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A NOTE ON MINORS DETERMINED BY CLONES OF SEMILATTICES

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Abstract. The $\mathcal{C}$-minor partial orders determined by the clones generated by a semilattice operation (and possibly the constant operations corresponding to its identity or zero elements) are shown to satisfy the descending chain condition.

1. Introduction

This paper is a study of substitution instances of functions of several arguments when the inner functions are taken from a prescribed set of functions. Such an idea has been studied by several authors. Henno [6] generalized Green's relations to Menger algebras (essentially, abstract clones) and described Green's relations on the set of all operations on A for each set A. Harrison [5] considered two Boolean functions to be equivalent if they are substitution instances of each other with respect to the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ or the affine linear group $\mathrm{AGL}(n, \mathbb{F}_2)$, where $\mathbb{F}_2$ denotes the two-element field. In [15, 16], a Boolean function f is defined to be a minor of another Boolean function g, if and only if f can be obtained from g by substituting for each variable of g a variable, a negated variable, or one of the constants 0 or 1. Further variants of the notion of minor can be found in [1, 3, 4, 13, 17].

These ideas are unified and generalized by the notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence, which first appeared in print in [8]. More precisely, let A be a nonempty set, and let $f \colon A^n \to A$ and $g \colon A^m \to A$ be operations on A. Let $\mathcal{C}$ be a set of operations on A. We say that f is a $\mathcal{C}$-minor of g, if $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in \mathcal{C}$, and we say that f and g are $\mathcal{C}$-equivalent if f and g are $\mathcal{C}$-minors of each other. If $\mathcal{C}$ is a clone, then the $\mathcal{C}$-minor relation is a preorder and it induces a partial order on the $\mathcal{C}$-equivalence classes. For background and basic results on $\mathcal{C}$-minors and $\mathcal{C}$-equivalences, see [8, 10, 11, 12].

In this paper, we study the $\mathcal{C}$-minors and $\mathcal{C}$-equivalences induced by the clones generated by semilattice opeations (and possibly some constants). Our main result (Theorem 3.1) asserts that if $(A; \wedge)$ is a semilattice, then for the clone $\mathcal{C} := \langle \wedge \rangle$ generated by $\wedge$, the induced $\mathcal{C}$-minor partial order satisfies the descending chain condition. Furthermore, if $(A; \wedge)$ has an identity element 1 and a zero element 0, then this property is enjoyed by all clones $\mathcal{C}$ such that $\langle \wedge \rangle \subseteq \mathcal{C} \subseteq \langle \wedge, 0, 1 \rangle$. These results find an application in [9], in which the clones of Boolean functions are classified according to certain order-theoretical properties of their induced $\mathcal{C}$-minor partial orders.

2. Clones, $\mathcal{C}$-minors and $\mathcal{C}$-decompositions

2.1. **Operations and clones.** Throughout this paper, for an integer $n \geq 1$, we denote $[n] := \{1, \dots, n\}$. Let A be a fixed nonempty base set. An *operation* on A is a map $f\colon A^n \to A$ for some integer $n \geq 1$, called the *arity* of f. We denote the set of all n-ary operations on A by $\mathcal{O}_A^{(n)}$, and we denote by $\mathcal{O}_A := \bigcup_{n \geq 1} \mathcal{O}_A^{(n)}$ the set of all operations on A. The i-th n-ary *projection* $(1 \leq i \leq n)$ is the operation $(a_1, \dots, a_n) \mapsto a_i$, and it is denoted by $x_i^{(n)}$, or simply by x_i when the arity is clear from the context.

If $f \in \mathcal{O}_A^{(n)}$ and $g_1, \dots, g_n \in \mathcal{O}_A^{(m)}$, then the *composition* of f with $g_1, \dots, g_n$, denoted $f(g_1, \dots, g_n)$ is the m-ary operation defined by

$$f(g_1, \dots, g_n)(\mathbf{a}) = f\bigl(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})\bigr)$$

for all $\mathbf{a} \in A^m$.

Let $\mathcal{C} \subseteq \mathcal{O}_A$. The *$n$-ary part of* $\mathcal{C}$ is the set $\mathcal{C}^{(n)} := \mathcal{C} \cap \mathcal{O}_A^{(n)}$ of n-ary members of $\mathcal{C}$. A *clone* on A is a subset $\mathcal{C} \subseteq \mathcal{O}_A$ that contains all projections and is closed under composition, i.e., $f(g_1, \dots, g_n) \in \mathcal{C}$ whenever $f, g_1, \dots, g_n \in \mathcal{C}$ and the composition is defined.

The clones on A constitute a complete lattice under inclusion. Therefore, for each set $F \subseteq \mathcal{O}_A$ of operations there exists a smallest clone that contains F, which will be denoted by $\langle F \rangle$ and called the *clone generated by F.* See [2, 7, 14] for general background on clones.

2.2. **$\mathcal{C}$-minors.** Let $\mathcal{C} \subseteq \mathcal{O}_A$, and let $f, g \in \mathcal{O}_A$. We say that f is a *$\mathcal{C}$-minor* of g, if $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in \mathcal{C}$, and we denote this fact by $f \leq_{\mathcal{C}} g$. We say that f and g are *$\mathcal{C}$-equivalent,* denoted $f \equiv_{\mathcal{C}} g$, if f and g are $\mathcal{C}$-minors of each other.

The $\mathcal{C}$-minor relation $\leq_{\mathcal{C}}$ is a preorder (i.e., a reflexive and transitive relation) on $\mathcal{O}_A$ if and only if $\mathcal{C}$ is a clone. If $\mathcal{C}$ is a clone, then the $\mathcal{C}$-equivalence relation $\equiv_{\mathcal{C}}$ is an equivalence relation on $\mathcal{O}_A$, and, as for preorders, $\leq_{\mathcal{C}}$ induces a partial order $\preccurlyeq_{\mathcal{C}}$ on the quotient $\mathcal{O}_A/{\equiv_{\mathcal{C}}}$. It follows from the definition of $\mathcal{C}$-minor, that if $\mathcal{C}$ and $\mathcal{K}$ are clones such that $\mathcal{C} \subseteq \mathcal{K}$, then ${\leq_{\mathcal{C}}} \subseteq {\leq_{\mathcal{K}}}$ and ${\equiv_{\mathcal{C}}} \subseteq {\equiv_{\mathcal{K}}}$. For further background and properties of $\mathcal{C}$-minor relations, see [8, 10, 11, 12].

2.3. **C-decompositions.** Let $\mathcal{C}$ be a clone on A, and let $f \in \mathcal{O}_A^{(n)}$. If $f = g(\phi_1, \dots, \phi_m)$ for some $g \in \mathcal{O}_A^{(m)}$ and $\phi_1, \dots, \phi_m \in \mathcal{C}$, then we say that the $(m+1)$-tuple $(g, \phi_1, \dots, \phi_m)$ is a *$\mathcal{C}$-decomposition* of f. We often avoid referring explicitly to the tuple and we simply say that $f = g(\phi_1, \dots, \phi_m)$ is a $\mathcal{C}$-decomposition. Clearly, there always exists a $\mathcal{C}$-decomposition of every operation f for every clone $\mathcal{C}$, because $f = f(x_1^{(n)}, \dots, x_n^{(n)})$ and projections are members of every clone. A $\mathcal{C}$-decomposition of a nonconstant function f is *minimal* if the arity m of g is the smallest possible among all $\mathcal{C}$-decompositions of f. This smallest possible m is called the *$\mathcal{C}$-degree* of f, denoted $\deg_{\mathcal{C}} f$. We agree that the $\mathcal{C}$-degree of every constant function is 0.

Lemma 2.1. *If $f \leq_{\mathcal{C}} g$, then* $\deg_{\mathcal{C}} f \leq \deg_{\mathcal{C}} g$.

Proof. Let $\deg_{\mathcal{C}} g = m$, and let $g = h(\gamma_1, \dots, \gamma_m)$ be a minimal $\mathcal{C}$-decomposition of g. Since $f \leq_{\mathcal{C}} g$, there exist $\phi_1, \dots, \phi_n \in \mathcal{C}$ such that $f = g(\phi_1, \dots, \phi_n)$. Then

$$f = h(\gamma_1, \dots, \gamma_m)(\phi_1, \dots, \phi_n) = h(\gamma_1(\phi_1, \dots, \phi_n), \dots, \gamma_m(\phi_1, \dots, \phi_n)),$$

and since $\gamma_i(\phi_1, \dots, \phi_n) \in \mathcal{C}$ for $1 \leq i \leq m$, we have that $(h, \gamma_1(\phi_1, \dots, \phi_n), \dots, \gamma_m(\phi_1, \dots, \phi_n))$ is a $\mathcal{C}$-decomposition of f, not necessarily minimal, so $\deg_{\mathcal{C}} f \leq m$. □

An immediate consequence of Lemma 2.1 is that $\mathcal{C}$-equivalent functions have the same $\mathcal{C}$-degree.

Let $(\phi_1, \dots, \phi_m)$ be an m-tuple ($m \geq 2$) of n-ary operations on A. If there is an $i \in \{1, 2, \dots, m\}$ and $g\colon A^{m-1} \to A$ such that

$$\phi_i = g(\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_m),$$

then we say that the m-tuple $(\phi_1, \dots, \phi_m)$ is *functionally dependent.* Otherwise we say that $(\phi_1, \dots, \phi_m)$ is *functionally independent.* We often omit the m-tuple notation and simply say that $\phi_1, \dots, \phi_m$ are functionally dependent or independent.

Remark 2.2. Every m-tuple containing a constant function is functionally dependent. Also if $f_i = f_j$ for some $i \neq j$, then $f_1, \dots, f_n$ are functionally dependent.

Lemma 2.3. *If $(g, \phi_1, \dots, \phi_m)$ is a minimal $\mathcal{C}$-decomposition of f, then $\phi_1, \dots, \phi_m$ are functionally independent.*

Proof. Suppose, on the contrary, that $\phi_1, \dots, \phi_m$ are functionally dependent. Then there is an i and an $h\colon A^{m-1} \to A$ such that $\phi_i = h(\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_m)$. Then

$$\begin{aligned} f &= g(\phi_1, \dots, \phi_{i-1}, h(\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_m), \phi_{i+1}, \dots, \phi_m) \\ &= g(x_1^{(m-1)}, \dots, x_{i-1}^{(m-1)}, h, x_i^{(m-1)}, \dots, x_{m-1}^{(m-1)})(\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_m), \end{aligned}$$

which shows that $(g(x_1, \dots, x_{i-1}, h, x_i, \dots, x_{m-1}), \phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_m)$ is a $\mathcal{C}$-decomposition of f, contradicting the minimality of $(g, \phi_1, \dots, \phi_m)$. □

3. $\mathcal{C}$-MINORS DETERMINED BY CLONES OF SEMILATTICES

In this section we will prove our main result, namely Theorem 3.1. It will find an application in [9] where the clones of Boolean functions are classified according to certain order-theoretical properties that their induced $\mathcal{C}$-minor partial orders enjoy.

A binary operation $\wedge$ on A is called a *semilattice operation,* if for all $x, y, z \in A$, the following identities hold:

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad x \wedge y = y \wedge x, \qquad x \wedge x = x,$$

i.e., $\wedge$ is associative, commutative and idempotent.

A partial order $(P; \leq)$ is said to satisfy the *descending chain condition,* or it is called *well-founded,* if it contains no infinite descending chains, i.e., given any sequence of elements of P

$$\cdots \leq a_3 \leq a_2 \leq a_1,$$

there exists a positive integer n such that

$$a_n = a_{n+1} = a_{n+2} = \cdots .$$

Theorem 3.1. *Let $\mathcal{S}$ be the clone generated by a semilattice operation $\wedge$ on A. Then the $\mathcal{S}$-minor partial order $\preccurlyeq_{\mathcal{S}}$ satisfies the descending chain condition.*

Proof. Let $(\phi_1, \dots, \phi_m) \in (\mathcal{S}^{(n)})^m$. Then, for $1 \leq j \leq m$, ϕ_j is of the form

$$\phi_j = \bigwedge_{i \in \Phi_j} x_i^{(n)} \tag{1}$$

for some $\emptyset \neq \Phi_j \subseteq [n]$. For $1 \leq i \leq n$, denote

$$X_i := \{j \in [m] : i \in \Phi_j\}, \tag{2}$$

and let $X(\phi_1, \dots, \phi_m) := \{X_1, \dots, X_n\} \subseteq \mathcal{P}([m])$. It follows from the definitions of Φ_j and X_i that

$$j \in X_i \iff i \in \Phi_j. \tag{3}$$

Correspondingly, for any $\emptyset \neq E \subseteq \mathcal{P}([m])$, denote $\Psi_E := (\psi_1, \dots, \psi_m)$, where $\psi_j \in \mathcal{S}^{(|E|)}$ is given by

$$\psi_j = \bigwedge_{j \in S \in E} x_{\sigma_E(S)},$$

where $\sigma_E \colon E \to [|E|]$ is any fixed bijection.

Let $(g, \phi_1, \dots, \phi_m)$ be an $\mathcal{S}$-decomposition of $f \colon A^n \to A$. Then each ϕ_j is of the form (1) for some $\emptyset \neq \Phi_j \subseteq [n]$. Let $E := X(\phi_1, \dots, \phi_m)$, $(\psi_1, \dots, \psi_m) := \Psi_E$, and let $f' = g(\psi_1, \dots, \psi_m)$. We will show that $f \equiv_{\mathcal{S}} f'$.

As in (2), for $1 \leq i \leq n$, let $X_i = \{j \in [m] : i \in \Phi_j\}$. Let $\pi \colon [n] \to [|E|]$ be defined as $\pi(i) := \sigma_E(X_i)$. Then

$$\begin{aligned} f(x_{\pi(1)}, \dots, x_{\pi(n)}) &= g(\phi_1, \dots, \phi_m)(x_{\pi(1)}, \dots, x_{\pi(n)}) \\ &= g(\phi_1(x_{\pi(1)}, \dots, x_{\pi(n)}), \dots, \phi_m(x_{\pi(1)}, \dots, x_{\pi(n)})) = g(\psi_1, \dots, \psi_m) = f', \end{aligned}$$

where the second last equality holds because for $1 \leq j \leq m$,

$$\phi_j(x_{\pi(1)}, \dots, x_{\pi(n)}) = \bigwedge_{i \in \Phi_j} x_{\pi(i)} = \bigwedge_{i \in \Phi_j} x_{\sigma(X_i)} = \bigwedge_{j \in S \in E} x_{\sigma(S)} = \psi_j.$$

Since all projections are members of $\mathcal{S}$, we have that $f' \leq_{\mathcal{C}} f$. On the other hand, for $1 \leq j \leq |E|$, let $\Xi_j := \{i \in [n] : X_i = \sigma_E^{-1}(j)\}$, and let

$$\xi_j := \bigwedge_{i \in \Xi_j} x_i.$$

It is easy to see that $\Xi_j \neq \emptyset$; hence $\xi_j \in \mathcal{S}$. Then

$$\begin{aligned} f'(\xi_1, \dots, \xi_{|E|}) &= g(\psi_1, \dots, \psi_m)(\xi_1, \dots, \xi_{|E|}) \\ &= g(\psi_1(\xi_1, \dots, \xi_{|E|}), \dots, \psi_m(\xi_1, \dots, \xi_{|E|})) = g(\phi_1, \dots, \phi_m) = f, \end{aligned}$$

where the second last equality holds because for $j = 1, \dots, m$,

$$\begin{aligned} \psi_j(\xi_1, \dots, \xi_{|E|}) &= \Big(\bigwedge_{j \in S \in E} x_{\sigma_E(S)} \Big)(\xi_1, \dots, \xi_{|E|}) = \bigwedge_{j \in S \in E} \xi_{\sigma_E(S)} \\ &= \bigwedge_{j \in S \in E} \Big(\bigwedge_{i \in \Xi_{\sigma_E(S)}} x_i \Big) = \bigwedge_{j \in S \in E} \Big(\bigwedge_{\substack{i \in [n] \\ X_i = S}} x_i \Big) = \bigwedge_{i \in \Phi_j} x_i = \phi_j. \end{aligned}$$

Here, the third last equality holds, because $\Xi_{\sigma_E(S)} = \{i \in [n] : X_i = S\}$, and the second last equality holds by (3) and the associativity, commutativity and idempotency of $\wedge$. Since $\xi_j \in \mathcal{S}$, we have that $f \leq_{\mathcal{C}} f'$. We conclude that $f \equiv_{\mathcal{C}} f'$, as desired.

Claim. If $f_1 = g(\phi_1, \dots, \phi_m)$ and $f_2 = g(\varphi_1, \dots, \varphi_m)$ are $\mathcal{S}$-decompositions and $X(\phi_1, \dots, \phi_m) = X(\varphi_1, \dots, \varphi_m)$, then $f_1 \equiv_{\mathcal{S}} f_2$.

Proof of the claim. Let $(\psi_1, \dots, \psi_m) := \Psi_{X(\phi_1,\dots,\phi_m)}$ $(= \Psi_{X(\varphi_1,\dots,\varphi_m)})$, and let $f' = g(\psi_1, \dots, \psi_m)$. It follows from what was shown above that $f_1 \equiv_{\mathcal{S}} f' \equiv_{\mathcal{S}} f_2$. The claim follows by the transitivity of $\equiv_{\mathcal{S}}$. $\diamond$

To finish the proof that $\preccurlyeq_{\mathcal{S}}$ satisfies the descending chain condition, assume that $f_1 <_{\mathcal{S}} f_2$, $f_2 = g(\phi_1, \dots, \phi_m)$ is a minimal $\mathcal{S}$-decomposition, and $f_1 = f_2(h_1, \dots, h_n)$ for some $h_1, \dots, h_n \in \mathcal{S}$. For $i = 1, \dots, m$, denote $\phi'_i = \phi_i(h_1, \dots, h_n)$, so that $f_1 = g(\phi'_1, \dots, \phi'_m)$. By Lemma 2.1, either $\deg_{\mathcal{S}} f_1 < \deg_{\mathcal{S}} f_2$, or $\deg_{\mathcal{S}} f_1 = \deg_{\mathcal{S}} f_2$ and $X(\phi_1, \dots, \phi_m) \neq X(\phi'_1, \dots, \phi'_m)$. Since $\mathcal{S}$-degrees are nonnegative integers and $\mathcal{P}([m])$ is a finite set, there are only a finite number of $\equiv_{\mathcal{S}}$-classes preceding the $\equiv_{\mathcal{S}}$-class of f_2 in the $\mathcal{S}$-minor partial order $\preccurlyeq_{\mathcal{S}}$. This completes the proof of the theorem. $\square$

Corollary 3.2. *Assume that a semilattice $(A; \wedge)$ has identity and zero elements* 1 *and* 0*, respectively. Let $\mathcal{C}$ be a clone on A such that $\langle \wedge \rangle \subseteq \mathcal{C} \subseteq \langle \wedge, 0, 1 \rangle$. Then the $\mathcal{C}$-minor partial order $\preccurlyeq_{\mathcal{C}}$ satisfies the descending chain condition.*

Proof. The proof of Theorem 3.1 in fact shows that $\preccurlyeq_{\mathcal{C}}$ satisfies the descending chain condition. For, in this case $\mathcal{C} \setminus \mathcal{S}$ contains only constant operations. Remark 2.2 and Lemma 2.3 guarantee that $f = g(h_1, \dots, h_m)$ is a minimal $\mathcal{S}$-decomposition if and only if it is a minimal $\mathcal{C}$-decomposition, and since $\mathcal{S} \subseteq \mathcal{C}$, $\mathcal{S}$-equivalence implies $\mathcal{C}$-equivalence. $\square$

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