

Approximations of Lovász extensions and their induced interaction index

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Abstract

The Lovász extension of a pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ is defined on each simplex of the standard triangulation of $[0, 1]^n$ as the unique affine function $\hat{f} : [0, 1]^n \rightarrow \mathbb{R}$ that interpolates f at the $n + 1$ vertices of the simplex. Its degree is that of the unique multilinear polynomial that expresses f . In this paper we investigate the least squares approximation problem of an arbitrary Lovász extension \hat{f} by Lovász extensions of (at most) a specified degree. We derive explicit expressions of these approximations. The corresponding approximation problem for pseudo-Boolean functions was investigated by Hammer and Holzman [14] and then solved explicitly by Grabisch, Marichal, and Roubens [11], giving rise to an alternative definition of Banzhaf interaction index. Similarly we introduce a new interaction index from approximations of \hat{f} and we present some of its properties. It turns out that its corresponding power index identifies with the power index introduced by Grabisch and Labreuche [10].

Key words: pseudo-Boolean function, Lovász extension, discrete Choquet integral, least squares approximation, interaction index.

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1 Introduction and problem setting

Real valued set functions are extensively used in both cooperative game theory and multicriteria decision making (see for instance [11] and the references therein). Indeed, in its characteristic form, a *cooperative game* on a finite set of players $N := \{1, \dots, n\}$, is a set function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$ which assigns to each coalition S of players a real number $v(S)$ representing the *worth* of S . In multicriteria decision making, when N represents a set of criteria, a *capacity* (also called *non-additive measure*) on N is a monotone set function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$ and $v(N) = 1$ which assigns to each subset S of criteria its weight $v(S)$.

There is a natural identification of real valued set functions $v : 2^N \rightarrow \mathbb{R}$ with pseudo-Boolean functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$ through the equality $v(S) = f(\mathbf{1}_S)$, where $\mathbf{1}_S$ denotes the characteristic vector of S in $\{0, 1\}^n$ (whose i th component is 1 if and only if $i \in S$). Moreover, it is known [15] that any pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$ has a unique expression as a multilinear polynomial in n variables

$$f(\mathbf{x}) = \sum_{S \subseteq N} a(S) \prod_{i \in S} x_i, \quad (1)$$

where the set function $a : 2^N \rightarrow \mathbb{R}$ is simply the Möbius transform (see for instance [21]) of v , which can be calculated through

$$a(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T). \quad (2)$$

Let \mathfrak{S}_n denote the family of all permutations σ on N . The *Lovász extension* $\hat{f} : [0, 1]^n \rightarrow \mathbb{R}$ of any pseudo-Boolean function f is defined on each n -simplex

$$S_\sigma := \{\mathbf{x} \in [0, 1]^n \mid x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}\} \quad (\sigma \in \mathfrak{S}_n),$$

as the unique affine function which interpolates f at the $n + 1$ vertices of S_σ (see Lovász [17, §3] and Singer [24, §2]).

The concept of Lovász extension was initially used in combinatorial optimization [8,17,24] and then in cooperative game theory [2,11]. In this paper we are mostly concerned by the interpretation of this concept in decision making, where it was observed in [18] that the Lovász extension of a monotone pseudo-Boolean function is nothing but a discrete *Choquet integral* [4,5]. Recall that discrete Choquet integrals were recently proposed in decision making as aggregation functions that generalize the weighted arithmetic means by the taking into consideration of the interaction phenomena among input variables; see for instance [12,19].

As pointed out in Grabisch et al. [11], the Lovász extension of any pseudo-Boolean function f can be expressed as the following min-polynomial

$$\hat{f}(\mathbf{x}) = \sum_{S \subseteq N} a(S) \min_{i \in S} x_i.$$

Its degree is that of the multilinear polynomial (1) that expresses f .

Hammer and Holzman [14] investigated the approximation of a pseudo-Boolean function f by a multilinear polynomial of (at most) a specified degree. More precisely, fixing $k \in \mathbb{N}$, with $k \leq n$, they defined the best k th approximation of f as the multilinear polynomial $f_k : \{0, 1\}^n \rightarrow \mathbb{R}$ of degree at most k which minimizes

$$\sum_{\mathbf{x} \in \{0,1\}^n} [f(\mathbf{x}) - g(\mathbf{x})]^2$$

among all multilinear polynomials g of degree at most k . They obtained an explicit form of the solution for $k = 1$ and $k = 2$. For a general k , the explicit form of the solution was obtained by Grabisch et al. [11, §7]. It is written as

$$f_k(\mathbf{x}) = \sum_{\substack{S \subseteq N \\ |S| \leq k}} a_k(S) \prod_{i \in S} x_i,$$

where

$$a_k(S) = a(S) + (-1)^{k+|S|} \sum_{\substack{T \supseteq S \\ |T| > k}} \binom{|T|-|S|-1}{k-|S|} 2^{|T|-|S|} a(T). \quad (3)$$

Hammer and Holzman [14] also noticed that $a_1(\{i\})$ identifies with the classical Banzhaf power index [3] related to the i th player. More generally, Grabisch et al. [11] noticed that, for any $S \subseteq N$, the coefficient $a_{|S|}(S)$ identifies with the Banzhaf interaction index [22] related to the coalition S .

In this paper we investigate the least squares approximation of a given Lovász extension \hat{f} by a min-polynomial of (at most) a specified degree. More precisely, fixing $k \in \mathbb{N}$, with $k \leq n$, we give explicit formulas (see Section 2) for the best k th approximation of \hat{f} , that is, the min-polynomial $\hat{f}_k : [0, 1]^n \rightarrow \mathbb{R}$ of degree at most k which minimizes

$$\int_{[0,1]^n} [\hat{f}(\mathbf{x}) - \hat{g}(\mathbf{x})]^2 d\mathbf{x}$$

among all min-polynomials \hat{g} of degree at most k .

It is important to emphasize that this least squares problem has a relevant application in multicriteria decision making. Indeed, consider a multicriteria decision making problem modelled by a Choquet integral and its underlying capacity on a set of n criteria. It is sometimes useful to approximate this Choquet integral (which is, actually, the Lovász extension of the capacity) by

simpler aggregation functions such as Lovász extensions of degree k ($k < n$). Indeed this subclass of functions includes the Choquet integrals constructed from k -order capacities introduced by Grabisch [9].

By analogy with the Banzhaf interaction index obtained from the approximations of f , we show (see Section 3) that the approximations of \hat{f} give rise to a new interaction index having appealing properties (see for instance [7]). Furthermore, it turns out that its corresponding power index identifies with a power index introduced axiomatically by Grabisch and Labreuche [10].

In order to avoid a heavy notation, cardinalities of subsets S, T, \dots are denoted whenever possible by the corresponding lower case letters s, t, \dots , otherwise by the standard notation $|S|, |T|, \dots$. Also, for any $S \subseteq N$ we define the function $\min_S : [0, 1]^n \rightarrow \mathbb{R}$ as

$$\min_S(\mathbf{x}) := \min_{i \in S} x_i,$$

with the convention that $\min_\emptyset \equiv 1$. Finally, we denote by \mathcal{F}_N the family of set functions $v : 2^N \rightarrow \mathbb{R}$.

2 Approximations of Lovász extensions

We first recall the setting of our approximation problem. For any $k \in \{0, \dots, n\}$, we denote by V_k the set of all min-polynomials $\hat{f}_k : [0, 1]^n \rightarrow \mathbb{R}$ of degree at most k , i.e., of the form

$$\hat{f}_k(\mathbf{x}) = \sum_{\substack{S \subseteq N \\ |S| \leq k}} a_k(S) \min_{i \in S} x_i.$$

Definition 1 *Let $\hat{f} \in V_n$ and $k \in \{0, \dots, n\}$. The best k th approximation of \hat{f} is the min-polynomial $\hat{f}_k \in V_k$ that minimizes*

$$\int_{[0,1]^n} [\hat{f}(\mathbf{x}) - \hat{g}(\mathbf{x})]^2 d\mathbf{x}$$

among all min-polynomials \hat{g} of V_k .

In this section we first discuss existence and uniqueness of the best k th approximation of any function $\hat{f} \in V_n$ (see Proposition 3). Then we provide a closed-form formula for this approximation (see Theorem 9).

We begin with the following straightforward proposition:

Proposition 2 *For any $k \in \{0, \dots, n\}$, the set V_k is a linear subspace of V_n , which in turn is a linear subspace of the space $C_0([0, 1]^n)$ of continuous*

functions on $[0, 1]^n$. Moreover, the set

$$B_k := \{\min_S \mid S \subseteq N, s \leq k\}, \quad (4)$$

forms a basis of V_k and hence $\dim(V_k) = \sum_{s=0}^k \binom{n}{s}$.

Consider the usual inner product $\langle \cdot, \cdot \rangle$ of $C_0([0, 1]^n)$ given by

$$\langle f_1, f_2 \rangle := \int_{[0,1]^n} f_1(\mathbf{x}) f_2(\mathbf{x}) d\mathbf{x}.$$

This clearly induces an inner product in V_n , which in turn allows to define a norm $\|\hat{f}\| := \langle \hat{f}, \hat{f} \rangle^{1/2}$ and then a distance $d(\hat{f}_1, \hat{f}_2) := \|\hat{f}_1 - \hat{f}_2\|$ in V_n .

According to this terminology, the best k th approximation of $\hat{f} \in V_n$ is the solution \hat{f}_k of the following least squares problem

$$\min\{\|\hat{f} - \hat{g}\|^2 \mid \hat{g} \in V_k\},$$

which amounts to a classical orthogonal projection problem.

From now on, for any $k \in \{0, \dots, n\}$, we denote by A_k the orthogonal projector from V_n onto V_k .

Proposition 3 *For any $\hat{f} \in V_n$ and any $k \in \{0, \dots, n\}$, the best k th approximation of \hat{f} exists and is uniquely given by $\hat{f}_k = A_k(\hat{f})$. In particular, it is characterized as the unique element \hat{f}_k of V_k such that $\hat{f} - \hat{f}_k$ is orthogonal to V_k , i.e., to each element of the basis B_k defined in (4).*

This projection problem can be simplified because of the following result, which immediately follows from the fact that the subspaces V_k are nested:

Proposition 4 *The operators A_k , $k = 0, \dots, n$, are such that*

$$k \leq k' \quad \Rightarrow \quad A_k(A_{k'}(\hat{f})) = A_k(\hat{f}). \quad (5)$$

We thus observe that $A_k(\hat{f})$ can be attained from \hat{f} by successively carrying out the projections $A_{n-1}, A_{n-2}, \dots, A_k$. We will therefore search for the relation that links any two consecutive projections. This relation will enable us to compute gradually $A_k(\hat{f})$ from \hat{f} .

Remark 5 *Fix $k \in \{0, \dots, n-1\}$ and assume that $\hat{f}_{k+1} = A_{k+1}(\hat{f})$ is given. By (5), $A_k(\hat{f})$ is the orthogonal projection of $A_{k+1}(\hat{f})$ onto V_k and by Proposition 3, it is given as the unique solution $\hat{f}_k \in V_k$ of the following system*

$$\langle \hat{f}_k, \min_S \rangle = \langle \hat{f}_{k+1}, \min_S \rangle \quad (S \subseteq N, s \leq k).$$

To solve this projection problem, we need the following two lemmas. The first one yields an explicit expression of the inner product $\langle \min_S, \min_T \rangle$ of any two elements of B_n . The second lemma, which makes use of the first one, yields an explicit expression of the projections onto V_k of the elements of $B_{k+1} \setminus B_k$.

Lemma 6 *For any $S, T \subseteq N$, there holds*

$$\int_{[0,1]^n} \min_S(\mathbf{x}) \min_T(\mathbf{x}) d\mathbf{x} = \frac{1}{|S \cup T| + 2} \left(\frac{1}{s+1} + \frac{1}{t+1} \right). \quad (6)$$

Proof. See Appendix A. \square

Lemma 7 *For any $S \subseteq N$, with $s = k + 1$, we have*

$$A_k(\min_S) = \sum_{T \not\subseteq S} (-1)^{k+t} \frac{\binom{k+t+1}{k+1}}{\binom{2k+2}{k+1}} \min_T.$$

Proof. See Appendix B. \square

Proposition 8 *Let $k \in \{0, \dots, n-1\}$. Given the coefficients $a_{k+1}(S)$ of $A_{k+1}(\hat{f})$, the coefficients $a_k(S)$ of $A_k(\hat{f})$ are given by*

$$a_k(S) = a_{k+1}(S) + (-1)^{k+s} \frac{\binom{k+s+1}{k+1}}{\binom{2k+2}{k+1}} \sum_{\substack{T \supseteq S \\ t=k+1}} a_{k+1}(T). \quad (7)$$

Proof. Using formula (5) and then Lemma 7, we obtain

$$\begin{aligned} A_k(\hat{f}) &= A_k(A_{k+1}(\hat{f})) = \sum_{\substack{S \subseteq N \\ s \leq k+1}} a_{k+1}(S) A_k(\min_S) \\ &= \sum_{\substack{S \subseteq N \\ s \leq k}} a_{k+1}(S) \min_S + \sum_{\substack{S \subseteq N \\ s=k+1}} a_{k+1}(S) A_k(\min_S) \\ &= \sum_{\substack{S \subseteq N \\ s \leq k}} a_{k+1}(S) \min_S + \sum_{\substack{S \subseteq N \\ s=k+1}} a_{k+1}(S) \sum_{T \not\subseteq S} (-1)^{k+t} \frac{\binom{k+t+1}{k+1}}{\binom{2k+2}{k+1}} \min_T, \end{aligned}$$

which proves the result. \square

We are now ready to derive the explicit form of the coefficients of $A_k(\hat{f})$ in terms of the coefficients of \hat{f} . It is worth comparing this formula with the solution (3) of the Hammer-Holzman approximation problem.

Theorem 9 Let $k \in \{0, \dots, n\}$. The coefficients of $A_k(\hat{f})$ are given from those of \hat{f} by

$$a_k(S) = a(S) + (-1)^{k+s} \sum_{\substack{T \supseteq S \\ t > k}} \frac{\binom{k+s+1}{k+1} \binom{t-s-1}{k-s}}{\binom{k+t+1}{k+1}} a(T). \quad (8)$$

Proof. We only have to prove that the coefficients $a_k(S)$ given in (8) fulfill the recurrence relation (7).

Let $k \in \{0, \dots, n\}$ and fix $S \subseteq N$ with $s \leq k$. By substituting (8) into (7), we obtain, after removing the common term $a(S)$,

$$\begin{aligned} & (-1)^{k+s} \sum_{\substack{R \supseteq S \\ r > k}} \frac{\binom{k+s+1}{k+1} \binom{r-s-1}{k-s}}{\binom{k+r+1}{k+1}} a(R) \\ &= (-1)^{k+s+1} \sum_{\substack{R \supseteq S \\ r > k+1}} \frac{\binom{k+s+2}{k+2} \binom{r-s-1}{k-s+1}}{\binom{k+r+2}{k+2}} a(R) \\ & \quad + (-1)^{k+s} \frac{\binom{k+s+1}{k+1}}{\binom{2k+2}{k+1}} \sum_{\substack{T \supseteq S \\ t=k+1}} \left[a(T) + \sum_{\substack{R \supseteq T \\ r > k+1}} \frac{\binom{2k+3}{k+2}}{\binom{k+r+2}{k+2}} a(R) \right]. \end{aligned}$$

Let us show that this equality holds. Dividing through by $(-1)^{k+s}$ and then removing the terms corresponding to $a(R)$, with $r = k + 1$, the equality becomes

$$\begin{aligned} & \sum_{\substack{R \supseteq S \\ r > k+1}} \frac{\binom{k+s+1}{k+1} \binom{r-s-1}{k-s}}{\binom{k+r+1}{k+1}} a(R) \\ &= - \sum_{\substack{R \supseteq S \\ r > k+1}} \frac{\binom{k+s+2}{k+2} \binom{r-s-1}{k-s+1}}{\binom{k+r+2}{k+2}} a(R) + \sum_{\substack{R \supseteq S \\ r > k+1}} \frac{\binom{k+s+1}{k+1} \binom{2k+3}{k+2} \binom{r-s}{k-s+1}}{\binom{2k+2}{k+1} \binom{k+r+2}{k+2}} a(R). \end{aligned}$$

Fix $R \supseteq S$, with $r > k + 1$, and consider the coefficients of $a(R)$ in the previous equality. By equating them, we obtain an identity which can be easily checked. \square

Example 10 Let $\hat{f} : [0, 1]^4 \rightarrow \mathbb{R}$ be given by

$$\hat{f}(\mathbf{x}) = \frac{3}{10} (x_1 + x_2 + x_3 + \min\{x_1, x_2\} + \min\{x_1, x_3\} + \min\{x_2, x_3\}) \\ - \frac{21}{25} \min\{x_1, x_2, x_3\} + \frac{1}{25} \min\{x_1, x_2, x_3, x_4\}.$$

The best constant approximation is given by

$$(A_0\hat{f})(\mathbf{x}) = \frac{137}{250},$$

the best linear approximation by

$$(A_1\hat{f})(\mathbf{x}) = \frac{1}{100} + \frac{89}{250} (x_1 + x_2 + x_3) + \frac{1}{125} x_4$$

and the best min-quadratic approximation by

$$(A_2\hat{f})(\mathbf{x}) = -\frac{27}{700} + \frac{803}{1750} (x_1 + x_2 + x_3) - \frac{8}{875} x_4 \\ - \frac{19}{175} (\min\{x_1, x_2\} + \min\{x_1, x_3\} + \min\{x_2, x_3\}) \\ + \frac{2}{175} (\min\{x_1, x_4\} + \min\{x_2, x_4\} + \min\{x_3, x_4\}).$$

Before closing this section, we show that every approximation preserves the symmetry properties of the functions. For instance, in the previous example, we observe that the function \hat{f} and all its approximations are symmetric in the variables x_1 , x_2 , and x_3 .

Let us establish this result in a more general setting. Let X be a nonempty closed convex set in a finite-dimensional inner product space V . For any $u \in V$, the distance between u and X is achieved at a unique point in X . We denote this point by $A_X(u)$ and call it the projection of u onto X (see for instance [16, Chap. 3, §3.1]).

Lemma 11 *Let $I : V \rightarrow V$ be an isometry such that $I(X) = X$. Then I and A_X commute, i.e., $I \circ A_X = A_X \circ I$.*

Proof. We clearly have $I[A_X(u)] \in X$ for any $u \in V$. Furthermore, for any $v \in X$, we have

$$\|I(u) - I[A_X(u)]\| = \|u - A_X(u)\| \leq \|u - v\| = \|I(u) - I(v)\|,$$

which shows that $I[A_X(u)]$ is the projection of $I(u)$ onto X . \square

Let us now apply this result to our least squares approximation problem.

Definition 12 For any σ in \mathfrak{S}_n , we define the linear operator P_σ of V_n as

$$P_\sigma \hat{f}(x_1, \dots, x_n) := \hat{f}(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We say that $\sigma \in \mathfrak{S}_n$ is a symmetry of \hat{f} if $P_\sigma(\hat{f}) = \hat{f}$.

Proposition 13 For any $\sigma \in \mathfrak{S}_n$, the operator P_σ is an isometry of V_n such that $P_\sigma(V_k) = V_k$. In particular, the operators P_σ and A_k commute and A_k preserves the symmetries of its arguments.

Proof. It is clear that $P_\sigma(V_k) = V_k$ and that P_σ is an isometry, i.e., it satisfies

$$\langle P_\sigma \hat{f}, P_\sigma \hat{g} \rangle = \langle \hat{f}, \hat{g} \rangle \quad (\hat{f}, \hat{g} \in V_n).$$

Therefore, by Lemma 11, P_σ and A_k commute. In particular, if $\sigma \in \mathfrak{S}_n$ is a symmetry of \hat{f} , then

$$P_\sigma[A_k(\hat{f})] = A_k[P_\sigma(\hat{f})] = A_k(\hat{f})$$

and hence σ is also a symmetry of $A_k(\hat{f})$. \square

3 A new interaction index

In cooperative game theory the concept of *power index* (or *value*) was introduced in the pioneering work of Shapley [23]. Roughly speaking, a power index on N is a function $\phi : \mathcal{F}_N \times N \rightarrow \mathbb{R}$ that assigns to every player $i \in N$ in a game $v \in \mathcal{F}_N$ his/her prospect $\phi(v, i)$ from playing the game. The *Shapley power index* of a player $i \in N$ in a game $v \in \mathcal{F}_N$ is given by

$$\phi_{Sh}(v, i) := \sum_{T \subseteq N \setminus \{i\}} \frac{1}{n} \binom{n-1}{t}^{-1} [v(T \cup \{i\}) - v(T)].$$

Another frequently used power index is the *Banzhaf power index* [3,6] which, for a player $i \in N$ in a game $v \in \mathcal{F}_N$, is defined by

$$\phi_B(v, i) := \sum_{T \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} [v(T \cup \{i\}) - v(T)].$$

The concept of *interaction index*, which is an extension of that of power index, was recently introduced axiomatically to measure the interaction phenomena among players. An interaction index on N is essentially a function $I : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ that assigns to every coalition $S \subseteq N$ of players in a game $v \in \mathcal{F}_N$

its interaction degree. Various interaction indices have been introduced thus far in the literature: the *Shapley interaction index* [9], the *Banzhaf interaction index* [13,22], and the *chaining interaction index* [20], which all belong to the class of *cardinal-probabilistic interaction indices* (see Definition 14 below) newly axiomatized in [7].

For instance, the *Banzhaf interaction index* on N , which extends the concept of Banzhaf power index on N , is the mapping $I_B : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ defined by

$$I_B(v, S) := \sum_{T \subseteq N \setminus S} \frac{1}{2^{n-s}} \Delta_S v(T),$$

where $\Delta_S v(T)$ is the S -derivative of v at T defined for any disjoint subsets $S, T \subseteq N$ by

$$\Delta_S v(T) := \sum_{R \subseteq S} (-1)^{s-r} v(R \cup T). \quad (9)$$

This index can be easily expressed in terms of the Möbius transform (2) of v as

$$I_B(v, S) = \sum_{T \supseteq S} \frac{1}{2^{t-s}} a(T).$$

It is noteworthy that, besides the axiomatic approach presented in [13,22], the Banzhaf interaction index can also be defined from the Hammer-Holzman approximation problem. Indeed, as pointed out in [14] and [11, §7], by considering the leading coefficients in (3) of the best s th approximation, for all $s = 0, \dots, n$, we immediately observe that

$$I_B(v, S) = a_s(S).$$

In this section we use the same approach to define a new interaction index from our approximation problem of Lovász extensions. In this sense this new index can be seen as an analog of the Banzhaf interaction index.

Before going on, we recall the concept of cardinal-probabilistic interaction index [7].

Definition 14 *A cardinal-probabilistic interaction index on N is a mapping $I_p : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ such that, for any $S \subseteq N$, there is a family of nonnegative real numbers $\{p_t^s(n)\}_{t=0, \dots, n-s}$ satisfying $\sum_{t=0}^{n-s} p_t^s(n) = 1$, such that*

$$I_p(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n) \Delta_S v(T). \quad (10)$$

It has been proved [7, §4] that a mapping $I_p : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ of the form (10) is a cardinal-probabilistic interaction index on N if and only if, for any integer

$s \in \{0, \dots, n\}$, there exists a uniquely determined cumulative distribution function F_s on $[0, 1]$ such that

$$p_t^s(n) = \int_0^1 x^t (1-x)^{n-s-t} dF_s(x). \quad (11)$$

Moreover, defining $q_t^s := p_{t-s}^s(t) = \int_0^1 x^{t-s} dF_s(x)$ for all integers s, t such that $0 \leq s \leq t \leq n$, we have

$$I_p(v, S) = \sum_{T \supseteq S} q_t^s a(T). \quad (12)$$

The following lemma provides conditions on arbitrary coefficients q_t^s so that a mapping of the form (12) can be a cardinal-probabilistic interaction index.

Lemma 15 *Consider an infinite sequence $\{q_t^s\}_{t \geq s \geq 0}$. Then the following three conditions are equivalent:*

- (i) *we have $\sum_{i=0}^m (-1)^i \binom{m}{i} q_{t+i}^s \geq 0$ for all $m \geq 0$ and all $t \geq s \geq 0$,*
- (ii) *there is a unique cumulative distribution function F_s on $[0, 1]$ such that, for any integer $t \geq s \geq 0$,*

$$q_t^s = \int_0^1 x^{t-s} dF_s(x),$$

- (iii) *for any $N = \{1, \dots, n\}$, the function $I_p : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$, defined in (12) is a cardinal-probabilistic interaction index.*

Moreover, when these conditions are satisfied, then the mapping $I_p : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$, defined in (12), is of the form (10) and the coefficients q_t^s and $p_t^s(n)$ are linked through $q_t^s = p_{t-s}^s(t)$ and

$$p_t^s(n) = \sum_{i=s+t}^n (-1)^{i-s-t} \binom{n-s-t}{i-s-t} q_i^s.$$

Proof. (i) \Leftrightarrow (ii) Follows immediately from the Hausdorff's moment problem (see for instance Akhiezer [1, Theorem 2.6.4]).

(ii) \Rightarrow (iii) Rewriting (12) in terms of v gives

$$I_p(v, S) = \sum_{J \supseteq S} q_j^s \sum_{L \subseteq J} (-1)^{j-l} v(L).$$

Partitioning $L \subseteq J$ into $R \subseteq S$ and $T \subseteq J \setminus S$ and then using (9), we obtain

$$\begin{aligned}
I_p(v, S) &= \sum_{T \subseteq N \setminus S} \left[\sum_{J \supseteq S \cup T} (-1)^{j-s-t} q_j^s \right] \sum_{R \subseteq S} (-1)^{s-r} v(R \cup T) \\
&= \sum_{T \subseteq N \setminus S} p_t^s(n) \Delta_S v(T),
\end{aligned}$$

where the coefficients $p_t^s(n)$ are given by

$$p_t^s(n) = \sum_{j=s+t}^n (-1)^{j-s-t} \binom{n-s-t}{j-s-t} q_j^s = \int_0^1 x^t (1-x)^{n-s-t} dF_s(x),$$

which shows that I_p is a cardinal-probabilistic interaction index.

(iii) \Rightarrow (ii) According to formula (12), we have

$$I_p(v, S) = \sum_{T \supseteq S} q_t^s a(T) = \sum_{T \supseteq S} \left[\int_0^1 x^{t-s} dF_s(x) \right] a(T),$$

which completes the proof by uniqueness of this decomposition. \square

We are now ready to introduce our new interaction index, namely the mapping

$$I_M : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R} : (v, S) \mapsto I_M(v, S) = a_s(S),$$

where $a_s(S)$ are the leading coefficients in (8) of the best s th approximation of \hat{f} , for all $s = 0, \dots, n$.

This immediately leads to the following equivalent definition, which makes use of the classical beta function

$$B(a, b) := \int_0^1 u^{a-1} (1-u)^{b-1} du \quad (a, b > 0).$$

Definition 16 For any $N = \{1, \dots, n\}$, we define the mapping $I_M : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ as

$$I_M(v, S) = \sum_{T \supseteq S} q_t^s a(T),$$

where $q_t^s := \frac{\binom{2s+1}{s+1}}{\binom{s+t+1}{s+1}} = \frac{B(t+1, s+1)}{B(s+1, s+1)}$ for all $t \geq s \geq 0$.

By using Lemma 15, we can easily show that I_M is a cardinal-probabilistic interaction index.

Proposition 17 The function $I_M : \mathcal{F}_N \times 2^N \rightarrow \mathbb{R}$ is a cardinal-probabilistic interaction index, given by

$$I_M(v, S) = \sum_{T \subseteq N \setminus S} p_t^s(n) \Delta_S v(T),$$

where $p_t^s(n) := \frac{B(n-t+1, s+t+1)}{B(s+1, s+1)}$.

Proof. We immediately observe that Lemma 15 applies with the beta distribution

$$F_s(x) = \frac{\int_0^x u^s (1-u)^s du}{B(s+1, s+1)}.$$

Indeed, we have

$$\int_0^1 x^{t-s} dF_s(x) = \frac{1}{B(s+1, s+1)} \int_0^1 x^t (1-x)^s dx = q_t^s.$$

The expression of the coefficients $p_t^s(n)$ then follows immediately from (11). \square

The following proposition shows that the mapping $v \mapsto (I_M(v, S))_{S \subseteq N}$ is invertible and yields the transformation formula from $(I_M(v, S))_{S \subseteq N}$ to $a = (a(S))_{S \subseteq N}$.

Proposition 18 *For any $S \subseteq N$, we have*

$$a(S) = \sum_{T \supseteq S} h_t^s I_M(v, T).$$

with

$$h_t^s = (-1)^{t-s} \frac{\binom{s+t}{t}}{\binom{2t}{t}}.$$

Proof. We only need to show that

$$\sum_{T \supseteq S} q_t^s \sum_{R \supseteq T} h_r^t I_M(v, R) = I_M(v, S).$$

We have

$$\sum_{T \supseteq S} q_t^s \sum_{R \supseteq T} h_r^t I_M(v, R) = \sum_{R \supseteq S} I_M(v, R) \sum_{T: S \subseteq T \subseteq R} q_t^s h_r^t,$$

where the inner sum equals 1 if $S = R$ and 0 if $S \subsetneq R$. Indeed, in the latter case, we have

$$\begin{aligned}
\sum_{T: S \subseteq T \subseteq R} q_t^s h_r^t &= \frac{\binom{2s+1}{s+1}}{\binom{2r}{r}} \frac{(s+1)!}{r!} \sum_{t=s}^r (-1)^{r-t} \binom{r-s}{t-s} \frac{(r+t)!}{(s+t+1)!} \\
&= \frac{\binom{2s+1}{s+1}}{\binom{2r}{r}} \frac{(s+1)!}{r!} \sum_{t=s}^r (-1)^{r-t} \binom{r-s}{t-s} \left[\frac{d^{r-s-1}}{dx^{r-s-1}} x^{r+t} \right]_{x=1} \\
&= \frac{\binom{2s+1}{s+1}}{\binom{2r}{r}} \frac{(s+1)!}{r!} \left[\frac{d^{r-s-1}}{dx^{r-s-1}} x^{r+s} (x-1)^{r-s} \right]_{x=1} \\
&= 0. \quad \square
\end{aligned}$$

In order to conclude, we focus on the power index associated to I_M . The following corollary shows that, incidentally, this power index identifies with the index $W_i(\hat{f})$ previously introduced axiomatically by Grabisch and Labreuche [10, Theorem 2] in the context of multicriteria decision making.

Corollary 19 *The restriction of $I_M(v, \cdot)$ to singletons is given by:*

$$I_M(v, \{i\}) = \sum_{T \subseteq N \setminus \{i\}} \frac{6(n-t)!(t+1)!}{(n+2)!} [v(T \cup \{i\}) - v(T)].$$

Appendix A: Proof of Lemma 6

Observe first that we can assume that S and T are such that $|S \cup T| = n$. Moreover, suppose that S and T are nonempty. Remark that $[0, 1]^n$ is equal almost everywhere to the disjoint union of the sets

$$S_\sigma^\circ := \{\mathbf{x} \in [0, 1]^n \mid x_{\sigma(1)} < \dots < x_{\sigma(n)}\} \quad (\sigma \in \mathfrak{S}_n).$$

Then we have

$$\int_{[0,1]^n} \min_S(\mathbf{x}) \min_T(\mathbf{x}) d\mathbf{x} = \sum_{\sigma \in \mathfrak{S}_n} I_\sigma, \quad (13)$$

where

$$I_\sigma := \int_{S_\sigma^\circ} \min_S(\mathbf{x}) \min_T(\mathbf{x}) d\mathbf{x} \quad (\sigma \in \mathfrak{S}_n),$$

Now, define for every $R \subseteq N$ and $p \in \{1, \dots, |R|\}$ the set

$$\mathfrak{S}_n^{(p)}(R) := \{\sigma \in \mathfrak{S}_n \mid \sigma(1), \dots, \sigma(p) \in R, \sigma(p+1) \notin R\}.$$

It is then clear that there is a decomposition of \mathfrak{S}_n into disjoint subsets

$$\mathfrak{S}_n = \left[\mathfrak{S}_n(S \cap T) \right] \cup \left[\bigcup_{p=1}^{n-t} \mathfrak{S}_n^{(p)}(S \setminus T) \right] \cup \left[\bigcup_{p=1}^{n-s} \mathfrak{S}_n^{(p)}(T \setminus S) \right], \quad (14)$$

where

$$\mathfrak{S}_n(S \cap T) := \{\sigma \in \mathfrak{S}_n \mid \sigma(1) \in S \cap T\}.$$

Moreover we have

$$\begin{cases} |\mathfrak{S}_n(S \cap T)| = |S \cap T| (n-1)! \\ |\mathfrak{S}_n^{(p)}(S \setminus T)| = \binom{n-t}{p} p! t (n-p-1)! . \end{cases}$$

Indeed, every element of $\mathfrak{S}_n(S \cap T)$ corresponds to a choice of $\sigma(1)$ in $S \cap T$, and a permutation of the remaining elements in N . In the same fashion, every element of $\mathfrak{S}_n^{(p)}(S \setminus T)$ is determined by the choice of p elements in $S \setminus T$, a permutation of these elements, the choice of a single element in T and a permutation of the remaining elements. Moreover, for any $\sigma \in \mathfrak{S}_n(S \cap T)$, we have

$$I_\sigma = \int_0^1 \int_0^{x_{\sigma(n)}} \cdots \int_0^{x_{\sigma(2)}} x_{\sigma(1)}^2 dx_{\sigma(1)} \cdots dx_{\sigma(n)} = \frac{2}{(n+2)!},$$

and

$$\sum_{\sigma \in \mathfrak{S}_n(S \cap T)} I_\sigma = \frac{2|S \cap T| (n-1)!}{(n+2)!} = \frac{2(s+t-n)}{n(n+1)(n+2)}. \quad (15)$$

In the same way, for any $\sigma \in \mathfrak{S}_n^{(p)}(S \setminus T)$, we have

$$\begin{aligned} I_\sigma &= \int_0^1 \int_0^{x_{\sigma(n)}} \cdots \int_0^{x_{\sigma(p+2)}} x_{\sigma(p+1)} \int_0^{x_{\sigma(p+1)}} \cdots \int_0^{x_{\sigma(2)}} x_{\sigma(1)} dx_{\sigma(1)} \cdots dx_{\sigma(n)} \\ &= \frac{p+2}{(n+2)!}, \end{aligned}$$

and

$$\begin{aligned} \sum_{p=1}^{n-t} \sum_{\sigma \in \mathfrak{S}_n^{(p)}(S \setminus T)} I_\sigma &= \frac{t}{(n+2)!} \sum_{p=1}^{n-t} \binom{n-t}{p} p! (n-p-1)! (p+2) \\ &= \frac{t! (n-t)!}{(n+2)!} \sum_{p=1}^{n-t} \binom{n-p-1}{t-1} (p+2) \\ &= \frac{t! (n-t)!}{(n+2)!} \left[(n+2) \sum_{p=1}^{n-t} \binom{n-p-1}{t-1} - t \sum_{p=1}^{n-t} \binom{n-p}{t} \right] \\ &= \frac{t! (n-t)!}{(n+2)!} \left[(n+2) \binom{n-1}{t} - t \binom{n}{t+1} \right], \end{aligned}$$

that is,

$$\sum_{p=1}^{n-t} \sum_{\sigma \in \mathfrak{S}_n^{(p)}(S \setminus T)} I_\sigma = \frac{(n-t)(n+2t+2)}{n(n+1)(n+2)(t+1)}. \quad (16)$$

Similarly, we can write

$$\sum_{p=1}^{n-s} \sum_{\sigma \in \mathfrak{S}_n^{(p)}(T \setminus S)} I_\sigma = \frac{(n-s)(n+2s+2)}{n(n+1)(n+2)(s+1)} \quad (17)$$

and by Eqs. (13)–(17), we have

$$\begin{aligned} \int_{[0,1]^n} \min_S(\mathbf{x}) \min_T(\mathbf{x}) d\mathbf{x} &= \sum_{\sigma \in \mathfrak{S}_n(S \cap T)} I_\sigma + \sum_{p=1}^{n-t} \sum_{\sigma \in \mathfrak{S}_n^{(p)}(S \setminus T)} I_\sigma + \sum_{p=1}^{n-s} \sum_{\sigma \in \mathfrak{S}_n^{(p)}(T \setminus S)} I_\sigma \\ &= \frac{s+t+2}{(n+2)(s+1)(t+1)} \end{aligned}$$

as desired. We can easily see that the result still holds if $S = \emptyset$ or $T = \emptyset$. \square

Appendix B: Proof of Lemma 7

We only have to prove that the explicit expression we give for $A_k(\min_S)$ fulfills the conditions of Proposition 3, namely,

$$\langle \min_S - A_k(\min_S), \min_T \rangle = 0 \quad (T \subseteq N, t \leq k),$$

or, equivalently,

$$\sum_{J \subseteq S} I_{J,T} (-1)^{k+j} \binom{k+j+1}{k+1} = 0 \quad (T \subseteq N, t \leq k), \quad (18)$$

where $I_{J,T} := \langle \min_J, \min_T \rangle$ is given explicitly in (6).

Partitioning $J \subseteq S$ into $P \subseteq S \setminus T$ and $Q \subseteq R$ with $R := S \cap T$ (hence $r \leq t \leq k$), the left-hand side of (18) becomes

$$\begin{aligned} & \sum_{P \subseteq S \setminus T} \sum_{Q \subseteq R} I_{P \cup Q, T} (-1)^{k+p+q} \binom{k+p+q+1}{k+1} \\ &= \sum_{p=0}^{k-r+1} \binom{k-r+1}{p} \sum_{q=0}^r \binom{r}{q} \frac{p+q+t+2}{(p+t+2)(p+q+1)(t+1)} (-1)^{k+p+q} \binom{k+p+q+1}{k+1} \\ &= - \sum_{p=0}^{k-r+1} \binom{k-r+1}{p} (-1)^{k-r+1-p} \frac{1}{p+t+2} \sum_{q=0}^r \binom{r}{q} (-1)^{r-q} g(p+q) \end{aligned}$$

where $g : \mathbb{N} \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned}
g(z) &= \frac{z+t+2}{(z+1)(t+1)} \binom{z+k+1}{k+1} \\
&= \frac{1}{k+1} \binom{z+k+1}{k} + \frac{1}{t+1} \binom{z+k+1}{k+1}.
\end{aligned}$$

Consider the difference operator

$$\Delta_n f(n) := f(n+1) - f(n)$$

for functions on \mathbb{N} . It is well-known that we have

$$\Delta_n^k f(n) = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f(n+j) \quad (k \in \mathbb{N}). \quad (19)$$

Applying this to g , we obtain

$$\sum_{q=0}^r \binom{r}{q} (-1)^{r-q} g(p+q) = \Delta_p^r g(p) = \frac{1}{k+1} \binom{p+k+1}{k-r} + \frac{1}{t+1} \binom{p+k+1}{k-r+1}$$

and the left-hand side of (18) becomes

$$- \sum_{p=0}^{k-r+1} \binom{k-r+1}{p} (-1)^{k-r+1-p} \frac{1}{p+t+2} \left(\frac{1}{k+1} \binom{p+k+1}{k-r} + \frac{1}{t+1} \binom{p+k+1}{k-r+1} \right).$$

Applying (19) again, we see that this latter expression can be written as

$$- \left[\Delta_z^{k-r+1} \frac{1}{z+t+2} \left(\frac{1}{k+1} \binom{z+k+1}{k-r} + \frac{1}{t+1} \binom{z+k+1}{k-r+1} \right) \right]_{z=0}.$$

We now show that the expression in brackets is identically zero. We do this by considering two cases:

- If $t = k$ then

$$\begin{aligned}
&\frac{1}{z+t+2} \left(\frac{1}{k+1} \binom{z+k+1}{k-r} + \frac{1}{t+1} \binom{z+k+1}{k-r+1} \right) \\
&= \frac{1}{(k+1)(z+k+2)} \binom{z+k+2}{k-r+1} = \frac{1}{(k+1)(k-r+1)} \binom{z+k+1}{k-r}
\end{aligned}$$

is a polynomial $P_{k-r}(z)$ of degree $k-r$, and $\Delta_z^{k-r+1} P_{k-r}(z) \equiv 0$.

- If $r \leq t \leq k-1$ then

$$\begin{aligned}
&\frac{1}{z+t+2} \left(\frac{1}{k+1} \binom{z+k+1}{k-r} + \frac{1}{t+1} \binom{z+k+1}{k-r+1} \right) \\
&= \frac{1}{z+t+2} \left(\frac{1}{(k+1)(k-r)!} \prod_{i=r+2}^{k+1} (z+i) + \frac{1}{(t+1)(k-r+1)!} \prod_{i=r+1}^{k+1} (z+i) \right)
\end{aligned}$$

is a polynomial $Q_{k-r}(z)$ of degree $k-r$, and $\Delta_z^{k-r+1} Q_{k-r}(z) \equiv 0$. \square

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