Axiomatizations of Lovász extensions of pseudo-Boolean functions

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Abstract

Three important properties in aggregation theory are investigated, namely horizontal min-additivity, horizontal max-additivity, and comonotonic additivity, which are defined by certain relaxations of the Cauchy functional equation in several variables. We show that these properties are equivalent and we completely describe the functions characterized by them. By adding some regularity conditions, these functions coincide with the Lovász extensions vanishing at the origin, which subsume the discrete Choquet integrals. We also propose a simultaneous generalization of horizontal min-additivity and horizontal max-additivity, called horizontal median-additivity, and we describe the corresponding function class. Additional conditions then reduce this class to that of symmetric Lovász extensions, which includes the discrete symmetric Choquet integrals.

Keywords: Aggregation function, discrete Choquet integral, discrete symmetric Choquet integral, Lovász extension, functional equation, Cauchy equation, comonotonic additivity, horizontal additivity, axiomatization.
2010 MSC: 39B22, 39B72 (primary) 26B35 (secondary)

1. Introduction

When we need to merge a set of numerical values into a single one, we usually make use of a so-called aggregation function, e.g., a mean or an averaging function. Various aggregation functions have been proposed in

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the literature, thus giving rise to the growing theory of aggregation which proposes, analyzes, and characterizes aggregation function classes. For recent references, see Beliakov et al. [2] and Grabisch et al. [6].

A noteworthy aggregation function is the so-called discrete Choquet integral, which has been widely investigated in aggregation theory, due to its many applications for instance in decision making (see the edited book [7]). A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $f:\mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\} \qquad (\sigma \in S_n)$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1, \ldots, n\}$. An n-place Choquet integral is simply a nondecreasing (in each variable) n-place Lovász extension which vanishes at the origin. For general background, see [6, §5.4].

The class of n-place Choquet integrals has been axiomatized independently by means of two noteworthy aggregation properties, namely comonotonic additivity (see, e.g., [4]) and horizontal min-additivity (originally called "horizontal additivity", see [3, §2.5]). Recall that a function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be comonotonically additive if, for every $\sigma \in S_n$, we have

$$f(\mathbf{x} + \mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}') \qquad (\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_\sigma).$$

To describe the second property, consider the horizontal min-additive decomposition of the n-tuple $\mathbf{x} \in \mathbb{R}^n$ obtained by "cutting" it with a real number c, namely

$$\mathbf{x} = (\mathbf{x} \wedge c) + (\mathbf{x} - (\mathbf{x} \wedge c)),$$

where $\mathbf{x} \wedge c$ denotes the n-tuple whose ith component is $x_i \wedge c = \min(x_i, c)$. A function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be horizontally min-additive if

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f(\mathbf{x} - (\mathbf{x} \wedge c)) \qquad (\mathbf{x} \in \mathbb{R}^n,\ c \in \mathbb{R}).$$

In this paper we completely describe the function classes axiomatized by each of these properties. More precisely, after recalling the definitions of Lovász extensions, discrete Choquet integrals, and their symmetric versions (Section 2), we show that comonotonic additivity and horizontal min-additivity (as well as its dual counterpart, namely horizontal max-additivity) are actually equivalent properties. We describe the function class axiomatized by these properties and we show that, up to certain regularity conditions (based on those we usually add to the Cauchy functional equation to get linear solutions only), these properties completely characterize those n-place Lovász extensions which vanish at the origin. Nondecreasing monotonicity is then added to characterize the class of n-place Choquet integrals (Section 3). We also introduce a weaker variant of the properties above, called horizontal median-additivity, and determine the function class axiomatized by this new property. Finally, by adding some natural properties, we characterize the class of n-place symmetric Lovász extensions and the subclass of n-place symmetric Choquet integrals (Section 4).

We employ the following notation throughout the paper. Let $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}_- =]-\infty, 0]$. We let I denote a nontrivial (i.e., of positive measure) real interval, possibly unbounded, containing the origin 0. We also introduce the notation $I_+ = I \cap \mathbb{R}_+$, $I_- = I \cap \mathbb{R}_-$, and $I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$. For every $A \subseteq [n]$, the symbol $\mathbf{1}_A$ denotes the n-tuple whose ith component is 1, if $i \in A$, and 0, otherwise. Let also $\mathbf{1} = \mathbf{1}_{[n]}$ and $\mathbf{0} = \mathbf{1}_\varnothing$. The symbols $\wedge$ and $\vee$ denote the minimum and maximum functions, respectively. For every $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{x}^+ = \mathbf{x} \vee 0$ and $\mathbf{x}^- = (-\mathbf{x})^+$. For every function $f\colon I^n \to \mathbb{R}$, we define its diagonal section $\delta_f\colon I \to \mathbb{R}$ by $\delta_f(x) = f(x\mathbf{1})$. More generally, for every $A \subseteq [n]$, we define the function $\delta^A_f\colon I \to \mathbb{R}$ by $\delta^A_f(x) = f(x\mathbf{1}_A)$.

In order not to restrict our framework to functions defined on $\mathbb{R}$, we consider functions defined on intervals I containing 0, in particular of the forms I_+, I_-, and those centered at 0.

It is important to notice that comonotonic additivity as well as horizontal min-additivity and horizontal max-additivity, when restricted to functions $f\colon I^n \to \mathbb{R}$, extend the classical additivity property defined by the Cauchy functional equation for n-place functions

$$f(\mathbf{x} + \mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}') \qquad (\mathbf{x}, \mathbf{x}', \mathbf{x} + \mathbf{x}' \in I^n). \tag{1}$$

In this regard, recall that the general solution $f\colon I^n \to \mathbb{R}$ of the Cauchy equation (1) is given by $f(\mathbf{x}) = \sum_{k=1}^n f_k(x_k)$, where the $f_k\colon I \to \mathbb{R}$ ($k \in [n]$) are arbitrary solutions of the basic Cauchy equation $f_k(x+x') = f_k(x)+f_k(x')$ (see [1, §2–4]). As the following theorem states, under some regularity conditions, each f_k is necessarily a linear function.

Theorem 1. *Let I be a nontrivial real interval, possibly unbounded, contain-*

ing 0. *If* $f\colon I \to \mathbb{R}$ *solves the basic Cauchy equation*

$$f(x+x') = f(x) + f(x') \qquad (x, x', x+x' \in I),$$

then either f *is of the form* $f(x) = cx$ *for some* $c \in \mathbb{R}$*, or the graph of* f *is everywhere dense in* $I \times \mathbb{R}$*. The latter case is excluded as soon as* f *is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure.*

Proof. See Appendix Appendix A. □

(We would like to acknowledge Professor Maksa at the Institute of Mathematics of the University of Debrecen, Hungary, for providing the proof of Theorem 1.)

As we will see in this paper, comonotonic additivity, horizontal min-additivity, and horizontal median-additivity of a function $f\colon I^n \to \mathbb{R}$ force the 1-place functions $\delta_f^A|_{I_+}$, and $\delta_f^A|_{I_-}$ ($A \subseteq [n]$) to solve the basic Cauchy equation. Theorem 1 will hence be useful to describe the corresponding function classes whenever the regularity conditions stated are assumed.

Recall that a function $f\colon I^n \to \mathbb{R}$ is said to be *homogeneous* (resp. *positively homogeneous*) *of degree one* if $f(c\mathbf{x}) = c\,f(\mathbf{x})$ for every $\mathbf{x} \in I^n$ and every $c \in \mathbb{R}$ (resp. every $c > 0$) such that $c\mathbf{x} \in I^n$.

2. Lovász extensions and symmetric Lovász extensions

We now recall the concept of Lovász extension and introduce that of symmetric Lovász extension.

Consider a *pseudo-Boolean function*, that is, a function $\phi\colon \{0,1\}^n \to \mathbb{R}$. The *Lovász extension* of ϕ is the function $f_\phi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ ($\sigma \in S_n$) is the unique affine function which agrees with ϕ at the $n+1$ vertices of the n-simplex $[0,1]^n \cap \mathbb{R}^n_\sigma$ (see [9, 11]). We then have $f_\phi|_{\{0,1\}^n} = \phi$.

We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a *Lovász extension* if there is a function $\phi\colon \{0,1\}^n \to \mathbb{R}$ such that $f = f_\phi$. For any Lovász extension $f\colon \mathbb{R}^n \to \mathbb{R}$, the function $f_0 = f - f(\mathbf{0})$ has the representation

$$f_0(\mathbf{x}) = x_{\sigma(1)}\,\delta_{f_0}(1) + \sum_{i=2}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\,\delta_{f_0}^{A_\sigma^\uparrow(i)}(1) \qquad (\mathbf{x} \in \mathbb{R}^n_\sigma), \tag{2}$$

where $A^{\uparrow}_{\sigma}(i) = \{\sigma(i),\ldots,\sigma(n)\}$, with $A^{\uparrow}_{\sigma}(n+1) = \varnothing$. Indeed, both sides of (2) agree at $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{1}_{A^{\uparrow}_{\sigma}(k)}$ for every $k \in [n]$. Thus we see that f_0 is positively homogeneous of degree one.

An n-place *Choquet integral* is a nondecreasing Lovász extension $f\colon\mathbb{R}^n \to \mathbb{R}$ such that $f(\mathbf{0}) = 0$. It is easy to see that a Lovász extension $f\colon\mathbb{R}^n \to \mathbb{R}$ is a Choquet integral if and only if its underlying pseudo-Boolean function $\phi = f|_{\{0,1\}^n}$ is nondecreasing and vanishes at the origin (see [6, §5.4]).

We now introduce the concept of symmetric Lovász extension. Here "symmetric" does not refer to invariance under a permutation of variables but rather to the role of the origin of $\mathbb{R}^n$ as a symmetry center with respect to the function values whenever the function vanishes at the origin.

The *symmetric Lovász extension* of a pseudo-Boolean function $\phi\colon\{0,1\}^n \to \mathbb{R}$ is the function $\check{f}_{\phi}\colon\mathbb{R}^n \to \mathbb{R}$ defined by

$$\check{f}_{\phi}(\mathbf{x}) = f_{\phi}(\mathbf{0}) + f_{\phi}(\mathbf{x}^+) - f_{\phi}(\mathbf{x}^-) \qquad (\mathbf{x} \in \mathbb{R}^n), \tag{3}$$

where f_{ϕ} is the Lovász extension of ϕ.

We say that a function $f\colon\mathbb{R}^n \to \mathbb{R}$ is a *symmetric Lovász extension* if there is a function $\phi\colon\{0,1\}^n \to \mathbb{R}$ such that $f = \check{f}_{\phi}$. For any symmetric Lovász extension $f\colon\mathbb{R}^n \to \mathbb{R}$, by (2) and (3) the function $f_0 = f - f(\mathbf{0})$ has the representation

$$\begin{aligned} f_0(\mathbf{x}) \;&=\; x_{\sigma(p+1)}\,\delta_{f_0}^{A^{\uparrow}_{\sigma}(p+1)}(1) + \sum_{i=p+2}^{n}\big(x_{\sigma(i)} - x_{\sigma(i-1)}\big)\,\delta_{f_0}^{A^{\uparrow}_{\sigma}(i)}(1) \\ &\quad + x_{\sigma(p)}\,\delta_{f_0}^{A^{\downarrow}_{\sigma}(p)}(1) + \sum_{i=1}^{p-1}\big(x_{\sigma(i)} - x_{\sigma(i+1)}\big)\,\delta_{f_0}^{A^{\downarrow}_{\sigma}(i)}(1) \qquad (\mathbf{x} \in \mathbb{R}^n_{\sigma}), \end{aligned} \tag{4}$$

where $A^{\downarrow}_{\sigma}(i) = \{\sigma(1),\ldots,\sigma(i)\}$, with $A^{\downarrow}_{\sigma}(0) = \varnothing$, and the integer $p \in \{0,1,\ldots,n\}$ is such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$. Thus we see that f_0 is homogeneous of degree one.

Nondecreasing symmetric Lovász extensions vanishing at the origin, also called *discrete symmetric Choquet integrals*, were introduced by Šipoš [12] (see also [6, §5.4]). We observe that a symmetric Lovász extension $f\colon\mathbb{R}^n \to \mathbb{R}$ is a symmetric Choquet integral if and only if its underlying pseudo-Boolean function $\phi = f|_{\{0,1\}^n}$ is nondecreasing and vanishes at the origin.

3. Axiomatizations of Lovász extensions

In the present section we show that, for a class of intervals I, comonotonic additivity is equivalent to horizontal min-additivity (resp. horizontal max-additivity) and we describe the corresponding function class. By adding certain regularity conditions, we then axiomatize the class of n-place Lovász extensions. We first recall these properties.

Let I be a nontrivial real interval, possibly unbounded, containing 0. Two n-tuples $\mathbf{x},\mathbf{x}'\in I^n$ are said to be *comonotonic* if there exists $\sigma\in S_n$ such that $\mathbf{x},\mathbf{x}'\in I^n_\sigma$. A function $f\colon I^n\to\mathbb{R}$ is said to be *comonotonically additive* if, for every comonotonic n-tuples $\mathbf{x},\mathbf{x}'\in I^n$ such that $\mathbf{x}+\mathbf{x}'\in I^n$, we have

$$f(\mathbf{x}+\mathbf{x}')=f(\mathbf{x})+f(\mathbf{x}'). \tag{5}$$

Given $\mathbf{x}\in I^n$ and $c\in I$, let $[\![\mathbf{x}]\!]_c=\mathbf{x}-\mathbf{x}\wedge c$ and $[\![\mathbf{x}]\!]^c=\mathbf{x}-\mathbf{x}\vee c$. We say that a function $f\colon I^n\to\mathbb{R}$ is

- *horizontally min-additive* if, for every $\mathbf{x}\in I^n$ and every $c\in I$ such that $[\![\mathbf{x}]\!]_c\in I^n$, we have

$$f(\mathbf{x})=f(\mathbf{x}\wedge c)+f([\![\mathbf{x}]\!]_c). \tag{6}$$

- *horizontally max-additive* if, for every $\mathbf{x}\in I^n$ and every $c\in I$ such that $[\![\mathbf{x}]\!]^c\in I^n$, we have

$$f(\mathbf{x})=f(\mathbf{x}\vee c)+f([\![\mathbf{x}]\!]^c). \tag{7}$$

We immediately observe that, since any $\mathbf{x}\in I^n$ decomposes into the sum of the comonotonic n-tuples $\mathbf{x}\wedge c$ and $[\![\mathbf{x}]\!]_c$ for every c (i.e., $\mathbf{x}=(\mathbf{x}\wedge c)+[\![\mathbf{x}]\!]_c$), any comonotonically additive function is necessarily horizontally min-additive. Dually, any comonotonically additive function is horizontally max-additive.

We also observe that if $f\colon I^n\to\mathbb{R}$ satisfies any of these properties, then necessarily $f(\mathbf{0})=0$ (just take $\mathbf{x}=\mathbf{x}'=\mathbf{0}$ and $c=0$ in (5)–(7)).

Lemma 2. *If $f\colon I^n\to\mathbb{R}$ is horizontally min-additive (resp. horizontally max-additive) then $\delta_f^A|_{I_+}$ (resp. $\delta_f^A|_{I_-}$) is additive for every $A\subseteq[n]$. Moreover, if I is centered at 0, then δ_f is additive and odd.*

Proof. We prove the result when f is horizontally min-additive; the other claim can be dealt with dually. If $x,x'\in I_+$ is such that $x+x'\in I_+$, then $x\leqslant x+x'$ and, using horizontal min-additivity with $c=x$, we get $\delta_f^A(x+x')=\delta_f^A(x)+\delta_f^A(x')$, which shows that $\delta_f^A|_{I_+}$ is additive.

Assume now that I is centered at 0. If $x<0$ and $x'\geqslant 0$ are such that $x,x'\in I$, then $x\leqslant x+x'<x'$ and, using horizontal min-additivity with $c=x$,

we get $\delta_f(x+x') = \delta_f(x) + \delta_f(x')$. In particular, taking $x' = -x$, we obtain $0 = \delta_f(0) = \delta_f(x-x) = \delta_f(x) + \delta_f(-x)$ and hence $\delta_f(-x) = -\delta_f(x)$.

If $x < 0$ and $x' < 0$ are such that $x, x' \in I$, then, using horizontal min-additivity with $c = x+x'$, we get $\delta_f(x) = \delta_f(x+x') + \delta_f(-x') = \delta_f(x+x') - \delta_f(x')$. Thus δ_f is additive and odd. □

Remark 1. For a horizontally min-additive or horizontally max-additive function $f\colon I^n \to \mathbb{R}$, the function δ_f^A need not be additive. For instance, consider the horizontally min-additive function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1 \wedge x_2$. For $x > 0$ and $x' = -x$, we have

$$\delta_f^{\{1\}}(x-x) = 0 > -x = \delta_f^{\{1\}}(x) + \delta_f^{\{1\}}(-x).$$

Theorem 3. *Assume $0 \in I \subseteq \mathbb{R}_+$ or $I = \mathbb{R}$. A function $f\colon I^n \to \mathbb{R}$ is horizontally min-additive if and only if there exists $g\colon I^n \to \mathbb{R}$, with δ_g and $\delta_g^A|_{I_+}$ additive for every $A \subseteq [n]$, such that, for every $\sigma \in S_n$,*

$$f(\mathbf{x}) = \delta_g(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_g^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \qquad (\mathbf{x} \in I_\sigma^n). \tag{8}$$

In this case, we can choose $g = f$.

Proof. (Necessity) Let $\sigma \in S_n$ and let $\mathbf{x} \in I_\sigma^n$. By repeatedly applying horizontal min-additivity with the successive cut levels

$$x_{\sigma(1)}, x_{\sigma(2)} - x_{\sigma(1)}, \ldots, x_{\sigma(n-1)} - x_{\sigma(n-2)},$$

we obtain

$$\begin{aligned}
f(\mathbf{x}) &= \delta_f(x_{\sigma(1)}) + f(0, x_{\sigma(2)} - x_{\sigma(1)}, \ldots, x_{\sigma(n)} - x_{\sigma(1)}) \\
&= \delta_f(x_{\sigma(1)}) + \delta_f^{A_\sigma^\uparrow(2)}(x_{\sigma(2)} - x_{\sigma(1)}) + f(0, 0, x_{\sigma(3)} - x_{\sigma(2)}, \ldots, x_{\sigma(n)} - x_{\sigma(2)}) \\
&= \cdots \\
&= \delta_f(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_f^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}).
\end{aligned}$$

Thus (8) holds with $g = f$. Moreover, by Lemma 2, δ_f and $\delta_f^A|_{I_+}$ are additive for every $A \subseteq [n]$.

(Sufficiency) Let $\mathbf{x} \in I^n$ and $c \in I$ such that $[\![\mathbf{x}]\!]_c \in I^n$. There is $\sigma \in S_n$ such that $\mathbf{x} \in I_\sigma^n$ and hence $f(\mathbf{x})$ is given by (8), where δ_g and $\delta_g^A|_{I_+}$ are additive for every $A \subseteq [n]$, which implies $g(\mathbf{0}) = 0$.

Suppose first that $c \leqslant x_{\sigma(1)}$. Then we have $f(\mathbf{x}\wedge c)=\delta_g(c)$ and

$$f([\![\mathbf{x}]\!]_c)=\delta_g(x_{\sigma(1)}-c)+\sum_{i=2}^{n}\delta_g^{A^{\uparrow}_{\sigma}(i)}(x_{\sigma(i)}-x_{\sigma(i-1)}).$$

Since δ_g is additive, we finally obtain $f(\mathbf{x}\wedge c)+f([\![\mathbf{x}]\!]_c)=f(\mathbf{x})$.

Now, suppose that there is $p\in[n]$ such that $x_{\sigma(p)}<c\leqslant x_{\sigma(p+1)}$, where $x_{\sigma(n+1)}=\infty$. Then

$$f(\mathbf{x}\wedge c)=\delta_g(x_{\sigma(1)})+\sum_{i=2}^{p}\delta_g^{A^{\uparrow}_{\sigma}(i)}(x_{\sigma(i)}-x_{\sigma(i-1)})+\delta_g^{A^{\uparrow}_{\sigma}(p+1)}(c-x_{\sigma(p)})$$

and

$$f([\![\mathbf{x}]\!]_c)=\delta_g^{A^{\uparrow}_{\sigma}(p+1)}(x_{\sigma(p+1)}-c)+\sum_{i=p+2}^{n}\delta_g^{A^{\uparrow}_{\sigma}(i)}(x_{\sigma(i)}-x_{\sigma(i-1)}).$$

Since $\delta_g^A|_{I_+}$ is additive for every $A\subseteq[n]$, we finally obtain $f(\mathbf{x}\wedge c)+f([\![\mathbf{x}]\!]_c)=f(\mathbf{x})$. □

Similarly, we obtain the following dual characterization.

Theorem 4. *Assume $0\in I\subseteq\mathbb{R}_-$ or $I=\mathbb{R}$. A function $f\colon I^n\to\mathbb{R}$ is horizontally max-additive if and only if there exists $h\colon I^n\to\mathbb{R}$, with δ_h and $\delta_h^A|_{I_-}$ additive for every $A\subseteq[n]$, such that, for every $\sigma\in S_n$,*

$$f(\mathbf{x})=\delta_h(x_{\sigma(n)})+\sum_{i=1}^{n-1}\delta_h^{A^{\downarrow}_{\sigma}(i)}(x_{\sigma(i)}-x_{\sigma(i+1)})\qquad(\mathbf{x}\in I^n_{\sigma}).$$

In this case, we can choose $h=f$.

Theorem 5. *Assume $0\in I\subseteq\mathbb{R}_+$ or $I=\mathbb{R}$ (resp. $0\in I\subseteq\mathbb{R}_-$ or $I=\mathbb{R}$). A function $f\colon I^n\to\mathbb{R}$ is comonotonically additive if and only if it is horizontally min-additive (resp. horizontally max-additive). In this case, δ_f and $\delta_f^A|_{I_+}$ (resp. δ_f and $\delta_f^A|_{I_-}$) are additive for every $A\subseteq[n]$.*

Proof. We already observed that the condition is necessary. Let us now prove that it is sufficient.

Let $\mathbf{x},\mathbf{x}'\in I^n$ be two comonotonic n-tuples, let $\sigma\in S_n$ be such that $\mathbf{x},\mathbf{x}'\in I^n_{\sigma}$, and suppose that $f\colon I^n\to\mathbb{R}$ is horizontally min-additive; the other case can be established dually. By Lemma 2 and Theorem 3, δ_f and $\delta_f^A|_{I_+}$ are additive for every $A \subseteq [n]$ and we have

$$\begin{aligned} & f(\mathbf{x}+\mathbf{x}') \\ = \; & \delta_f(x_{\sigma(1)}+x'_{\sigma(1)}) + \sum_{i=2}^{n} \delta_f^{A_\sigma^{\uparrow}(i)}\big((x_{\sigma(i)}+x'_{\sigma(i)}) - (x_{\sigma(i-1)}+x'_{\sigma(i-1)})\big) \\ = \; & \delta_f(x_{\sigma(1)}) + \delta_f(x'_{\sigma(1)}) + \sum_{i=2}^{n} \delta_f^{A_\sigma^{\uparrow}(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) + \sum_{i=2}^{n} \delta_f^{A_\sigma^{\uparrow}(i)}(x'_{\sigma(i)} - x'_{\sigma(i-1)}) \\ = \; & f(\mathbf{x}) + f(\mathbf{x}'), \end{aligned}$$

which shows that f is comonotonically additive. □

Remark 2. (a) Theorems 3 and 4 provide two equivalent representations of comonotonically additive functions $f\colon \mathbb{R}^n \to \mathbb{R}$ (see Theorem 5). For instance, for a binary comonotonically additive function $f\colon \mathbb{R}^2 \to \mathbb{R}$, we have the representations

$$f(x_1,x_2) = \begin{cases} g(x_1,x_1) + g(0,x_2-x_1), & \text{if } x_1 \leqslant x_2, \\ g(x_2,x_2) + g(x_1-x_2,0), & \text{if } x_1 \geqslant x_2, \end{cases}$$

and

$$f(x_1,x_2) = \begin{cases} h(x_2,x_2) + h(x_1-x_2,0), & \text{if } x_1 \leqslant x_2, \\ h(x_1,x_1) + h(0,x_2-x_1), & \text{if } x_1 \geqslant x_2, \end{cases}$$

where δ_g, $\delta_g^A|_{\mathbb{R}_+}$, δ_h, and $\delta_h^A|_{\mathbb{R}_-}$ are additive for every $A \subseteq [2]$. Figure 1 illustrates both representations in the region $x_1 \leqslant x_2$, which recalls the standard "parallelogram rule" for vector addition. Thus, f is completely determined by its values on the x_1-axis, the x_2-axis, and the line $x_2 = x_1$.

(b) More generally, every comonotonically additive function $f\colon \mathbb{R}^n \to \mathbb{R}$ is completely determined by its values on the lines $\{x\mathbf{1}_A : x \in \mathbb{R}\}$ $(A \subseteq [n])$.

We now axiomatize the class of n-place Lovász extensions. A function $f\colon I^n \to \mathbb{R}$ is a Lovász extension if it is the restriction to I^n of a Lovász extension on $\mathbb{R}^n$.

Theorem 6. *Assume $[0,1] \subseteq I \subseteq \mathbb{R}_+$ or $I = \mathbb{R}$. Let $f\colon I^n \to \mathbb{R}$ be a function and let $f_0 = f - f(\mathbf{0})$. Then f is a Lovász extension if and only if the following conditions hold:*

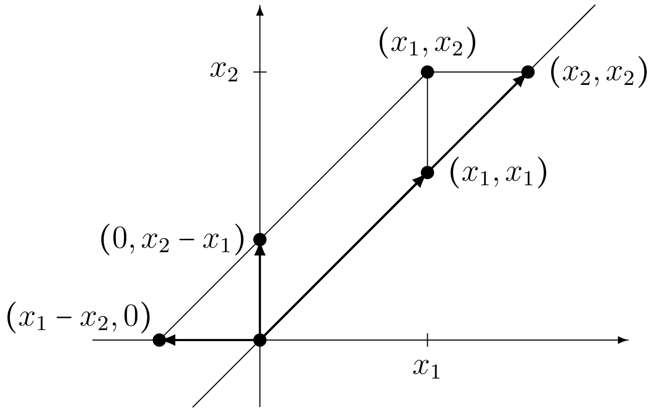

Figure 1: Representations in the region $x_1 \leqslant x_2$

(i) *f_0 is comonotonically additive or horizontally min-additive (or horizontally max-additive if $I = \mathbb{R}$).*

(ii) *Each of the maps δ_{f_0} and $\delta_{f_0}^A|_{I_+}$ ($A \subseteq [n]$) is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure.*

If $I = \mathbb{R}$, then the set I_+ can be replaced by I_- in (ii). Finally, Condition (ii) holds whenever Condition (i) holds and $\delta_{f_0}^A$ is positively homogeneous of degree one for every $A \subseteq [n]$.

Proof. (Necessity) Follows from (2) and Theorems 3 and 5.

(Sufficiency) By Theorems 3 and 5, if (i) holds, then δ_{f_0} and $\delta_{f_0}^A|_{I_+}$ are additive for every $A \subseteq [n]$ and, for every $\sigma \in S_n$ and every $\mathbf{x} \in I_\sigma^n$, we have

$$f_0(\mathbf{x}) = \delta_{f_0}(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_{f_0}^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}).$$

By Theorem 1, if (ii) holds, then $\delta_{f_0}(x) = x\,\delta_{f_0}(1)$ for every $x \in I$ and $\delta_{f_0}^A(x) = x\,\delta_{f_0}^A(1)$ for every $x \in I_+$. By (2) it follows that f_0 is a Lovász extension such that $f_0(\mathbf{0}) = 0$.

Now suppose that $I = \mathbb{R}$ and that I_+ is replaced by I_- in (ii). Then by Theorems 4 and 5, δ_{f_0} and $\delta_{f_0}^A|_{I_-}$ are additive for every $A \subseteq [n]$ and, for every $\sigma \in S_n$ and every $\mathbf{x} \in I_\sigma^n$, we have

$$f_0(\mathbf{x}) = \delta_{f_0}(x_{\sigma(n)}) + \sum_{i=1}^{n-1} \delta_{f_0}^{A_\sigma^\downarrow(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}). \tag{9}$$

By Theorem 1, if (ii) holds, then $\delta_{f_0}(x) = x\,\delta_{f_0}(1)$ for every $x \in I$ and $\delta^A_{f_0}(x) = -x\,\delta^A_{f_0}(-1)$ for every $x \in I_-$. Thus (9) becomes

$$f_0(\mathbf{x}) = x_{\sigma(n)}\,\delta_{f_0}(1) + \sum_{i=1}^{n-1}(x_{\sigma(i+1)} - x_{\sigma(i)})\,\delta_{f_0}^{A^{\downarrow}_{\sigma}(i)}(-1)$$

from which we derive $\delta_{f_0}^{A^{\uparrow}_{\sigma}(i+1)}(1) = \delta_{f_0}(1) + \delta_{f_0}^{A^{\downarrow}_{\sigma}(i)}(-1)$ and hence

$$f_0(\mathbf{x}) = x_{\sigma(n)}\,\delta_{f_0}(1) + \sum_{i=2}^{n}(x_{\sigma(i)} - x_{\sigma(i-1)})\,\big(\delta_{f_0}^{A^{\uparrow}_{\sigma}(i)}(1) - \delta_{f_0}(1)\big).$$

Again, we retrieve (2), thus showing that f_0 is a Lovász extension such that $f_0(\mathbf{0}) = 0$.

Finally, if (i) holds and $\delta^A_{f_0}$ is positively homogeneous of degree one for every $A \subseteq [n]$, then we have $\delta_{f_0}(x) = x\,\delta_{f_0}(1)$ for every $x \in I_+$ and even for every $x \in I$ if $I = \mathbb{R}$ since then δ_{f_0} is odd by Lemma 2. Also, we have $\delta^A_{f_0}(x) = x\,\delta^A_{f_0}(1)$ for every $x \in I_+$ and, if $I = \mathbb{R}$, $\delta^A_{f_0}(x) = -x\,\delta^A_{f_0}(-1)$ for every $x \in I_-$. □

Remark 3. (a) Since any Lovász extension vanishing at the origin is positively homogeneous of degree one, Condition (ii) of Theorem 6 can be replaced by the stronger condition: f_0 is positively homogeneous of degree one.

(b) Axiomatizations of the class of n-place Choquet integrals can be immediately derived from Theorem 6 by adding nondecreasing monotonicity. Similar axiomatizations using comonotonic additivity (resp. horizontal min-additivity) were obtained by de Campos and Bolaños [4] (resp. by Benvenuti et al. [3, §2.5]).

(c) The concept of comonotonic additivity appeared first in Dellacherie [5] and then in Schmeidler [10]. The concept of horizontal min-additivity was previously considered by Šipoš [12] and then by Benvenuti et al. [3, §2.3] where it was called “horizontal additivity”.

4. Axiomatizations of symmetric Lovász extensions

In this final section we introduce a simultaneous generalization of horizontal min-additivity and horizontal max-additivity, called horizontal median-additivity, and we describe the corresponding function class. By adding further conditions, we then axiomatize the class of n-place symmetric Lovász extensions.

Horizontal median-additivity in a sense combines horizontal min-additivity and horizontal max-additivity by using two cut levels that are symmetric with respect to the origin. Formally, assuming that I is centered at 0, we say that a function $f\colon I^n \to \mathbb{R}$ is *horizontally median-additive* if, for every $\mathbf{x} \in I^n$ and every $c \in I_+$, we have

$$f(\mathbf{x}) = f\big(\mathrm{med}(-c, \mathbf{x}, c)\big) + f\big([\![\mathbf{x}]\!]_c\big) + f\big([\![\mathbf{x}]\!]^{-c}\big), \tag{10}$$

where $\mathrm{med}(-c, \mathbf{x}, c)$ is the n-tuple whose ith component is the middle value of $\{-c, x_i, c\}$.

Since any $\mathbf{x} \in I^n$ decomposes into the sum of the comonotonic n-tuples $\mathrm{med}(-c, \mathbf{x}, c) + [\![\mathbf{x}]\!]^{-c} = \mathbf{x} \wedge c$ and $[\![\mathbf{x}]\!]_c$ for every $c \in I_+$ (i.e., $\mathbf{x} = \mathrm{med}(-c, \mathbf{x}, c) + [\![\mathbf{x}]\!]_c + [\![\mathbf{x}]\!]^{-c}$), any comonotonically additive function is necessarily horizontally median-additive. However, we will see (see Proposition 12 below) that the converse claim is not true.

We also observe that if $f\colon I^n \to \mathbb{R}$ is horizontally median-additive, then necessarily $f(\mathbf{0}) = 0$ (take $\mathbf{x} = \mathbf{0}$ and $c = 0$ in (10)). We then see that

$$f(\mathbf{x}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-) \qquad (\mathbf{x} \in I^n) \tag{11}$$

(take $c = 0$ in (10)). This observation motivates the following definitions.

Assume that I is centered at 0. We say that a function $f\colon I^n \to \mathbb{R}$ is

- *positively comonotonically additive* if (5) holds for every comonotonic n-tuples $\mathbf{x}, \mathbf{x}' \in I_+^n$ such that $\mathbf{x} + \mathbf{x}' \in I_+^n$.
- *negatively comonotonically additive* if (5) holds for every comonotonic n-tuples $\mathbf{x}, \mathbf{x}' \in I_-^n$ such that $\mathbf{x} + \mathbf{x}' \in I_-^n$.
- *positively horizontally min-additive* if (6) holds for every $\mathbf{x} \in I_+^n$ and every $c \in I_+$.
- *negatively horizontally max-additive* if (7) holds for every $\mathbf{x} \in I_-^n$ and every $c \in I_-$.

We observe immediately that if $f\colon I^n \to \mathbb{R}$ satisfies any of the four properties above, then $f(\mathbf{0}) = 0$.

Lemma 7. *Assume that I is centered at 0. For any function $f\colon I^n \to \mathbb{R}$, the following assertions are equivalent.*

(i) *f is horizontally median-additive.*

(ii) *f is positively horizontally min-additive, negatively horizontally max-additive, and satisfies (11).*

(iii) *There exists a positively horizontally min-additive function $g\colon I^n \to \mathbb{R}$ and a negatively horizontally max-additive function $h\colon I^n \to \mathbb{R}$ such that $f(\mathbf{x}) = g(\mathbf{x}^+) + h(-\mathbf{x}^-)$ for every $\mathbf{x} \in I^n$.*

Proof. $(i) \Rightarrow (ii)$ If f is horizontally median-additive, then f satisfies (11) and $f(\mathbf{0}) = 0$. Also, f is positively horizontally min-additive. Indeed, for every $\mathbf{x} \in I_+^n$ and every $c \in I_+$, we have $f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f([\![\mathbf{x}]\!]_c) + f(\mathbf{0})$. Dually, we can also show that f is negatively horizontally max-additive.

$(ii) \Rightarrow (iii)$ It suffices to take $g = h = f$.

$(iii) \Rightarrow (i)$ Assume that (iii) holds. Then, for every $\mathbf{x} \in I^n$, we have $f(\mathbf{x}^+) = g(\mathbf{x}^+)$ and $f(-\mathbf{x}^-) = h(-\mathbf{x}^-)$ and hence f satisfies (11). Let $\mathbf{x} \in I^n$ and $c \in I_+$. Applying (11) to $\mathrm{med}(-c, \mathbf{x}, c)$, we obtain

$$f\big(\mathrm{med}(-c,\mathbf{x},c)\big) = f(\mathbf{x}^+ \wedge c) + f\big((-\mathbf{x}^-) \vee (-c)\big). \tag{12}$$

By (11) and positive horizontal min-additivity and negative horizontal max-additivity, we finally have

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}^+) + f(-\mathbf{x}^-) \\ &= f(\mathbf{x}^+ \wedge c) + f([\![\mathbf{x}^+]\!]_c) + f\big((-\mathbf{x}^-) \vee (-c)\big) + f([\![-\mathbf{x}^-]\!]^{-c}) \\ &= f\big(\mathrm{med}(-c,\mathbf{x},c)\big) + f([\![\mathbf{x}]\!]_c) + f([\![\mathbf{x}]\!]^{-c}) \qquad \text{(by (12))}, \end{aligned}$$

which shows that f is horizontally median-additive. □

By Lemma 7, to describe the class of horizontally median-additive functions, it suffices to describe the class of positively horizontally min-additive functions and that of negatively horizontally max-additive functions. These descriptions are given in the following two theorems. The proofs are similar to those of Theorems 3, 4, and 5 and hence are omitted.

Theorem 8. *Assume that I is centered at 0. For any function $f\colon I^n \to \mathbb{R}$, the following assertions are equivalent.*

(i) *f is positively horizontally min-additive.*

(ii) *f is positively comonotonically additive.*

(*iii*) *There exists* $g: I^n \to \mathbb{R}$*, with* $\delta_g^A|_{I_+}$ *additive for every* $A \subseteq [n]$*, such that, for every* $\sigma \in S_n$*,*

$$f(\mathbf{x}) = \delta_g(x_{\sigma(1)}) + \sum_{i=2}^{n} \delta_g^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \qquad (\mathbf{x} \in I_\sigma^n \cap I_+^n).$$

In this case, we can choose $g = f$*.*

Theorem 9. *Assume that* I *is centered at* 0*. For any function* $f: \mathbb{R}^n \to \mathbb{R}$*, the following assertions are equivalent.*

(*i*) f *is negatively horizontally max-additive.*

(*ii*) f *is negatively comonotonically additive.*

(*iii*) *There exists* $h: I^n \to \mathbb{R}$*, with* $\delta_h^A|_{I_-}$ *additive for every* $A \subseteq [n]$*, such that, for every* $\sigma \in S_n$*,*

$$f(\mathbf{x}) = \delta_h(x_{\sigma(n)}) + \sum_{i=1}^{n-1} \delta_h^{A_\sigma^\downarrow(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}) \qquad (\mathbf{x} \in I_\sigma^n \cap I_-^n).$$

In this case, we can choose $h = f$*.*

The following theorem gives a description of the class of horizontally median-additive functions.

Theorem 10. *Assume that* I *is centered at* 0*. For any function* $f: I^n \to \mathbb{R}$*, the following assertions are equivalent.*

(*i*) f *is horizontally median-additive.*

(*ii*) f *is positively horizontally min-additive (or positively comonotonically additive), negatively horizontally max additive (or negatively comonotonically additive), and satisfies (11).*

(*iii*) *There exist* $g: I^n \to \mathbb{R}$ *and* $h: I^n \to \mathbb{R}$*, with* $\delta_g^A|_{I_+}$ *and* $\delta_h^A|_{I_-}$ *additive for every* $A \subseteq [n]$*, such that, for every* $\sigma \in S_n$*,*

$$\begin{aligned} f(\mathbf{x}) \ = \ & \delta_g^{A_\sigma^\uparrow(p+1)}(x_{\sigma(p+1)}) + \sum_{i=p+2}^{n} \delta_g^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \\ & + \delta_h^{A_\sigma^\downarrow(p)}(x_{\sigma(p)}) + \sum_{i=1}^{p-1} \delta_h^{A_\sigma^\downarrow(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}) \qquad (\mathbf{x} \in I_\sigma^n) \end{aligned} \tag{13}$$

where $p \in \{0, \ldots, n\}$ *is such that* $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$*. In this case, we can choose* $g = h = f$*.*

Proof. $(i) \Leftrightarrow (ii) \Rightarrow (iii)$ Follows from Lemma 7 and Theorems 8 and 9.

$(iii) \Rightarrow (ii)$ Considering (13) with $g = h = f$, we see immediately that f satisfies (11). We then conclude by Theorems 8 and 9. □

We now axiomatize the class of n-place symmetric Lovász extensions. A function $f\colon I^n \to \mathbb{R}$ is a symmetric Lovász extension if it is the restriction to I^n of a symmetric Lovász extension on $\mathbb{R}^n$.

Theorem 11. *Assume that I is centered at 0 with $[-1,1] \subseteq I$. Let $f\colon I^n \to \mathbb{R}$ be a function and let $f_0 = f - f(\mathbf{0})$. Then f is a symmetric Lovász extension if and only if the following conditions hold:*

- (i) *f_0 is horizontally median-additive.*
- (ii) *Each of the maps $\delta_{f_0}^A|_{I_+}$ and $\delta_{f_0}^A|_{I_-}$ $(A \subseteq [n])$ is continuous at a point or monotonic or Lebesgue measurable or bounded from one side on a set of positive measure.*
- (iii) *$\delta_{f_0}^A(-1) = -\delta_{f_0}^A(1)$ for every $A \subseteq [n]$.*

Conditions (ii) and (iii) hold together if and only if $\delta_{f_0}^A$ is homogeneous of degree one for every $A \subseteq [n]$.

Proof. (Necessity) Follows from (4) and Theorem 10.

(Sufficiency) By Theorem 10, if (i) holds, then $\delta_{f_0}^A|_{I_+}$ and $\delta_{f_0}^A|_{I_-}$ are additive for every $A \subseteq [n]$ and, for every $\sigma \in S_n$ and every $\mathbf{x} \in I_\sigma^n$, we have

$$\begin{aligned} f_0(\mathbf{x}) &= \delta_{f_0}^{A_\sigma^\uparrow(p+1)}(x_{\sigma(p+1)}) + \sum_{i=p+2}^{n} \delta_{f_0}^{A_\sigma^\uparrow(i)}(x_{\sigma(i)} - x_{\sigma(i-1)}) \\ &\quad + \delta_{f_0}^{A_\sigma^\downarrow(p)}(x_{\sigma(p)}) + \sum_{i=1}^{p-1} \delta_{f_0}^{A_\sigma^\downarrow(i)}(x_{\sigma(i)} - x_{\sigma(i+1)}), \end{aligned}$$

where $p \in \{0,\dots,n\}$ is such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$.

By Theorem 1, if (ii) holds, then $\delta_{f_0}^A(x) = x\,\delta_{f_0}^A(1)$ for every $x \in I_+$ and $\delta_{f_0}^A(x) = -x\,\delta_{f_0}^A(-1)$ for every $x \in I_-$. Thus we have

$$\begin{aligned} f_0(\mathbf{x}) &= x_{\sigma(p+1)}\,\delta_{f_0}^{A_\sigma^\uparrow(p+1)}(1) + \sum_{i=p+2}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)})\,\delta_{f_0}^{A_\sigma^\uparrow(i)}(1) \\ &\quad - x_{\sigma(p)}\,\delta_{f_0}^{A_\sigma^\downarrow(p)}(-1) - \sum_{i=1}^{p-1} (x_{\sigma(i)} - x_{\sigma(i+1)})\,\delta_{f_0}^{A_\sigma^\downarrow(i)}(-1). \end{aligned}$$

Using (iii) and (4), it follows that f_0 is a symmetric Lovász extension such that $f_0(\mathbf{0}) = 0$.

Finally, (ii) and (iii) imply that $\delta_{f_0}^A(x) = x\,\delta_{f_0}^A(1)$ and $\delta_{f_0}^A(-x) = x\,\delta_{f_0}^A(-1) = -x\,\delta_{f_0}^A(1)$ for every $x \in I_+$, which means that $\delta_{f_0}^A$ is homogeneous of degree one. Conversely, if $\delta_{f_0}^A$ is homogeneous of degree one for every $A \subseteq [n]$, then (ii) and (iii) hold trivially. $\square$

Remark 4. (a) Since any symmetric Lovász extension vanishing at the origin is homogeneous of degree one, Conditions (ii) and (iii) of Theorem 11 can be replaced by the stronger condition: f_0 is homogeneous of degree one.

(b) Axiomatizations of the class of n-place symmetric Choquet integrals can be immediately derived by adding nondecreasing monotonicity.

The following proposition gives a condition for a symmetric Lovász extension to be a Lovász extension. This clearly shows that horizontal median-additivity does not imply comonotonic additivity.

Proposition 12. *Let $f:\mathbb{R}^n \to \mathbb{R}$ be a Lovász extension and let $g:\mathbb{R}^n \to \mathbb{R}$ be a symmetric Lovász extension such that $f|_{\{0,1\}^n} = g|_{\{0,1\}^n}$. Let also $f_0 = f - f(\mathbf{0})$. Then we have $f = g$ if and only if $f_0(-\mathbf{x}) = -f_0(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}_+^n$ (or equivalently, for every $\mathbf{x} \in \mathbb{R}_-^n$).*

Proof. Since f_0 is comonotonically additive, for every $\mathbf{x} \in \mathbb{R}^n$ we have

$$f(\mathbf{x}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-) - f(\mathbf{0}).$$

Combining this with (3), that is

$$g(\mathbf{x}) = f(\mathbf{x}^+) - f(\mathbf{x}^-) + f(\mathbf{0}),$$

we see that $f = g$ if and only if $f_0(-\mathbf{x}^-) = -f_0(\mathbf{x}^-)$, which completes the proof. $\square$

Acknowledgments

The authors would like to thank Professor Maksa for useful discussions and his precious comments. This work was supported by the internal research project F1R-MTH-PUL-09MRDO of the University of Luxembourg.

Appendix A. Proof of Theorem 1

We first observe that Theorem 1 holds when $I = \mathbb{R}$; see [1, §2]. To see that the result still holds for any nontrivial interval I containing 0, we consider the following two lemmas.

Lemma 13. *Let J be a nontrivial real interval containing 0, and let $I = J+J = \{x+y : x, y \in J\}$. If a function $f: I \to \mathbb{R}$ satisfies $f(x+x') = f(x)+f(x')$ for $x, x' \in J$, then f can be uniquely extended onto $\mathbb{R}$ to an additive function.*

Proof. See [8, Theorem 13.5.3]. □

Lemma 14. *Let J be a nontrivial real interval containing 0, and let $I = J+J = \{x+y : x, y \in J\}$. If a function $f: I \to \mathbb{R}$ satisfies $f(x+x') = f(x)+f(x')$ for $x, x' \in J$ and there is no $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in J$, then the graph of f is everywhere dense in $I \times \mathbb{R}$.*

Proof. By Lemma 13, there is an additive function $g: \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for $x \in I$. If there existed $c \in \mathbb{R}$ such that $g(x) = cx$ for all $x \in \mathbb{R}$ then $f(x) = cx$ would follow for all $x \in I$. Since $0 \in J$, we would have $J \subseteq I$ and hence $f(x) = cx$ for all $x \in J$, a contradiction. Therefore, since Theorem 1 holds when $I = \mathbb{R}$, the graph G_g of g must be dense in $\mathbb{R}^2$. Hence, for any $(x, y) \in I \times \mathbb{R}$ there is a sequence $(x_n, g(x_n)) \in G_g$ such that $(x_n, g(x_n)) \to (x, y)$ as $n \to \infty$. since I is an interval we may (and do) assume that $x_n \in I$ for all n. Thus $(x_n, f(x_n)) = (x_n, g(x_n)) \to (x, y)$ as $n \to \infty$, which proves that the graph of f is dense in $I \times \mathbb{R}$. □

Proof of Theorem 1. Let $J = \{x/2 : x \in I\}$. By definition, f satisfies $f(x + x') = f(x) + f(x')$ for $x, x' \in J$. By Lemma 14, if there is no $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in J$, then the graph of f is everywhere dense in $I \times \mathbb{R}$. Since Theorem 1 holds when $I = \mathbb{R}$, the second claim follows trivially. □

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