f|_F$, and hence $f\mathbf{M} = f|_F\mathbf{M} = g|_F\mathbf{M} = g\mathbf{M} \notin S$, which contradicts the fact that $f \triangleright (\phi, S)$.

(i) $\Rightarrow$ (ii): It is easy to verify that $\emptyset$ and $\mathcal{F}_{AB}$ are characterized by $\mathcal{G}_{AB}$ and $\emptyset$, respectively, so we assume that $\emptyset \neq \mathcal{F} \neq \mathcal{F}_{AB}$. We need to show that for every function $g \in \mathcal{F}_{AB} \setminus \mathcal{F}$, there exists a generalized constraint that is preserved by every function in $\mathcal{F}$ but not by g. The set of all such "separating" generalized constraints, for each $g \in \mathcal{F}_{AB} \setminus \mathcal{F}$, characterizes $\mathcal{F}$.

Suppose that $g \in \mathcal{F}_{AB} \setminus \mathcal{F}$ is n-ary. Since $\mathcal{F}$ is locally closed, there is a finite subset $F \subseteq A^n$ such that $g|_F \neq f|_F$ for every $f \in \mathcal{F}^{(n)}$. Clearly F is nonempty. Let $\mathbf{M}$ be a $|F| \times n$ matrix whose rows are the elements of F in some fixed order, and consider the generalized constraint $(\chi_\mathbf{M}, \mathcal{F}\mathbf{M})$. By Lemma 3.2, every function in $\mathcal{F}$ preserves $(\chi_\mathbf{M}, \mathcal{F}\mathbf{M})$. But $g\mathbf{M} = g|_F\mathbf{M} \notin \mathcal{F}\mathbf{M}$, and hence $g \not\triangleright (\chi_\mathbf{M}, \mathcal{F}\mathbf{M})$. □

4. Closure conditions for generalized constraints

In this section, we will describe the Galois closed sets of generalized constraints. Hellerstein [6] introduced the following eight operations on $\mathcal{G}_{AB}$.

1. Let $(\phi, S), (\phi', S') \in \mathcal{G}_{AB}^{(m)}$, $m \in \omega \setminus \{0\}$. Let π be a permutation of $\{1, \dots, m\}$. If

$$\phi(a_1, \dots, a_m) = \phi'(a_{\pi(1)}, \dots, a_{\pi(m)})$$

for all $(a_1, \dots, a_m) \in A^m$ and

$$(b_1, \dots, b_m) \in S \iff (b_{\pi(1)}, \dots, b_{\pi(m)}) \in S'$$

for all $(b_1, \dots, b_m) \in A^m$, then we say that (ϕ', S') is obtained from (ϕ, S) by *permutation of arguments.*

2. Let $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ and $(\phi', S') \in \mathcal{G}_{AB}^{(m-1)}$, $m \geq 2$. If
$$\phi'(a_1, \ldots, a_{m-1}) = \sum_{d \in A} \phi(a_1, \ldots, a_{m-1}, d)$$
for all $(a_1, \ldots, a_{m-1}) \in A^{m-1}$ and
$$S' = \{(b_1, \ldots, b_{m-1}) \in B^{m-1} \mid \exists b_m \in B \colon (b_1, \ldots, b_m) \in S\},$$
then we say that (ϕ', S') is obtained from (ϕ, S) by *projection.*
3. Let $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ and $(\phi', S') \in \mathcal{G}_{AB}^{(m-1)}$, $m \geq 2$. If
$$\phi'(a_1, \ldots, a_{m-1}) = \phi(a_1, \ldots, a_{m-2}, a_{m-1}, a_{m-1})$$
for all $(a_1, \ldots, a_{m-1}) \in A^{m-1}$ and
$$S' = \{(b_1, \ldots, b_{m-1}) \in B^{m-1} \mid (b_1, \ldots, b_{m-2}, b_{m-1}, b_{m-1}) \in S\},$$
then we say that (ϕ', S') is obtained from (ϕ, S) by *identification of arguments* (or *diagonalization*).
4. Let $(\phi, S) \in \mathcal{G}_{AB}^{(m-1)}$ and $(\phi', S') \in \mathcal{G}_{AB}^{(m)}$, $m \geq 2$. If
$$\phi(a_1, \ldots, a_{m-1}) = \sum_{d \in A} \phi'(a_1, \ldots, a_{m-1}, d)$$
for all $(a_1, \ldots, a_{m-1}) \in A^{m-1}$ and
$$S' = \{(b_1, \ldots, b_m) \in B^{m-1} \mid \exists b_m \in B \colon (b_1, \ldots, b_{m-1}) \in S\},$$
then we say that (ϕ', S') is obtained from (ϕ, S) by *addition of a dummy argument.*
5. Let $a_0, a_1, a_2, \ldots$ be a sequence of natural numbers such that $a_i \leq a_{i+1}$ for all $i \in \omega$. If the sequence contains a maximum element, we define the limit of the sequence to be the value of that element. Otherwise we define the limit of the sequence to be ω. This limit is denoted by $\lim_{i \to \infty} a_i$.

   For any fixed domain S, we define a partial order $\leq$ on the set of all functions $\phi \colon S \to \omega \cup \{\omega\}$ as follows: $\phi \leq \phi'$ if and only if for all $x \in S$, $\phi(x) \leq \phi'(x)$. Let $\phi_0, \phi_1, \phi_2, \ldots$ be a sequence of functions $S \to \omega \cup \{\omega\}$ such that $\phi_i \leq \phi_{i+1}$ for all $i \in \omega$. The limit of the sequence is defined to be the function $\phi \colon S \to \omega \cup \{\omega\}$ such that for all $x \in S$, $\phi(x) = \lim_{i \to \infty} \phi_i(x)$.

   If $(\phi_i, S)_{i \in \omega}$ is a family of members of $\mathcal{G}_{AB}^{(m)}$ such that $\phi_i \leq \phi_{i+1}$ for all $i \in \omega$, then we say that the generalized constraint $(\lim_{i \to \infty} \phi_i, S)$ is obtained from $(\phi_i, S)_{i \in \omega}$ by *taking the limit of antecedents.*
6. If $(\phi, S), (\phi', S) \in \mathcal{G}_{AB}^{(m)}$ are such that $\phi' \leq \phi$, then we say that (ϕ', S) is obtained from (ϕ, S) by *restricting the antecedent.*
7. If $(\phi, S), (\phi, S') \in \mathcal{G}_{AB}^{(m)}$ are such that $S' \supseteq S$, then we say that (ϕ, S') is obtained from (ϕ, S) by *extending the consequent.*
8. If $(\phi, S), (\phi, S') \in \mathcal{G}_{AB}^{(m)}$, then we say that $(\phi, S \cap S')$ is obtained from (ϕ, S) and (ϕ, S') by *intersecting the consequents.*

If (ϕ', S') is obtained from (ϕ, S) by restricting the antecedent or by extending the consequent or by a combination of the two, i.e., $\phi' \leq \phi$ and $S' \supseteq S$, then we say that (ϕ', S') is a *relaxation* of (ϕ, S).

The m-ary *generalized equality constraint* is defined to be the generalized constraint $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ such that $\phi(\mathbf{a}) = \omega$ if all components of $\mathbf{a} \in A^m$ are equal and $\phi(\mathbf{a}) = 0$ otherwise, and such that the elements of S are exactly those m-tuples

$\mathbf{b} \in B^m$ in which all components are equal. The m-ary *generalized empty constraint* is defined to be the generalized constraint $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ where $\phi(\mathbf{a}) = 0$ for all $\mathbf{a} \in A^m$ and $S = \emptyset$. The m-ary *generalized trivial constraint* is defined to be the generalized constraint $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ where $\phi(\mathbf{a}) = \omega$ for all $\mathbf{a} \in A^m$ and $S = B^m$.

A set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints is *minor-closed* if it is closed under the eight operations defined above and it contains the unary generalized empty constraint and the binary generalized equality constraint.

Theorem 4.1 (Hellerstein [6])**.** *Let A and B be finite nonempty sets. A set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints is characterized by some set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions if and only if it is minor-closed.*

In order to extend Theorem 4.1 to arbitrary, possibly infinite domains, we need a more general closure condition. We will follow Couceiro and Foldes's [2] proof techniques and adapt their notion of conjunctive minor to generalized constraints. We first introduce several technical notions and definitions that will be needed in the statement of the extension of Theorem 4.1 and in its proof.

For maps $f\colon A \to B$ and $g\colon C \to D$, the composition $g \circ f$ is defined only if $B = C$. Removing this restriction, the *concatenation* of f and g is defined to be the map $gf\colon f^{-1}[B \cap C] \to D$ given by the rule $(gf)(a) = g\big(f(a)\big)$ for all $a \in f^{-1}[B \cap C]$. Clearly, if $B = C$, then $gf = g \circ f$; thus functional composition is subsumed and extended by concatenation. Concatenation is associative, i.e., for any maps f, g, h, we have $h(gf) = (hg)f$.

For a family $(g_i)_{i \in I}$ of maps $g_i\colon A_i \to B_i$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, we define the (*piecewise*) *sum of the family* $(g_i)_{i \in I}$ to be the map $\sum_{i \in I} g_i\colon \bigcup_{i \in I} A_i \to \bigcup_{i \in I} B_i$ whose restriction to each A_i coincides with g_i. If I is a two-element set, say $I = \{1, 2\}$, then we write $g_1 + g_2$. Clearly, this operation is associative and commutative.

Concatenation is distributive over summation, i.e., for any family $(g_i)_{i \in I}$ of maps on disjoint domains and any map f,

$$\big(\sum_{i \in I} g_i\big) f = \sum_{i \in I} (g_i f) \qquad \text{and} \qquad f\big(\sum_{i \in I} g_i\big) = \sum_{i \in I} (f g_i).$$

In particular, if g_1 and g_2 are maps with disjoint domains, then

$$(g_1 + g_2) f = (g_1 f) + (g_2 f) \qquad \text{and} \qquad f(g_1 + g_2) = (f g_1) + (f g_2).$$

Let $g_1, \dots, g_n$ be maps from A to B. The n-tuple $(g_1, \dots, g_n)$ determines a *vector-valued map* $g\colon A \to B^n$, given by $g(a) := \big(g_1(a), \dots, g_n(a)\big)$ for every $a \in A$. For $f\colon B^n \to C$, the composition $f \circ g$ is a map from A to C, denoted by $f(g_1, \dots, g_n)$, and called the *composition of f with* $g_1, \dots, g_n$. Suppose that $A \cap A' = \emptyset$ and $g'_1, \dots, g'_n$ are maps from A' to B. Let g and g' be the vector-valued maps determined by $(g_1, \dots, g_n)$ and $(g'_1, \dots, g'_n)$, respectively. We have that $f(g + g') = (fg) + (fg')$, i.e.,

$$f\big((g_1 + g'_1), \dots, (g_n + g'_n)\big) = f(g_1, \dots, g_n) + f(g'_1, \dots, g'_n).$$

For $B \subseteq A$, ι_{AB} denotes the canonical injection (inclusion map) from B to A. Thus the *restriction* $f|_B$ of any map $f\colon A \to C$ to the subset B is given by $f|_B = f \iota_{AB}$.

Remark 4.2. Observe that the notation $f\mathbf{M}$ introduced in Section 2 is in accordance with the notation for concatenation of mappings. Since a matrix $\mathbf{M} :=$

$(\mathbf{a}^1, \dots, \mathbf{a}^n)$ is an n-tuple of m-tuples $\mathbf{a}^i \colon m \to A$, $1 \leq i \leq n$, the composition of the vector-valued map $(\mathbf{a}^1, \dots, \mathbf{a}^n) \colon m \to A^n$ with $f \colon A^n \to B$ gives rise to the m-tuple $f(\mathbf{a}^1, \dots, \mathbf{a}^n) \colon m \to B$.

Let m and n be positive integers (viewed as ordinals, i.e., $m = \{0, \dots, m-1\}$). Let $h \colon n \to m \cup V$ where V is an arbitrary set of symbols disjoint from the ordinals, called *existentially quantified indeterminate indices,* or simply *indeterminates,* and let $\sigma \colon V \to A$ be any map, called a *Skolem map.* Then each m-tuple $\mathbf{a} \in A^m$, being a map $\mathbf{a} \colon m \to A$, gives rise to an n-tuple $(\mathbf{a} + \sigma)h =: (b_0, \dots, b_{n-1}) \in A^n$, where

$$b_i := \begin{cases} a_{h(i)}, & \text{if } h(i) \in \{0, 1, \dots, m-1\}, \\ \sigma(h(i)), & \text{if } h(i) \in V. \end{cases}$$

Let $H := (h_j)_{j \in J}$ be a nonempty family of maps $h_j \colon n_j \to m \cup V$, where each n_j is a positive integer. Then H is called a *minor formation scheme* with *target* m, *indeterminate set* V, and *source family* $(n_j)_{j \in J}$. Let $(R_j)_{j \in J}$ be a family of relations (of various arities) on the same set A, each R_j of arity n_j, and let R be an m-ary relation on A. We say that R is a *restrictive conjunctive minor* of the family $(R_j)_{j \in J}$ *via* H, if for every m-tuple $\mathbf{a} \in A^m$,

$$\mathbf{a} \in R \Longrightarrow [\exists \sigma \in A^V \ \forall j \in J \colon (\mathbf{a} + \sigma)h_j \in R_j].$$

On the other hand, if for every $\mathbf{a} \in A^m$,

$$[\exists \sigma \in A^V \ \forall j \in J \colon (\mathbf{a} + \sigma)h_j \in R_j] \Longrightarrow \mathbf{a} \in R,$$

then we say that R is an *extensive conjunctive minor* of the family $(R_j)_{j \in J}$ *via* H. If R is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(R_j)_{j \in J}$ via H, i.e., for every $\mathbf{a} \in A^m$,

$$\mathbf{a} \in R \Longleftrightarrow [\exists \sigma \in A^V \ \forall j \in J \colon (\mathbf{a} + \sigma)h_j \in R_j],$$

then R is said to be a *tight conjunctive minor* of the family $(R_j)_{j \in J}$ *via* H. For a scheme H and a family $(R_j)_{j \in J}$ of relations, there is a unique tight conjunctive minor of the family $(R_j)_{j \in J}$ via H.

We adapt these notions to repetition functions. Let $(\phi_j)_{j \in J}$ be a family of repetition functions (of various arities) on A, each ϕ_j of arity n_j, and let ϕ be an m-ary repetition function on A. We say that ϕ is a *restrictive conjunctive minor* of the family $(\phi_j)_{j \in J}$ *via* H, if, for every $m \times n$ matrix $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n) \in A^{m \times n}$,

$$\mathbf{M} \prec \phi \Longrightarrow \big[\exists \sigma_1, \dots, \sigma_n \in A^V \ \forall j \in J \colon \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) \prec \phi_j\big].$$

On the other hand, if, for every $m \times n$ matrix $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n) \in A^{m \times n}$,

$$\big[\exists \sigma_1, \dots, \sigma_n \in A^V \ \forall j \in J \colon \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) \prec \phi_j\big] \Longrightarrow \mathbf{M} \prec \phi,$$

then we say that ϕ is an *extensive conjunctive minor* of the family $(\phi_j)_{j \in J}$ *via* H. If ϕ is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(\phi_j)_{j \in J}$ via H, i.e., for every $m \times n$ matrix $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n) \in A^{m \times n}$,

$$\mathbf{M} \prec \phi \Longleftrightarrow \big[\exists \sigma_1, \dots, \sigma_n \in A^V \ \forall j \in J \colon \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) \prec \phi_j\big],$$

then ϕ is said to be a *tight conjunctive minor* of the family $(\phi_j)_{j \in J}$ *via* H.

Remark 4.3. If ϕ is a restrictive conjunctive minor of the family $(\phi_j)_{j \in J}$ of repetition functions via the scheme $(h_j)_{j \in J}$, then it holds for every $\mathbf{a} \in A^m$ that, for all $j \in J$,

$$\sum_{\mathbf{b} \in \langle \mathbf{a} \rangle} \phi(\mathbf{b}) \leq \sum_{\mathbf{c} \in S_j^{\mathbf{a}}} \phi_j(\mathbf{c}),$$

where $\langle \mathbf{a} \rangle := \{\mathbf{b} \in A^m \mid (\mathbf{b} + \sigma)h_j = (\mathbf{a} + \sigma)h_j\}$ for some Skolem map $\sigma \colon V \to A$, and $S_j^{\mathbf{a}} := \{(\mathbf{a} + \sigma)h_j \in A^{n_j} \mid \sigma \in A^V\}$. Note that the definition of $\langle \mathbf{a} \rangle$ does not depend on the choice of σ. Also, $S_j^{\mathbf{a}} = S_j^{\mathbf{b}}$ for every $\mathbf{b} \in \langle \mathbf{a} \rangle$.

Similarly, if ϕ is an extensive conjunctive minor of $(\phi_j)_{j \in J}$ via $(h_j)_{j \in J}$, then it holds for every $\mathbf{a} \in A^m$ that, for all $j \in J$,

$$\sum_{\mathbf{b} \in \langle \mathbf{a} \rangle} \phi(\mathbf{b}) \geq \sum_{\mathbf{c} \in S_j^{\mathbf{a}}} \phi_j(\mathbf{c}).$$

Consequently, for a tight conjunctive minor ϕ of $(\phi_j)_{j \in J}$ via $(h_j)_{j \in J}$, we have the equality

$$\sum_{\mathbf{b} \in \langle \mathbf{a} \rangle} \phi(\mathbf{b}) = \sum_{\mathbf{c} \in S_j^{\mathbf{a}}} \phi_j(\mathbf{c}),$$

but tight conjunctive minors of families of repetition functions are not unique.

If $(\phi_j, S_j)_{j \in J}$ is a family of members of $\mathcal{G}_{AB}$ (of various arities) and ϕ is a restrictive conjunctive minor of $(\phi_j)_{j \in J}$ via a scheme H and S is an extensive conjunctive minor of $(S_j)_{j \in J}$ via the same scheme H, then the generalized constraint $(\phi, S) \in \mathcal{G}_{AB}$ is said to be a *conjunctive minor* of the family $(\phi_j, S_j)_{j \in J}$ *via* H. If both ϕ and S are tight conjunctive minors of the respective families via H, then (ϕ, S) is said to be a *tight conjunctive minor* of the family $(\phi_j, S_j)_{j \in J}$ *via* H. Tight conjunctive minors of families of generalized constraints are not unique, but if both (ϕ, S) and (ϕ', S') are tight conjunctive minors of the same family of generalized constraints via the same scheme, then $S = S'$. If the minor formation scheme $H := (h_j)_{j \in J}$ and the family $(\phi_j, S_j)_{j \in J}$ are indexed by a singleton $J := \{0\}$, then a tight conjunctive minor (ϕ, S) of a family consisting of a single generalized constraint (ϕ_0, S_0) is called a *simple minor* of (ϕ_0, S_0).

Lemma 4.4. *Assume that (ϕ, S) is a conjunctive minor of a nonempty family $(\phi_j, S_j)_{j \in J}$ of members of $\mathcal{G}_{AB}$, and let $f \in \mathcal{F}_{AB}$. If $f \rhd (\phi_j, S_j)$ for all $j \in J$, then $f \rhd (\phi, S)$.*

Proof. Let (ϕ, S) be an m-ary conjunctive minor of the family $(\phi_j, S_j)_{j \in J}$ via the scheme $H := (h_j)_{j \in J}$, $h_j \colon n_j \to m \cup V$. Let $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n)$ be an arbitrary $m \times n$ matrix such that $\mathbf{M} \prec \phi$. We need to prove that $f\mathbf{M} \in S$. Since ϕ is a restrictive conjunctive minor of $(\phi_j)_{j \in J}$ via $H = (h_j)_{j \in J}$, there are Skolem maps $\sigma_i \colon V \to A$, $1 \leq i \leq n$, such that for every $j \in J$, $\mathbf{M}_j := \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) \prec \phi_j$.

Since S is an extensive conjunctive minor of $(S_j)_{j \in J}$ via the same scheme $H = (h_j)_{j \in J}$, to prove that $f\mathbf{M} \in S$, it suffices to give a Skolem map $\sigma \colon V \to B$ such that, for all $j \in J$, $(f\mathbf{M} + \sigma)h_j \in S_j$. Let $\sigma := f(\sigma_1, \dots, \sigma_n)$. We have that, for each $j \in J$,

$$\begin{aligned}
(f\mathbf{M} + \sigma)h_j &= \big(f(\mathbf{a}^1, \dots, \mathbf{a}^n) + f(\sigma_1, \dots, \sigma_n)\big)h_j \\
&= \big(f(\mathbf{a}^1 + \sigma_1, \dots, \mathbf{a}^n + \sigma_n)\big)h_j \\
&= f\big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) = f\mathbf{M}_j.
\end{aligned}$$

By our assumption $f \rhd (\phi_j, S_j)$, so we have $f\mathbf{M}_j \in S_j$. $\square$

We say that a set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints is *closed under formation of conjunctive minors* if whenever $(\phi_j, S_j)_{j \in J}$ is a nonempty family of members of $\mathcal{G}$, all conjunctive minors of the family $(\phi_j, S_j)_{j \in J}$ are also in $\mathcal{G}$.

The formation of conjunctive minors subsumes the formation of simple minors as well as the operations of restricting the antecedent, extending the consequent, and intersecting the consequents. Simple minors in turn subsume permutation of arguments, projection, identification of arguments, and addition of a dummy argument.

Lemma 4.5. *Let $\mathcal{G} \subseteq \mathcal{G}_{AB}$ be a set of generalized constraints that contains the binary generalized equality constraint and the unary generalized empty constraint. If $\mathcal{G}$ is closed under formation of conjunctive minors, then it contains all generalized trivial constraints, all generalized equality constraints, and all generalized empty constraints.*

Proof. The unary generalized trivial constraint is a simple minor of the binary generalized equality constraint via the scheme $H := \{h\}$, where $h \colon 2 \to 1$ is given by $h(0) = h(1) = 0$ (by identification of arguments). The m-ary generalized trivial constraint is a simple minor of the unary generalized trivial constraint via the scheme $H := \{h\}$, where $h \colon 1 \to m$ is given by $h(0) = 0$ (by addition of $m - 1$ dummy arguments).

For $m \geq 2$, the m-ary generalized equality constraint is a conjunctive minor of the binary generalized equality constraint via the scheme $H := (h_i)_{i \in m-1}$, where $h_i \colon 2 \to m$ is given by $h_i(0) = i$, $h_i(1) = i + 1$ (by addition of $n - 2$ dummy arguments, restricting the antecedents and intersecting the consequents).

The m-ary generalized empty constraint is a simple minor of the unary generalized empty constraint via the scheme $H := \{h\}$, where $h \colon 1 \to m$ is given by $h(0) = 0$ (by addition of $m - 1$ dummy arguments). $\square$

Let $(\phi, S) \in \mathcal{G}_{AB}^{(m)}$ and $\phi' \colon A^m \to \omega \cup \{\omega\}$. If $\phi' \leq \phi$ and the set $\{\mathbf{a} \in A^m \mid \phi'(\mathbf{a}) \neq 0\}$ is finite, then we say that the generalized constraint (ϕ', S) is obtained from (ϕ, S) by a *finite restriction of the antecedent.* We say that a set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ is *locally closed* if for every $(\phi, S) \in \mathcal{G}_{AB}$, it holds that $(\phi, S) \in \mathcal{G}$ whenever every generalized constraint obtained from (ϕ, S) by a finite restriction of the antecedent belongs to $\mathcal{G}$.

Theorem 4.6. *Let A and B be arbitrary, possibly infinite nonempty sets. For any set $\mathcal{G} \subseteq \mathcal{G}_{AB}$ of generalized constraints, the following two conditions are equivalent:*

(i) $\mathcal{G}$ is locally closed and contains the binary generalized equality constraint and the unary generalized empty constraint, and it is closed under formation of conjunctive minors and taking the limit of antecedents.

(ii) $\mathcal{G}$ is characterized by some set $\mathcal{F} \subseteq \mathcal{F}_{AB}$ of functions.

In order to prove Theorem 4.6, we need to extend the notions of relation and generalized constraint and allow them to have infinite arities, as will be explained below. Functions remain finitary. These extended definitions have no bearing on Theorem 4.6 itself; they are only needed as a tool in its proof.

For any nonzero, possibly infinite ordinal m (an ordinal m is the set of lesser ordinals), an m-tuple $\mathbf{a} \in A^m$ is formally a map $\mathbf{a} \colon m \to A$. The arities of relations and generalized constraints are thus allowed to be arbitrary nonzero, possibly infinite ordinals. In minor formation schemes, the target m and the members n_j of the source family are also allowed to be arbitrary nonzero, possibly infinite ordinals. For relations and repetition functions, we shall use the terms *restrictive conjunctive ∞-minor* and *extensive conjunctive ∞-minor* to indicate a restrictive or an extensive conjunctive minor via a scheme whose target and source ordinals may be infinite or finite. Similarly, for generalized constraints, we will use the terms *conjunctive ∞-minor* and *simple ∞-minor* to indicate conjunctive minors and simple minors via a scheme whose target and source ordinals may be infinite or finite. Thus in the sequel the use of the term "minor" without the prefix "∞" continues to mean the respective minor via a scheme whose target and source ordinals are all finite. Matrices can also have infinitely many rows but only a finite number of columns; an $m \times n$ matrix $\mathbf{M} \in A^{m \times n}$, where n is finite but m may be finite or infinite, is an n-tuple of m-tuples $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n)$ where $\mathbf{a}^i \colon m \to A$ for $1 \leq i \leq n$.

In order to discuss the formation of repeated ∞-minors, we need the following definition. Let $H := (h_j)_{j \in J}$ be a minor formation scheme with target m, indeterminate set V and source family $(n_j)_{j \in J}$, and, for each $j \in J$, let $H_j := (h_j^i)_{i \in I_j}$ be a scheme with target n_j, indeterminate set V_j and source family $(n_j^i)_{i \in I_j}$. Assume that V is disjoint from the V_j's, and for distinct j's the V_j's are also pairwise disjoint. Then the *composite scheme* $H(H_j \mid j \in J)$ is the scheme $K := (k_j^i)_{j \in J, i \in I_j}$ defined as follows:

(i) the target of K is the target m of H,
(ii) the source family of K is $(n_j^i)_{j \in J, i \in I_j}$,
(iii) the indeterminate set of K is $U := V \cup (\bigcup_{j \in J} V_j)$,
(iv) $k_j^i \colon n_j^i \to m \cup U$ is defined by $k_j^i := (h_j + \iota_{UV_j}) h_j^i$, where ι_{UV_j} is the canonical injection (inclusion map) from V_j to U.

Lemma 4.7. *If (ϕ, S) is a conjunctive ∞-minor of a nonempty family $(\phi_j, S_j)_{j \in J}$ of generalized constraints from A to B via the scheme H, and, for each $j \in J$, (ϕ_j, S_j) is a conjunctive ∞-minor of a nonempty family $(\phi_j^i, S_j^i)_{i \in I_j}$ via the scheme H_j, then (ϕ, S) is a conjunctive ∞-minor of the nonempty family $(\phi_j^i, S_j^i)_{j \in J, i \in I_j}$ via the composite scheme $K := H(H_j \mid j \in J)$.*

Proof. First, we need to see that ϕ is a restrictive conjunctive ∞-minor of the family $(\phi_j^i)_{j \in J, i \in I_j}$ via K. Let $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n)$ be an $m \times n$ matrix such that $\mathbf{M} \prec \phi$. This implies that there are Skolem maps $\sigma_i \colon V \to A$, $1 \leq i \leq n$, such that for all $j \in J$ we have $\bigl((\mathbf{a}^1 + \sigma_1) h_j, \dots, (\mathbf{a}^n + \sigma_n) h_j\bigr) \prec \phi_j$. This in turn implies that for all $j \in J$ there exist Skolem maps $\sigma_j^p \colon V_j \to A$, $1 \leq p \leq n$, such that for all $i \in I_j$ we have

$$\bigl(\bigl((\mathbf{a}^1 + \sigma_1) h_j + \sigma_j^1\bigr) h_j^i, \dots, \bigl((\mathbf{a}^n + \sigma_n) h_j + \sigma_j^n\bigr) h_j^i\bigr) \prec \phi_j^i.$$

Define the Skolem maps $\tau_p \colon U \to A$, $1 \leq p \leq n$, by $\tau_p := \sigma_p + \sum_{q \in J} \sigma_q^p$. Then for every $j \in J$ and $i \in I_j$, we have for $1 \leq p \leq n$,

(4.1)
$$\begin{aligned}
(\mathbf{a}^p + \tau_p) k_j^i &= (\mathbf{a}^p + \sigma_p + \sum_{q \in J} \sigma_q^p)(h_j + \iota_{UV_j}) h_j^i \\
&= \Bigl((\mathbf{a}^p + \sigma_p) h_j + \bigl(\sum_{q \in J} \sigma_q^p\bigr) h_j + (\mathbf{a}^p + \sigma_p)\iota_{UV_j} + \bigl(\sum_{q \in J} \sigma_q^p\bigr)\iota_{UV_j}\Bigr) h_j^i \\
&= \bigl((\mathbf{a}^p + \sigma_p) h_j + \sigma_j^p\bigr) h_j^i,
\end{aligned}$$

and hence

$$\big((\mathbf{a}^1 + \tau_1)k_j^i, \dots, (\mathbf{a}^n + \tau_n)k_j^i\big) \prec \phi_j^i.$$

Second, we need to show that S is an extensive conjunctive ∞-minor of the family $(S_j^i)_{j \in J, i \in I_j}$ via K. Let $\mathbf{b} \in B^m$ and assume that there is a Skolem map $\tau \colon U \to B$ such that for every $j \in J$ and $i \in I_j$, the n_j^i-tuple $(\mathbf{b} + \sigma)k_j^i$ is in S_j^i. We need to show that $\mathbf{b} \in S$. Define the Skolem maps $\sigma \colon V \to B$ and $\sigma_j \colon V_j \to B$ for every $j \in J$ such that each of these functions coincides with the restriction of τ to the respective domain, i.e., $\tau = \sigma + \sum_{j \in J} \sigma_j$. As in (4.1), we can derive

$$(\mathbf{b} + \tau)k_j^i = \big((\mathbf{b} + \sigma)h_j + \sigma_j\big)h_j^i.$$

Since S_j is an extensive conjunctive ∞-minor of the family $(S_j^i)_{j \in J, i \in I_j}$ via the scheme H_j, we have $(\mathbf{b} + \sigma)h_j \in S_j$. Since the condition $(\mathbf{b} + \sigma)h_j \in S_j$ holds for all $j \in J$ and S is an extensive conjunctive ∞-minor of the family $(S_j)_{j \in J}$ via H, we have that $\mathbf{b} \in S$. □

For a set $\mathcal{G}$ of generalized constraints from A to B of arbitrary, possibly infinite arities, we denote by $\mathcal{G}^\infty$ the set of those generalized constraints which are conjunctive ∞-minors of families of members of $\mathcal{G}$. This set $\mathcal{G}^\infty$ is the smallest set of generalized constraints containing $\mathcal{G}$ which is closed under formation of conjunctive ∞-minors, and it is called the *conjunctive ∞-minor closure* of $\mathcal{G}$. In the sequel, we will make use of the following corollary of Lemma 4.7:

Corollary 4.8. *Let $\mathcal{G} \subseteq \mathcal{G}_{AB}$ be a set of finitary generalized constraints, and let $\mathcal{G}^\infty$ be its conjunctive ∞-minor closure. If $\mathcal{G}$ is closed under formation of conjunctive minors, then $\mathcal{G}$ is the set of all finitary generalized constraints belonging to $\mathcal{G}^\infty$.*

Lemma 4.9 (Hellerstein [6])**.** *Let S be a finite set. Let $\mathcal{Q}$ be a set of functions $\phi \colon S \to \omega \cup \{\omega\}$. Then the number of maximal elements of $\mathcal{Q}$ is finite.*

Lemma 4.10 (Hellerstein [6])**.** *Let S be a finite set. Let $\mathcal{Q}$ be a set of functions $\phi \colon S \to \omega \cup \{\omega\}$ such that $\mathcal{Q}$ contains the limits of all sequences of functions in $\mathcal{Q}$. Then for each element $\phi \in \mathcal{Q}$, there exists a maximal element ϕ' of $\mathcal{Q}$ such that $\phi \leq \phi'$.*

Lemma 4.11. *Let $\mathcal{G} \subseteq \mathcal{G}_{AB}$ be a locally closed set of finitary generalized constraints which contains the binary generalized equality constraint and the unary generalized empty constraint and is closed under formation of conjunctive minors and taking the limit of antecedents. Let $\mathcal{G}^\infty$ be the conjunctive ∞-minor closure of $\mathcal{G}$. Let $(\phi, S) \in \mathcal{G}_{AB} \setminus \mathcal{G}$ be finitary. Then there exists a function in $\mathcal{F}_{AB}$ which preserves every generalized constraint in $\mathcal{G}^\infty$ but does not preserve (ϕ, S).*

Proof. We shall construct a function g which preserves all generalized constraints in $\mathcal{G}^\infty$ but does not preserve (ϕ, S).

Note that, by Corollary 4.8, (ϕ, S) cannot be in $\mathcal{G}^\infty$. Let m be the arity of (ϕ, S). Since $\mathcal{G}$ is locally closed and $(\phi, S) \notin \mathcal{G}$, there exists a repetition function $\phi_1 \colon A^m \to \omega \cup \{\omega\}$ such that $\phi_1 \leq \phi$, the set $F := \{\mathbf{a} \in A^m \mid \phi_1(\mathbf{a}) \neq 0\}$ is finite and $(\phi_1, S) \notin \mathcal{G}$. Observe that $S \neq B^m$, because otherwise (ϕ, S) would be a conjunctive minor of the m-ary generalized trivial constraint (by restricting the antecedent), which is in $\mathcal{G}$ by Lemma 4.5. Also, ϕ_1 is not identically 0, because otherwise (ϕ_1, S) would be a conjunctive minor of the m-ary generalized empty constraint (by extending the consequent), which is in $\mathcal{G}$ by Lemma 4.5. The set $\mathcal{G}$

cannot contain $(\phi_1, B^m \setminus \{\mathbf{s}\})$ for every $\mathbf{s} \in B^m \setminus S$, because if it did, then (ϕ_1, S) would be a conjunctive minor of the family $(\phi_1, B^m \setminus \{\mathbf{s}\})_{\mathbf{s} \in B^m \setminus S}$ (by intersecting consequents). Choose some $\mathbf{s} \in B^m \setminus S$ such that $(\phi_1, B^m \setminus \{\mathbf{s}\}) \notin \mathcal{G}$.

Consider the set Q consisting of all repetition functions $\phi' \colon A^m \to \omega \cup \{\omega\}$ such that the restriction of ϕ' to $A^m \setminus F$ is identically 0 and $(\phi', B^m \setminus \{\mathbf{s}\}) \in \mathcal{G}$. By the assumption that $\mathcal{G}$ is closed under taking the limit of antecedents, Q contains the limits of all sequences of functions in Q. Since the functions in Q are completely determined by their restrictions to the finite set F (they are all identically 0 outside of F), we can apply Lemmas 4.9 and 4.10 to conclude that the set $Q_{\max}$ of maximal elements in Q is finite, and for all $\phi' \in Q$, there exists a $\phi'' \in Q_{\max}$ such that $\phi' \leq \phi''$.

Note that $(\phi_1, B^m \setminus \{\mathbf{s}\}) \notin \mathcal{G}$, and for all $\phi'' \in Q_{\max}$, $(\phi'', B^m \setminus \{\mathbf{s}\}) \in \mathcal{G}$. Therefore, for all $\phi'' \in Q_{\max}$, $\phi'' \ngeq \phi_1$.

The set Q (and hence the set $Q_{\max}$) is not empty, because $\mathcal{G}$ contains $(\phi', B^m \setminus \{\mathbf{s}\})$ where ϕ' is identically 0, which is a conjunctive minor of the m-ary generalized empty constraint (by extending the consequent), which is in $\mathcal{G}$ by Lemma 4.5.

Let $X := \{\mathbf{a} \in A^m \mid \phi_1(\mathbf{a}) \neq \omega\}$. Define $\beta \colon A^m \to \omega \cup \{\omega\}$ by the rule: for all $\mathbf{a} \in A^m$,

- $\beta(\mathbf{a}) = \phi_1(\mathbf{a}), \quad$ if $\mathbf{a} \in X$,
- $\beta(\mathbf{a}) = 0, \quad$ if $\mathbf{a} \notin X$ and $\phi''(\mathbf{a}) = \omega$ for all $\phi'' \in Q_{\max}$,
- $\beta(\mathbf{a}) = \max\{\phi''(\mathbf{a}) + 1 \mid \phi'' \in Q_{\max}$ such that $\phi''(\mathbf{a}) \neq \omega\}, \quad$ otherwise.

In the third case, the value of β is finite because $Q_{\max}$ is a finite set.

We claim that $(\beta, B^m \setminus \{\mathbf{s}\}) \notin \mathcal{G}$. To prove the claim, consider any $\phi'' \in Q_{\max}$. Since $\phi'' \ngeq \phi_1$, there exists an $\mathbf{a} \in A^m$ such that $\phi''(\mathbf{a}) < \phi_1(\mathbf{a})$, and hence $\phi''(\mathbf{a}) < \beta(\mathbf{a})$. Thus, there is no $\phi'' \in Q_{\max}$ such that $\beta \leq \phi''$, implying that $\beta \notin Q$. Therefore, $(\beta, B^m \setminus \{\mathbf{s}\}) \notin \mathcal{G}$.

Let $n := \sum_{\mathbf{a} \in A^m} \beta(\mathbf{a})$. Consider any $\phi'' \in Q_{\max}$. Because $\phi'' \ngeq \phi_1$, there exists an $\mathbf{a} \in A^m$ such that $\phi''(\mathbf{a}) < \phi_1(\mathbf{a})$ and hence $\beta(\mathbf{a}) > 0$. Therefore $n > 0$.

Let $\mathbf{D} := (\mathbf{d}^1, \dots, \mathbf{d}^n)$ be an $m \times n$ matrix whose columns consist of $\beta(\mathbf{a})$ copies of $\mathbf{a}$ for each $\mathbf{a} \in A^m$. Let $\mathbf{M} := (\mathbf{m}^1, \dots, \mathbf{m}^n)$ be a $\mu \times n$ matrix whose first m rows are the rows of $\mathbf{D}$ (i.e., $\big(\mathbf{m}^1(i), \dots, \mathbf{m}^n(i)\big) = \big(\mathbf{d}^1(i), \dots, \mathbf{d}^n(i)\big)$ for every $i \in m$) and whose other rows are the remaining distinct n-tuples in A^n; every n-tuple in A^n is a row of $\mathbf{M}$, and any repetition of rows can only occur in the first m rows of $\mathbf{M}$. Note that $m \leq \mu$ and that μ is infinite if and only if A is infinite. Let $\chi_{\mathbf{M}}$ be the characteristic function of $\mathbf{M}$, and let $S_{\mathbf{M}}$ be the μ-ary relation consisting of those μ-tuples $\mathbf{b} := (b_t \mid t \in \mu)$ in B^μ such that $(b_t \mid t \in m)$ belongs to $B^m \setminus \{\mathbf{s}\}$.

Observe that $(\chi_{\mathbf{M}}, S_{\mathbf{M}}) \notin \mathcal{G}^\infty$, because $(\beta, B^m \setminus \{\mathbf{s}\})$ is a simple ∞-minor of $(\chi_{\mathbf{M}}, S_{\mathbf{M}})$, and if $(\chi_{\mathbf{M}}, S_{\mathbf{M}}) \in \mathcal{G}^\infty$, we would conclude, from Corollary 4.8, that $(\beta, B^m \setminus \{\mathbf{s}\}) \in \mathcal{G}$. Furthermore, there must exist a μ-tuple $\mathbf{u} := (u_t \mid t \in \mu)$ in B^μ such that $(u_t \mid t \in m) = \mathbf{s}$ and $(\chi_{\mathbf{M}}, B^\mu \setminus \{\mathbf{u}\}) \notin \mathcal{G}^\infty$; otherwise by arbitrary intersections of consequents we would conclude that $(\chi_{\mathbf{M}}, S_{\mathbf{M}}) \in \mathcal{G}^\infty$.

We can define a function $g \colon A^n \to B$ by the condition $g\mathbf{M} = \mathbf{u}$. This definition is valid, because the set of rows of $\mathbf{M}$ is the set of all n-tuples in A^n, and if two rows of $\mathbf{M}$ coincide, then the corresponding components of $\mathbf{u}$ also coincide. For, suppose, on the contrary, that $\big(\mathbf{m}^1(i), \dots, \mathbf{m}^n(i)\big) = \big(\mathbf{m}^1(j), \dots, \mathbf{m}^n(j)\big)$ but $\mathbf{u}(i) \neq \mathbf{u}(j)$.

Consider the μ-ary generalized constraint $(\phi^=, S^=)$ from A to B defined by

$$\phi^=(\mathbf{a}) = \begin{cases} \omega, & \text{if } a_i = a_j, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad S^= = \{(b_t \mid t \in \mu) \in B^\mu \mid b_i = b_j\}.$$

The generalized constraint $(\phi^=, S^=)$ is a simple ∞-minor of the binary generalized equality constraint and therefore belongs to $\mathcal{G}^\infty$. On the other hand, $(\chi_\mathbf{M}, B^\mu \setminus \{\mathbf{u}\})$ is a relaxation of $(\phi^=, S^=)$ and should also belong to $\mathcal{G}^\infty$, yielding the intended contradiction.

By the definition of $\mathbf{u}$, $g \not\triangleright (\chi_\mathbf{M}, S_\mathbf{M})$, and it is easily seen that $g \not\triangleright (\beta, B^m \setminus \{\mathbf{s}\})$. Since $\mathbf{N} \prec \phi$, we also have that $g \not\triangleright (\phi, S)$.

We then show that g preserves every generalized constraint in $\mathcal{G}^\infty$. Suppose, on the contrary, that there is a ρ-ary generalized constraint $(\phi_0, S_0) \in \mathcal{G}^\infty$, possibly infinitary, which is not preserved by g. Thus, for some $\rho \times n$ matrix $\mathbf{M}_0 := (\mathbf{c}^1, \dots, \mathbf{c}^n) \prec \phi_0$ we have $g\mathbf{M}_0 \notin S_0$. Define $h \colon \rho \to \mu$ to be any map such that

$$\bigl(\mathbf{c}^1(i), \dots, \mathbf{c}^n(i)\bigr) = \bigl((\mathbf{m}^1 h)(i), \dots, (\mathbf{m}^n h)(i)\bigr)$$

for every $i \in \rho$, i.e., row i of $\mathbf{M}_0$ is the same as row $h(i)$ of $\mathbf{M}$, for each $i \in \rho$. Let (ϕ_h, S_h) be a μ-ary simple ∞-minor of (ϕ_0, S_0) via $H := \{h\}$. Note that $(\phi_h, S_h) \in \mathcal{G}^\infty$.

We claim that $\chi_\mathbf{M} \leq \phi_h$. This will follow if we show that $\mathbf{M} \prec \phi_h$. By the definition of simple ∞-minor, it is enough to show that $(\mathbf{m}^1 h, \dots, \mathbf{m}^n h) \prec \phi_0$. In fact, we have, for $1 \leq j \leq n$,

$$\mathbf{m}^j h = (\mathbf{m}^j h(i) \mid i \in \rho) = (\mathbf{c}^j(i) \mid i \in \rho) = \mathbf{c}^j,$$

and $(\mathbf{c}^1, \dots, \mathbf{c}^n) \prec \phi_0$.

Next we claim that $B^\mu \setminus \{\mathbf{u}\} \supseteq S_h$, i.e., $\mathbf{u} \notin S_h$. By the definition of simple ∞-minor, it is enough to show that $\mathbf{u}h \notin S_0$. For every $i \in \rho$ we have

$$\begin{aligned} (\mathbf{u}h)(i) &= \bigl(g(\mathbf{m}^1, \dots, \mathbf{m}^n)h\bigr)(i) \\ &= g\bigl((\mathbf{m}^1 h)(i), \dots, (\mathbf{m}^n h)(i)\bigr) = g\bigl(\mathbf{c}^1(i), \dots, \mathbf{c}^n(i)\bigr). \end{aligned}$$

Thus $\mathbf{u}h = g\mathbf{M}_0$. Since $g\mathbf{M}_0 \notin S_0$, we conclude that $\mathbf{u} \notin S_h$.

So $(\chi_M, B^\mu \setminus \{\mathbf{u}\})$ is a relaxation of (ϕ_h, S_h), and we conclude that $(\chi_M, B^\mu \setminus \{\mathbf{u}\}) \in \mathcal{G}^\infty$. By the definition of $\mathbf{u}$, this is impossible, and we have reached a contradiction. □

Proof of Theorem 4.6. (ii) ⇒ (i): It is obvious that every function in $\mathcal{F}_{AB}$ preserves the generalized equality and empty constraints. It follows from Lemma 4.4 that if a function preserves every member of a nonempty family $(\phi_j, S_j)_{j \in J}$ of generalized constraints, then it preserves every conjunctive minor of the family. It is also clear that if a function preserves every member of a family $(\phi_i, S)_{i \in \omega}$ of generalized constraints such that $\phi_i \leq \phi_{i+1}$ for all $i \in \omega$, then it also preserves $(\lim_{i \to \infty} \phi_i, S)$.

It remains to show that $\mathcal{G}$ is locally closed. It is clear that $\mathcal{G}_{AB}$ is locally closed, so we can assume that $\mathcal{G} \neq \mathcal{G}_{AB}$. Suppose on the contrary that there is a generalized constraint $(\phi, S) \in \mathcal{G}_{AB} \setminus \mathcal{G}$, say of arity m, such that every generalized constraint obtained from (ϕ, S) by a finite restriction of the antedecent is in $\mathcal{G}$. By (ii), there is a function $f \in \mathcal{F}_{AB}^{(n)}$ that preserves every generalized constraint in $\mathcal{G}$ but does not preserve (ϕ, S). Thus, there is an $m \times n$ matrix $\mathbf{M} \prec \phi$ such that $f\mathbf{M} \notin S$. The generalized constraint $(\chi_\mathbf{M}, S)$ is obtained from (ϕ, S) by a finite restriction of the antecedent, and hence $(\chi_\mathbf{M}, S) \in \mathcal{G}$ by our assumption. We have that $\mathbf{M} \prec \chi_\mathbf{M}$ but

$f\mathbf{M} \notin S$, which is a contradiction to the fact that $f \rhd (\chi_{\mathbf{M}}, S)$. This completes the proof of the implication (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): By Lemma 4.11, for every generalized constraint $(\phi, S) \in \mathcal{G}_{AB} \setminus \mathcal{G}$, there is a function which preserves every generalized constraint in $\mathcal{G}$ but does not preserve (ϕ, S). The set of all such "separating" functions, for each $(\phi, S) \in \mathcal{G}_{AB} \setminus \mathcal{G}$, characterizes $\mathcal{G}$. □

5. Sets of operations closed under permutation of variables, addition of dummy variables, and composition

We now consider the problem of characterizing the sets of operations on an arbitrary nonempty set A that are closed under permutation of variables, addition of dummy variables, and composition (but not necessarily under identification of variables). In other words, we are going to characterize the subalgebras of $(\mathcal{O}_A; \zeta, \tau, \nabla, *)$. We confine ourselves to dealing only with sets that contain all projections. Of course, every clone on A is such a closed set. Examples of sets that are closed under the operations considered but not under identification of variables are given below.

Example 5.1. Let $\mathcal{C}$ be a clone on A, and let $m \geq 2$ be an integer. The set $\mathcal{C}^{(\geq m)} := \bigcup_{n \geq m} \mathcal{C}^{(n)}$ is clearly closed under permutation of variables, addition of dummy variables and composition, but it is not closed under identification of variables. Note, however, that $\mathcal{C}^{(\geq m)}$ does not contain all projections – the unary projections are missing.

Example 5.2. The set of surjective operations on A is closed under permutation of variables, addition of dummy variables and composition, but it is not closed under identification of variables (unless $|A| = 1$). Consider, for example, the operation $f \colon A^2 \to A$ defined by

$$f(x, y) = \begin{cases} x, & \text{if } x \neq y, \\ a, & \text{if } x = y, \end{cases}$$

where $a \in A$ is a fixed element. Clearly f is surjective, but Δf is a constant map and hence not surjective (unless $|A| = 1$).

Example 5.3. Let $(A; \leq)$ be a partially ordered set. We say that an operation $f \in \mathcal{O}_A^{(n)}$ is *order-preserving* (with respect to the partial order $\leq$) *in the i-th variable* ($1 \leq i \leq n$), if for all $a_1, \dots, a_n, b \in A$, we have that $a_i \leq b$ implies

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \leq f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

We say that f is *order-reversing in the i-th variable* ($1 \leq i \leq n$), if for all $a_1, \dots, a_n, b \in A$, we have that $a_i \leq b$ implies

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \geq f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

The set $\mathcal{M}_{\leq}$ of all operations on A that are order-preserving or order-reversing with respect to $\leq$ in each variable is clearly closed under permutation of variables, addition of dummy variables (every function is both order-preserving and order-reversing in a dummy variable) and composition. If the poset $(A; \leq)$ contains a three-element chain $a < b < c$, then the class $\mathcal{M}_{\leq}$ is not closed under identification of variables. Consider, for example, the operation $f\colon A^2 \to A$ defined as follows: let $\leq^*$ be a linear extension of $\leq$, and let

$$f(x, y) = \begin{cases} b, & \text{if } b \leq^* x \text{ and } y \leq^* b, \\ a, & \text{otherwise.} \end{cases}$$

It is easy to verify that f is order-preserving in the first variable and order-reversing in the second variable. However, the operation Δf, given by

$$(\Delta f)(x) = \begin{cases} b, & \text{if } x = b, \\ a, & \text{otherwise,} \end{cases}$$

is neither order-preserving nor order-reversing in its only variable.

Example 5.4. Let $(A; +, \cdot)$ be a field, and for a fixed integer $p \geq 2$, let $\mathcal{L}_p$ be the class of all linear functions $f(x_1, \dots, x_n) = \sum_{i=1}^{n} c_i x_i$, where $c_i \in A$ for $1 \leq i \leq n$, such that $|\{i \mid c_i \neq 0\}| \equiv 1 \pmod{p}$. $\mathcal{L}_p$ is closed under permutation of variables, addition of dummy variables and composition, but it is not closed under identification of variables (unless $p = 2$ and A is a two-element field).

Example 5.5. Let $\mathbf{A} := (A; F)$ be an algebra, and let $\mathcal{T}_{\mathbf{A}}^{\text{lin}}$ be the the set of term functions of $\mathbf{A}$ induced by *linear terms,* i.e., terms with no repeated variables; such operations are sometimes called *read-once functions* of the algebra $\mathbf{A}$. It is clear that $\mathcal{T}_{\mathbf{A}}^{\text{lin}}$ is closed under permutation of variables, addition of dummy variables and composition, but it is not in general closed under identification of variables.

A *finite multiset* S on a set A is a map $\nu_S\colon A \to \omega$, called a *multiplicity function,* such that the set $\{x \in A \mid \nu_S(x) \neq 0\}$ is finite. Then the sum $\sum_{x \in A} \nu_S(x)$ is a well-defined natural number, and it is called the *cardinality* of S and denoted by $|S|$. The number $\nu_S(x)$ is called the *multiplicity* of x in S. We may represent a finite multiset S by giving a list enclosed in set brackets, i.e., $\{a_1, \dots, a_n\}$, where each element $x \in A$ occurs $\nu_S(x)$ times. If S' is another multiset on A corresponding to $\nu_{S'}\colon A \to \omega$, then we say that S' is a *submultiset* of S, denoted $S' \subseteq S$, if $\nu_{S'}(x) \leq \nu_S(x)$ for all $x \in A$. We denote the set of all finite multisets on A by $\mathcal{M}(A)$. The set $\mathcal{M}(A)$ is partially ordered by the multiset inclusion relation "$\subseteq$". The *join* $S \uplus S'$ and the *difference* $S \setminus S'$ of multisets S and S' are determined by the multiplicity functions $\nu_{S \uplus S'}(x) := \nu_S(x) + \nu_{S'}(x)$ and $\nu_{S \setminus S'}(x) := \max\{\nu_S(x) - \nu_{S'}(x), 0\}$, respectively. The *empty multiset* on A is the zero function, and it is denoted by ε. A *partition* of a finite multiset S on A is a multiset $\{S_1, \dots, S_n\}$ (on the set of all finite multisets on A) of nonempty finite multisets on A such that $S = S_1 \uplus \cdots \uplus S_n$.

For an $m \times n$ matrix $\mathbf{M} \in A^{m \times n}$, the *multiset of columns* of $\mathbf{M}$ is the multiset $\mathbf{M}^*$ on A^m defined by the characteristic function $\chi_{\mathbf{M}}$ of $\mathbf{M}$, which maps each m-tuple $\mathbf{a} \in A^m$ to the number of times $\mathbf{a}$ occurs as a column of $\mathbf{M}$. A matrix $\mathbf{N} \in A^{m \times n'}$ is a *submatrix* of $\mathbf{M} \in A^{m \times n}$ if $\mathbf{N}^* \subseteq \mathbf{M}^*$, i.e., $\chi_{\mathbf{N}}(\mathbf{a}) \leq \chi_{\mathbf{M}}(\mathbf{a})$ for all $\mathbf{a} \in A^m$.

For an integer $m \geq 1$, an m-ary *cluster* on A is an initial segment Φ of the set $\mathcal{M}(A^m)$ of all finite multisets on A^m, partially ordered by multiset inclusion "$\subseteq$", i.e., a subset Φ of $\mathcal{M}(A^m)$ such that, for all $S, T \in \mathcal{M}(A^m)$, if $S \in \Phi$ and $T \subseteq S$, then also $T \in \Phi$. The number $\max\{|S| \mid S \in \Phi\}$, if it exists, is called the *breadth* of Φ; if the maximum does not exist, then Φ is said to have *infinite breadth.* For $m \geq 1$, we denote

$$\mathcal{K}_A^{(m)} := \{\Phi \in \mathcal{P}(\mathcal{M}(A^m)) \mid \Phi \text{ is an initial segment of } \mathcal{M}(A^m)\}$$

and

$$\mathcal{K}_A := \bigcup_{m \geq 1} \mathcal{K}_A^{(m)}.$$

If $\mathbf{M} \in A^{m \times n}$ and $\Phi \in \mathcal{K}_A^{(m)}$, we write $\mathbf{M} \prec \Phi$ to mean that the multiset $\mathbf{M}^*$ of columns of $\mathbf{M}$ is an element of Φ. If $f \in \mathcal{O}_A^n$ and $\Phi \in \mathcal{K}_A^{(m)}$, we say that f *preserves* Φ, denoted $f \rhd \Phi$, if for every matrix $\mathbf{M} \in A^{m \times p}$ for some $p \geq 0$, it holds that whenever $\mathbf{M} \prec \Phi$ and $\mathbf{M} = [\mathbf{M}_1 | \mathbf{M}_2]$ where $\mathbf{M}_1$ has n columns and $\mathbf{M}_2$ may be empty, we have that $[f\mathbf{M}_1 | \mathbf{M}_2] \prec \Phi$.

In light of Theorem 2.1, the relation $\rhd$ establishes a Galois connection between the sets $\mathcal{O}_A$ and $\mathcal{K}_A$. We say that a set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations on A is *characterized* by a set $\mathcal{K} \subseteq \mathcal{K}_A$ of clusters, if $\mathcal{F} = \{f \in \mathcal{O}_A \mid \forall \Phi \in \mathcal{K} \colon f \rhd \Phi\}$, i.e., $\mathcal{F}$ is precisely the set of operations on A that preserve every cluster in $\mathcal{K}$. Similarly, we say that $\mathcal{K}$ is *characterized* by $\mathcal{F}$, if $\mathcal{K} = \{\Phi \in \mathcal{K}_A \mid \forall f \in \mathcal{F} \colon f \rhd \Phi\}$, i.e., $\mathcal{K}$ is precisely the set of clusters that are preserved by every operation in $\mathcal{F}$. Thus, the Galois closed sets of operations (clusters) are exactly those that are characterized by clusters (operations, respectively).

Remark 5.6. Recall that a finite multiset S on A^m is a map $\nu_S \colon A^m \to \omega$; hence it is in fact an m-ary repetition function on A. Thus, adopting the notation for matrices and repetition functions introduced in Section 3, for a matrix $\mathbf{M} \in A^{m \times n}$, we write $\mathbf{M} \prec S$ to mean that each m-tuple $\mathbf{a} \in A^m$ occurs as a column of $\mathbf{M}$ at most $\nu_S(\mathbf{a})$ times, i.e., $\chi_{\mathbf{M}}(\mathbf{a}) \leq \nu_S(\mathbf{a})$ for all $\mathbf{a} \in A^m$, i.e., $\mathbf{M}^* \subseteq S$.

Remark 5.7. Alternatively, we can define an m-ary cluster Φ on A to be a set of m-ary repetition functions on A. Then $\mathbf{M} \prec \Phi$ means that $\mathbf{M} \prec \phi$ for some $\phi \in \Phi$ (see Section 3). To see that these definitions are equivalent, observe first that every set of finite multisets is in fact itself a set of repetition functions (cf. Remark 5.6). On the other hand, a set Φ_{R} of repetition functions corresponds to the downward closed set Φ_{F} of finite multisets S satisfying $\nu_S(\mathbf{a}) \leq \phi(\mathbf{a})$ for all $\mathbf{a} \in A^m$, for some $\phi \in \Phi_{\mathrm{R}}$. It can be easily shown that if Φ_{F} is a downward closed set of multisets and Φ_{R} is a set of repetition functions such that Φ_{F} and Φ_{R} correspond to each other under the two alternative definitions of cluster, then $\mathbf{M} \prec \Phi_{\mathrm{F}}$ if and only if $\mathbf{M} \prec \Phi_{\mathrm{R}}$.

While we keep to the original definition of cluster when we prove our theorems, we may sometimes find it simpler to represent clusters in terms of repetition functions in the subsequent examples.

Example 5.8. An m-ary relation R on A is equivalent to the m-ary cluster

$$\Phi_R := \{S \in \mathcal{M}(A^m) \mid \forall \mathbf{a} \in A^m \colon [\nu_S(\mathbf{a}) > 0 \Rightarrow \mathbf{a} \in R]\}.$$

Thus every locally closed clone can be characterized by a set of clusters of this kind.

Using the alternative definition of cluster, Φ_R is equivalent to the cluster $\{\phi_R\}$, where the repetition function ϕ_R is defined by the rule $\phi_R(\mathbf{a}) = \omega$ if $\mathbf{a} \in R$ and $\phi_R(\mathbf{a}) = 0$ otherwise.

Example 5.9. Let $\leq$ be a partial order on A. A function $f \colon A^n \to A$ is not order-preserving nor order-reversing in its i-th variable if and only if there exist elements $a_1, \dots, a_n, a'_i, b_1, \dots, b_n, b'_i \in A$ such that $a_i < a'_i$, $b_i < b'_i$ and

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \not\leq f(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n),$$
$$f(b_1, \dots, b_{i-1}, b_i, b_{i+1}, \dots, b_n) \not\geq f(b_1, \dots, b_{i-1}, b'_i, b_{i+1}, \dots, b_n).$$

Thus, it is easy to see that the class of operations on A that are order-preserving or order-reversing in each variable with respect to $\leq$ is characterized by the quaternary cluster $\Phi_{\leq}$ consisting precisely of those finite multisets S on A^4 that satisfy the conditions

- $\nu_S(a, b, c, d) = 0$ whenever $a < b$ and $c > d$, or $a > b$ and $c < d$, or a and b are incomparable, or c and d are incomparable; and
- $\sum_{\mathbf{a} \in X} \nu_S(\mathbf{a}) \leq 1$, where

$$X := \{(a, b, c, d) \in A^4 \mid \\ ((a \leq b) \wedge (c \leq d)) \vee ((a \geq b) \wedge (c \geq d)) \wedge ((a \neq b) \vee (c \neq d))\}.$$

Lemma 5.10. *Let $\mathcal{F} \subseteq \mathcal{O}_A$ be a locally closed set of operations that contains all projections and is closed under permutation of variables, addition of dummy variables, and composition. Then for every $g \in \mathcal{O}_A \setminus \mathcal{F}$, there exists a cluster $\Phi \in \mathcal{K}_A$ that is preserved by every operation in $\mathcal{F}$ but not by g.*

Proof. Since $\mathcal{F}$ contains all projections, $\mathcal{F} \neq \emptyset$. Suppose that $g \in \mathcal{O}_A \setminus \mathcal{F}$ is n-ary. Since $\mathcal{F}$ is locally closed, there is a finite subset $F \subseteq A^n$ such that $g|_F \neq f|_F$ for every $f \in \mathcal{F}^{(n)}$. Clearly F is nonempty. Let $\mathbf{M}$ be a $|F| \times n$ matrix whose rows are the elements of F in some fixed order.

Let X be any submultiset of $\mathbf{M}^*$. (Recall that $\mathbf{M}^*$ denotes the multiset of columns of $\mathbf{M}$.) Let $\Pi := (\mathbf{M}_1, \dots, \mathbf{M}_q)$ be a sequence of submatrices of $\mathbf{M}$ such that $\{\mathbf{M}_1^*, \dots, \mathbf{M}_q^*\}$ is a partition of $\mathbf{M}^* \setminus X$. For $1 \leq i \leq q$, let $\mathbf{d}^i \in \mathcal{F}\mathbf{M}_i$, and let $\mathbf{D} := (\mathbf{d}^1, \dots, \mathbf{d}^q)$. (Note that each $\mathcal{F}\mathbf{M}_i$ is nonempty, because $\mathcal{F}$ contains all projections. Observe also that each $\mathcal{F}\mathbf{M}_i$ is a subset of $\mathcal{F}\mathbf{M}$, because $\mathcal{F}$ is closed under addition of dummy variables.) Denote $\langle X, \Pi, \mathbf{D} \rangle := X \uplus \mathbf{D}^*$.

We define Φ to be the set of all submultisets of the multisets $\langle X, \Pi, \mathbf{D} \rangle$ for all possible choices of X, Π, and $\mathbf{D}$. Observe first that $g \not\triangleright \Phi$. For, it holds that $\mathbf{M} \prec \Phi$, because $\mathbf{M}^* = \langle \mathbf{M}^*, (), () \rangle \in \Phi$. On the other hand, since $g\mathbf{M} \notin \mathcal{F}\mathbf{M}$, we have that $g\mathbf{M} \notin \langle X, \Pi, \mathbf{D} \rangle$ for all X, Π, $\mathbf{D}$, and hence $g\mathbf{M} \not\prec \Phi$.

It remains to show that $f \triangleright \Phi$ for all $f \in \mathcal{F}$. Assume that f is n-ary. If $\mathbf{N} := [\mathbf{N}_1|\mathbf{N}_2] \prec \Phi$, where $\mathbf{N}_1$ has n columns, then $\mathbf{N} \prec \langle X, \Pi, \mathbf{D} \rangle$ for some X, $\Pi := (\mathbf{M}_1, \dots, \mathbf{M}_q)$, $\mathbf{D} := (\mathbf{d}^1, \dots, \mathbf{d}^q)$, where $\{\mathbf{M}_1^*, \dots, \mathbf{M}_q^*\}$ is a partition of $\mathbf{M}^* \setminus X$ and for $1 \leq i \leq q$, $\mathbf{d}^i \in \mathcal{F}\mathbf{M}_i$. We will show by induction on q that there exist X', Π', $\mathbf{D}'$ such that $[f\mathbf{N}_1|\mathbf{N}_2] \prec \langle X', \Pi', \mathbf{D}' \rangle$ and hence $[f\mathbf{N}_1|\mathbf{N}_2] \prec \Phi$.

If $q = 0$, then $X = \mathbf{M}^*$, $\Pi = ()$, $\mathbf{D} = ()$, and $\langle X, \Pi, \mathbf{D} \rangle = \mathbf{M}^*$, and the condition $\mathbf{N} = [\mathbf{N}_1|\mathbf{N}_2] \prec \langle \mathbf{M}^*, (), () \rangle$ means that $\mathbf{N}$ is a submatrix of $\mathbf{M}$. Then $f\mathbf{N}_1 \in \mathcal{F}\mathbf{N}_1$ and $[f\mathbf{N}_1|\mathbf{N}_2] \prec \langle \mathbf{M}^* \setminus \mathbf{N}_1^*, (\mathbf{N}_1), (f\mathbf{N}_1) \rangle$.

Assume that the claim holds for $q = k \geq 0$, and consider the case that $q = k + 1$. Let $\mathbf{N} = [\mathbf{N}_1|\mathbf{N}_2] \prec \langle X, \Pi, \mathbf{D} \rangle$. If $\mathbf{N}_1 \prec X$, then $f\mathbf{N}_1 \in \mathcal{F}\mathbf{N}_1$ and

$$[f\mathbf{N}_1|\mathbf{N}_2] \prec \langle X \setminus \mathbf{N}_1^*, (\mathbf{M}_1, \dots, \mathbf{M}_{k+1}, \mathbf{N}_1), (\mathbf{d}^1, \dots, \mathbf{d}^{k+1}, f\mathbf{N}_1) \rangle.$$

Otherwise, for some $i \in \{1, \dots, k+1\}$, $\mathbf{d}^i$ is a column of $\mathbf{N}_1$. Denote by $\mathbf{N}_1'$ the matrix obtained from $\mathbf{N}_1$ by deleting the column $\mathbf{d}^i$. Since $\mathcal{F}$ is closed under permutation of variables, there is an operation $f' \in \mathcal{F}^{(n)}$ such that $f\mathbf{N}_1 = f'[\mathbf{d}^i|\mathbf{N}_1']$. By the definition of $\mathbf{d}^i$, there is an operation $h \in \mathcal{F}$ such that $h\mathbf{M}_i = \mathbf{d}^i$, and we have that

$$f'[\mathbf{d}^i|\mathbf{N}_1'] = f'[h\mathbf{M}_i|\mathbf{N}_1'] = (f' * h)[\mathbf{M}_i|\mathbf{N}_1'].$$

Since $\mathcal{F}$ is closed under composition, $f' * h \in \mathcal{F}$. Furthermore,

$$[\mathbf{M}_i|\mathbf{N}'_1|\mathbf{N}_2] \prec \langle X \uplus \mathbf{M}_i^*, (\mathbf{M}_1, \dots, \mathbf{M}_{i-1}, \mathbf{M}_{i+1}, \dots, \mathbf{M}_{k+1}), (\mathbf{d}^1, \dots, \mathbf{d}^{i-1}, \mathbf{d}^{i+1}, \dots, \mathbf{d}^{k+1})\rangle.$$

By the induction hypothesis, there exist X', Π', $\mathbf{D}'$ such that

$$\big[(f' * h)[\mathbf{M}_i|\mathbf{N}'_1]\big|\mathbf{N}_2\big] \prec \langle X', \Pi', \mathbf{D}'\rangle,$$

and hence $[f\mathbf{N}_1|\mathbf{N}_2] \prec \langle X', \Pi', \mathbf{D}'\rangle$. □

Theorem 5.11. *Let A be an arbitrary, possibly infinite nonempty set. For any set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations, the following two conditions are equivalent:*

(i) $\mathcal{F}$ is locally closed, contains all projections, and is closed under permutation of variables, addition of dummy variables, and composition.
(ii) $\mathcal{F}$ is characterized by a set $\mathcal{K} \subseteq \mathcal{K}_A$ of clusters.

Proof. (ii) ⇒ (i): It is straightforward to verify that the set of operations preserving a set of clusters is closed under permutation of variables and addition of dummy variables, and it contains all projections. To see that it is closed under composition, let $f \in \mathcal{F}^{(n)}$ and $g \in \mathcal{F}^{(p)}$, and consider $f * g\colon A^{n+p-1} \to A$. Let $\Phi \in \mathcal{K}$, and let $\mathbf{M} := [\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_3] \prec \Phi$, where $\mathbf{M}_1$ has p columns and $\mathbf{M}_2$ has $n-1$ columns. Since $g \rhd \Phi$, we have that $[g\mathbf{M}_1|\mathbf{M}_2|\mathbf{M}_3] \prec \Phi$. Then $[g\mathbf{M}_1|\mathbf{M}_2]$ has n columns, and since $f \rhd \Phi$, we have that $\big[f[g\mathbf{M}_1|\mathbf{M}_2]\big|\mathbf{M}_3\big] \prec \Phi$. But $f[g\mathbf{M}_1|\mathbf{M}_2] = (f * g)[\mathbf{M}_1|\mathbf{M}_2]$, so $\big[(f * g)[\mathbf{M}_1|\mathbf{M}_2]\big|\mathbf{M}_3\big] \prec \Phi$, and we conclude that $f * g \rhd \Phi$.

It remains to show that $\mathcal{F}$ is locally closed. It is clear that $\mathcal{O}_A$ is locally closed, so we may assume that $\mathcal{F} \neq \mathcal{O}_A$. Suppose on the contrary that there is a $g \in \mathcal{O}_A \setminus \mathcal{F}$, say of arity n, such that for every finite subset $F \subseteq A^n$, there is an $f \in \mathcal{F}^{(n)}$ such that $g|_F = f|_F$. Since $\mathcal{F}$ is characterized by $\mathcal{K}$ and $g \notin \mathcal{F}$, there is a cluster $\Phi \in \mathcal{K}$ such that $g \not\rhd \Phi$, and hence for some matrix $\mathbf{M} := [\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$ where $\mathbf{M}_1$ has n columns, we have that $[g\mathbf{M}_1|\mathbf{M}_2] \not\prec \Phi$. Let F be the finite set of rows of $\mathbf{M}_1$. By our assumption, there is an $f \in \mathcal{F}^{(n)}$ such that $g|_F = f|_F$, and hence $f\mathbf{M}_1 = f|_F\mathbf{M}_1 = g|_F\mathbf{M}_1 = g\mathbf{M}_1$, and so $[f\mathbf{M}_1|\mathbf{M}_2] \not\prec \Phi$, which contradicts the fact that $f \rhd \Phi$.

(i) ⇒ (ii): It follows from Lemma 5.10 that for every operation $g \in \mathcal{O}_A \setminus \mathcal{F}$, there exists a cluster $\Phi \in \mathcal{K}_A$ that is preserved by every operation in $\mathcal{F}$ but not by g. The set of all such "separating" clusters, for each $g \in \mathcal{O}_A \setminus \mathcal{F}$, characterizes $\mathcal{F}$. □

6. Closure conditions for clusters

In order to describe the sets of clusters that are characterized by sets of operations, we need to introduce a number of operations on clusters. First, we will adapt Couceiro and Foldes's [2] notion of conjunctive minor to clusters.

Let $H := (h_j)_{j \in J}$ be a minor formation scheme with target m, indeterminate set V, and source family $(n_j)_{j \in J}$ (see the definition in Section 4). Let $(\Phi_j)_{j \in J}$ be a family of clusters on A, each Φ_j of arity n_j, and let Φ be an m-ary cluster on A. We say that Φ is a *conjunctive minor* of the family $(\Phi_j)_{j \in J}$ *via* H, if, for every $m \times n$ matrix $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^n) \in A^{m \times n}$,

$$\mathbf{M} \prec \Phi \iff \big[\exists \sigma_1, \dots, \sigma_n \in A^V \; \forall j \in J\colon \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) \prec \Phi_j\big].$$

If the minor formation scheme $H := (h_j)_{j \in J}$ and the family $(\Phi_j)_{j \in J}$ are indexed by a singleton $J := \{0\}$, then a conjunctive minor Φ of a family consisting of a single cluster Φ_0 is called a *simple minor* of Φ_0.

The formation of conjunctive minors subsumes the formation of simple minors and the intersection of clusters. Simple minors in turn subsume permutation of arguments, projection, identification of arguments, and addition of a dummy argument, operations which can be defined for clusters in an analogous way as for generalized constraints (see Section 4).

Lemma 6.1. *Let Φ be a conjunctive minor of a nonempty family $(\Phi_j)_{j \in J}$ of clusters on A. If $f \colon A^n \to A$ preserves Φ_j for all $j \in J$, then f preserves Φ.*

Proof. Let Φ be an m-ary conjunctive minor of the family $(\Phi_j)_{j \in J}$ via the scheme $H := (h_j)_{j \in J}$, $h_j \colon n_j \to m \cup V$. Let $\mathbf{M} := (\mathbf{a}^1, \dots, \mathbf{a}^{n'})$ be an $m \times n'$ matrix ($n' \geq n$) such that $\mathbf{M} \prec \Phi$, and denote $\mathbf{M}_1 := (\mathbf{a}^1, \dots, \mathbf{a}^n)$, $\mathbf{M}_2 := (\mathbf{a}^{n+1}, \dots, \mathbf{a}^{n'})$, so $\mathbf{M} = [\mathbf{M}_1|\mathbf{M}_2]$. We need to prove that $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$.

Since Φ is a conjunctive minor of $(\Phi_j)_{j \in J}$ via $H := (h_j)_{j \in J}$, there are Skolem maps $\sigma_i \colon V \to A$, $1 \leq i \leq n'$, such that for every $j \in J$, we have

$$\big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^{n'} + \sigma_{n'})h_j\big) \prec \Phi_j.$$

Denote

$$\begin{aligned} \mathbf{M}_1^j &:= \big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big), \\ \mathbf{M}_2^j &:= \big((\mathbf{a}^{n+1} + \sigma_{n+1})h_j, \dots, (\mathbf{a}^{n'} + \sigma_{n'})h_j\big). \end{aligned}$$

By the assumption that $f \rhd \Phi_j$, we have $[f\mathbf{M}_1^j|\mathbf{M}_2^j] \prec \Phi_j$ for each $j \in J$.

Let $\sigma := f(\sigma_1, \dots, \sigma_n)$. We have that, for each $j \in J$,

$$\begin{aligned} (f\mathbf{M}_1 + \sigma)h_j &= \big(f(\mathbf{a}^1, \dots, \mathbf{a}^n) + f(\sigma_1, \dots, \sigma_n)\big)h_j \\ &= \big(f(\mathbf{a}^1 + \sigma_1, \dots, \mathbf{a}^n + \sigma_n)\big)h_j \\ &= f\big((\mathbf{a}^1 + \sigma_1)h_j, \dots, (\mathbf{a}^n + \sigma_n)h_j\big) = f\mathbf{M}_1^j. \end{aligned}$$

Since Φ is a conjunctive minor of $(\Phi_j)_{j \in J}$ via $H = (h_j)_{j \in J}$ and

$$\big((f\mathbf{M}_1 + \sigma)h_j, (\mathbf{a}^{n+1} + \sigma_{n+1})h_j, \dots, (\mathbf{a}^{n'} + \sigma_{n'})h_j\big) = [f\mathbf{M}_1^j|\mathbf{M}_2^j] \prec \Phi_j$$

for each $j \in J$, we have that $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$. Thus $f \rhd \Phi$. ☐

Lemma 6.2. *Let $(\Phi_j)_{j \in J}$ be a nonempty family of m-ary clusters on A. If $f \colon A^n \to A$ preserves Φ_j for all $j \in J$, then f preserves $\bigcup_{j \in J} \Phi_j$.*

Proof. Let $\mathbf{M} := [\mathbf{M}_1|\mathbf{M}_2] \prec \bigcup_{j \in J} \Phi_j$. Then $\mathbf{M} \prec \Phi_j$ for some $j \in J$. By the assumption that $f \rhd \Phi_j$, we have $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi_j$, and hence $[f\mathbf{M}_1|\mathbf{M}_2] \prec \bigcup_{j \in J} \Phi_j$. ☐

The *quotient* of an m-ary cluster Φ on A with a multiset $S \in \mathcal{M}(A^m)$ is defined as

$$\Phi/S := \{S' \in \mathcal{M}(A^m) \mid S \uplus S' \in \Phi\}.$$

It is easy to see that Φ/S is a cluster and $\Phi/S \subseteq \Phi$ for any S.

Lemma 6.3. *Let $\Phi, \Phi' \in \mathcal{K}_A^{(m)}$ and $S \in \mathcal{M}(A^m)$. Then*

(i) *$X \in \Phi/S$ if and only if $X \uplus S \in \Phi$;*

(ii) *$(\Phi \cup \Phi')/S = (\Phi/S) \cup (\Phi'/S)$.*

Proof. (i) Immediate from the definition.

(ii) By part (i) and the definition of union, we have

$$X \in (\Phi \cup \Phi')/S \iff X \uplus S \in \Phi \cup \Phi' \iff X \uplus S \in \Phi \vee X \uplus S \in \Phi' \\ \iff X \in \Phi/S \vee X \in \Phi'/S \iff X \in (\Phi/S) \cup (\Phi'/S).$$

The claimed equality thus follows. □

Lemma 6.4. *Let Φ be an m-ary cluster on A. If $f\colon A^n \to A$ preserves Φ, then f preserves Φ/S for every multiset $S \in \mathcal{M}(A^m)$.*

Proof. Let $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi/S$. Let $\mathbf{N}$ be a matrix such that $\mathbf{N}^* = S$. Then $[\mathbf{M}_1|\mathbf{M}_2|\mathbf{N}] \prec \Phi$. By our assumption that $f \rhd \Phi$, we have $[f\mathbf{M}_1|\mathbf{M}_2|\mathbf{N}] \prec \Phi$. Thus, $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi/S$, and we conclude that $f \rhd \Phi/S$. □

Lemma 6.5. *Assume that Φ is an m-ary cluster on A that contains all multisets on A^m of cardinality at most p. If $f\colon A^n \to A$ preserves all quotients Φ/S where $|S| \geq p$, then f preserves Φ.*

Proof. Let $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$, where $\mathbf{M}_1$ has n columns and $\mathbf{M}_2$ has n' columns. If $n' < p$, then the number of columns of $[f\mathbf{M}_1|\mathbf{M}_2]$ is $n'+1 \leq p$, and hence $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$. Otherwise $n' \geq p$ and, by our assumption, $f \rhd \Phi/\mathbf{M}_2^*$. Thus, since $\mathbf{M}_1 \prec \Phi/\mathbf{M}_2^*$, we have that $f\mathbf{M}_1 \prec \Phi/\mathbf{M}_2^*$. Therefore $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$, and we conclude that $f \rhd \Phi$. □

For $p \geq 0$, the m-ary *trivial cluster of breadth p*, denoted $\Omega_m^{(p)}$, is the set of all finite multisets on A^m of cardinality at most p. The m-ary *empty cluster* on A is the empty set $\emptyset$. Note that $\Omega_m^{(0)} \neq \emptyset$, because the empty multiset ε on A^m is the unique member of $\Omega_m^{(0)}$. The binary *equality cluster* on A, denoted E_2, is the set of all finite multisets S on A^2 for which it holds that $\nu_S(\mathbf{a}) = 0$ whenever $\mathbf{a} = (a, b)$ with $a \neq b$.

For $p \geq 0$, we say that the cluster $\Phi^{(p)} := \Phi \cap \Omega_m^{(p)}$ is obtained from the m-ary cluster Φ by *restricting the breadth to p.*

Lemma 6.6. *Let Φ be an m-ary cluster on A. Then $f\colon A^n \to A$ preserves Φ if and only if f preserves $\Phi^{(p)}$ for all $p \geq 0$.*

Proof. Assume first that $f \rhd \Phi$. Let $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi^{(p)}$. Since $\Phi^{(p)} \subseteq \Phi$, we have that $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$, and hence $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$ by our assumption. The number of columns of $[f\mathbf{M}_1|\mathbf{M}_2]$ is at most p, so we have that $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi^{(p)}$. Thus, $f \rhd \Phi^{(p)}$.

Assume then that $f \rhd \Phi^{(p)}$ for all $p \geq 0$. Let $\mathbf{M} := [\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$, and let q be the number of columns in $\mathbf{M}$. Then $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi^{(q)}$, and hence $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi^{(q)}$ by our assumption. Since $\Phi^{(p)} \subseteq \Phi$, we have that $[f\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$, and we conclude that $f \rhd \Phi$. □

We say that a set $\mathcal{K} \subseteq \mathcal{K}_A$ of clusters is *closed under quotients,* if for any $\Phi \in \mathcal{K}$, every quotient Φ/S is also in $\mathcal{K}$. We say that $\mathcal{K}$ is *closed under dividends,* if for every cluster $\Phi \in \mathcal{K}_A$, say of arity m, it holds that $\Phi \in \mathcal{K}$ whenever $\Omega_m^{(p)} \subseteq \Phi$ and $\Phi/S \in \mathcal{K}$ for every multiset S on A^m of cardinality at least p. We say that $\mathcal{K}$ is *locally closed,* if $\Phi \in \mathcal{K}$ whenever $\Phi^{(p)} \in \mathcal{K}$ for all $p \geq 0$. We say than $\mathcal{K}$ is *closed under unions,* if $\bigcup_{j \in J} \Phi_j \in \mathcal{K}$ whenever $(\Phi_j)_{j \in J}$ is a nonempty family of m-ary clusters in $\mathcal{K}$. We say that $\mathcal{K}$ is *closed under formation of conjunctive minors*, if all conjunctive minors of nonempty families of members of $\mathcal{K}$ are members of $\mathcal{K}$.

Theorem 6.7. *Let A be an arbitrary, possibly infinite nonempty set. For any set $\mathcal{K} \subseteq \mathcal{K}_A$ of clusters on A, the following two conditions are equivalent:*

(i) $\mathcal{K}$ is locally closed and contains the binary equality cluster, the unary empty cluster, and all unary trivial clusters of breadth $p \geq 0$, and it is closed under formation of conjunctive minors, unions, quotients, and dividends.

(ii) $\mathcal{K}$ is characterized by some set $\mathcal{F} \subseteq \mathcal{O}_A$ of operations.

Similarly to the proof of Theorem 4.6, we extend the notion of cluster to arbitrary, possibly infinite arities. This extended notion has no bearing on Theorem 6.7 but is only used as a tool in its proof. We use the terms *conjunctive ∞-minor* and *simple ∞-minor* to refer to conjunctive minors and simple minors via a scheme whose target and source ordinals may be infinite or finite. The use of the term "minor" without the prefix "∞" continues to mean the respective minor via a scheme whose target and source ordinals are all finite. Matrices can also have infinitely many rows but only a finite number of columns.

For a set $\mathcal{K}$ of clusters on A of arbitrary, possibly infinite arities, we denote by $\mathcal{K}^\infty$ the set of those clusters which are conjunctive ∞-minors of families of members of $\mathcal{K}$. This set $\mathcal{K}^\infty$ is the smallest set of clusters containing $\mathcal{K}$ which is closed under formation of conjunctive ∞-minors, and it is called the *conjunctive ∞-minor closure* of $\mathcal{K}$. Considering the formation of repeated conjunctive ∞-minors, we can show that the following analogues of Lemma 4.7 and Corollary 4.8 hold.

Lemma 6.8. *If Φ is a conjunctive ∞-minor of a nonempty family $(\Phi_j)_{j \in J}$ of clusters on A via the scheme H, and, for each $j \in J$, Φ_j is a conjunctive ∞-minor of a nonempty family $(\Phi_j^i)_{i \in I_j}$ via the scheme H_j, then Φ is a conjunctive ∞-minor of the nonempty family $(\Phi_j^i)_{j \in J,\, i \in I_j}$ via the composite scheme $K := H(H_j \mid j \in J)$.*

Corollary 6.9. *Let $\mathcal{K} \subseteq \mathcal{K}_A$ be a set of finitary clusters, and let $\mathcal{K}^\infty$ be its conjunctive ∞-minor closure. If $\mathcal{K}$ is closed under formation of conjunctive minors, then $\mathcal{K}$ is the set of all finitary clusters belonging to $\mathcal{K}^\infty$.*

Lemma 6.10. *Let A be an arbitrary, possibly infinite nonempty set. Let $\mathcal{K} \subseteq \mathcal{K}_A$ be a locally closed set of finitary clusters that contains the binary equality cluster, the unary empty cluster, and all unary trivial clusters of breadth $p \geq 0$, and is closed under formation of conjunctive minors, unions, quotients, and dividends. Let $\mathcal{K}^\infty$ be the conjunctive ∞-minor closure of $\mathcal{K}$. Let $\Phi \in \mathcal{K}_A \setminus \mathcal{K}$ be finitary. Then there exists a function in $\mathcal{O}_A$ which preserves every cluster in $\mathcal{K}^\infty$ but does not preserve Φ.*

Proof. We shall construct a function g that preserves all clusters in $\mathcal{K}^\infty$ but does not preserve Φ.

Note that, by Corollary 6.9, Φ cannot be in $\mathcal{K}^\infty$. Let m be the arity of Φ. Since $\mathcal{K}$ is locally closed and $\Phi \notin \mathcal{K}$, there is an integer p such that $\Phi^{(p)} := \Phi \cap \Omega_m^{(p)} \notin \mathcal{K}$; let n be the smallest such integer. By Lemma 6.6, every function not preserving $\Phi^{(n)}$ does not preserve Φ either, so we can consider $\Phi^{(n)}$ instead of Φ. Due to the minimality of n, the breadth of $\Phi^{(n)}$ is n. Observe that Φ is not the trivial cluster of breadth n nor the empty cluster, because these are members of $\mathcal{K}$. Thus, $n \geq 1$.

We can assume that Φ is a minimal nonmember of $\mathcal{K}$ with respect to identification of rows, i.e., every simple minor of Φ obtained by identifying some rows of Φ is a member of $\mathcal{K}$. If this is not the case, then we can identify some rows of Φ to obtain a minimal nonmember Φ' of $\mathcal{K}$ and consider the cluster Φ' instead of Φ. Note that by Lemma 6.1, every function not preserving Φ' does not preserve Φ either.

We can also assume that Φ is a minimal nonmember of $\mathcal{K}$ with respect to taking quotients, i.e., whenever $S \neq \varepsilon$, we have that $\Phi/S \in \mathcal{K}$. If this is not the case, then consider a minimal nonmember Φ/S of $\mathcal{K}$ instead of Φ. By Lemma 6.4, every function not preserving Φ/S does not preserve Φ either.

The fact that Φ is a minimal nonmember of $\mathcal{K}$ with respect to taking quotients implies that $\Omega_m^{(1)} \not\subseteq \Phi$. For, suppose, on the contrary, that $\Omega_m^{(1)} \subseteq \Phi$. Since all quotients Φ/S where $|S| \geq 1$ are in $\mathcal{K}$ and $\mathcal{K}$ is closed under dividends, we have that $\Phi \in \mathcal{K}$, a contradiction.

Let $\Psi := \bigcup\{\Phi' \in \mathcal{T} \mid \Phi' \subseteq \Phi\}$, i.e., Ψ is the largest cluster in $\mathcal{T}$ such that $\Psi \subseteq \Phi$. Note that this is not the empty union, because the empty cluster is a member of $\mathcal{K}$. It is clear that $\Psi \neq \Phi$. Since n was chosen to be the smallest integer satisfying $\Phi^{(n)} \notin \mathcal{K}$, we have that $\Phi^{(n-1)} \in \mathcal{K}$ and since $\Phi^{(n-1)} \subseteq \Phi^{(n)}$, it holds that $\Phi^{(n-1)} \subseteq \Psi$. Thus there is a multiset $Q \in \Phi \setminus \Psi$ with $|Q| = n$. Let $\mathbf{D} := (\mathbf{d}^1, \dots, \mathbf{d}^n)$ be an $m \times n$ matrix whose multiset of columns equals Q.

The rows of $\mathbf{D}$ are pairwise distinct. Suppose, for the sake of contradiction, that rows i and j of $\mathbf{D}$ coincide. Since Φ is a minimal nonmember of $\mathcal{K}$ with respect to identification of rows, by identifying rows i and j of Φ, we obtain a cluster Φ' that is in $\mathcal{K}$. By adding a dummy row in the place of the row that got deleted when we identified rows i and j, and finally by intersecting with the conjunctive minor of the binary equality cluster whose rows i and j are equal (the overall effect of the operations performed above is the selection of exactly those multisets in Φ whose rows i and j coincide), we obtain a cluster in $\mathcal{K}$ that contains Q and is a subset of Φ. But this is impossible by the choice of Q.

Let $\Upsilon := \bigcap\{\Phi' \in \mathcal{K} \mid Q \in \Phi'\}$, i.e., Υ is the smallest cluster in $\mathcal{K}$ that contains Q as an element. Note that this is not the empty intersection, because the trivial cluster $\Omega_m^{(n)}$ is a member $\mathcal{K}$ and contains Q. By the choice of Q, $\Upsilon \not\subseteq \Phi$.

Let $\hat{\Phi} := \Phi \cup \Omega_m^{(1)}$. We claim that for $S \neq \varepsilon$, $\hat{\Phi}/S = \Phi/S$ or $\hat{\Phi}/S = \Omega_m^{(0)} = \{\varepsilon\}$. For, we have $(\Phi \cup \Omega_m^{(1)})/S = (\Phi/S) \cup (\Omega_m^{(1)}/S)$ by Lemma 6.3. If $|S| > 1$, then $\Omega_m^{(1)}/S = \emptyset$ and hence we have $\hat{\Phi}/S = \Phi/S$ in this case. If $|S| = 1$, then $\Omega_m^{(1)}/S = \{\varepsilon\}$, and hence $\hat{\Phi}/S = \Phi/S \cup \{\varepsilon\}$. Since an m-ary cluster is an initial segment of $\mathcal{M}(A^m)$, we have either $\Phi/S = \emptyset$ or $\varepsilon \in \Phi/S$. In the former case, we have $\hat{\Phi}/S = \{\varepsilon\}$, and in the latter case we have $\hat{\Phi}/S = \Phi/S$. Thus, the claim follows.

Since Φ is a minimal nonmember of $\mathcal{K}$ with respect to quotients and $\Omega_m^{(0)} \in \mathcal{K}$, by the above claim we have that $\hat{\Phi}/S \in \mathcal{K}$ whenever $|S| \geq 1$. Since $\mathcal{K}$ is closed under dividends, we have that $\hat{\Phi} \in \mathcal{K}$, and hence $\Upsilon \subseteq \hat{\Phi}$. Thus, there exists an m-tuple $\mathbf{s} \in A^m$ such that $\{\mathbf{s}\} \in \Upsilon \setminus \Phi$.

Let $\mathbf{M} := (\mathbf{m}^1, \dots, \mathbf{m}^n)$ be a $\mu \times n$ matrix whose first m rows are the rows of $\mathbf{D}$ (i.e., $\bigl(\mathbf{m}^1(i), \dots, \mathbf{m}^n(i)\bigr) = \bigl(\mathbf{d}^1(i), \dots, \mathbf{d}^n(i)\bigr)$ for every $i \in m$) and whose other rows are the remaining distinct n-tuples in A^n; every n-tuple in A^n is a row of $\mathbf{M}$ and there is no repetition of rows in $\mathbf{M}$. Note that $m \leq \mu$ and μ is infinite if and only if A is infinite.

Let $\Theta := \bigcap\{\Phi' \in \mathcal{T}^\infty \mid \mathbf{M} \prec \Phi'\}$. There must exist a μ-tuple $\mathbf{u} := (u_t \mid t \in \mu)$ in A^μ such that $\mathbf{u}(i) = \mathbf{s}(i)$ for all $i \in m$ and $\{\mathbf{u}\} \in \Theta$. For, if this is not the case, then the projection of Θ to its first m coordinates would be a member of $\mathcal{K}$ containing Q but not containing $\{\mathbf{s}\}$, contradicting the choice of $\mathbf{s}$.

We can now define a function $g\colon A^n \to A$ by the rule $g\mathbf{M} = \mathbf{u}$. The definition is valid, because every n-tuple in A^n occurs exactly once as a row of $\mathbf{M}$. It is clear that $g \not\triangleright \Phi$, because $\mathbf{D} \prec \Phi$ but $g\mathbf{D} = \mathbf{s} \not\prec \Phi$.

We need to show that every cluster in $\mathcal{K}^\infty$ is preserved by g. Suppose, on the contrary, that there is a ρ-ary cluster $\Phi_0 \in \mathcal{K}^\infty$, possibly infinitary, which is not preserved by g. Thus, for some $\rho \times n'$ matrix $\mathbf{N} := (\mathbf{c}^1, \dots, \mathbf{c}^{n'}) \prec \Phi_0$, with $\mathbf{N}_0 := (\mathbf{c}^1, \dots, \mathbf{c}^n)$, $\mathbf{N}_1 := (\mathbf{c}^{n+1}, \dots, \mathbf{c}^{n'})$, we have $[g\mathbf{N}_0|\mathbf{N}_1] \not\prec \Phi_0$. Let $\Phi_1 := \Phi_0/\mathbf{N}_1^*$. Since $\mathcal{K}$ is closed under quotients, $\Phi_1 \in \mathcal{K}$. We have that $\mathbf{N}_0 \prec \Phi_1$ but $g\mathbf{N}_0 \not\prec \Phi_1$, so g does not preserve Φ_1 either. Define $h\colon \rho \to \mu$ to be any map such that

$$\bigl(\mathbf{c}^1(i), \dots, \mathbf{c}^n(i)\bigr) = \bigl((\mathbf{m}^1 h)(i), \dots, (\mathbf{m}^n h)(i)\bigr)$$

for every $i \in \rho$, i.e., row i of $\mathbf{N}_0$ is the same as row $h(i)$ of $\mathbf{M}$, for each $i \in \rho$. Let Φ_h be the μ-ary simple ∞-minor of Φ_1 via $H := \{h\}$. Note that $\Phi_h \in \mathcal{K}^\infty$.

We claim that $\mathbf{M} \prec \Phi_h$. To prove this, by the definition of simple ∞-minor, it is enough to show that $(\mathbf{m}^1 h, \dots, \mathbf{m}^n h) \prec \Phi_1$. In fact, we have for $1 \leq j \leq n$,

$$\mathbf{m}^j h = (\mathbf{m}^j h(i) \mid i \in \rho) = (\mathbf{c}^j(i) \mid i \in \rho) = \mathbf{c}^j,$$

and $(\mathbf{c}^1, \dots, \mathbf{c}^n) = \mathbf{N}_0 \prec \Phi_1$.

Next we claim that $\{\mathbf{u}\} \notin \Phi_h$. For this, by the definition of simple ∞-minor, it is enough to show that $\mathbf{u}h \not\prec \Phi_1$. For every $i \in \rho$, we have

$$\begin{aligned}(\mathbf{u}h)(i) &= \bigl(g(\mathbf{m}^1, \dots, \mathbf{m}^n)h\bigr)(i) \\ &= g\bigl((\mathbf{m}^1 h)(i), \dots, (\mathbf{m}^n h)(i)\bigr) = g\bigl(\mathbf{c}^1(i), \dots, \mathbf{c}^n(i)\bigr).\end{aligned}$$

Thus $\mathbf{u}h = g\mathbf{N}_0$. Since $g\mathbf{N}_0 \not\prec \Phi_1$, we conclude that $\{\mathbf{u}\} \notin \Phi_h$.

Thus, Φ_h is a cluster in $\mathcal{K}^\infty$ that contains $\mathbf{M}$ but does not contain $\{\mathbf{u}\}$. By the choice of $\mathbf{u}$, this is impossible, and we have reached a contradiction. □

Proof of Theorem 6.7. (ii) ⇒ (i): It is clear that every function preserves the equality, empty, and trivial clusters. By Lemmas 6.1, 6.2, 6.4, and 6.5, $\mathcal{K}$ is closed under formation of conjunctive minors, unions, quotients, and dividends.

It remains to show that $\mathcal{K}$ is locally closed. Suppose on the contrary that there is a cluster $\Phi \in \mathcal{K}_A \setminus \mathcal{K}$, say of arity m, such that $\Phi^{(p)} = \Phi \cap \Omega_m^{(p)} \in \mathcal{K}$ for all $p \geq 0$. By (ii), there is an operation $f\colon A^n \to A$ that preserves every cluster in $\mathcal{K}$ but does not preserve Φ. Thus, there is a $p \geq 0$ and an $m \times p$ matrix $\mathbf{M} := [\mathbf{M}_1|\mathbf{M}_2] \prec \Phi$ such that $[f\mathbf{M}_1|\mathbf{M}_2] \not\prec \Phi$. By our assumption, $\Phi^{(p)} \in \mathcal{K}$, but we have that $[\mathbf{M}_1|\mathbf{M}_2] \prec \Phi^{(p)}$ and $[f\mathbf{M}_1|\mathbf{M}_2] \not\prec \Phi^{(p)}$, which is a contradiction to the fact that $f \triangleright \Phi^{(p)}$.

(i) ⇒ (ii): By Lemma 6.10, for every cluster $\Phi \in \mathcal{K}_A \setminus \mathcal{K}$, there is a function in $\mathcal{O}_A$ which preserves every cluster in $\mathcal{K}$ but does not preserve Φ. The set of these "separating" functions, for each $\Phi \in \mathcal{K}_A \setminus \mathcal{K}$, characterizes $\mathcal{K}$. □

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