

# QUASI-FUCHSIAN MANIFOLDS WITH PARTICLES

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**ABSTRACT.** We consider 3-dimensional hyperbolic cone-manifolds which are “convex co-compact” in a natural sense, with cone singularities along infinite lines. Such singularities are sometimes used by physicists as models for massive spinless point particles. We prove an infinitesimal rigidity statement when the angles around the singular lines are less than  $\pi$ : any infinitesimal deformation changes either these angles, or the conformal structure at infinity with marked points corresponding to the endpoints of the singular lines. Moreover, any small variation of the conformal structure at infinity and of the singular angles can be achieved by a unique small deformation of the cone-manifold structure. These results hold also when the singularities are along a graph, i.e., for “interacting particles”.

## 1. INTRODUCTION

*Quasi-Fuchsian hyperbolic 3-manifolds.* Let  $M$  be the interior of a compact manifold with boundary. A complete hyperbolic metric  $g$  on  $M$  is *convex co-compact* if  $M$  contains a compact subset  $K$  which is *convex*: any geodesic segment  $c$  in  $(M, g)$  with endpoints in  $K$  is contained in  $K$ . Such convex co-compact metrics (considered up to isotopy) determine a conformal structure on the boundary at infinity of  $M$  (also considered up to isotopy), i.e., an element of the Teichmüller space of  $\partial M$ . According to a celebrated theorem of Ahlfors and Bers (see e.g., [2, 1]), convex co-compact metrics are uniquely determined by the induced conformal structure at infinity, and all conformal structures on  $\partial M$  can be achieved in this way.

A topologically simple but already interesting instance is obtained when  $M$  is the product of a closed surface  $\Sigma$  by an interval. The space of convex co-compact metrics on  $\Sigma \times \mathbb{R}$ , which are called “quasi-Fuchsian” metrics, is parametrized by the product of two copies of the Teichmüller space  $\mathcal{T}_\Sigma$  of  $\Sigma$ , one corresponding to each boundary component of  $M$ . In this manner the geometry of quasi-Fuchsian manifolds has much to say on the Teichmüller theory of  $\Sigma$ ; among many examples we can mention the fact that the renormalized volume of quasi-Fuchsian metrics provides a Kähler potential for the Weil-Petersson metric on Teichmüller space, see [29].

*Teichmüller theory with marked points.* The main motivation here is to extend these ideas by replacing the Teichmüller space  $\mathcal{T}_\Sigma$  of  $\Sigma$  by its Teichmüller space with  $N$  marked points,  $\mathcal{T}_{\Sigma, N}$ , and by attaching to each marked point an angle in  $(0, \pi)$ . The quasi-Fuchsian metrics

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on  $\Sigma \times \mathbb{R}$  are then replaced by hyperbolic metrics with conical singularities along infinite lines going from one connected component of the boundary to the other; the marked points on each boundary component are the endpoints of those infinite lines, and the numbers attached to the marked points are the angles around the singular lines. We require that the total angle around each singular curve is less than  $\pi$ , a restriction which appears naturally at different stages. In the limit case where those angles tend to 0 we obtain geometrically finite hyperbolic manifolds with rank one cusps.

The main result of this paper is the first step one has to take when extending the quasi-Fuchsian theory to encompass those manifolds with conical singularities along infinite lines: we prove a local deformation result, namely that the small deformations of the “quasi-Fuchsian cone-manifolds” described above are parametrized by the small variations of the angles at the singular lines and of the conformal structures at infinity, marked by the endpoints of the singular lines. The results are actually stated in a more general context of “convex co-compact cone-manifolds”, again with “particles” – cone singularities along infinite arcs. Note that some results in this direction, albeit in special cases of manifolds with finite volume, were obtained by Weiss [34].

Our results actually hold for cone-manifolds with singularities along graphs which have a finite number of vertices, still under the condition that the cone angle at each singular curve is less than  $\pi$  (as in [33, 34]). Under this condition the singular graph has valence 3. The vertices can be understood heuristically as “interactions” of “particles”.

We now describe in a more detailed way the content of the paper.

*Hyperbolic cone-manifolds.* Hyperbolic cone-manifolds were introduced by Thurston (see [30]). They are basically hyperbolic manifolds which are singular along a stratified subset. In the special case of 3-dimensional cone-manifolds with a singular set which is a disjoint union of curves, a simple definition can be used (and is given at the beginning of section 3). In this case, the behavior of the metric in the neighborhood of a point of the singular locus is entirely determined by a real number, the total angle around the singularity, which is locally constant on the singular locus. When the singularity is along a graph, the behavior of the metric close to the vertices is more complicated. However, under the condition that the cone angles are less than  $\pi$ , the valence of the singular graph is 3, and it remains true that the cone angles determine completely a local model of the metric.

Hodgson and Kerckhoff [14] considered compact such hyperbolic cone-manifolds, for which the singular set is a disjoint union of closed curves. They showed that, when the total angle around each singular curve is less than  $2\pi$ , those manifolds are infinitesimally rigid: any non-trivial small deformation induces a deformation of the complex angle around at least one of the connected curves in the singular locus. Weiss [33] showed that the same rigidity result holds when the singular locus is a graph, under the condition that the angles at the singular curves are less than  $\pi$ .

The rigidity result of Hodgson and Kerckhoff was extended by Bromberg [6], who considered complete, non-compact hyperbolic cone-manifolds, again with singular locus a disjoint union of closed curves, but also with some non-singular infinite ends similar to the ends of convex co-compact hyperbolic 3-manifolds. In this more general case, any non-trivial

infinitesimal deformation of the hyperbolic metric induces a non-trivial deformation either of the conformal structure at infinity, or of the angle around at least one of the connected curves in the singular locus.

*Convex co-compact manifolds with particles.* We consider in this paper complete hyperbolic cone-manifolds, with singularities along a disjoint union of open curves, or possibly along a graph. The difference with the situation considered by Hodgson and Kerckhoff [14] or by Bromberg [6] is that the curves in the singular locus are not compact, but are instead complete, with endpoints on the boundary at infinity. A precise definition is given in section 3, it includes a description of a neighborhood of the endpoints, ensuring in particular that two singular curves can not be asymptotic to each other.

It might be possible to extend the setting considered here to include hyperbolic cone-manifolds of finite volume, with cusps. This more general setting is left for further investigations.

We will use the following definition of convexity, which is stronger than the condition of having locally convex boundary.

**Definition 1.1.** Let  $M$  be a hyperbolic cone-manifold. A subset  $C \subset M$  is *convex* if it is non-empty and any geodesic segment in  $M$  with endpoints in  $C$  is contained in  $C$ .

For instance, with this definition, points are not convex — unless  $M$  is topologically trivial. It follows from the definition that the intersection of two convex subsets of  $M$  is either empty or convex. We show in the appendix that, when the angles at the singular curves of  $M$  are less than  $\pi$  and under some weak topological assumptions on  $M$ , any closed geodesic in  $M$  is contained in any convex subset. It follows that the intersection of two convex subsets of  $M$  is convex (unless  $\pi_1(M) = 0$ ).

**Definition 1.2.** Let  $M$  be a complete, non-compact hyperbolic cone-manifold, with singular locus a graph with a finite number of vertices.  $M$  is *convex co-compact with particles* if the angles at each singular line is less than  $\pi$  and if  $M$  contains a compact subset  $C$  which is convex.

It follows from this definition, and more precisely from Lemma A.11 in the appendix, that  $M$  is then homeomorphic to the interior of a compact manifold with boundary which we will call  $N$  ( $N$  is actually homeomorphic to the compact convex subset  $C$  in the definition). The singular set of  $M$  corresponds under the homeomorphism with a graph  $\Gamma$  embedded in  $N$ , such that vertices of  $\Gamma$  adjacent to only one edge are in the boundary of  $N$ .

We will in particular use the term *interacting particles* when the singular locus has at least one vertex, and *non-interacting particles* when the singular locus is a disjoint union of curves.

A special case is of interest to us, although it does not play a central role here (except in the title).

**Definition 1.3.** A convex co-compact manifold with particles  $M$  is called a *quasifuchsian manifold with particles* if it is homeomorphic to  $\Sigma \times \mathbb{R}$ , where  $\Sigma$  is a closed surface, with the singular locus corresponding to lines  $\{x_i\} \times \mathbb{R}$ , for  $x_1, \dots, x_n \in \Sigma$ .

We hope that the local rigidity result proved here, along with some compactness statements that will be stated elsewhere, can be used to extend to quasifuchsian manifolds with particles certain results which are either classical or known for convex co-compact (non-singular) hyperbolic manifolds: a Bers-type theorem on the possible conformal structures at infinity, and statements on what induced metrics or measured bending laminations can be prescribed on the boundary of the convex core. Proving those results in the general setting of convex co-compact hyperbolic cone-manifolds appears to be more difficult.

Given a hyperbolic manifold (which is not necessarily complete, so that this includes the complement of the singular set in the cone-manifolds considered here) there is a basic setting, recalled in section 2, which can be used to understand its infinitesimal deformations. It uses a description of those deformations as closed 1-forms with values in a vector bundle of “local” Killing fields defined on the manifold, called  $E$  here, an idea going back to Weil [31] and recently used for cone-manifolds by Hodgson and Kerckhoff [14].

Among those deformations, some do not change the underlying geometry of the manifold; they are the differentials (with respect to a natural flat connection on  $E$ ) of sections of  $E$ , they are called *trivial* deformations.

*Main statements.* The first result of this paper is an infinitesimal rigidity result, stating that infinitesimal deformations of one of the cone-manifolds considered here always induces a infinitesimal variation of one of the “parameters”: the conformal structure at infinity, or the angle around the singular curves.

**Theorem 1.4.** *Let  $(M, g)$  be a convex co-compact manifold with particles. Any non-trivial infinitesimal deformation of the hyperbolic cone-metric  $g$  induces a non-trivial deformation of the conformal structure with marked points at infinity or of one of the angles around the singular lines.*

The second, related, result is that the small deformations of these “parameters” are actually in one-to-one correspondence with the small deformations of the cone-manifolds. Let  $\mathcal{R}(M_r)$  be the representation variety of  $\pi_1(M_r)$  into  $PSL_2(\mathbb{C})$  and  $\rho$  the holonomy representation of  $M_r$ . We call  $\mathcal{R}_{cone}(M_r)$  the subset of representations for which the holonomy of meridians of the singular curves have no translation component, that is, the holonomy of the meridians are rotations. Thus  $\rho \in \mathcal{R}_{cone}(M_r)$ , and, in the neighborhood of  $\rho$ , the points of  $\mathcal{R}_{cone}(M_r)$  are precisely the holonomies of cone-manifolds.

**Theorem 1.5.** *Let  $(M, g)$  be a convex co-compact manifold with particles. Let  $c$  be the conformal structure at infinity, and let  $\theta_1, \dots, \theta_N \in (0, \pi)$  be the angles around the singular lines. In the neighborhood of  $\rho$ , the quotient of  $\mathcal{R}_{cone}(M_r)$  by  $PSL(2, \mathbb{C})$  is parameterized by small deformations of  $c, \theta_1, \dots, \theta_n$ .*

Note that these results could be somewhat extended, at the cost of more complicated statements but without any significant change in the proof; it should be possible to include singularities along closed curves, still under the hypothesis that the angles around those singularities are less than  $\pi$  (or perhaps even  $2\pi$  as in [14]). On the other hand, the condition that the angle around the “open” singular curves is less than  $\pi$  appears to be

necessary, at least it occurs at several distinct points in the proof given here, and it also comes up naturally in other properties of those cone-manifolds “with singular infinity” that will not be treated here (in particular the geometry of the boundary of their convex cores).

*More about the motivations.* It was mentioned above that the main motivation for our work is the search for a generalization to manifolds with particles of the classical result, due to Ahlfors and Bers, describing convex co-compact hyperbolic metrics in terms of the conformal structure on their boundary at infinity.

It appears conceivable that a proof of such a statement could follow a “deformation” approach: proving that, given the angles around the singular lines, the natural map sending a cone-manifold to its conformal structure at infinity (marked by the position of the endpoints of the singular lines) is a homeomorphism. The topology that one should consider on the space of cone-manifolds would then come from the associated representations of the fundamental group of the complement of the singular locus. Three main difficulties would arise:

- showing that the map is a local homeomorphism — this is precisely the content of Theorem 1.5,
- showing that the map is proper — which translates as a compactness question for convex co-compact manifolds with particles (see [19]),
- showing that some conformal data have a unique inverse image — a point which appears not to be difficult for some particular values of the angles (of the form  $2\pi/k$ ,  $k \in \mathbb{N}$ ) for which finite coverings can be used.

So, given the results presented here, a kind of “double uniformization” theorem for manifolds with particles would follow from some compactness results. Since such statements depend on geometric methods which are completely different from the more analytic tools used here, we have decided not to include any developments concerning them, and hope to treat them in a subsequent work.

This line of arguments also leads to applications to Teichmüller theory, in particular for the Teichmüller space of hyperbolic metrics with cone singularities of prescribed angle on a closed surface. Having a Bers-type theorem for quasifuchsian manifolds with particles would make it possible to use in this context renormalized volume arguments as those used in [17] to recover results of [29], and to show that the natural Weil-Petersson metric on those Teichmüller spaces is Kähler, and has the renormalized volume as a Kähler potential.

*The geometry of the convex core.* It is possible to define the convex core of hyperbolic cone-manifold as for non-singular convex co-compact hyperbolic 3-manifolds. This appears natural but proving it properly leads to some technical considerations which have been moved to the appendix to keep the main part of the paper focused. With respect to the properties of the convex core, the “convex co-compact” manifolds with particles that we consider here appear to share some important properties of (non-singular) convex co-compact hyperbolic manifolds (the hypothesis that the cone angles are less than  $\pi$  is relevant here). This is beyond the scope of this paper, however we do need some definitions, since they will be helpful in the geometric constructions of sections 3 and 4.

**Definition 1.6.** The *convex core*  $CC(M)$  is the smallest non-empty subset of  $M$  which is convex.

The term “convex” should be understood here as in Definition 1.1, and “smallest” is for the inclusion. The existence of  $CC(M)$  is clear as soon as  $M$  is non-contractible. Indeed,  $M$  itself is convex, while the intersection of two convex subsets of  $M$  is convex and contains any closed geodesic of  $M$  (see Lemma A.12).

It follows from this definition that  $CC(M)$  is a convex set without extremal points (outside the singular locus of  $M$ ), and therefore that the intersection of its boundary with the regular set in  $M$  is a “pleated surface” as for (non-singular) quasi-Fuchsian manifolds (see [30]). When the angles around the singular lines are less than  $\pi$ , a simple but interesting phenomenon occurs: the convex core  $CC(M)$  contains all the vertices, and its boundary is “orthogonal” to the singular locus of  $M$ , so that its induced metric is a hyperbolic metric with cone singularities (at the intersection with the singular locus) of angle equal to the angle of the corresponding curve of the singular locus. Moreover, still under the hypothesis that the singular angles are less than  $\pi$ , the support of the bending lamination of  $\partial CC(M)$  does not contain its intersection with the singular lines.

These aspects of the geometry of quasi-Fuchsian cone-manifolds, which will not be developed much here, are important as motivations since they appear to indicate that several interesting questions concerning quasi-Fuchsian manifolds can also be asked for quasi-Fuchsian manifolds with particles as defined here, for instance whether any couple of hyperbolic metrics with cone singularities of prescribed angles can be uniquely obtained as the induced metric on the boundary of the convex core, or whether any couple of “reasonable” measured laminations, on a surface with some marked points, can be uniquely obtained as the bending lamination of the boundary of the convex core (for non-singular quasi-Fuchsian manifolds, see [4, 18]). Other similar questions concerning domains with smooth boundary can also be considered (see [28] for the non-singular analog).

*AdS manifolds and 3d gravity.* G. Mess [23] discovered that there is a class of anti-de Sitter manifolds, sometimes called “globally hyperbolic maximal compact” (GHMC), which is in many ways analogous to the quasi-Fuchsian hyperbolic manifolds. One such analogy is the fact that the space of GHMC AdS manifolds of given topology is parametrized by the product of two copies of Teichmüller space, and the geometry of the convex core presents striking similarities with the quasifuchsian case.

Cone singularities along time-like lines are quite natural in the context of those AdS manifolds, since they are used in the physics literature to model point particles. It appears (see [16, 5]) that some properties of hyperbolic and AdS manifolds with cone singularities along open lines (which are time-like in the AdS case) are quite parallel.

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2. LOCAL DEFORMATIONS

We recall the link between infinitesimal deformations of hyperbolic metrics and the first cohomology group of the bundle of infinitesimal Killing fields.

**2.1. The developing map of a hyperbolic metric.** Let  $M_r$  be a connected 3-manifold, with a hyperbolic metric  $g$  (i.e., Riemannian metric with constant sectional curvature  $-1$ ); this metric does not have to be complete, we are interested in the regular set of a hyperbolic cone-manifold. Each point  $x \in M_r$  has a neighborhood which is isometric to an open subset of hyperbolic 3-space  $H^3$ . This isometry can be extended uniquely to a local isometry from the universal cover  $(\tilde{M}_r, g)$  to  $H^3$ , called the *developing map* of  $(M_r, g)$ . We denote it by  $\text{dev}_g$ , it is well defined up to composition on the left by a global isometry of  $H^3$ .

If  $(M_r, g)$  is the regular part of a hyperbolic cone-manifold  $M$ , then  $\text{dev}_g$  is defined outside the singular set of  $M$ . It is usually not injective.

*Deformations of hyperbolic metrics.* Let  $\dot{g}$  be a infinitesimal deformation of the hyperbolic metric  $g$ ;  $\dot{g}$  is a section of the bundle of symmetric bilinear forms over  $M$ . We suppose that  $\dot{g}$  is such that the metric remains hyperbolic, i.e., the infinitesimal variation of the sectional curvature of  $g$  induced by  $\dot{g}$  vanishes.

One obvious way to define such “hyperbolic” deformations of  $g$  is by considering the Lie derivative of  $g$  under the action of a vector field  $u$  on  $M$ . We call such infinitesimal deformations *trivial*.

*The vector field associated to a deformation.* Consider the germ at  $t = 0$  of a smooth 1-parameter family  $(g_t)_{0 \leq t < \epsilon}$  of hyperbolic metrics on  $M$  with  $g_0 = g$  and  $(\partial_t g_t)_{t=0} = \dot{g}$ . Choose a smooth 1-parameter family of developing maps  $\text{dev}_{g_t}$  for the metrics  $g_t$ . One way to do this is as follows: fix a point  $x_0$  in  $\tilde{M}$ , a point  $p_0$  in  $H^3$  and an isometry  $I$  between  $T_{x_0} \tilde{M}$  and  $T_{p_0} H^3$ , then there exists a unique  $\text{dev}_{g_t}$  with the property

$$\text{dev}_{g_t}(x_0) = p_0, \quad (\text{dev}_{g_t})_*(x_0) = I.$$

Any other choice must be of the form

$$\text{dev}'_{g_t} = a_t \text{dev}_{g_t}$$

for some smooth family  $(a_t)_{0 \leq t < \epsilon}$  of isometries of  $H^3$ .

For each  $x \in \tilde{M}$  the curve

$$t \mapsto \text{dev}_g^{-1}(\text{dev}_{g_t}(x))$$

is well-defined for some positive time, in particular it defines a vector at  $x$ . Denote by  $u$  the vector field on  $\tilde{M}$  obtained in this way.

Let  $G$  be the group of deck transformations of  $\tilde{M}$ . Then  $u$  is *automorphic* with respect to the action of  $G$ , in the sense that for all  $\gamma \in G$ , the vector field  $\gamma_* u - u$  is Killing (we follow here the terminology used in [14]). Indeed, by definition  $\gamma_* g_t = g_t$ , so there exists an isometry  $a_\gamma(t)$  of  $H^3$  such that

$$\text{dev}_{g_t} \circ \gamma = a_\gamma(t) \circ \text{dev}_{g_t}.$$

By differentiation at  $t = 0$  this implies

$$u_{\gamma x} = \gamma_* u_x + \text{dev}_g^* \dot{\alpha}_\gamma.$$

**2.2. The bundle  $E$  of germs of Killing fields.** Over an arbitrary Riemannian manifold  $M$  consider the vector bundle

$$E := TM \oplus \Lambda^2 T^* M$$

with connection  $D$  given by

$$D_V(u, \alpha) = (\nabla_V u + V \lrcorner \alpha, \nabla_V \alpha - R_{uV})$$

where  $R$  is the curvature tensor (we identify vectors and 1-forms using the Riemannian metric). Define a differential operator  $\mathfrak{s} : \mathcal{C}^\infty(M, TM) \rightarrow \mathcal{C}^\infty(M, E)$  by the formula

$$u \mapsto \mathfrak{s}_u := \left( u, -\frac{1}{2}(\nabla u)_{\text{anti-sym}} \right).$$

The operator  $s$  is called the *canonical lift*, see [14]. The following elementary lemma is well-known (see [15]):

**Lemma 2.1.** *On every Riemannian manifold, the canonical lift operator induces an isomorphism between the space of Killing vector fields and the space of parallel sections of  $E$ .*

We specialize now to  $M$  orientable of dimension 3, so we identify  $\Lambda^2 T^* M$  with  $TM$  via the Hodge star and duality. Keeping into account that the sectional curvature is  $-1$ ,  $R_{uV}$  is mapped under this identification to  $u \times V$ . Let  $v$  be the vector corresponding to the 2-form  $\alpha$ , then  $V \lrcorner \alpha$  is dual to  $v \times V$ . Hence under this identification of  $E$  with  $TM \oplus TM$ , the connection  $D$  becomes

$$D_V(u, v) = (\nabla_V u + v \times V, \nabla_V v - u \times V).$$

To simplify even further, note that  $E$  is isomorphic to the complexified tangent bundle  $T_{\mathbb{C}}M$  via

$$(u, v) \mapsto u + iv.$$

We extend by linearity the Levi-Civita connection and the vector product to  $T_{\mathbb{C}}M$ . Hence the bundle with connection  $(E, D)$  is isomorphic to  $T_{\mathbb{C}}M$  with the connection (again denoted by  $D$ ) given by

$$(1) \quad D_V \phi = \nabla_V \phi + iV \times \phi.$$

Clearly  $D$  commutes with complex multiplication. In this framework, the canonical lift operator is given by the expression

$$(2) \quad \mathfrak{s}_u = u - i \frac{\text{curl}(u)}{2}.$$

Using the fact that  $M$  is hyperbolic, a straightforward computation shows that  $D$  is flat. Note that in general  $D$  is flat if and only if  $M$  has constant sectional curvature.



**2.3. The closed 1-form associated to an infinitesimal deformation of a hyperbolic metric.** Starting from a 1-parameter family of hyperbolic metrics on  $M$  we have constructed above an automorphic vector field  $u$  on  $\tilde{M}$ . Let  $\mathfrak{s}_u$  be its canonical lift. We claim that  $\mathfrak{s}_u$  is itself automorphic, in the sense that  $\gamma^*\mathfrak{s}_u - \mathfrak{s}_u$  is a parallel section in  $E$ . Indeed, since the group of deck transformations acts by isometries on  $\tilde{M}$ , it commutes with curl, hence from (2) it also commutes with the linear operator  $\mathfrak{s}$ :

$$\mathfrak{s}_{\gamma^*u} = \gamma^*\mathfrak{s}_u, \text{ for all } \gamma \in \pi_1(M).$$

We have seen in Section 2.1 that  $\kappa := \gamma^*u - u$  is Killing. Thus by Lemma 2.1

$$\gamma^*\mathfrak{s}_u - \mathfrak{s}_u = \mathfrak{s}_{\gamma^*u} - \mathfrak{s}_u = \mathfrak{s}_\kappa$$

is parallel as claimed.

Let  $d_D$  be the de Rham differential twisted by the flat connection  $D$ . Let  $\tilde{\omega}$  be the 1-form

$$\tilde{\omega} = d_D\mathfrak{s}_u.$$

Since  $d_D$  commutes with the action of  $G$ , we see that  $\tilde{\omega}$  is  $G$ -invariant on  $\tilde{M}$ :

$$\gamma^*\tilde{\omega} = d_D\gamma^*\mathfrak{s}_u = d_D(\mathfrak{s}_u + \mathfrak{s}_\kappa) = \tilde{\omega}.$$

Thus  $\tilde{\omega}$  descends to a 1-form  $\omega$  on  $M$  with values in  $E = T_{\mathbb{C}}M$ . This form is closed since by construction it is locally exact.

**2.4. Link between infinitesimal deformations and  $H^1(M, E)$ .** Let us gather below a few facts about  $\omega$ .

*The closed 1-form  $\omega$  does not depend on the choice of the family of developing maps  $\text{dev}_{g_t}$ .* Indeed, if we replace  $\text{dev}_{g_t}$  by  $\text{dev}'_{g_t} = a_t\text{dev}_{g_t}$ , then  $u' = u + \kappa$  where  $\kappa = \text{dev}_{g_0}^*\dot{a}_t$  is a Killing field, so

$$d_D\mathfrak{s}_{u'} = d_D\mathfrak{s}_u + d_D\mathfrak{s}_\kappa = \omega.$$

*The 1-form  $\omega$  is exact if and only if the infinitesimal deformation  $\dot{g}$  of the hyperbolic metric is trivial.* In one direction this is clear: a vector field  $u$  on  $M$  determines a germ of a 1-parameter group of local diffeomorphisms  $\Phi_t$ ; choose  $g_t := \Phi_t^*g$ . Then  $\text{dev}_{g_t}$  may be chosen as  $\text{dev}_0 \circ \tilde{\Phi}$  so the vector field of the deformation will be precisely the lift of  $u$  to  $\tilde{M}$ . Thus  $\mathfrak{s}_{\tilde{u}}$  is the lift to  $\tilde{M}$  of the section  $\mathfrak{s}_u$  in  $E$  over  $M$  defined by (2), in other words  $\omega$  is exact already on  $M$ . Conversely, assume that there exists  $\alpha \in \mathcal{C}^\infty(M, E)$  with  $d_D\alpha = \omega$ . Lifting to  $\tilde{M}$  we get

$$d_D\tilde{\alpha} = d_D\mathfrak{s}_u$$

so by Lemma 2.1, there exists a Killing vector field  $\kappa$  on  $\tilde{M}$  with  $\tilde{\alpha} - \mathfrak{s}_u = \mathfrak{s}_\kappa$ . Replace  $\text{dev}_{g_t}$  by  $\text{dev}'_{g_t} = a_t\text{dev}_{g_t}$ , where  $a_t = \exp(t\text{dev}_{g_t}\kappa)$  is a family of isometries of  $H^3$ . Thus  $\tilde{\alpha} = \mathfrak{s}_{u+\kappa} = \mathfrak{s}_{u'}$ . Since  $\tilde{\alpha}$  is  $G$ -invariant, so must be  $u'$ , therefore  $u'$  defines a vector field on  $M$ , which by definition means that the deformation is trivial.

Any closed form  $\alpha \in \Lambda^1(M, E)$  is cohomologous to  $d_D \mathfrak{s}_u$  for an automorphic vector field  $u$  on  $\tilde{M}$ . Indeed, the lift of  $\alpha$  to  $\tilde{M}$  is exact, since  $\tilde{M}$  is simply connected. Thus  $\tilde{\alpha} = d_D a$ . Now decompose  $\mathcal{C}^\infty(\tilde{M}, E)$  as follows:

$$\mathcal{C}^\infty(\tilde{M}, E) = \text{Range}(\mathfrak{s}) \oplus i\mathcal{C}^\infty(\tilde{M}, T\tilde{M})$$

where  $\mathfrak{s}$  is the differential operator (2). Note that both spaces in the right-hand side are  $G$ -invariant. With respect to this decomposition we write  $a = \mathfrak{s}_u + iv$ , so

$$\tilde{\alpha} = d_D a = d_D \mathfrak{s}_u + id_D v.$$

Let  $\gamma \in G$ ; since  $\tilde{\alpha} = d_D a$  is invariant, it follows that  $a$  is automorphic, thus there exists a Killing vector field  $\kappa$  with  $\gamma^* a - a = \mathfrak{s}_\kappa$ . Put this together with

$$a = \mathfrak{s}_u + iv, \quad \gamma^* a = \mathfrak{s}_{\gamma^* u} + i\gamma^* v.$$

Since  $\text{Range}(s)$  and  $i\mathcal{C}^\infty(\tilde{M}, T\tilde{M})$  are transversal in  $\mathcal{C}^\infty(\tilde{M}, E)$  we deduce

$$\gamma^* u - u = \kappa, \quad \gamma^* v - v = 0.$$

Hence  $iv$  descends to a section of  $E$  on  $M$ ; by subtracting  $d_D$  of this section from  $\alpha$  we get the cohomologous form  $d_D \mathfrak{s}_u$  with  $u$  automorphic as required.

In summary, we have shown that the application

$$\dot{g} \mapsto [\omega]_{H^1(M, E)}$$

is a well-defined isomorphism between the space of infinitesimal deformations of the hyperbolic structures on  $M$  modulo trivial deformations, and  $H^1(M, E)$ . Note that the argument holds more generally for deformations of metrics of constant sectional curvature of any sign.

**2.5. The variety of representations.** To go from infinitesimal deformations (as in Theorem 1.4) to small deformations (as in Theorem 1.5) it is necessary to understand the structure of the space of representations of  $\pi_1(M_r)$  in  $PSL(2, \mathbb{C})$  in the neighborhood of the holonomy representation  $\rho$  of a convex co-compact manifold with particles  $M$ .

We call  $R(M_r)$  the space of representations of  $\pi_1(M_r)$  in  $PSL(2, \mathbb{C})$ , that is, the space of homomorphisms of  $\pi_1(M_r)$  in  $PSL(2, \mathbb{C})$ . The representation  $\rho$  is irreducible: note (as in [7]) that the restriction of  $\rho$  to each boundary component of  $M$  is the holonomy representation of a complex projective structure on the complement of the singular points, and as such it is irreducible because any reducible representation fixes a point in  $\mathbb{C}P^1$  and is therefore the holonomy representation of an affine structure.

It then follows from a result of Thurston [30, 9] (see [14], Theorem 4.3) that the irreducible component of  $R(M_r)$  containing  $\rho$  is a complex variety.

Following a classical construction of Weil [32], one can associate to  $R(N)$  a scheme  $\mathcal{R}(N)$ , based on the choice of a presentation of  $\pi_1(N)$ . Then the Zariski tangent space of  $\mathcal{R}(N)$  at  $\rho$  is naturally associated to the space of 1-cocycles  $Z^1(\pi_1(N); Ad(\rho))$  (see [14], Proposition 4.1).

We will see in subsection 6.3 that  $R(M_r)/PSL(2, \mathbb{C})$  is actually a smooth complex manifold in the neighborhood of  $\rho$ , and that its tangent space is canonically identified with  $H^1(M, E)$ .

### 3. THE GEOMETRY OF CONVEX CO-COMPACT MANIFOLDS WITH PARTICLES

The goal of the next two sections is to find a convenient way to “normalize” infinitesimal deformations of convex co-compact manifolds with particles close to infinity. This will then be used to prove an infinitesimal rigidity statement for hyperbolic manifolds with particles, Theorem 1.4. Here “normalize” means to write them as sections of a certain bundle which are in  $L^2$ . Theorem 1.4 will then follow from an analytical argument; this argument was originally due to Calabi [8] and Weil [31], and has been extended recently to hyperbolic cone-manifolds by Hodgson and Kerckhoff [14]. The treatment here of deformations close to infinity is inspired by the recent work of Bromberg [6, 7], while the general approach is related to the argument used by Weiss [33]. It would also be interesting to compare the methods used here to the ones developed by Montcouquiol [25, 24, 26] to treat similar questions in higher dimensions, in the setting of Einstein manifolds with conical singularities.

#### 3.1. Hyperbolic cone-manifolds.

*Definitions.* Hyperbolic cone-manifolds were defined by Thurston [30], using a recursive definition. We define first the special case when the singular locus is a disjoint union of lines (i.e. a graph without vertices). Consider a fixed, oriented hyperbolic geodesic  $\Delta_0$  in  $H^3$ , and let  $U$  be the universal cover of the complement of  $\Delta_0$  in  $H^3$ . Let  $V$  be the metric completion of  $U$ , so that  $V \setminus U$  is canonically identified with  $\Delta_0$ ; it will be called the *singular set* of  $V$ . For each  $\alpha > 0$ , let  $V_\alpha$  be the quotient of  $V$  by the rotation of angle  $\alpha$  around  $\Delta_0$ ; the image under this quotient of the singular set of  $V$  is called the singular set of  $V_\alpha$ . Another description of  $V_\alpha$  is as the hyperbolic cone over the spherical surface  $S_\alpha$  with two cone singularities, both of angle  $\alpha$ :

$$V_\alpha = (S_\alpha \times \mathbb{R}_{>0}, dt^2 + \sinh^2(t)h) ,$$

where  $h$  is the metric on  $S_\alpha$ .

A *hyperbolic cone-manifold* with singular locus a union of lines is a complete metric space for which each point has a neighborhood which is isometric of an open subset of  $V_\alpha$ , for some  $\alpha > 0$ . The points which have a neighborhood isometric to an open subset of the complement of the singular set in  $V_\alpha$  are called *regular points*, and the others *singular points*. The set of regular points of a hyperbolic cone-manifold is a (non-complete) hyperbolic 3-manifold.

We are interested here in a more general notion of cone-manifolds, for which the singular set is a graph, that is, three singular lines can meet at a “vertex”. We require however the number of vertices and edges to be finite. Such cone-manifolds are made of three kinds of points: in addition to the regular points and to the points of the singular lines, already described above, there can be “vertices”, i.e., points which have a neighborhood isometric to a hyperbolic cone over a 2-dimensional spherical cone-manifold (see [30]). Given such a vertex  $v$  in a cone-manifold  $M$ , the 2-dimensional spherical cone-manifold over which the neighborhood of  $v$  in  $M$  is “built” is called the *link* of  $v$ . Each singular point of the link of

$v$  corresponds to one of the singular lines ending at  $v$ , and the angle around the singular point in the link is equal to the angle around the corresponding singular curve.

Here we suppose that the angle at each singular line is less than  $\pi$ , it follows that the same condition holds at each singular point of the link of  $v$ , so that the corresponding singular curvature is larger than  $\pi$ . So it follows from the Gauss-Bonnet theorem that the link of each vertex can have at most 3 cone points. The picture is further simplified by the fact that a spherical cone-manifold with 3 singular points where the singular curvature is positive is the double cover of a spherical triangle (this is a special case of a theorem of Alexandrov, see [3, 21]).

Let  $M$  be a hyperbolic cone-manifold. Each singular point  $x$  of  $M$  other than the vertices has a neighborhood which is isometric to a subset of  $V_\alpha$  for a unique  $\alpha > 0$ ; we call  $\alpha$  the *angle* of  $M$  at  $x$ . By construction, the angle is locally constant on the singular lines of  $M$ .

We will consider here only hyperbolic cone-manifolds which are homeomorphic to the interior of a compact manifold with boundary, with the singular set sent by this homeomorphism to an embedded graph, with a finite number of vertices, and with the exterior vertices on the boundary.

We can follow the definition of  $V_\alpha$  above using the Poincaré model of  $H^3$ , taking as  $\Delta_0$  the intersection with the ball of a line  $D_0$  going through the origin. This leads to a conformal model of  $V_\alpha$ :  $V_\alpha$  is conformal to the quotient by a rotation of angle  $\alpha$  of the universal cover of the complement of  $D_0$  in the Euclidean ball of radius 1.

*Hyperbolic manifolds with particles.* The specific class of hyperbolic cone-manifolds that we consider contains the convex co-compact hyperbolic manifolds, as well as analogous cone-manifolds, see Definition 1.2. Those convex co-compact cone-manifolds are required to contain a non-empty, compact, geodesically convex subset.

The properties of convex subsets in hyperbolic manifolds with particles are quite reminiscent of the corresponding properties in non-singular hyperbolic manifolds. Some considerations on this can be found in the appendix. One key property, which we will need here, is Lemma A.11.

Consider a convex subset  $K$  in a convex co-compact cone-manifold  $M$ , let  $N^1(K)$  be the unit normal bundle of  $K$ , as defined in the Appendix.  $N^1K$  contains all unit vectors in  $TM$  which are orthogonal to a support plane of  $K$  (and oriented towards the exterior), as well as some vectors based at the intersection of  $\partial K$  with the singular set of  $M$  (see Definition A.8). When  $\partial K$  is smooth and “orthogonal” to the singular locus,  $N^1K$  is homeomorphic to  $\partial K$ , and diffeomorphic outside the singular points, but in general  $N^1K$  is only a  $C^{1,1}$  surface, it has one singular point for each intersection point between  $\partial K$  and the singular set of  $M$ . (The  $C^{1,1}$  structure on  $N^1K$  is clear if  $K$  is the convex core of  $M$  and the support of its bending lamination is a disjoint union of closed curves. It is almost as clear if  $\partial K$  is polyhedral. In the general case its existence follows from a limiting argument.)

We consider the restriction of the exponential map to the normal bundle of  $K$ , as the map

$$\exp_K : N^1K \times (0, \infty) \rightarrow M$$

such that  $\exp_K(v, s) = \exp(sv)$ , where  $\exp$  is the usual exponential map.

The content of Lemma A.11 is that, for any non-empty, convex, compact subset  $K$  of  $M$ ,  $\exp_K$  is a homeomorphism from  $N^1K \times (0, \infty)$  to  $M \setminus K$ .

**3.2. Induced structures at infinity.** Let  $M$  be a hyperbolic manifold with particles, let  $M_r$  and  $M_s$  be the subsets of its regular and of its singular points, respectively.  $M_r$  has a natural (non-complete) hyperbolic metric, and its universal cover  $\tilde{M}_r$  has a locally isometric projection  $\text{dev}$  to  $H^3$  which is unique up to composition on the left by an isometry of  $H^3$ .

The metric completion of  $\tilde{M}_r$  is the union of  $\tilde{M}_r$  with a union of connected sets, each of which projects to a connected component of  $M_s$  and also, by  $\text{dev}$ , to a complete graph in  $H^3$ .

Let  $\partial_\infty H^3$  be the boundary at infinity of  $H^3$ . Then  $\text{dev}$  has a natural extension as a local homeomorphism:

$$\tilde{\text{dev}} : \tilde{M}_r \cup \partial_\infty \tilde{M}_r \rightarrow H^3 \cup \partial_\infty H^3 ,$$

where  $\partial_\infty \tilde{M}_r$  can be defined, as  $\partial_\infty H^3$ , as the space of equivalence classes of geodesic rays in  $\tilde{M}_r$ , for which the distance to the singular locus is bounded from below by a positive constant, where two rays are in the same class if and only if they are asymptotic.

The boundary at infinity of  $H^3$  can be canonically identified to  $\mathbb{C}P^1$ , so that  $\tilde{\text{dev}}$  induces on  $\partial_\infty \tilde{M}_r$  a complex projective structure. We get the same  $\mathbb{C}P^1$ -structure if we compose  $\tilde{\text{dev}}$  to the left by an isometry. Furthermore, since the hyperbolic isometries act on  $\partial_\infty H^3$  by complex projective transformations, the fundamental group of  $M_r$  acts on  $\tilde{M}_r$  by hyperbolic isometries which extend to  $\partial_\infty \tilde{M}_r$  as complex projective transformations. Therefore,  $M_r$  has a well-defined boundary at infinity, which is the quotient of  $\partial_\infty \tilde{M}_r$  by the fundamental group of  $M_r$ , and which carries a canonical complex projective structure.

Let  $K \subset M$  be a compact convex subset. The map  $\exp : N^1K \times (0, \infty) \rightarrow M \setminus K$  can be used to define a “limit”  $\exp_\infty : N^1K \rightarrow \partial_\infty M$  (technically, the image of a point  $(x, v) \in N^1K$  is the equivalence class of the geodesic ray  $t \mapsto \exp((x, v), t)$ ). Lemma A.11 shows that this map is a homeomorphism from  $N^1K$  to  $\partial_\infty M$ , by construction it sends the singular points of  $N^1K$  to the endpoints at infinity of the singular curves of  $M$ . This shows in particular that two cone singularities in  $M$  end at different points in  $\partial_\infty M$  (i.e., they can not be asymptotic). The following statement is a consequence.

**Lemma 3.1.** *Each point  $x \in \partial_\infty M$  has a neighborhood which is isometric either to a half of the hyperbolic space  $H^3$  (when  $x$  is not an endpoint of one of the singular curves) or to a neighborhood of one of the endpoints of  $D_0$  in the Poincaré model of  $V_\alpha$  described above (when  $x$  is an endpoint of one of the singular curves, of angle  $\alpha$ ).*

*Proof.* Suppose that  $x$  is the endpoint at infinity of a singular ray  $p$  in the singular set of  $M$ . Since, by Lemma A.11,  $K$  intersects all singular rays in  $M_s$ ,  $p$  intersects  $\partial K$  at a point  $y$ . Let  $n$  be the singular point in  $N^1K$  corresponding to the intersection with  $p$ , so that the projection of  $n$  to  $\partial K$  is  $y$  and  $n$  is directed along  $p$ .

Let  $U$  be a neighborhood of  $n$  in  $N^1K$ , and let  $C = \exp(U \times (0, \infty))$ . Then  $C$  contains a cylinder of exponentially expanding radius around  $p$  – this follows from standard arguments

on the normal exponential map of a convex surface in  $H^3$  – and the statement of the lemma follows.  $\square$

Clearly the complex projective structure at infinity is defined on  $\partial_\infty M_r$  only, and does not extend to the endpoints at infinity of the singular curves in  $M$ . An extension is however possible. For this note that  $\partial_\infty M_r$  is projectively equivalent, in the neighborhood of an endpoint at infinity  $x$  of a cone singularity, to a neighborhood in the boundary at infinity of  $V_\theta$  of one of the endpoints of the singular line (here  $\theta$  has to be equal to the angle at the singular line ending at  $x$ ). Considering such model neighborhoods leads to a natural notion of “complex projective structure with cone singularities”.

One can also consider the conformal structure underlying the complex projective structure at infinity ; we will call it the conformal structure at infinity of  $M$ . It is defined in the complement, in  $\partial N$ , of the points which are the endpoints of the singular graph. (Recall that  $N$  is the compact manifold with boundary introduced after Definition 1.2). We will see in Remark 4.9 that this conformal structure can be extended to the singular points, hence it can also be considered as a conformal structure on  $\partial N$  with some marked points.

**3.3. The  $L^2$  deformations.** The regular set  $M_r$  of a hyperbolic cone-manifold  $M$  carries by definition a (non-complete) hyperbolic metric. The deformation theory outlined in section 2 for hyperbolic manifolds therefore applies to this setting. There is a natural vector bundle over  $M_r$ , which we still call  $E$ , with fiber at a point the vector space of Killing fields in a neighborhood of this point. Moreover  $E$  can be identified with  $T_{\mathbb{C}}M_r$  with its natural metric and the flat connection (1) (see also [14]), which we still call  $D$ , with flat sections the sections corresponding to a fixed Killing field. Finally, the infinitesimal deformations of the hyperbolic cone-manifold structure are associated to closed 1-forms with values in  $E$ , with two 1-forms corresponding to equivalent deformations if and only if the difference is the differential of a section of  $E$ .

Let  $\omega$  be a closed 1-form on  $M_r$  with values in  $E$ . Then  $\omega$  is in  $L^2$  if:

$$\int_{M_r} \|\omega\|_E^2 dv < \infty ,$$

where the norm of  $\omega$  is measured with respect to the hyperbolic metric on  $M$  and the natural metric, at each point of  $M$ , on  $E$ . The tensor product connection  $\nabla \otimes D$ , where  $\nabla$  is the Levi-Civita connection of  $M$ , can be applied to  $\omega$ , to obtain a tensor  $D\omega$  whose norm can also be measured with respect to the same metrics; again,  $D\omega$  is  $L^2$  if the integral of the square of its norm converges over  $M$ .

The following lemma is a key point of this paper. It is proved at the end of the next section, after some preliminary constructions, since it uses some details on the normalization of a deformation near the singular set and in the neighborhood of infinity.

**Lemma 3.2.** *Let  $(M, g)$  be a hyperbolic manifold with particles. Let  $\dot{g}$  be a infinitesimal deformation of  $g$ , among hyperbolic manifolds with particles, which changes neither the conformal structure at infinity nor the angles at the singular arcs. Then there is a deformation 1-form  $\omega$  associated to  $\dot{g}$  which is  $L^2$  and such that  $D\omega$  is  $L^2$ .*



Here again, the conformal structure at infinity which is considered is the conformal structure with marked points corresponding to the endpoints of the singular lines. Note that the fact that  $D\omega$  is in  $L^2$  is not used in the sequel, it is included in the lemma since it follows from the proof and because it could be of interest in different situations.

*Convex surfaces close to infinity.* It will be useful, in order to obtain a good normalization of the infinitesimal deformations of the hyperbolic metrics close to infinity, to find a foliation of the ends by convex surfaces which are “orthogonal” to the singularities. We first consider another notion of “convex core” containing the singular locus of  $M$ . We suppose from here on that  $M$  is not one of the model spaces  $V_\theta$  defined above.

**Definition 3.3.** The smallest convex subset of  $M$  containing  $M_s$  is called the *singular convex core* of  $M$  and is denoted by  $C_S(M)$ .

Here “smallest” should be understood for the inclusion; the existence of  $C_S(M)$  is clear since  $M$  itself is convex, and the intersection of two convex subsets of  $M$  containing  $M_s$  is itself convex in the sense of Definition 1.1, it cannot be empty since it always contains  $M_s$ .

Close to infinity,  $C_S(M)$  is “thin” and concentrated near the singular locus, as stated in the next proposition. For each  $r > 0$ , we define  $CC_r(M)$  as the set of points of  $M$  which are at distance at most  $r$  from the convex core  $CC(M)$ . It is not difficult to check that, for any  $r > 0$ ,  $CC_r(M)$  is convex (this follows from the arguments in the appendix).

**Proposition 3.4.** *There exists a constant  $C > 0$  such that, for each  $r > 0$ , any point  $x \in C_S(M)$  which is not in  $CC_r(M)$  is at distance at most  $Ce^{-r}$  from the singular locus.*

*Proof.* In the Poincaré model of  $V_\alpha$  described above, the intersection of the model with a Euclidean ball, with boundary orthogonal to the boundary of the model, which does not intersect the singular segment, is isometric to a hyperbolic half-space. Considering such balls which are tangent to the singular segment at its endpoint, and which are small enough to fit in the neighborhood of the endpoint which appears in Lemma 3.1, we can find for each endpoint  $x_\infty$  of the singular graph  $M_s$  in  $M$  a finite set of half-spaces  $H_1, \dots, H_p \subset M$ , disjoint from the singular set  $M_s$ , such that any point  $y$  which is at distance at most  $ce^r$  from  $M_s \setminus CC_r(M)$  but not in  $\cup_{i=1}^p H_i$  is actually at distance at most  $c'e^{-r}$  from  $M_s$  (for some  $c, c' > 0$ ).

By construction,  $C_S(M)$  is contained in the complement of the  $H_i, 1 \leq i \leq p$ . It follows that, maybe after changing the constants  $c$  and  $c'$ , any point in  $C_S(M) \cap (M \setminus CC_r(M))$  which is at distance at most  $ce^r$  from one of the singular curves is actually at distance at most  $c'e^{-r}$  from this singular curve.

However, for  $r$  large enough, a point  $y \in M \setminus CC_r(M)$  which is at distance at least  $ce^r$  from all the singular curves can not be contained in  $C_S(M)$ , since one can construct a half-space in  $M$ , disjoint from the singular locus, which contains it. The statement follows.  $\square$

**Definition 3.5.** Let  $\Sigma$  be a surface in  $M_r$ , and let  $\bar{\Sigma}$  be its closure as a subset of  $M$ ; suppose that  $\bar{\Sigma} \setminus \Sigma \subset M_s$ . We say that  $\bar{\Sigma}$  is *orthogonal to the singular locus* if, for each

$x \in \overline{\Sigma} \setminus \Sigma$  and each sequence  $(x_n)_{n \in \mathbb{N}}$  of points of  $\overline{\Sigma}$  converging to  $x$ , the ratio between the distance from  $x_n$  to  $x$  in  $\overline{\Sigma}$  and the distance from  $x_n$  to  $M_s$  in  $M$  converges to 1.

**Lemma 3.6.** *There exists a constant  $k_0 > 0$  (e.g.  $k_0 = 2$ ) and a compact, convex subset  $K \subset M$ , with  $K \supset CC(M)$ , such that the complement of  $K$  in  $M$  is foliated by equidistant surfaces, which are smooth and locally convex outside the singular locus, with principal curvatures at most equal to  $k_0$ , and orthogonal to the singular locus.*

*Proof.* Choose  $r > 0$ , on which more details will be given below. Let  $M_s$  be the singular graph of  $M$ , and let  $x$  be a point in  $M_s$  at distance  $r$  from  $CC(M)$ . By the definition of a cone-manifold given above, there is a neighborhood  $\Omega$  of  $x$  in  $M$  which is isometric to a ball  $\Omega'$  centered at a point  $y$  of  $\Delta_0$  in  $V_\alpha$ , for some  $\alpha \in \mathbb{R}_+$ .

Recall that the universal cover of  $V_\alpha \setminus \Delta_0$  has a canonical projection to the complement of a line (which we also call  $\Delta_0$ ) in  $H^3$ . The metric completion of the universal cover of  $V_\alpha \setminus \Delta_0$  is obtained by adding a line, which we still call  $\Delta_0$ , which contains a unique point  $y'$  corresponding to  $y$ .

Let  $y''$  be the image of  $y'$  in  $H^3$ , and let  $Q$  be the plane orthogonal to  $\Delta_0$  at  $y''$ . Then the lift of  $Q$  to the universal cover of  $V_\alpha \setminus \Delta_0$  is a totally geodesic subspace  $Q'$  which is orthogonal to  $\Delta_0$  at  $y'$ .  $Q'$  projects to  $V_\alpha$  as a totally geodesic subset  $Q$  which is also orthogonal to  $\Delta_0$  at  $y$ .

We call  $P$  the subset of  $\Omega \subset M$  which corresponds to the subset  $Q \cap \Omega'$  of  $\Omega' \subset V_\alpha$ . If  $r$  is large enough, Proposition 3.4 indicates that  $C_S(M) \setminus P$  has two connected components, one of which is contained in an  $\epsilon$ -neighborhood of the subset of  $\gamma$  which is bounded by  $x$  on the side opposite to  $CC(M)$ .

Since the same construction can be done for each of the points at distance  $r$  from  $CC(M)$  in the singular locus of  $M$ , we can “cut out” the neighborhoods in  $C_S(M)$  of the parts of the singular curves which are at distance more than  $r + 1$  from  $CC(M)$ . Since this is done by cutting along totally geodesic surfaces which are orthogonal to the singular locus, we obtain in this way a compact subset  $K'$  of  $M$ , contained in  $C_S(M)$  and in  $C_{r+1}(M)$ , which is convex. However the boundary of  $K'$  is not smooth.

We can now call  $K$  the set of points of  $M$  at distance at most 1 from  $K'$ ; it is again compact and convex, and its boundary is  $C^{1,1}$  smooth and strictly convex. Smoothing this boundary surface by any of the classical techniques – without changing it in a neighborhood of its intersections with the singular curves, where it is totally umbilic – yields a convex, compact subset  $K$  of  $M$  with a boundary which is smooth and orthogonal to the singular locus. The statement is then obtained by considering the foliation of the complement of  $K$  by the surfaces at constant distance from  $K$ .

Consider an integral curve of the unit vector field orthogonal to these surfaces, towards infinity. Since the surfaces are equidistant, this integral curve is a geodesic, and a classical computation (see e.g. [12]) shows that, along it, the second fundamental form of the surfaces satisfies a Riccati equation:

$$B' = I - B^2 .$$

It follows that the principal curvatures of the equidistant surfaces converge to 1 close to infinity in each of the ends of  $M$ . Therefore, replacing  $K$  by a larger compact subset if necessary, we obtain that the principal curvatures of the equidistant surfaces are at most  $k_0$ , for any choice of  $k_0 > 1$ .  $\square$

In the sequel, for each end  $e$  of  $M$ , we call  $(S_{e,t})_{t \in \mathbb{R}_+}$  the family of surfaces obtained in the previous lemma, which foliates a neighborhood of infinity in the end  $e$ .

*The metric at infinity associated to an equidistant foliation.* Such an equidistant foliation  $(S_{e,t})_{t \in \mathbb{R}_+}$  determines a natural metric  $g_{\infty,e}$  on the connected component of  $\partial_\infty M$  corresponding to  $e$ , it is defined as:

$$g_{\infty,e} = \lim_{t \rightarrow \infty} e^{-2t} I_t ,$$

where  $I_t$  is the induced metric on  $S_{e,t}$ . The surfaces  $(S_{e,t})_{t \in \mathbb{R}_+}$ , and the boundary at infinity, are identified through the orthogonal projections on the  $S_{e,t}$ . The homothety factor  $e^{-2t}$  is designed to compensate the divergence of  $I_t$  as  $t \rightarrow \infty$ .

Clearly, the conformal structure of  $g_{\infty,e}$  is equal to the conformal structure underlying the  $\mathbb{C}P^1$ -structure on  $\partial_\infty M$  which was already mentioned above. It also follows quite directly from its definition that  $g_{\infty,e}$  is a smooth metric with conical singularities at the endpoints of the singular lines of  $M$ , where its singular angle is equal to the singular angles around the corresponding singular lines of  $M$ .

Note that  $g_{\infty,e}$  is not in general hyperbolic, it depends on the choice of the equidistant foliation  $(S_{e,t})_{t \in \mathbb{R}_+}$ . Actually it is possible to choose  $g_{\infty,e}$  and deduce from it an equidistant foliation, which might however only be defined for  $t \geq t_0$ , for some  $t_0 \in \mathbb{R}$  (see e.g. [11, 13] where related questions are treated in the more general context of conformally compact Einstein manifolds, but without singularities, or [17] for the 3-dimensional hyperbolic case).

It is perhaps worth noting that there is another possible definition of the metric at infinity  $g_{e,\infty}$ : it is equal to  $e^{-2t} I_t^*$ , where  $I_t^*$  is the ‘‘horospherical metric’’ of  $S_{e,t}$ , i.e.,  $I_t + 2II_t + III_t$  (where  $I_t$ ,  $II_t$  and  $III_t$  are the induced metric, second and third fundamental forms of  $S_{e,t}$ , respectively) for any choice of  $t$  — the result does not depend on the choice of  $t$ . Details on this can be found in [10, 27].

*Geodesics close to infinity.* A direct consequence of the existence of the foliation by parallel, convex surfaces orthogonal to the singular locus, obtained in the previous paragraph, is the existence of another foliation, by geodesics going to infinity and normal to those surfaces.

**Lemma 3.7.** *For each end  $e$  of  $M$ , for each  $x \in S_{e,0}$ , there exists a geodesic ray  $h_{e,x}$  with endpoint  $x$  which is orthogonal to the surfaces  $S_{e,t}$ ,  $t \in \mathbb{R}_+$ . The geodesic rays  $h_{e,x}$ ,  $x \in S_{e,0}$ , foliate  $\cup_{t \in \mathbb{R}_+} S_{e,t}$ , and, for each  $x \in S_{e,0}$ , the point at distance  $t$  from  $x$  in  $h_{e,x}$ , called  $h_{e,x}(t)$ , is in  $S_{e,t}$ .*

*Proof.* This is a direct consequence of the previous lemma, taking as the  $h_{e,x}$  the curve orthogonal to the equidistant surfaces and starting from  $x$ .  $\square$

**Lemma 3.8.** *For each end  $e$  of  $M$ , there exists a constant  $C_e > 0$  with the following property. Let  $\gamma : [0, 1] \rightarrow S_{e,0}$  be a smooth curve, then, for each  $t \in \mathbb{R}_+$ :*

$$\left\| \frac{\partial h_{e,\gamma(s)}(t)}{\partial s} \right\| \geq C_e \|\gamma'(s)\| e^t .$$

*Proof.* The  $h_{e,\gamma(s)}$  are geodesics, and are orthogonal to  $\gamma$ . Moreover:

$$\frac{\partial}{\partial t} \left\| \frac{\partial h_{e,\gamma(s)}(t)}{\partial s} \right\|_{t=0} \geq 0 ,$$

because  $S_{e,0}$  is convex. So the estimate follows directly from classically known estimates on the behavior of Jacobi fields along a geodesic, see e.g., [12].  $\square$

#### 4. THE NORMALIZATION OF INFINITESIMAL DEFORMATIONS

The goal of this section is to prove Lemma 3.2. The argument uses some additional notations, which we first introduce. We denote by  $\tilde{M}$  the universal cover of  $M$ , with its singular locus. So  $\tilde{M}$  is a quotient of the metric completion of the universal cover  $\tilde{M}_r$  of  $M_r$ , already defined above, with ramification at the lift to  $\tilde{M}$  of the singular locus  $M_s$ . The boundary at infinity  $\partial_\infty \tilde{M}$  of  $\tilde{M}$  can be defined in the same way as the boundary at infinity of  $\tilde{M}_r$  (as the space of geodesic rays up to the equivalence relation “being asymptotic”). The complement of the singular points in  $\partial_\infty \tilde{M}$  is the quotient of the complement of the endpoints of the singular curves in the boundary at infinity of  $\tilde{M}_r$  by the group acting on  $\tilde{M}_r$  with quotient the complement of the singular curves in  $\tilde{M}$ .

There are three main steps in the proof of Lemma 3.2. The first is to normalize a family of hyperbolic cone-metrics  $g_s$  with cone angles constant in  $s$  by a family of isotopies, so that the automorphic vector field  $v$  on  $\tilde{M}$  associated to the deformation extends to an automorphic vector field  $V$  on  $\partial_\infty \tilde{M}$ . Moreover,  $V$  will turn out to be equivalent to a holomorphic vector field  $V + W'$ , where  $W'$  is the lift to  $\partial_\infty \tilde{M}$  of a vector field defined on  $\partial_\infty M$ , and the behavior of  $W'$  near the singular points of  $\partial_\infty M$  can be understood thoroughly.

The second step is to construct from  $V + W'$  a section  $F$  of a bundle of quadratic polynomials on  $\partial_\infty \tilde{M}$ , which is strongly related to the bundle  $E$  of local Killing fields on  $\tilde{M}$ , and use the description of  $W'$  at the singular points to show that  $F$  also behaves rather nicely close to the singular points.

Finally the third step uses the section  $F$  to construct a deformation 1-form  $\omega$  in  $M$  equivalent to the initial deformation. The estimates on  $F$  then translate as the required estimates on  $\omega$ .

##### 4.1. The vector field at infinity.

**Lemma 4.1.** *Let  $g_s$  be a 1-parameter family of hyperbolic cone-metrics on  $M$  with constant angles at the singular graph. There exists a 1-parameter family  $\Phi_s$  of isotopies of  $M$  such that for all  $s$ ,*

- (1) the hyperbolic cone-metric  $g'_s := \Phi_s^* g_s$  coincides with  $g_0$  in a model neighborhood of the singular locus near infinity, and near the vertices;
- (2) the geodesic half-lines  $h_{e,x}(t)$ , defined using  $g_0$ , are also geodesic for  $g'_s$ .
- (3) the deformation 1-form for the family  $g'_s$  is uniformly bounded in norm near the singular graph, and vanishes in a model neighborhood of the singular half-lines outside the convex core.

*Proof.* From Lemma 3.1, if the cone angles are fixed then the metrics  $g_s$  are all isometric in a neighborhood of the singular lines near infinity to a subset in a fixed model neighborhood  $V_\alpha$ . Also near the vertices of the singular graphs, hyperbolic metrics with fixed cone angles are rigid (see Proposition A.2). However the metrics  $g_s$  may vary in a neighborhood of the singular graph, even if the angles are fixed: there may appear an elongation of the singular segments, and also a twist of the graph along such a segment.

If we fix these lengths and the twists of the graph along segments, the hyperbolic cone-metric is clearly rigid near the singular graph. We construct now some explicit metrics  $g'_s$  with the same lengths and twists as  $g_s$ , which are therefore isometric to  $g_s$  on the  $\epsilon$ -neighborhood  $U_\epsilon$  of the singular graph. Let  $l_{g_s}(e)$  be the length of the edge  $e$  with respect to  $g_s$ , and  $\theta_{g_s}(e)$  the additional twisting angle of  $g_s$  along  $e$ , as compared to  $g_0$ . Choose  $\epsilon$  sufficiently small so that the singular graph is a deformation-retract of its  $\epsilon$ -neighborhood  $U_\epsilon$ . Cut this neighborhood into pieces using totally geodesic disks orthogonal to the singular graph at some fixed distance  $\delta$  from the vertices. We obtain in this way for each edge  $e$  a finite-length cone-manifold  $C_e$  of angle  $\alpha(e)$  around a singular curve of length  $l_{g_0}(e) - 2\delta$ . Replace this cylinder by the cylinder of the same type of length  $l_{g_s}(e) - 2\delta$ , and glue it back with a twist of angle  $\theta_s$ .

One can realize this metric on  $U_\epsilon$  (outside the singular locus) as follows: fix an edge  $e$ , let  $l_s$  be the length of  $e$  with respect to  $g_s$ , also  $\theta_s$  the twist of  $g_s$  along  $e$  (relative to  $g_0$ ). Let  $(x, z)$  be coordinates adapted to  $g_0$  on the cylinder corresponding to  $e$ , where  $x \in [\delta, l_0 - \delta]$  is the height function and  $z \in \mathbb{C}$  is a complex variable in the disk of radius  $\epsilon$ , written  $z = (r, \theta)$  in polar coordinates:

$$g_0 = dr^2 + \cosh(r)^2 dx^2 + \sinh(r)^2 d\theta^2.$$

Let  $\phi$  be a cut-off function on  $[0, l_0]$ , which vanishes for  $x < \delta$ , is increasing, and equals 1 for  $x > l_0 - \delta$ . Pull back (at time  $s$ ) the metric  $g_0$  through the map

$$(3) \quad (x, r, \theta) \mapsto (x + \phi(x)(l_s - l_0), r, \theta + \phi(x)\theta_s).$$

These maps for different edges do not glue nicely to  $U_\epsilon$  (they do not agree near the vertices) but the pulled-back metric does extend to  $U_\epsilon$ , and is isometric to  $g_s$  since they have the same elongation and twist along each edge.

Since the metrics  $g_s$  and  $g'_s$  are isometric on  $U_\epsilon$ , we can pull-back  $g_s$  through a family of isotopies of  $M$  starting from the identity at  $s = 0$ , such that the resulting metrics are equal to  $g'_s$  near the singular locus. The surface  $S_{e,0}$  constructed in Section 3 is convex also for the metrics  $g'_s$  for sufficiently small  $s$ . Choose a second family of isotopies which is the identity on a neighborhood of the singular locus near infinity and on the convex core, and which maps the normal geodesics flow from  $S_{e,0}$  (with respect to  $g_s$ ) onto the corresponding flow

with respect to  $g_0$ . The points where the surfaces  $S_{e,0}$  (for different values of  $s$ ) intersect the singular lines may lay at varying distance from the convex core; we choose this second family of isotopies to be given by (3) between these intersection points and the convex core, with  $\theta_s = 0$  and the necessary elongation  $l_s$ .

The metrics  $g'_s$  coincide near the singular locus at infinity and near the vertices, hence the associated deformation vector field  $v$  is Killing so the deformation 1-form  $\omega$  vanishes in the above region as claimed. It remains to check that  $\omega$  is bounded on the cylinders near each singular segment  $e$ . Recall that  $\omega = d_D \mathfrak{s}_v$ , where  $\mathfrak{s}$  is the canonical lift operator (2), and  $v$  is the vector field tangent in  $s = 0$  to the 1-parameter family of maps (3):

$$v = \phi(x)(\dot{l}\partial_x + \dot{\theta}\partial_\theta)$$

where the constants  $\dot{l}, \dot{\theta}$  are the infinitesimal variations of the length and twist of the edge  $e$ . It is not hard to see that  $\partial_x$  and  $\partial_\theta$  are Killing fields, which correspond to translations along  $e$ , respectively rotations around  $e$ , therefore  $d_D \mathfrak{s}_{\partial_x} = d_D \mathfrak{s}_{\partial_\theta} = 0$ . Also one sees easily that

$$\mathfrak{s}_{\phi\partial_x} = \phi\mathfrak{s}_{\partial_x}, \quad \mathfrak{s}_{\phi\partial_\theta} = \phi\mathfrak{s}_{\partial_\theta} - \frac{i}{2}\phi'(x)\tanh(r)\partial_r.$$

It follows that

$$\omega = \phi'(x)dx \otimes (i\mathfrak{s}_{\partial_x} + \dot{\theta}\mathfrak{s}_{\partial_\theta}) - \frac{i\dot{\theta}}{2}d_D(\phi'(x)\tanh(r)\partial_r).$$

The volume form is  $\sinh(r)\cosh(r)drd\theta dx$ . A straightforward computation shows that  $\omega$  is bounded uniformly near  $e$ .  $\square$

*Remark 4.2.* The same computation shows that  $Dd_D \mathfrak{s}_v$  is also uniformly bounded at finite distance from the convex core.

*The deformation vector field in the ends.* Let  $v$  be the automorphic vector field on  $\tilde{M}_r$  defined as in section 2 from a family of hyperbolic cone-metrics with constant cone angles, normalized as in Lemma 4.1. Note that the normalization from Lemma 4.1 gives in particular an identification of the boundaries at infinity for the different metrics.

Let  $E$  be an end of  $M$ , i.e., a connected component of the complement in  $M$  of a non-empty, compact, convex subset  $K$ . The singular set of  $E$  is a disjoint union of singular rays  $p_1, \dots, p_N$ . The boundary at infinity  $\partial_\infty \tilde{M}_r$  is the disjoint union of (possibly countably) copies of the  $\partial_\infty E_r$ , where  $E$  runs through all the ends of  $M$ . Consider  $v$  on each such copy of  $\tilde{E}_r$ ; as can be checked locally in hyperbolic space  $H^3$ , it extends smoothly to  $\partial_\infty \tilde{E}_r$ . Moreover  $v$  is Killing in a neighborhood of the singular locus near infinity, since the family  $g_s$  was normalized to be constant there. We call  $V$  the automorphic vector field on  $\partial_\infty \tilde{M}_r$  obtained by extending  $v$  in this manner. Since  $v$  is Killing in a neighborhood of the singular lines near infinity, it follows that  $V$  is locally a projective vector field near the singular points.



A holomorphic vector field on  $\partial_\infty \tilde{M}$ . We will need an elementary and well-known statement: given an automorphic vector field on a Riemann surface, it is equivalent to a holomorphic automorphic vector field if and only if the induced infinitesimal variation of the complex structure (considered up to isotopy) vanishes.

**Lemma 4.3.** *Let  $\Sigma$  be a closed surface with marked points  $x_1, \dots, x_n$ , endowed with a  $\mathbb{C}P^1$ -structure  $\sigma$  with singularities at the  $x_i$ . Set  $\Sigma_r := \Sigma \setminus \{x_1, \dots, x_n\}$ . Let  $\phi : \tilde{\Sigma}_r \rightarrow \mathbb{C}P^1$  be the developing map of  $\sigma$ , and let  $V$ , a section of  $T\tilde{\Sigma}_r$ , be an automorphic vector field corresponding to a infinitesimal variation of  $\sigma$  (among the  $\mathbb{C}P^1$ -structures). Suppose that the infinitesimal variation of the complex structure on  $\Sigma$ , marked by the position of the  $x_i$ , vanishes up to isotopy. Moreover, suppose that  $V$  is projective in the lift of a uniform neighborhood of the singular points. Then there exists a smooth vector field  $W$  on  $\Sigma$  (i.e., smooth at the  $x_i$ ), vanishing at the  $x_i$ , such that if  $W'$  is the lift of  $W$  to  $\tilde{\Sigma}_r$ , then  $V + W'$  is a holomorphic vector field.*

The smoothness of  $W$  at  $x_i$  is to be understood for the underlying complex structure on  $\Sigma$ . Since  $V$  is projective, in particular it does not change, at first order, the angle around the singular points.

*Proof.* Let  $J$  be the complex structure underlying the  $\mathbb{C}P^1$ -structure  $\sigma$ . By our hypothesis,  $V$  does not change the complex structure — marked by the position of the singular points — on  $\Sigma$ , considered up to diffeomorphisms isotopic to the identity. This means precisely that the action of  $V$  on the complex structure is the same as the action of a vector field defined on  $\Sigma$ , which we call  $-W$ , which vanishes at the singular points. Calling  $W'$  the lift of  $W$  to  $\tilde{\Sigma}$ , it is clear that  $V + W'$  does not change pointwise the complex structure on  $\Sigma$  (again, marked by the position of the singular points) so that  $V + W'$  is a holomorphic vector field. It follows that  $W$  is holomorphic in the neighborhood where  $V$  is projective.  $\square$

Since  $v$  was normalized to be Killing near infinity in a neighborhood of the singular locus, it follows that  $V$  is indeed projective near the singular points. It follows from the previous lemma that we can replace the vector field  $V$  on  $\partial_\infty \tilde{M}$  by another vector field  $V + W'$ , corresponding to the same infinitesimal variation of the  $\mathbb{C}P^1$ -structure, but which is holomorphic.

**4.2. A vector bundle of quadratic polynomials.** We recall here some well-known notions on a natural bundle of polynomials of degree at most 2 on a surface with a complex projective structure.

*Complex polynomials and Killing fields.* It is necessary to understand the relationship (partly based on the Poincaré half-space model) between hyperbolic Killing fields, projective vector fields on  $\mathbb{C}P^1$ , and polynomials of degree at most 2 over  $\mathbb{C}$  (or in other terms holomorphic vector fields over  $\mathbb{C}P^1$ ).

*Remark 4.4.* Let  $\kappa$  be a Killing field on  $H^3$ . Let  $\bar{\kappa}$  be the image of  $\kappa$  in the Poincaré half-space model. Then  $\bar{\kappa}$  has a continuous extension as a vector field on the closed half-space  $\{z \geq 0\}$ . On the boundary  $\{z = 0\}$ , the restriction of this extension is tangential to the

boundary plane, and its coordinates are given — after identification of  $\{z = 0\}$  with  $\mathbb{C}$  — by a polynomial of degree at most 2.

*Proof.* Let  $(\Phi_t)_{t \in [0,1]}$  be a one-parameter family of hyperbolic isometries, with  $\Phi_0 = I$ . For all  $t \in [0, 1]$ , let  $\phi_t$  be the action of  $\Phi_t$  on the boundary at infinity, identified with  $\overline{\mathbb{C}}$ . Then, for all  $t \in [0, 1]$ ,  $\phi_t$  acts on  $\mathbb{C}$  as:

$$\phi_t(z) = \frac{a(t)z + b(t)}{c(t)z + d(t)},$$

with  $a(0) = d(0) = 1, b(0) = c(0) = 0$ . Taking the derivative at  $t = 0$ , we find that:

$$\left( \frac{\partial}{\partial t} \phi_t(z) \right)_{|t=0} = (a'(0)z + b'(0)) - z(c'(0)z + d'(0)),$$

and the result follows.  $\square$

In other words, the hyperbolic Killing fields act on the boundary at infinity of  $H^3$ , identified with  $\mathbb{C}P^1$ , as holomorphic vector fields. Moreover, given any point  $z_0 \in \mathbb{C}P^1$ ,  $\mathbb{C}P^1 \setminus \{z_0\}$  can be identified with  $\mathbb{C}$ , and can therefore be given a complex coordinate  $z$ . The action at infinity of the Killing fields are of the form:

$$v(z) = P(z)\partial_z,$$

where  $P$  is a polynomial of degree at most 2. The set of these polynomials is invariant under the action of the Möbius transformations, so that the notion of polynomial of degree at most 2 makes sense on any surface endowed with a  $\mathbb{C}P^1$ -structure. More details on the relation between quadratic polynomials and Killing vector fields can be found in [6, 7].

*Estimates on Killing fields in terms of polynomials.* The different monomials have a simple interpretation in terms of hyperbolic Killing fields:

- Polynomials of degree 0 correspond to Killing fields that vanish at the point at infinity in  $\overline{\mathbb{C}}$ , and fix (globally) the horospheres “centered” at this point at infinity.
- Homogeneous polynomials of degree 1 correspond to Killing fields that fix (globally) the hyperbolic geodesic corresponding, in the Poincaré half-space model, to the vertical line containing 0. They are sums of infinitesimal rotations around this geodesic and infinitesimal translations along it.
- Homogeneous polynomials of degree 2 correspond to Killing fields that vanish at the origin, and fix (globally) the horospheres “centered” at this point.

These three types of Killing fields, and their interpretation, have a direct generalization to the more general situation of a hyperbolic 3-manifold  $M$ , in terms of the behavior at infinity, near a point  $z_0 \in \partial_\infty \tilde{M}$ , of the Killing vector fields defined on  $\tilde{M}$ . We consider an affine complex coordinate  $z$  defined in the neighborhood of  $z_0$ , i.e., the actions at infinity of the Killing vector fields are of the form  $P(z)\partial_z$ , where  $P$  is a polynomial of degree at most 2.

**Lemma 4.5.** *There exists a constant  $C > 0$  with the following property. Let  $x \in H^3$  and let  $N \in T_x H^3$  be a unit vector such that  $\lim_{s \rightarrow \infty} \exp_x(sN) = z_0 \in \partial_\infty H^3$ . Let  $P$  be the totally geodesic plane orthogonal to  $N$  at  $x$ , let  $g_0$  be the induced metric on  $P$ , and let  $G : P \rightarrow \partial_\infty H^3$  be the hyperbolic Gauss map. Suppose that, at  $z_0$ ,  $G_* g_0 = e^{2r} |dz|^2$ . Then:*

- *the Killing vector field  $\kappa_1$  corresponding to the polynomial  $(z - z_0)\partial_z$ , considered as a flat section of  $E$ , has norm bounded, at  $x$ , by  $C$ .*
- *the Killing vector field  $\kappa_2$  corresponding to the polynomial  $(z - z_0)^2\partial_z$  has norm bounded, at  $x$ , by  $Ce^{-r}$ .*

The norm which is considered here is not the norm of Killing fields, considered as vector fields on  $H^3$ , but rather their norm considered as (flat) sections of the vector bundle  $E$ ; recall that this norm depends on the point of  $H^3$  where they are considered.

*Proof.* Both statements follow from a direct computation, for instance using the Poincaré half-space model. □

Clearly the previous statement could be extended to include Killing vector fields corresponding to polynomials of degree 0, however this will not be of any use here. It is also worth noting that a possible proof uses the invariance under the multiplication of  $z - z_0$  by a constant  $\lambda$ ; then  $(z - z_0)\partial_z$  does not change, while  $(z - z_0)^2\partial_z$  is multiplied by  $\lambda$ . Under the same homothety,  $\kappa_1$  does not change along the “vertical” geodesic ending at  $z_0$ , while  $\kappa_2$  is multiplied by  $\lambda$  because it corresponds to a parabolic isometry fixing the horospheres “centered” at  $z_0$ .

*The vector bundle of quadratic polynomials.* The remarks in the previous paragraph lead naturally to define a bundle over  $\mathbb{C}P^1$ , which is strongly related to the bundle of local Killing fields, which is used on  $H^3$  or on any hyperbolic manifolds. Although the definition is given here on  $\mathbb{C}P^1$ , it should be clear that it is of a local nature, and makes sense for any surface with a  $\mathbb{C}P^1$ -structure.

**Definition 4.6.** We call  $P$  the trivial bundle over  $\mathbb{C}P^1$ , with fiber at each point the vector space of holomorphic vector fields on  $\mathbb{C}P^1$ .

Clearly  $P$  has a natural flat connection  $D^P$ , such that the flat sections are those which correspond, at each point of  $\mathbb{C}P^1$ , to the same holomorphic vector field. In other terms,  $D^P$  is the trivial connection on the trivial bundle  $P$ .

*The section of  $P$  associated to a vector field.* Given a vector field on  $\mathbb{C}P^1$ , or more generally on a surface with a  $\mathbb{C}P^1$ -structure, one can associate to it a section of the bundle  $P$ , defined by taking at each point the “best approximation” by polynomial vector fields of degree at most 2.

**Definition 4.7.** Given a holomorphic vector field  $v$  defined on an open subset  $\Omega \subset \mathbb{C}P^1$ , there is a section  $F$  of  $P$  which is naturally associated to  $v$ ; at each point  $z_0 \in \Omega$ ,  $F_{z_0}$  is equal to the holomorphic vector field on  $\mathbb{C}P^1$  which best approximates  $f$ . Given any affine

identification of  $\mathbb{C}P^1$  (minus a point) with  $\mathbb{C}$ , if  $v := f\partial_z$ , this translates in  $\mathbb{C}$  as:

$$F_{z_0}(z) = (f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2}f''(z_0))\partial_z .$$

**Lemma 4.8.** *Let  $v$  be a holomorphic vector field on  $\Omega \subset \mathbb{C}P^1$ . Let  $F$  be the associated section of  $P$ . Then, at each point  $z_0 \in \Omega$ ,  $D^P F$  has values in the subspace of  $P_{z_0}$  of vector fields which vanish, along with their first derivatives, at  $z_0$ .*

*Proof.* Since the statement is local, the proof takes place in  $\mathbb{C}$ , and we write  $v = f(z)\partial_z$ . Let  $z_0 \in \Omega$ , and let  $Z \in T_{z_0}\mathbb{C}$ . We identify vector fields on  $\mathbb{C}$  with complex functions on  $\mathbb{C}$  and obtain, using the definition of the flat connection  $D^P$ , that for all  $z$  in some open subset of  $\mathbb{C}$ :

$$\begin{aligned} (D_Z^P F)(z_0) &= (Z\partial_{z_0} F_{z_0}(z))\partial_z \\ &= Z\partial_{z_0} \left( f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2}f''(z_0) \right) \partial_z \\ &= Z \left( f'(z_0) - f'(z_0) + (z - z_0)f''(z_0) - (z - z_0)f''(z_0) + \frac{(z - z_0)^2}{2}f'''(z_0) \right) \partial_z, \end{aligned}$$

so that:

$$(4) \quad (D_Z^P F)(z_0) = \left( Z \frac{(z - z_0)^2}{2} f'''(z_0) \right) \partial_z .$$

This shows that  $(D^P F)(z_0)$  takes its values in the vector space of homogeneous polynomials of degree 2, as needed.  $\square$

**4.3. The geometry of  $\partial_\infty M$  near the singular points.** We now concentrate on an explicit description of the complex structure and complex projective structure on  $\partial_\infty M$  near its singular points, which will be necessary in estimates below.

*The boundary at infinity of  $\tilde{M}$ .* We have already noted that the boundary at infinity of  $\tilde{M}$  carries a  $\mathbb{C}P^1$  structure, with singular points corresponding to the endpoints of the singular arcs. It also carries a vector bundle,  $P$ , with fiber at each point  $x$  the vector space of vector fields in the neighborhood of  $x$  which are obtained as continuous extensions to the boundary (for instance in a local Poincaré model) of hyperbolic Killing vector fields.

By the (local) considerations above, the fiber of  $P$  at  $x$  can also be identified with the vector space of projective vector fields in a neighborhood of  $x$ . Again,  $P$  has a natural flat connection, still called  $D^P$ , with flat sections the sections corresponding to a given projective vector field. Since its statement is of a local nature, Lemma 4.8 still holds on  $\partial_\infty M_r$ .

*Special coordinates near the singular points.* We now consider more carefully what happens on the boundary at infinity of  $M$  in the neighborhood of a singular point. Let  $x_1, \dots, x_n$  be the singular points on  $\partial_\infty M$ , i.e., the endpoints of the singular arcs. For each  $i \in \{1, \dots, n\}$ , the  $\mathbb{C}P^1$ -structure of  $\partial_\infty M_r$  in the neighborhood of  $x_i$  is projectively equivalent to a neighborhood of the vertex in a “complex cone” which we call  $\mathbb{C}_{\theta_i}$ : it is the quotient

of the universal cover of the complement of 0 in  $\mathbb{C}$  by a rotation of center 0 and angle  $\theta_i$ , where  $\theta_i$  is the angle at  $x_i$ .

We choose a neighborhood  $\Omega_i$  of  $x_i$ , and a complex projective map  $u : \Omega_i \rightarrow \mathbb{C}_{\theta_i}$  sending  $x_i$  to 0, which is a diffeomorphism from  $\Omega_i \setminus \{x_i\}$  to its image. The map  $u$  is uniquely determined (by the complex projective structure) up to composition on  $\mathbb{C}_{\theta_i}$  with a rotation and a homothety; we choose this homothety so that, as  $x \rightarrow x_i$ , the metric  $g_\infty$  on  $T_x \partial_\infty M$  behaves as  $|du|^2$ . It follows that there is a constant  $C$ , independent of  $i$ , such that, on the  $\Omega_i \setminus \{x_i\}$ :

$$\frac{|du|^2}{C} \leq g_\infty \leq C|du|^2 .$$

There is a natural holomorphic local diffeomorphism from  $\mathbb{C}_{\theta_i}$  to  $\mathbb{C}$ . With obvious notations, it is defined by sending a point  $u \in \mathbb{C}_{\theta_i}$  to  $u^{2\pi/\theta_i}$ . With the same notations we set  $z := u^{2\pi/\theta_i}$ , this defines a complex coordinate  $z$  on  $\Omega_i$ . Therefore we have proved:

*Remark 4.9.* The boundary at infinity  $\partial_\infty M$  can be canonically considered as a smooth surface, with a smooth complex structure; only the complex projective structure and the metric at infinity,  $g_\infty$ , have singularities at the endpoints of the singular arcs of  $M$ . (The ‘‘singularities’’ of the complex projective structure are in the sense explained just before subsection 3.3.)

*Estimates on the deformation field at infinity.* We now have most of the tools necessary to ‘‘normalize’’ the infinitesimal deformations of the hyperbolic structure of a manifold with particles. This means that, given a infinitesimal deformation  $\dot{g}$  of the metric  $g$  keeping the cone angles and the conformal structure at infinity fixed, we will show that it is associated to a 1-form  $\omega$  with values in the bundle  $E$  which is of a very special form. It will then follow that  $\omega$  and  $D\omega$  are in  $L^2$ .

The first step is to associate to  $\dot{g}$  an automorphic vector field  $v$ , along the ideas at the end of section 2, using the infinitesimal deformation of the development map. We have seen that  $v$  can be chosen to have a continuous extension to the boundary at infinity of the universal cover of  $M$ . We call  $V$  the automorphic vector field on  $\partial_\infty \tilde{M}$  obtained by extending  $v$  in this manner. Since  $v$  can be chosen to be Killing near the singular curves, it follows that  $V$  is projective near the singular points of  $\partial_\infty \tilde{M}$ .

According to Lemma 4.3, there exists a holomorphic vector field  $V + W'$  on  $\partial_\infty \tilde{M}_r$  such that  $W'$  is the lift to  $\partial_\infty \tilde{M}_r$  of a vector field  $W$  defined on  $\partial_\infty M$ . It is then clear that  $V + W'$  is automorphic. Note that  $W'$  is holomorphic in a neighborhood of the singular points and vanishes at the singular points. Indeed,  $V$  is projective, hence holomorphic, in a small enough neighborhood, therefore  $W'$  itself must be holomorphic. Moreover, by construction  $W$  preserves the *marked* smooth structure of  $\partial_\infty M$ , which means that it vanishes at the singular points.

Choose  $i \in \{1, \dots, n\}$ . Let  $F$  be the section of  $P$ , associated by Definition 4.7 to  $V + W'$ . We use the coordinate  $u$  on the  $\Omega_i$  defined above, so a vector tangent to  $\partial_\infty M$  can be identified with a complex number.

**Lemma 4.10.** *Let  $i \in \{1, \dots, n\}$  and choose  $u_0 \in \Omega_i \setminus \{x_i\}$ . For all vectors fields  $U, U'$  defined in a neighbourhood of  $u_0 \in \partial_\infty \tilde{M}$  we have:*

$$D_U^P F(u_0) = U\alpha(u_0)(u - u_0)^2 \partial_u ,$$

while

$$D_{U'}^P D_U^P F(u_0) = UU'(\beta(u_0)(u - u_0)^2 + \gamma(u_0)(u - u_0)) \partial_u .$$

Moreover,  $\alpha$  and  $\gamma$  are bounded by a constant  $C > 0$ , and there exists another constant  $\epsilon_0 \in (0, 1)$  such that:

$$(5) \quad |\beta(u_0)| \leq C/|u_0|^{1-\epsilon_0} .$$

*Proof.* On compact sets disjoint from the singular points, the estimates follow directly from Lemma 4.8. Thus we consider only points  $u_0$  in a neighborhood of  $x_i$  where  $V$  is projective. Now  $W'$  is holomorphic near  $x_i$  and vanishes at  $x_i$ , so it admits a Taylor series decomposition

$$W' = (w_1 z + w_2 z^2 + w_3 z^3 \dots) \partial_z$$

in a neighborhood of  $x_i$ .

Let  $u$  be an ‘‘affine coordinate’’ at  $x_i$  for the  $\mathbb{C}P^1$ -structure induced on  $\partial_\infty M$ , as defined above. Let  $\mu := 2\pi/\theta$ , then  $\mu > 2$  since  $\theta \in (0, \pi)$ . Then  $dz = \mu u^{\mu-1} du$ , so that  $\partial_z = \mu^{-1} u^{1-\mu} \partial_u$ , and it follows that:

$$W' = (w_1 u + w_2 u^{1+\mu} + w_3 u^{1+2\mu} + \dots) \partial_u .$$

The section of  $P$  associated to  $V$  is parallel by definition on the set where  $V$  is projective, so it is enough to estimate the covariant derivatives of the section associated to the vector field  $W'$ . Thus we may assume that  $F$  is the section in  $P$  associated to  $W'$ . Equation (4) shows that:

$$D_U^P F_{u_0} = \frac{U}{2} (w_2 \mu (\mu^2 - 1) u_0^{\mu-2} + 2w_3 \mu (4\mu^2 - 1) u_0^{2\mu-2} + \dots) (u - u_0)^2 \partial_u .$$

Taking one more differential leads to:

$$\begin{aligned} D_{U'}^P D_U^P F_{u_0} &= -UU' (w_2 \mu (\mu^2 - 1) u_0^{\mu-2} + 2w_3 \mu (4\mu^2 - 1) u_0^{2\mu-2} + \dots) (u - u_0) \partial_u \\ &+ \frac{UU'}{2} (w_2 \mu (\mu^2 - 1) (\mu - 2) u_0^{\mu-3} + 2w_3 \mu (4\mu^2 - 1) (2\mu - 2) u_0^{2\mu-3} + \dots) (u - u_0)^2 \partial_u . \end{aligned}$$

Moreover,  $\mu > 2$ , so the estimates announced in the lemma follow directly from the two previous equations, by taking  $\epsilon_0 := \mu - 2$ .  $\square$

It follows by compactness from this statement that the same estimates hold on  $\partial_\infty M$ , with Eq. (5) replaced by  $|\beta(u_0)| \leq C$  in the complement of the union of the  $\Omega_i$ ,  $1 \leq i \leq n$ , with  $u$  taken to be an ‘‘affine’’ coordinate, compatible with the complex projective structure, defined on a finite number of compact domains covering  $\partial_\infty M$ .

**4.4. The deformation 1-form in the ends.** It remains now to define from the automorphic section  $F$  a section of  $E$  over  $\tilde{M}_r$  which is also well-mannered close to infinity, in the same way as in [6].



*Normalization from infinity.* By Lemma 3.8, there is a compact subset  $K \subset M$  whose complement is foliated by geodesic rays  $h_{e,x}$ , with  $x \in \partial K$ . These rays lift to geodesic rays  $\tilde{h}_{e,x}$  in  $\tilde{M}_r$ , where  $e$  is a lift of an end of  $M_r$  and  $x \in \partial\tilde{K}_r$ .

We now define a section  $\kappa$  of  $E$  over  $\tilde{M}_r \setminus \tilde{K}$  as follows: for each  $y \in \tilde{M}_r \setminus \tilde{K}$ , let  $e, x$  be the unique elements such that  $y \in \tilde{h}_{e,x}$ . Then  $\kappa_x$  is the Killing field (defined in the neighborhood of  $h_{e,x}$ ) with extension at infinity (in the neighborhood of the endpoint  $z$  of  $\tilde{h}_{e,x}$ ) the projective vector field corresponding to  $F_z$ . Clearly the section  $\kappa$  of  $E$  defined in this way over  $\tilde{M}_r \setminus \tilde{K}$  is smooth. Let  $\mathfrak{s}_v$  be the canonical lift of the deformation vector field  $v$  to a section of  $E$ . By lemma 4.3,  $\mathfrak{s}_v - \kappa$  is  $G$ -invariant on the ends. Let  $\phi(t)$  be a cut-off function depending on the distance function  $t$  to the convex core, which vanishes for  $t \leq 1$  and equals 1 for large  $t$ . Then  $\phi(t)(\mathfrak{s}_v - \kappa)$  is well-defined and  $G$ -invariant on  $\tilde{M}_r$ . Thus  $\mathfrak{s}_v - \phi(t)(\mathfrak{s}_v - \kappa)$  is automorphic, differs from  $\mathfrak{s}_v$  by a  $G$ -invariant section and behaves near infinity like  $\kappa$ . Consider the invariant 1-form  $\omega := d_D(\mathfrak{s}_v - \phi(t)(\mathfrak{s}_v - \kappa))$  on  $\tilde{M}_r$  with values in  $E$ . By construction, this form and the initial 1-form  $d_D(\mathfrak{s}_v)$  correspond to equivalent infinitesimal deformations of the hyperbolic cone-manifold structure on  $M$ . Moreover, both  $\omega$  and  $D\omega$  vanish in the direction of the lines  $h_{e,x}$  near infinity.

*Different metrics on  $\partial_\infty M$ .* It is natural to consider, on the boundary at infinity of  $M$ , the metric  $g_\infty$  which was already defined above in terms of the foliation of the ends near infinity. On the leaves of this foliation, however, there are two metrics which are quite natural:

- the ‘‘horospherical metric’’  $I_t^* := I_t + 2II_t + III_t$ . It is conformal — through the Gauss map — to the metric  $g_\infty$  at infinity,
- the metric  $g_t$  which is defined as follows. For each  $x \in S_{e,t}$ , let  $P_x$  be the totally geodesic plane tangent to  $S_{e,t}$  at  $x$ , then  $T_x S_{e,t} = T_x P_x$ , and the metric  $g_t$ , on  $T_x S_{e,t}$ , is equal to the pull-back of  $g_\infty$  to  $P_x$  through the Gauss map  $G : P_x \rightarrow \partial_\infty M$ .

Note that  $g_t$  is not equal to the pull-back to  $S_{e,t}$  of  $g_\infty$  by the Gauss map  $G : S_{e,t} \rightarrow \partial_\infty M$ . However each is bounded by a constant times the other. Recall that  $k_0$  was defined above as an upper bound on the principal curvatures of the surfaces  $S_{e,t}$ .

*Remark 4.11.* For all  $t \in \mathbb{R}_+$ , we have:

- (1) for all  $x \in S_{e,t}$ , if  $G : S_{e,t} \rightarrow \partial_\infty M$  is the hyperbolic Gauss map, then:

$$g_t \leq G^* g_\infty \leq (1 + k_0)^2 g_t ,$$

- (2)  $I_t^* = e^{2t} G^* g_\infty$ , where  $G : S_{e,t} \rightarrow \partial_\infty M$  is the hyperbolic Gauss map.

*Proof.* In the first point, the first inequality follows from the convexity of  $S_{e,t}$ , because the differential on  $T_x S_{e,t}$  of the Gauss map of  $S_{e,t}$  is ‘‘larger’’ than the differential of the Gauss map of the totally geodesic plane tangent to  $S_{e,t}$  at  $x$ . The second inequality follows in the same way from the fact that the principal curvatures of  $S_{e,t}$  are bounded by  $k_0$ .

The second point is a direct consequence of the fact, already mentioned above, that the horospherical metric changes in a very simple way along an equidistant foliation (see [27]).  $\square$

Moreover, the metric  $g_t$  is the one appearing in Lemma 4.5, which yields an estimate in terms of  $t$  of the Killing vector fields associated to special quadratic polynomials on  $\partial_\infty M$ .

**Corollary 4.12.** *There exists a constant  $C' > 0$  as follows. Let  $x \in S_{e,t}$ , and let  $u_0 := G(x)$ , where  $G$  is the hyperbolic Gauss map of  $S_{e,t}$ . Let  $u$  be an affine coordinate system in the neighborhood of  $u_0$  (for the  $\mathbb{C}P^1$ -structure on  $\partial_\infty M$ ), chosen so that  $|du|^2 = g_\infty$  on  $T_{u_0}\partial_\infty M$ . Then the Killing vector field  $\kappa_1$  corresponding to the vector field  $(u - u_0)\partial_u$  (considered as a flat section of  $E$ ) has norm, at  $x$ , bounded by  $C'$ , while the Killing vector field  $\kappa_2$  corresponding to the vector field  $(u - u_0)^2\partial_u$  has norm bounded, at  $x$ , by  $C'e^{-t}$ .*

*Proof.* Direct consequence of Lemma 4.5.  $\square$

*Estimates on  $\omega$  and  $D\omega$ .* It is now possible to estimate the  $L^2$  norm of  $\omega$ , and then of  $D\omega$ , so as to prove Lemma 3.2. Let  $x \in S_{e,t}$ , and let  $X \in T_x M$ . We are interested in  $\omega(X)$ , and we already know that  $\omega$  vanishes along the lines orthogonal to the surfaces  $S_{e,t}$ , so we suppose that  $X \in T_x S_{e,t}$ .

Let  $U := G_*X$ , where  $G : S_{e,t} \rightarrow \partial_\infty M$  is the Gauss map. Remark 4.11 shows that:

$$\|U\|_{g_\infty} \leq ce^{-t}\|X\|_M ,$$

for some constant  $c > 0$ .

Recall that  $\omega(X) = D_X\kappa$ , where  $\kappa$  is the section of  $E$  corresponding to  $F$ . So  $\omega(X)$  corresponds to the vector field  $D_U^P F$  on  $\partial_\infty M$ . According to Lemma 4.10,  $D_U^P F = U\alpha(u_0)(u - u_0)^2\partial_u$ , where  $\alpha$  is bounded and  $u$  is an affine coordinate system near  $u_0$ . Using Corollary 4.12 we see that  $\omega(X)$  has norm (at  $x$ ) bounded by  $C''e^{-t}\|U\|_{g_\infty}$ , or in other terms by  $C''e^{-2t}\|X\|$ , where  $C'' > 0$  is some constant.

This means that, at  $x$ ,  $\|\omega\| \leq C''e^{-2t}$ , so that by Fubini:

$$\int_0^\infty \int_{S_{e,t}} \|\omega\|^2 dadt \leq \int_0^\infty (C'')^2 e^{-4t} A(S_{e,t}) dt \leq C_3 \int_0^\infty e^{-2t} dt \leq C_3/2 ,$$

where  $C_3 > 0$  is yet another constant, and  $A(S_{e,t})$  denotes the area of the surface  $S_{e,t}$  (for the ambient metric). Since the same estimate applies to each end of  $M$ , we conclude that  $\omega \in L^2$  near infinity.

A similar argument can be used to estimate  $D\omega$ . Let  $X, X' \in T_x S_{e,t}$ , and let  $U := G_*X, U' := G_*X'$ . As above we have:

$$\|U\|_{g_\infty} \leq ce^{-t}\|X\|, \quad \|U'\|_{g_\infty} \leq ce^{-t}\|X'\| .$$

But  $(D_X\omega)(X')$  corresponds to the vector field

$$D_U^P D_{U'}^P F - D_{G_*(\nabla_X X')}^P F$$

on  $\partial_\infty M$ , which itself can be written using Lemma 4.10 as:

$$UU'(\beta(u_0)(u - u_0)^2 + \gamma(u_0)(u - u_0))\partial_u ,$$

where  $\gamma$  is bounded and where  $\beta(u_0)$  is bounded by  $C/|u_0|$  in the  $\Omega_i$  and by  $C$  in their complement,  $C$  being a constant.

This can be written, using Corollary 4.12, as the following estimates when  $u_0$  is in one of the  $\Omega_i$ ,  $1 \leq i \leq n$ :

$$\begin{aligned} \|(D_X \omega)(X')\|_E &\leq \|U\|_{g_\infty} \|U'\|_{g_\infty} \left( \frac{C_4 e^{-t}}{|u_0|^{1-\epsilon_0}} + C_5 \right) \\ &\leq \|X\| \|X'\| e^{-2t} \left( \frac{C_4 e^{-t}}{e^{-t} d_{S_{e,t}}(x, S_{e,t} \cap M_s)^{1-\epsilon_0}} + C_5 \right) \end{aligned}$$

which translates as:

$$(6) \quad \|D\omega\| \leq e^{-2t} \left( \frac{C_4}{d_{S_{e,t}}(x, S_{e,t} \cap M_s)^{1-\epsilon_0}} + C_5 \right) .$$

The same estimates can be used when  $u_0$  is not in one of the  $\Omega_i$  and yields:

$$\begin{aligned} \|(D_X \omega)(X')\|_E &\leq \|U\|_{g_\infty} \|U'\|_{g_\infty} (C_4 e^{-t} + C_5) \\ &\leq \|X\| \|X'\| e^{-2t} (C_4 e^{-t} + C_5) . \end{aligned}$$

and a simple compactness argument then shows that, perhaps after changing the constants  $C_4$  and  $C_5$ , Eq. (6) holds on all  $S_{e,t}$ .

We can now integrate the square of this norm over the ends of  $M$ , and obtain that:

$$\int_0^\infty \int_{S_{e,t}} \|D\omega\|^2 dx dt \leq \int_0^\infty \int_{S_{e,t}} e^{-4t} \left( \frac{C_4}{d_{S_{e,t}}(x, S_{e,t} \cap M_s)^{1-\epsilon_0}} + C_5 \right)^2 dx dt .$$

Using the comparison between the induced metric  $I_t$  on  $S_{e,t}$  and the metric at infinity  $g_\infty$ , the previous equation translates as:

$$\int_0^\infty \int_{S_{e,t}} \|D\omega\|^2 dx dt \leq \int_0^\infty \int_{\partial_\infty M} e^{-2t} \left( \frac{C'_4}{d_{g_\infty}(x, S_\infty)^{1-\epsilon_0}} + C'_5 \right)^2 dx dt ,$$

for some constants  $C'_4, C'_5 > 0$ , so that, calling  $S_\infty = \{x_1, \dots, x_n\}$  the set of singular points on  $\partial_\infty M$ :

$$\int_0^\infty \int_{S_{e,t}} \|D\omega\|^2 dx dt \leq \int_0^\infty e^{-2t} dt \int_{\partial_\infty M} \left( \frac{(C'_4)^2}{d_{g_\infty}(x, S_\infty)^{2-2\epsilon_0}} + \frac{2C'_4 C'_5}{d_{g_\infty}(x, S_\infty)^{1-\epsilon_0}} + (C'_5)^2 \right) dx .$$

Note that the area element of  $g_\infty$ , close to the singular points, behaves as  $\rho d\rho d\theta$  (where  $\rho$  is again the distance to the singular points considered). So  $1/\rho^{2-2\epsilon_0}$  is integrable, and it follows that all the terms in the integral over  $\partial_\infty M$  converge, with the contribution of the terms in  $1/d_{g_\infty}(x, S_\infty)^{2-2\epsilon_0}$  and  $1/d_{g_\infty}(x, S_\infty)^{1-\epsilon_0}$  bounded for each singular point of  $S_\infty$ . (Note that the hypothesis that the angles around the singular lines are less than  $\pi$  is used here.) This shows that, for each end of  $M$ , the integral of  $\|D\omega\|^2$  is bounded. Since this holds for all ends of  $M$ , the integral of  $\|D\omega\|^2$  is bounded near infinity.

Now by Lemma 4.1 and Remark 4.2, both  $d_D \mathfrak{s}_v$  and  $Dd_D \mathfrak{s}_v$  are bounded (hence  $L^2$ ) near the convex core. We still need to check the integrability of  $\phi'(t)dt \otimes (\mathfrak{s}_v - \kappa)$  and similarly for the  $D$ -covariant derivative of this form. The support of  $\phi'$  is compact and contained in the ends; thus it is enough to check that  $\mathfrak{s}_v - \kappa$  and  $D(\mathfrak{s}_v - \kappa)$  are bounded in norm on the ends. Recall that in a neighborhood of a singular line,  $\mathfrak{s}_v - \kappa$  is the  $G$ -invariant section in  $E$

corresponding to the holomorphic vector field  $W$  on the boundary at infinity. Therefore the required estimate follows again (as in the arguments right above) from the behavior of the section  $F$  on  $\partial_\infty \tilde{M}$  as described in Lemma 4.10 and from the relation between projective vector fields at infinity and Killing fields in  $\tilde{M}$  which can be read from Corollary 4.12.

This finishes the proof of Lemma 3.2.

## 5. INFINITESIMAL RIGIDITY

In this section we first prove a general result about  $L^2$  cohomology and then we show how to apply it in our setting.

**5.1. A general argument.** Let  $E \rightarrow M$  be a vector bundle over a Riemannian manifold together with a flat connection  $D$  and a Riemannian metric along the fibers. Let

$$d_D : \Lambda^*(M, E) \rightarrow \Lambda^*(M, E)$$

denote the twisted de Rham differential with coefficients in  $E$ , and  $\delta_D$  its formal adjoint. Consider the symmetric operator  $P := d_D + \delta_D : \Lambda^*(M, E) \rightarrow \Lambda^*(M, E)$ . Let  $L^2$  denote the Hilbert space of square-integrable sections in  $\Lambda^*(M, E)$ . We view  $P$  as an unbounded operator with domain  $\mathcal{C}_c^\infty$ , the space of smooth compactly supported  $E$ -valued forms on  $M$ , which is dense in  $L^2$ .

The elements of  $L^2$  act as distributions on  $\mathcal{C}_c^\infty$  and thus they can be differentiated. For  $k \in \mathbb{Z}$  define the Sobolev space  $H^k$  as the space of those section  $\phi \in L^2$  such that  $P^k \phi \in L^2$  in the sense of distributions (or equivalently,  $H^k = \text{Dom}(P^k)^*$ ). This is a Hilbert space with the graph norm

$$\|\phi\|_{H^k}^2 := \|\phi\|^2 + \|(P^*)^k \phi\|^2.$$

Define also  $H_{\min}^k$  as the completion of  $\mathcal{C}_c^\infty$  with respect to the norm  $\|\cdot\|_{H^k}$ . It is a small lemma that  $H_{\min}^k$  injects naturally in  $H^k$ .

By the Friedrichs extension theorem, the operator

$$P^* \overline{P} : H_{\min}^1 \cap H^2 \rightarrow L^2$$

is self-adjoint and non-negative. Note that a form belongs to  $H_{\min}^1 \cap H^2$  if and only if its components in all degrees do.

**Lemma 5.1.** *Let  $\alpha \in L^2(M, \Lambda^k M \otimes E)$ . Assume that the inequality*

$$(7) \quad (d_D + \delta_D)^2 - 1 \geq 0$$

*holds on  $\mathcal{C}_c^\infty(M, \Lambda^k \otimes E)$ . Then there exists a unique  $\gamma \in H_{\min}^1 \cap H^2$  of degree  $k$  such that  $P^* \overline{P} \gamma = \alpha$ . Moreover, if  $\alpha$  is smooth then  $\gamma$  is also smooth.*

*Proof.* Let  $(P^* \overline{P})_k$  denote the restriction of  $P^* \overline{P}$  to  $k$ -forms. By continuity, Eq. (7) implies that  $(P^* \overline{P})_k \geq 1$ , therefore 0 does not belong to its spectrum. In other words,  $(P^* \overline{P})_k$  is invertible from the  $k$ -form part of  $H_{\min}^1 \cap H^2$  to  $L^2$ . Finally, if  $\alpha$  is smooth then  $\gamma$  is also smooth by elliptic regularity.  $\square$

Of course, the lemma holds for any strictly positive constant instead of 1.

We make now the assumption that  $H_{\min}^1 = H^1$ , which is another way of saying that  $P = d_D + \delta_D$  is essentially self-adjoint. Let  $d_{\min}, d_{\max}, \delta_{\min}, \delta_{\max}$  denote the minimal, respectively maximal extensions of the operators  $d_D$  and  $\delta_D$ . An  $L^2$  form  $\alpha$  is called *closed* if  $d_{\max}\alpha = 0$ , and *exact* if there exists  $\gamma \in \text{Dom}(d_{\min})$  with  $d_{\min}\gamma = \alpha$ .

**Proposition 5.2.** *Assume that*

- *the inequality (7) holds on  $\mathcal{C}_c^\infty(M, \Lambda^k M \otimes E)$  for  $k = 0, \dots, \dim(M)$ ;*
- *the operator  $P$  is essentially self-adjoint.*

*Then for each  $k$ , every closed form  $\alpha$  in  $L^2(M, \Lambda^k M \otimes E)$  must be exact. Moreover, there exists a primitive  $\gamma$  which is in  $H_{\min}^1$ , coexact, and which is smooth if  $\alpha$  is smooth.*

*Proof.* The second hypothesis, which translates to  $\text{Dom}(P^*) = H_{\min}^1$ , will be used throughout the proof without further explanation.

We first claim that  $L^2(M, \Lambda^k M \otimes E)$  decomposes orthogonally into

$$L^2(M, \Lambda^k M \otimes E) = \text{Ran}(d_{\min}^{k-1}) \oplus \text{Ran}(\delta_{\min}^{k+1}).$$

Let  $\alpha \in L^2(M, \Lambda^k M \otimes E)$ . By Lemma 5.1 and the first hypothesis, there exists  $\beta \in H_{\min}^1 \cap H^2(M, \Lambda^k M \otimes E)$  such that  $P^*\overline{P}\beta = \alpha$ . Clearly  $\text{Dom}(\overline{P}) \subset \text{Dom}(d_{\min}) \cap \text{Dom}(\delta_{\min})$ , so  $d_{\min}\beta, \delta_{\min}\beta$  are both well-defined. Furthermore,  $d_{\min}\beta + \delta_{\min}\beta$  belongs to  $H^1 = H_{\min}^1$ , which implies that both  $d_{\min}\beta$  and  $\delta_{\min}\beta$  belong to  $H_{\min}^1$  and so  $P^*d_{\min}\beta = \delta_{\min}d_{\min}\beta$ ,  $P^*\delta_{\min}\beta = d_{\min}\delta_{\min}\beta$  are both  $L^2$ . Hence

$$\alpha = d_{\min}(\delta_{\min}\beta) + \delta_{\min}(d_{\min}\beta)$$

where  $\delta_{\min}\beta, d_{\min}\beta$  are both in  $H_{\min}^1$ , which proves the claim.

Next, we claim that  $\ker d_{\max} = \text{Ran}(d_{\min})$ . The inclusion  $\text{Ran}(d_{\min}) \subset \ker d_{\max}$  is clear from the definitions. Let  $\alpha = d_{\min}(\delta_{\min}\beta) + \delta_{\min}(d_{\min}\beta)$  be such that  $d_{\max}\alpha = 0$ . Since  $d_{\min}(\delta_{\min}\beta)$  is already in  $\ker d_{\max}$ , we deduce that  $d_{\max}\delta_{\min}d_{\min}\beta = 0$ . Also  $\text{Ran}(\delta_{\min}) \subset \ker \delta_{\max}$ , so  $\delta_{\max}\delta_{\min}d_{\min}\beta = 0$ . Thus,  $\delta_{\min}d_{\min}\beta$  belongs to  $\text{Dom}(d_{\max}) \cap \text{Dom}(\delta_{\max}) \subset \text{Dom}(P^*) = H_{\min}^1$  and  $P^*(\delta_{\min}d_{\min}\beta) = 0$ . The kernel of  $P^* = \overline{P}$  vanishes since  $P^*\overline{P}$  is invertible (by the first hypothesis), therefore  $\delta_{\min}d_{\min}\beta = 0$  and so  $\alpha = d_{\min}(\delta_{\min}\beta)$  is contained in the range of  $d_{\min}$ .

The primitive  $\delta_{\min}\beta$  is  $L^2$  by construction. Moreover, it belongs to  $\text{Dom}(d_{\min})$  and to  $\ker \delta_{\max}$ . Thus it belongs to  $\text{Dom}(d_{\max}) \cap \text{Dom}(\delta_{\max}) \subset \text{Dom}(P^*) = H_{\min}^1$  and  $P^*(\delta_{\min}\beta) = \alpha$ . If  $\alpha$  is smooth then by elliptic regularity,  $\delta_{\min}\beta$  is also smooth.  $\square$

We actually proved in particular that the  $L^2$  cohomology of  $M$  twisted by  $E$  vanishes.

**5.2. Application to cone-manifolds.** Consider now the bundle  $E \simeq TM \oplus TM \simeq T_{\mathbb{C}}M$  of infinitesimal Killing vector fields on the conical hyperbolic 3-manifold  $M$ . Let  $i : E \rightarrow E$  be the complex structure,  $i(u, v) := (-v, u)$ . Define an endomorphism-valued 1-form  $T : TM \otimes E \rightarrow E$  by

$$T_V\phi := V \times i\phi$$

where  $\times$  denotes the vector product in  $TM$ , acting on each component of  $\phi$ . The flat connexion  $D$  on  $E$  is given by the explicit formula

$$D_V(u, v) = (\nabla_V u + V \times v, \nabla_V v - V \times u)$$

where  $\nabla$  is the Levi-Civita connexion on  $TM$ . We write this as  $D\phi = (\nabla + T)\phi$ . We extend  $T$  and  $\nabla$  to  $\Lambda^*(M, E)$  as the identity, resp. the de Rham differential on the form factor. We endow  $E$  with the direct sum Riemannian metric, and  $\Lambda^*M$  with its usual metric.

**Proposition 5.3** (Matsushima & Murakami [22]). *The Laplacian of the twisted de Rham differential  $d_D$  satisfies Eq. (7) on  $\Lambda_c^k(M, E)$  for  $k = 0, \dots, 3$ .*

*Proof.* By definition,

$$\Delta = (\nabla^* \nabla + \nabla \nabla^*) + (T^* T + T T^*) + T \nabla^* + \nabla^* T + T^* \nabla + \nabla T^*.$$

It was observed by Matsushima & Murakami that  $T \nabla^* + \nabla^* T + T^* \nabla + \nabla T^* = 0$ . Indeed, let  $(x_j)_{1 \leq j \leq 3}$  be geodesic normal coordinates at a point  $x \in M$ ,  $(e_j)$  the coordinate vector fields and  $(e^j)$  the dual basis. Let  $\Phi = \alpha \otimes \phi \in \Lambda^*(M, E)$ . We have:

$$\begin{aligned} \nabla \Phi &= d\alpha \otimes \phi + \sum e^j \wedge \alpha \otimes \nabla_{e_j} \phi & T\Phi &= \sum e^j \wedge \alpha \otimes e_j \times i\phi \\ \nabla^* \Phi &= \delta\alpha \otimes \phi - \sum e^k \lrcorner \alpha \otimes \nabla_{e_k} \phi & T^* \Phi &= \sum e^k \lrcorner \alpha \otimes e^k \times i\phi \end{aligned}$$

where the contraction uses the metric on forms. Since  $\times$  and  $i$  commute with  $\nabla$ , we get at  $x$ ,

$$\begin{aligned} (\nabla T^* + T^* \nabla)\Phi &= \sum L_{e_j} \alpha \otimes e_j \times i\phi + \alpha \otimes e_j \times i \nabla_{e_j} \phi \\ &= - (T \nabla^* + \nabla^* T)\Phi. \end{aligned}$$

The Laplacian  $\nabla^* \nabla + \nabla \nabla^*$  is non-negative. We claim that  $T^* T + T T^* \geq 1$  pointwise. We work at  $x \in M$  where the basis  $e_j$  is orthonormal.

Let us first examine the action of  $T^* T$  on 0-forms. It is immediate that

$$T^* T \phi = \sum_{k=1}^3 e_k \times i(e_k \times i\phi) = 2\phi.$$

We focus now on 1-forms. Notice that  $T^* T$  and  $T T^*$  act diagonally on  $E$  with respect to the splitting  $E = TM \oplus TM$ , so it is enough to prove the claim on a real section

$$\Phi = \sum a_{ki} e^k \otimes e_i.$$

Set  $\Phi_{il} = e^i \otimes e_l$ . Then

$$\begin{aligned} (T^* T + T T^*) \Phi_{il} &= - \sum_{j,k} e^k \lrcorner (e^j \wedge e^i) \otimes e_k \times (e_j \times e_l) + e^j \wedge (e^k \lrcorner e_i) \otimes e_j \times (e_k \times e_l) \\ &=: \sum' \Phi_{il} + \sum'' \Phi_{il} \end{aligned}$$

where the two sums group the terms with  $j = k$ , resp.  $j \neq k$ . Since  $e^j \wedge e_j \lrcorner + e_j \lrcorner e^j \wedge = 1$ ,  $i^2 = -1$  and  $\sum e_j \times (e_j \times \phi) = -2\phi$ , we find that  $\sum' \Phi_{il} = 2\Phi_{il}$ .



For  $j \neq k$  notice that  $e^j \wedge e^k \lrcorner + e^k \lrcorner e^j \wedge = 0$ . Therefore

$$\begin{aligned} \sum''_{j \neq i} \Phi_{il} &= \sum_{j \neq i} e^j \otimes (e_i \times (e_j \times e_l) - e_j \times (e_i \times e_l)) \\ &= \begin{cases} \sum_{j \neq i} \Phi_{jj} & \text{if } i = l; \\ -\Phi_{li} & \text{if } i \neq l. \end{cases} \end{aligned}$$

In conclusion, for  $\Phi = \sum a_{ki} \Phi_{ki}$  we obtain

$$\langle (T^*T + TT^*)\Phi, \Phi \rangle = |\Phi|^2 + \left( \sum_i a_{ii} \right)^2 + \sum_{i \neq k} (a_{ik} - a_{ki})^2 \geq |\Phi|^2.$$

Note that the equality is obtained precisely for traceless, symmetric  $\Phi$ .

For  $k = 2, 3$  we remark that the Hodge  $*$  operator commutes with the Laplacian, and acts isometrically from  $\Lambda^k M \otimes E$  to  $\Lambda^{3-k} M \otimes E$ . Thus the result follows from what we proved above for  $k = 1, 0$ .  $\square$

**5.3. Cone angles and essential self-adjointness.** The aim of this subsection is to prove that when the cone angles of our hyperbolic cone manifold are smaller than  $\pi$ , the third hypothesis of Proposition 5.2 is satisfied. The proof is based on the analysis from [33].

**Theorem 5.4.** *Let  $M$  be a hyperbolic manifold with particles. Assume that all cone angles belong to the interval  $[0, \pi]$ . Then the twisted Hodge-de Rham operator  $P = d_D + \delta_D$  acting in  $L^2(M, \Lambda^* M \otimes E)$  is essentially self-adjoint.*

*Proof.* We must show that if  $u \in L^2$  and  $Pu \in L^2$  then  $u \in H_{\min}^1$ .

We first localize  $u$  near the singular locus. Let  $\psi_1 : M \rightarrow [0, 1]$  be a smooth function which equals 1 near the singular graph, and which vanishes outside the  $\epsilon$ -neighborhood of the singular set ( $\epsilon$  is chosen sufficiently small so that this is a tubular neighborhood). We also need  $|d\psi_1|$  to be uniformly bounded; actually we can choose  $\psi_1$  such that  $|d\psi| \rightarrow 0$  at infinity, by asking that  $\psi_1$  only depends on the distance function  $r$  to the singular set. Then  $\psi_1 u$  is clearly in  $L^2$ ; moreover,

$$(8) \quad P(\psi_1 u) = \psi_1 Pu + c(d\psi_1)u$$

is also in  $L^2$ , where  $c$  denotes Clifford multiplication, i.e.,

$$c(\alpha)u = \alpha \wedge u - \alpha \lrcorner u.$$

Set  $\psi_2 := 1 - \psi_1$ .

**Lemma 5.5.** *The form  $\psi_2 u$  belongs to  $H_{\min}^1$ .*

*Proof.* We rely on the results of Weiss [33]. We follow partly the proof of the fact that on a complete manifold, all “geometric” differential operators are self-adjoint. For  $n \rightarrow \infty$  let  $f_n$  be a smooth function on  $M$ , equal to 1 on the  $n$ -neighborhood of the convex core, supported on the  $2n$ -neighborhood of the convex core, and such that  $|df_n| \leq 2/n$ . We can choose such a function to depend only on the variable  $t$  which parametrizes the families of

equidistant surfaces on the ends from Lemma 3.6. Clearly  $f_n\psi_2u$  converges in  $L^2$  to  $\psi_2u$  as  $n \rightarrow \infty$ . By (8),

$$P(f_n\psi_2u) = f_nP(\psi_2u) + c(df_n)\psi_2u.$$

By Lebesgue dominated convergence, the first term converges in  $L^2$  to  $P(\psi_2u)$  while the second converges to 0; thus  $\psi_2u$  can be approximated by compactly-supported forms in  $H^1$  sense, as claimed.  $\square$

It is left to prove that  $\psi_1u$  belongs to  $H_{\min}^1$ . Without loss of generality we can therefore assume that  $u$  lives in a tubular neighborhood  $U_\epsilon$  of the singular graph.

Again by a partition of unity using the cut-off function  $f_1$ , it is enough to prove the statement separately for  $u$  supported at finite distance from the singular line in a model conical set  $V_\alpha$ , and for  $u$  supported near the convex core.

In the first case, we use the Poincaré ball model of  $H^3$ . Let  $g_n$  be a sequence of cut-off functions on the unit interval (i.e.,  $g_n : [0, 1] \rightarrow [0, 1]$  is smooth, equals 1 near  $R = 0$  and has compact support). Denote by  $R$  the radial function on the disk. Then  $g_n(R)$  is rotation-invariant, so it descends to a function on  $V_\alpha$ . We can choose  $g_n$  to converge to 1 on each compact set; moreover, since the metric  $dR^2/(1 - R^2)^2$  on  $(-1, 1)$  is complete, we can impose that

$$(9) \quad |dg_n(R)| \leq 1/n.$$

From (8) we see that  $g_nu$  and  $P(g_nu)$  are both in  $L^2$ , in other words  $g_nu$  belongs to  $H^1$ , the *maximal* domain of  $P$ . Now  $g_nu$  has support inside a ball (depending on  $g_n$ ).

From the results of [33, Sections 4 and 5], we claim that  $g_nu$  must be in the *minimal* domain of  $P$ , provided that  $\alpha$  is smaller than  $\pi$ . Indeed, Weiss shows in [33, Proposition 5.10] that the bundle  $E$  with its connection is *cone-admissible*; this is a technical condition which implies [33, Corollary 4.34] that on a compact hyperbolic cone-3-manifold the operator  $P$  is essentially self-adjoint. Finally, the proof of this last Corollary is local in nature, and amounts to proving, after the use of a partition of unity, exactly the above claim.

As in the proof of Lemma 5.5, we have obtained a sequence  $(g_nu)$  in  $H_{\min}^1$  which converges to  $u$  in  $L^2$  and which is Cauchy in the  $H^1$  norm by (9); thus  $u$  is itself in  $H_{\min}^1$ .

We use the same argument for the remaining case, namely where  $u \in H^1$  is supported near the convex core. By the local result [33, Corollary 4.34], again  $u$  must belong to  $H_{\min}^1$ , provided the angles are all bounded above by  $\pi$ .  $\square$

## 6. PROOFS OF THE MAIN RESULTS

**6.1. Proof of Theorem 1.4.** Let  $\dot{g}$  be a infinitesimal deformation of a hyperbolic metric  $g$  with particles among metrics of the same type, which fixes both the cone angles and the conformal structure at infinity. Let  $\omega$  be the closed  $E$ -valued deformation 1-form associated to  $\dot{g}$  by Lemma 3.2. We thus know that  $\omega$  is square-integrable. The first hypothesis of Proposition 5.2 holds by Proposition 5.3. The second hypothesis is fulfilled by Theorem 5.4 if all cone angles are at most  $\pi$ . Thus, by Proposition 5.2,  $\omega$  is exact as a smooth form. By the results of Section 2, the infinitesimal deformation  $\dot{g}$  is trivial.

**6.2. Cohomological arguments.** A curious phenomenon is that sometimes, uniqueness implies existence. Something similar happens here as we explain below. The arguments used here are somewhat similar to those in [14].

Let  $V \subset M_r$  be a compact manifold with boundary which is a deformation retract of  $M_r$ .  $V$  can be obtain e.g., by smoothing the boundary of the complement, inside the convex core, of the  $\epsilon$ -neighborhood of the singular locus. Let  $U$  denote the closure inside  $M_r$  of the complement of  $V$ , in particular  $U$  is an incomplete manifold with boundary. Note that the natural inclusion map on the level of forms induces isomorphisms

$$H_c^k(M_r) \cong H^k(M_r, U).$$

Note that all cohomology groups in this section are twisted by the flat bundle  $E$ , unless otherwise specified; we suppressed  $E$  from the notation. Consider the long exact cohomology sequence of the pair  $(M_r, U)$  twisted by  $E$ :

$$H^1(M_r, U) \rightarrow H^1(M_r) \rightarrow H^1(U) \xrightarrow{\delta} H^2(M_r, U) \rightarrow H^2(M_r).$$

*Remark 6.1.* The class of a closed 1-form  $\omega$  on  $U$  is contained in the image of  $H^1(M_r)$  if and only if  $\omega$  can be extended to a closed 1-form on  $M_r$ . This happens because the restriction map  $\mathcal{C}^\infty(M_r) \rightarrow \mathcal{C}^\infty(U)$  is surjective.

We claim that the first and last maps are zero. Indeed, a compactly-supported form is in particular  $L^2$ , hence it has a smooth  $L^2$  primitive by Proposition 5.2. Thus its cohomology class on  $M_r$  is zero. The long exact sequence therefore simplifies to

$$(10) \quad 0 \rightarrow H^1(M_r) \xrightarrow{i^*} H^1(U) \xrightarrow{\delta} H^2(M_r, U) \rightarrow 0$$

where  $i^*$  is the restriction map. The bundle with connection  $(E, D)$  does not preserve the natural hermitian metric on  $E = T_{\mathbb{C}}M$ . The dual of  $(E, D)$  is isomorphic to  $(E, \overline{D})$  where  $\overline{D}$  is the complex conjugate of  $D$  from Eq. (1). This is isomorphic to  $(E, D)$  (as real bundles) via complex conjugation. Thus  $(E, D)$  is isomorphic to its dual. Hence Poincaré duality gives

$$H^k(M_r) \cong H^{3-k}(M_r, U)^*.$$

For  $k = 1$ , it follows from (10) that the (real) dimensions satisfy

$$(11) \quad \dim H^1(M_r) = \dim H^2(M_r, U) = \frac{\dim H^1(U)}{2}.$$

Let us introduce the following notation for the trivalent graph  $M_s$ :

- $a$  is the number of complete geodesics (i.e., lines without vertices);
- $b$  is the number of half-lines;
- $c$  is the number of closed geodesics (loops);
- $v$  is the number of vertices;
- $l$  is the number of segments;
- $g_1, \dots, g_N$  are the genera of the connected components of  $\partial M$ ;
- $n_i$  is the number of singular points on the  $i$ -th component of  $\partial M$ .

**Lemma 6.2.** *The dimension of  $H^1(U, E)$  equals*

$$12 \sum_{i=1}^N (g_i - 1) + 12a + 8b + 4l + 2k,$$

where  $k$  is the number of independent Killing fields on  $U$ .

*Proof.* Let  $S$  be the boundary of a tubular neighborhood of the singular graph  $M_s$ , viewed inside the manifold with boundary  $N$  from the paragraph following Definition 1.2. The surface with boundary  $S$  has  $c$  connected components homeomorphic to a torus, and  $2a + b$  boundary components. The closed surface  $\Sigma := \partial U = \partial V$  is obtained from  $\partial M$  by removing small disks around the singular points and gluing the remainder with  $S$  along their common boundary circles. We use the fact that  $\Sigma$  is a deformation retract of  $U$ , so they have the same (twisted and un-twisted) Betti numbers. From the Mayer-Vietoris sequence, the (untwisted) Euler characteristic of  $\Sigma$  is:

$$\chi(\Sigma) = \sum_{i=1}^N \chi(\Sigma_i) + \chi(S) = \sum_{i=1}^N (2 - 2g_i - n_i) - v$$

(it is easy to see, again from the Mayer-Vietoris sequence, that  $\chi(S) = -v$ ). Note also the combinatorial identities

$$\sum n_i = 2a + b, \quad 3v = 2l + b.$$

By lemma 2.1, the (twisted) Betti number  $h^0(U)$  equals  $k$ , the number of Killing vector fields on  $U$ . By Poincaré duality, since  $(E, D)$  is isomorphic to  $(E, \overline{D})$ , we also have  $h^2(\Sigma) = h^0(\Sigma)$ , therefore  $h^0(U) = h^2(U) = k$ . The claim follows from the formula

$$(12) \quad \chi(\Sigma, E) = \dim(E)\chi(\Sigma)$$

(where  $\chi(\Sigma, E)$  is the twisted Euler characteristic) and from the fact that  $\dim E = 6$ .

Eq. (12) (which is well-known) is proved as follows: the complex bundle  $E \rightarrow \Sigma$  is flat, so its Chern character vanishes, hence  $E$  represents a torsion class in  $K$ -theory. This means that  $aE \oplus \mathbb{C}^b$  is trivial for some  $a > 0, b \geq 0$ . Endow  $\mathbb{C}^b$  with the trivial connection, and  $aE$  with the direct sum connection. By definition,  $\chi(\Sigma, aE) = a\chi(\Sigma, E)$  while  $\chi(\Sigma, \mathbb{C}^b) = 2b\chi(\Sigma)$ . Now deform the connection on  $aE \oplus \mathbb{C}^b$  to the trivial connection. The Euler characteristic is constant (the index of an elliptic complex is always homotopy-invariant). At the end of the deformation we get  $\chi(aE \oplus \mathbb{C}^b) = (a \dim(E) + 2b)\chi(\Sigma)$  from which (12) follows.  $\square$

**6.3. The local structure of the variety of representations.** To prove Theorem 1.5 we need to go from an understanding of infinitesimal deformations, in terms of  $H^1(M_r, E)$ , to a statement on small deformations. This is based on the inverse function theorem, applied to a natural function sending a hyperbolic metric with cone singularities – or more generally a representation of the fundamental group of  $M_r$  in  $PSL(2, \mathbb{C})$  – to the induced conformal structure at infinity and cone angles. In this respect it is necessary to prove that

the variety of representations of  $\pi_1(M_r)$  is a smooth manifold in the neighborhood of the holonomy representation of a convex co-compact hyperbolic metric with particles.

We outline here an argument from [14, 7], also related to earlier work of Thurston [30, 9] which can easily be extended to our context. This argument also provides the dimension of the variety of representations, obtained above in a different way in Lemma 6.2.

Keeping close to the notations in [14] we call  $R(M_r)$  the variety of representations of the fundamental group of  $M_r$  into  $PSL(2, \mathbb{C})$ . There is a scheme associated to  $R(M_r)$  by the choice of a presentation (see [32, 20]), we denote it by  $\mathcal{R}(M_r)$ .

We first recall Theorem 5.2 of [7], which was extending a similar statement in [14], itself related to a fundamental result of Thurston [30, 9].

**Theorem 6.3.** *Let  $M$  be a compact, connected 3-manifold with non-empty boundary consisting of  $t$  tori and higher genus surfaces. Let  $\rho \in \mathcal{R}(M)$  be an irreducible representation such that, if  $T$  is a torus component of  $\partial M$ , the image of  $\rho(\pi_1(T))$  is neither trivial nor  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . If the natural map*

$$i : H^1(M, \partial M; E) \mapsto H^1(M; E)$$

*is zero, then, in the neighborhood of  $\rho$ ,  $\mathcal{R}(M)$  is a smooth manifold of complex dimension  $t - 3\chi(M) + 3$ .*

We will also need a close analog of Proposition 5.3 of Bromberg's paper [7], which is proved in the same way.

**Proposition 6.4.** *Let  $M$  be a convex co-compact hyperbolic 3-manifold and let  $\rho$  be its holonomy representation.*

- (1) *The restriction of  $\rho$  to each end is irreducible.*
- (2)  *$\rho$  is irreducible.*
- (3) *Let  $T$  be a tubular neighborhood of a closed curve which is a connected component of the singular locus  $M_s$ . Then  $\rho(\pi_1(T))$  is infinite and non-parabolic.*

*Sketch of the proof.* The first two points were already noted in subsection 2.5. The last point follows from the fact that the holonomy of any closed curve in  $T$  which is not a multiple of a meridian has to be loxodromic.  $\square$

We can now check that Theorem 6.3 can be applied in our context. The tori boundary components of  $N$  correspond to the closed curves in the singular locus of  $M$ , so that the condition that  $\rho(\pi_1(T)) \notin \{1, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$  follows from point (3) of the previous proposition. Moreover, Theorem 1.4 immediately implies that the map  $i$  appearing in the statement of Theorem 6.3 is zero: if  $\omega \in H^1(N, \partial N; E)$  then  $\omega$  does not change either the conformal structure at infinity or the angles at the cone singularities, so that  $\omega = 0$ . So we can conclude that, if  $M$  is a convex co-compact hyperbolic manifold with particles, then  $R(\pi_1(M_r))$  is a smooth manifold in the neighborhood of the holonomy representation  $\rho$  of  $M_r$ .

Note that the dimension of  $R(\pi_1(M_r))$  from Theorem 6.3 is not the same as the dimension of the space of deformations of  $M$  among hyperbolic cone-manifolds (plus the dimension of

$PSL(2, \mathbb{C})$ , because some representations close to  $\rho$  in  $R(M_r)$  do not come from hyperbolic metrics with cone singularities – this happens precisely when the holonomy of the meridian of the tubular neighborhood of a cone singularity is not elliptic but has a translation component (this condition appears in the definition of  $\mathcal{R}_{cone}(M_r)$  just before Theorem 1.5).

**6.4. Proof of Theorem 1.5.** We now have the tools necessary to prove Theorem 1.5. First note the following

**Lemma 6.5.** *There do not exist non-zero Killing vector fields on the connected components of  $U$  others than those corresponding to the closed geodesic loops, therefore the number  $k$  in Lemma 6.2 is  $2c$ .*

*Proof.* Let  $\kappa$  be such a Killing field on a connected component of  $U$  which contains a geodesic line or half-line from  $M_s$ . Then  $\kappa$  would have an extension as a holomorphic vector field  $v_\kappa$  on the boundary at infinity of  $M$ . Moreover, since the angles at the singular arcs are less than  $\pi$ , any Killing field has to behave, near each singular arc, as a Killing field in  $V_\alpha$  with axis  $\Delta_0$  — indeed the only Killing fields on  $V_\alpha$ ,  $0 < \alpha < \pi$ , are induced by Killing fields on  $H^3$  with axis  $\Delta_0$ . It follows that  $v_\kappa$  has zeros at the singular points of  $\partial_\infty M$ , i.e., at the endpoints of the singular arcs.

Consider a connected component  $\partial_{\infty,0}M$  of  $\partial_\infty M$ , and the corresponding connected component  $\partial_0 CC(M)$  of the boundary of the convex core of  $M$ .  $\partial_0 CC(M)$  is ruled and convex, therefore hyperbolic, outside its intersections with the singular locus of  $M$ , where it has singular points of singular curvature less than  $2\pi$ . It follows from the Gauss-Bonnet theorem that if  $\partial_0 CC(M)$  is a torus, it intersects at least one singular arc, while if it is a sphere, it intersects at least 3 singular arcs.

The vector field  $v_\kappa$  considered above is holomorphic, and it has at least 3 zeros on  $\partial_{\infty,0}M$  if  $\partial_{\infty,0}M$  is a sphere, and at least one if  $\partial_{\infty,0}M$  is a torus. Therefore it vanishes. So  $v_\kappa$  vanishes on  $\partial_\infty M$ , and it follows that  $\kappa$  is zero.

For a component which contains a trivalent vertex, it is geometrically obvious that there are no local isometries near that vertex.  $\square$

Let  $\mathcal{D}$  be the space of data appearing in Theorem 1.5, with in addition, for each singular arc (either a segment, a circle, a half-line or a line), a number corresponding to the translation component of the holonomy along that singularity. Thus:

$$\mathcal{D} = \left( \prod_{i=1}^N \mathcal{T}_{g_i, n_i} \right) \times (\mathbb{R}_+)^{a+b+c+l} \times \mathbb{R}^{a+b+c+l},$$

where, for each  $i \in \{1, \dots, N\}$ ,  $n_i$  is the number of endpoints of the singular arcs on  $\partial_i M$ , the factor  $(\mathbb{R}_+)^{a+b+c+l}$  corresponds to the angles around the singular arcs, and the term  $\mathbb{R}^{a+b+c+l}$  corresponds to the translation component along the singular arcs of the corresponding holonomy. The factors  $\mathcal{T}_{g_i, n_i}$  contain the conformal structure at infinity on  $\partial_i M$ , with marked points corresponding to the endpoints of the singular arcs.

*Remark 6.6.*  $\dim(\mathcal{D}) = 6 \sum_{i=1}^N (g_i - 1) + 6a + 4b + 2c + 2l$ .



*Proof.* For each  $i \in \{1, \dots, N\}$ ,  $\dim(\mathcal{T}_{g_i, n_i}) = 6g_i - 6 + 2n_i$ , so the formula follows from the fact that  $\sum_{i=1}^N n_i = 2a + b$  because each singular line has two endpoints, each singular half-line has one endpoint on  $\partial M$ , while the edges and the circles have no such endpoint.  $\square$

From this remark, from Lemma 6.2, Lemma 6.5 and Eq. (11), we deduce the following

**Corollary 6.7.** *The dimensions of  $H^1(M_r, E)$  and of  $\mathcal{D}$  are equal.*

Let  $g$  be a convex co-compact hyperbolic singular metric on  $M$ , as in Theorem 1.5, and let  $c(g)$  be the induced element of  $\mathcal{D}$ . By definition the last term in  $c(g)$ , in  $\mathbb{R}^a$ , is equal to 0, since  $g$  is a cone-manifold (so that the translation component of the holonomy is 0 for each singular arc). Each element of  $H^1(M_r)$  induces an infinitesimal variation of the holonomy of  $(M_r, g)$ , and therefore an element of the tangent space  $T_{c(g)}\mathcal{D}$  of  $\mathcal{D}$  at  $c(g)$ , and this defines a linear map:

$$\gamma : H^1(U, E) \rightarrow T_{c(g)}\mathcal{D} .$$

Since  $H^1(M_r, E)$  and  $\mathcal{D}$  have the same dimension, and  $\gamma \circ i^* : H^1(M_r, E) \rightarrow T\mathcal{D}$  is injective by Theorem 1.4, it follows that it is surjective. But  $H^1(M_r, E)$  is the tangent space of  $R(M_r)/PSL(2, \mathbb{C})$  at the holonomy representation of the hyperbolic metric with cone singularities considered, and  $\gamma$  is the differential of the (smooth) map sending a holonomy representation of a hyperbolic metric with cone singularities to its cone angles and conformal structure at infinity. It is therefore possible to apply the inverse function theorem, which yields Theorem 1.5.

## APPENDIX A. CONVEX SUBSETS IN HYPERBOLIC MANIFOLDS WITH PARTICLES

*Definitions, outline.* This appendix contains some basic information on the geometry of convex subsets in hyperbolic manifolds with particles. The term “convex” should be understood here as in Definition 1.1: a non-empty subset  $K$  is convex if any geodesic segment with endpoints in  $K$  is entirely contained in  $K$ .

It follows directly from this definition that the intersection of two convex subsets is either empty or convex. Our main goal here is to show that, under weak topological assumptions on  $M$ , the intersection of two non-empty convex subsets cannot be empty. It will follow that it is possible to define the convex core of a hyperbolic manifold with particles, and we will then point out some of its elementary properties.

As in the body of the paper we consider here a hyperbolic manifold with particles  $M$ , and denote by  $M_r$  and by  $M_s$  its regular and singular set, respectively. By definition  $M_s$  is a finite graph, and the angle at each of its edges is less than  $\pi$ .

*Links of points in  $M$ .* It is useful to consider the set of unit vectors based at a point of  $M$ . For regular points of  $M$  this is just the unit tangent bundle, however for singular points this notion is more interesting.

**Definition A.1.** Let  $x \in M$ , we call  $L_x(M)$  (or simply  $L_x$ ) the set of unit vectors at  $x$ .  $L_x(M)$  is the *link* of  $M$  at  $x$ .

Note that a unit vector can be defined (in a general setting) as the speed at  $x$  of a geodesic ray starting at  $x$  with velocity 1; two unit vectors are identical if the corresponding geodesic rays are equal in an interval containing 0. There is a natural distance on  $L_x$ , defined by the angle between two unit vectors. It follows from the definition of a cone-manifold that  $L_x$ , with this distance, is a spherical surface with cone singularities. The cone points correspond to the singular segments containing  $x$ , and the angle at those cone points in  $L_x$  is equal to the angle at the corresponding singular arc in  $M$ .

There is a particular kind of spherical cone-manifolds which plays an important role here. Let  $\theta \in (0, \pi)$ , consider the universal cover of the complement of the two ‘‘poles’’ in the unit sphere, and then its quotient by the rotation of angle  $\theta$  fixing the two poles. This quotient is denoted here by  $S_\theta^2$ , it is a spherical surface with two cone singularities where the angle is equal to  $\theta$ .

**Proposition A.2.** *Let  $x \in M$ , then:*

- *if  $x \in M_r$  then  $L_x$  is isometric to the unit sphere  $S^2$ ,*
- *if  $x \in M_s$  is contained in the interior of a singular edge  $e$ , then  $L_x$  is isometric to  $S_\theta^2$ , where  $\theta$  is the angle at  $e$ ,*
- *if  $x \in M_s$  is a vertex, then  $L_x$  is isometric to the metric space obtained by gluing along their common boundary two copies of a spherical triangle with acute angles.*

*Proof.* If  $x \in M_r$  the statement is quite obvious. In the second case the result can be obtained from the definition of the hyperbolic metric in the neighborhood of a point in the interior of a singular segment.

In the third case the link of  $x$  is by definition a spherical metric with cone singularities. Moreover, the angle at each cone point is less than  $\pi$ , so that the corresponding singular curvature is larger than  $\pi$ . So it follows from the Gauss-Bonnet theorem that there are at most 3 cone points. But it follows from a theorem of Alexandrov (see [3, 21]) that such a metric is the double cover of a spherical triangle – this can also be proved directly, without reference to Alexandrov’s much more general theorem. Finally, since the angle at each cone point is less than  $\pi$ , the spherical triangle has acute angles.  $\square$

Note that in the third case the angles at the three singular segments arriving at  $x$  are twice the angles of a spherical triangle (with acute angles). These angles are equal to the edge lengths of the dual spherical triangle (for the polar duality in the sphere) so they satisfy the triangle inequality. It follows that the angles at the three singular arcs containing  $x$  also satisfy the triangle inequality, and the same line of reasoning shows that any triple of angles in  $(0, \pi)$  satisfying the triangle inequality can be realized in this manner.

*The link of a convex subset at a point.* Now let  $K$  be a convex subset of  $M$ .

**Definition A.3.** Let  $x \in K$ , we call  $L_x(K)$  the set of unit vectors  $v \in L_x(M)$  such that the geodesic ray starting from  $x$  in the direction of  $v$  is contained (on some interval containing 0) in  $K$ .  $L_x(K)$  is the link of  $K$  at  $x$ .

Clearly  $L_x(K) = \emptyset$  when  $x$  is not contained in  $K$ , while  $L_x(K) = L_x$  when  $x$  is contained in the interior of  $K$ . The most interesting case is when  $x \in \partial K$ , then  $L_x(K)$  is a subset of  $L_x(M)$ . This subset is almost always geodesically convex in the following sense.

**Lemma A.4.** *Let  $x \in \partial K$ , and let  $\gamma$  be a geodesic segment in  $L_x(M)$  of length less than  $\pi$ , with endpoints in  $L_x(K)$ . Then  $\gamma$  is contained in  $L_x(K)$ .*

*Proof.* Note that no cone point of  $L_x(M)$  is contained in  $\gamma$ , except perhaps at its endpoints. Let  $\epsilon > 0$  be small enough, and let  $\Omega_\epsilon$  be the union of the geodesic segments of length  $\epsilon$  starting from  $x$  in the directions of  $\gamma$ . Then  $\Omega_\epsilon$  is a plane sector of angle less than  $\pi$  at  $x$ .

Let  $s \in (0, \epsilon)$ , consider the geodesic segment  $c_s$  in  $\Omega_\epsilon$  with endpoints the points at distance  $s$  from  $x$  in  $\Omega_\epsilon$  in the segments starting from  $x$  in the direction of the endpoints of  $\gamma$ . By definition of a convex subset,  $c_s$  is contained in  $K$ . This shows that a neighborhood of  $x$  in  $\Omega_\epsilon$  is contained in  $K$ , and therefore that  $\gamma$  is contained in  $L_x(K)$ .  $\square$

**Corollary A.5.** *If  $L_x(K)$  has dimension 2, then it has locally convex boundary in  $L_x(M)$ .*

**Corollary A.6.** *Let  $K \subset M$  be a convex subset, which is not reduced to one point. Then  $\partial K$  contains no vertex of  $M$ .*

*Proof.* Suppose that  $v$  is a vertex of  $M$ , then  $L_v(M)$  is obtained by “doubling” a spherical triangle with acute angles, we call  $c_1, c_2$  and  $c_3$  its cone singularities. Suppose now that  $v \in \partial K$ , and that  $K$  is not reduced to one point, so that  $L_v(K) \neq \emptyset$ . Since  $L_v(K)$  is the double of a spherical triangle with acute angles, its diameter is less than  $\pi$ , it then follows from Lemma A.4 that  $L_v(K)$  is connected.

Note that  $L_v(K)$  cannot be reduced to only one point which is a cone singularity of  $L_v(M)$ . Indeed, suppose for instance that  $L_v(K) = \{c_1\}$ , let  $\gamma$  be a simple loop based at  $c_1$  with  $c_2$  on one side and  $c_3$  on the other. Since the cone angles are less than  $\pi$ ,  $\gamma$  can be deformed (in the complement of the singular points) to a geodesic loop based at  $c_1$ . A standard argument in the geometry of spherical surfaces shows that this geodesic loop has length less than  $\pi$  (this uses the fact that the cone angles are less than  $\pi$ ). So  $\gamma \subset L_v(K)$  by Lemma A.4, and therefore  $L_v(K)$  contains points of  $L_v(M)$  other than  $c_1$ .

Let  $x$  be a point of  $L_v(K)$  which is not a cone singularity of  $L_v(M)$ . Let  $\gamma_1, \gamma_2$  and  $\gamma_3$  be simple loops based at  $x$ , and going around  $c_1, c_2$  and  $c_3$ , respectively. Since the angles at the  $c_i$  are less than  $\pi$ , the curves  $\gamma_i$  can be deformed (in the complement of the singular points) to minimizing geodesic loops (based at  $x$ ), and those curves, being minimizing, are disjoint.

As already used above, the lengths of the  $\gamma_i$  are less than  $\pi$ , so Lemma A.4 shows that the  $\gamma_i$  are contained in  $L_v(K)$ . A simple convexity argument then shows that  $L_v(K)$  contains a neighborhood of  $x$ , so that  $L_v(K)$  is non-degenerate (it has dimension 2).

The complement of  $\gamma_1 \cup \gamma_2 \cup \gamma_3$  is composed of four topological disks, three containing one of the cone singularities of  $L_v(M)$ , and the last one not containing any. Since  $L_v(K)$  has locally convex boundary, the Gauss-Bonnet formula shows that it has positive Euler characteristic, so  $L_v(K)$  has to contain at least 3 of the 4 disks in the complement of  $\gamma_1 \cup \gamma_2 \cup \gamma_3$  because it has genus 0. But then  $L_v(K)$  contains at least two cone singularities, and the sum of their singular curvatures is more than  $2\pi$ , so that the Euler characteristic

of  $L_v(K)$  has to be at least 2 by the Gauss-Bonnet formula. So  $L_v(K) = L_v(M)$ , and this contradicts the fact that  $v \in \partial K$ .  $\square$

The same kind of arguments can be used to understand the link of  $K$  at a boundary point which is contained in a singular segment, but is not a vertex of the singular set of  $M$ .

**Corollary A.7.** *Let  $v \in \partial K$  be contained in a singular arc of  $M$  with angle  $\theta$ , then  $L_v(K)$  is a subset of  $L_v = S_\theta^2$ . If  $L_v(K)$  contains a point which is not one of the cone singularities of  $S_\theta^2$ , then*

- *either  $L_v(K)$  is non-degenerate (i.e., it has dimension 2) and it contains exactly one of the cone singularities of  $S_\theta^2$ ,*
- *or  $L_v(K)$  is a closed geodesic, and it contains no cone singularity.*

*Proof.* Let  $x \in L_v(K)$  be a point which is not one of the cone points of  $L_v(M)$ . Let  $\gamma$  be a simple loop based at  $x$ , not homotopically trivial in the complement of the cone points of  $L_v(M)$ . Then  $\gamma$  can be deformed to a geodesic loop, of length less than  $\pi$  (because the cone angles of  $L_v(M)$  are less than  $\pi$ ) so it is contained in  $L_v(K)$ .

This geodesic loop can be a closed geodesic, in this case it can be equal to  $L_v(K)$ . This corresponds to the second case in the statement of the corollary. We now suppose that we are not in this case. Then  $L_v(K)$  contains a neighborhood of  $x$ , so it is non-degenerate. Since  $L_v(K)$  has locally geodesic boundary, it has positive Euler characteristic by the Gauss-Bonnet Theorem, so  $L_v(K)$  must contain one of the disks bounded by  $\gamma$ , so one of the cone singularities of  $L_v(M)$ .

But  $L_v(K)$  cannot contain both cone singularities of  $L_v(M)$ , otherwise its Euler characteristic would be at least 2, again by the Gauss-Bonnet Theorem because the sum of the singular curvatures of the cone singularities is larger than  $2\pi$ . Thus  $L_v(K)$  would be equal to  $L_v(M)$ , this is impossible since  $v \in \partial K$ .  $\square$

*The normal unit bundle.* Here we consider a convex subset  $K$  in  $M$  (the definitions given here make sense for other subsets).

**Definition A.8.** Let  $x \in \partial K$ , the unit normal subset of  $K$  at  $x$ , called  $N_x^1 K$ , is the set of points  $v \in L_x(M)$  which are at distance at least  $\pi/2$  from  $L_x(K)$ .

For instance:

- If  $x \in M_r$ , then  $L_x(K)$  is a subset of  $L_x$  with locally convex boundary, and  $N_x^1(K)$  is the dual of  $L_x(K)$ .
- In particular, if  $\partial K$  is smooth at  $x$ , then  $N_x^1 K$  has only one point, which is the unit normal of  $\partial K$  at  $x$ .
- If  $K = \{x\}$ , then  $L_x(K) = \emptyset$  and  $N_x^1 K = L_x$ .
- If  $x \in K \subset M_s$  and  $K$  contains a segment of  $M_s$  around  $x$ , then  $L_x(K)$  is made of the two singular points of  $L_x$ , and  $N_x^1 K$  is the “equator” of  $L_x$  (the set of points at distance  $\pi/2$  from both singular points).

*Remark A.9.* Let  $y \in M \setminus K$ , and let  $x$  be a point in  $\partial K$  such that  $d(x, y) = d(K, y)$ . Let  $c$  be a minimizing geodesic segment between  $x$  and  $y$ . Then the unit vector at  $x$  in the direction of  $c$  is contained in  $N_x^1(K)$ .

*Proof.* Clearly, otherwise it would be possible to find another point of  $K$ , close to  $x$ , closer to  $y$  than  $x$ .  $\square$

We call  $N^1(K)$  the disjoint union of the sets  $N_x^1(K)$  over the points  $x \in \partial K$ .

*The normal exponential map.* Let  $x \in M$ , let  $v \in L_x$  and let  $t \in \mathbb{R}_+$ , we denote by  $\exp_x(tv)$  the point of  $M$  which is at distance  $t$  from  $x$  on the geodesic ray starting from  $x$  with speed  $v$ . Note that, given  $x$  and  $v$ ,  $\exp_x(tv)$  is well-defined for  $t$  small enough (if  $v$  is a regular point of  $L_x$ , until the geodesic ray starting from  $x$  in the direction of  $v$  arrives at the singular set of  $M$ , and, if  $v$  is a cone point of  $L_x$ , until that geodesic arrives at a vertex of  $M$ ).

Note that it is not clear at this point that  $\exp$  is defined at all points of  $N^1(K) \times \mathbb{R}_+$  since some geodesic rays could run into the singular set of  $M$ . We will see below that this can not happen. In the meantime we call  $R$  the length of the smallest geodesic segment where this phenomenon happens; thus it will be shown below that  $R = \infty$ .

**Lemma A.10.** *The map:  $\exp : N^1K \times (0, R) \rightarrow M$  is a homeomorphism from  $N^1K \times (0, R)$  to the set of points at distance less than  $R$  from  $K$  in  $M \setminus K$ . It sends the complement of the points  $(x, v, t)$  where  $x \in \partial K$  and  $v$  is a singular point of  $L_x$  to the complement of the singular locus in  $M \setminus K$ .*

*Proof.* By construction the restriction of  $\exp$  to  $N^1(K) \times (0, r)$  is a homeomorphism onto its image for  $r$  small enough. Moreover Remark A.9 shows that its image is exactly the set of points at distance less than  $r$  from  $K$ . The shape operators of the surfaces  $\exp(N^1(K) \times \{s\})$ , for  $s \in (0, r)$ , satisfy a Riccati equation, and an argument which is classical in hyperbolic geometry shows that these surfaces are locally convex. It also follows from the definition that they are orthogonal to the singular locus.

Suppose that  $\exp$  is not injective on  $N^1(K) \times (0, R)$ . Let  $r_M$  be the supremum of the  $r \in (0, R)$  such that the restriction of  $\exp$  to  $N^1(K) \times (0, r)$  is injective. Since  $\exp$  remains a local homeomorphism at  $r$ , there are two points  $(x, v), (x', v') \in N^1(K)$  such that  $\lim_{r \rightarrow r_M} \exp_x(rv) = \lim_{r \rightarrow r_M} \exp_{x'}(rv')$ . But then the set:

$$\exp_x([0, r_M]v) \cup \exp_{x'}([0, r_M]v')$$

is a geodesic segment (otherwise an intersection would appear before  $r = r_M$ ) with endpoints in  $K$  but which is not contained in  $K$ , a contradiction.  $\square$

Now suppose that  $R < \infty$ , then there is a geodesic segment  $c$  of length  $R$  starting from a point  $x \in \partial K$ , with direction given by a vector  $v \in N_x^1(K)$ , and ending at a point  $y \in M_s$ . Moreover, either  $c$  is contained in  $M_r$  (except for its endpoints) and  $y$  is in an arc  $e$  of  $M_s$ , or  $c$  is contained in an arc of  $M_s$ , and  $y$  is a vertex of  $M$ . Let  $w$  be the unit vector at  $y$  in the direction of  $c$ .

In the first case,  $w$  is a point of  $L_y$  at distance  $\pi/2$  of both cone points of  $L_y$ . Let  $\gamma$  be a geodesic ray starting from  $y$  in a direction  $w'$  which is not one of the cone points of  $L_y$ . Then the distance between  $w$  and  $w'$  in  $L_y$  is less than  $\pi/2$ , so the derivative of the distance to  $K$  is negative along  $\gamma$ . So for  $r \in (0, R)$  close enough to  $R$ , the set of points at distance less than  $r$  from  $K$  contains the complement of the  $M_s$  in a neighborhood of  $y$ , so it does not retract on  $K$ . This contradicts the previous lemma.

In the second case,  $y$  is a vertex of  $M_s$ , and all points of  $L_y$  are at distance less than  $\pi/2$  from  $w$  (this follows from the description of  $L_y$  as obtained by gluing two copies of a spherical triangle with acute angles). So  $y$  is a local maximum of the distance to  $K$ , and this yields again a contradiction with the previous lemma. So  $R = \infty$ . This argument shows the following statement.

**Lemma A.11.** *The map  $\exp : N^1(K) \times (0, \infty) \rightarrow M \setminus K$  is a homeomorphism. It sends the complement of the points  $(x, v, t)$  where  $x \in \partial K$  and  $v$  is a singular point of  $L_x$  to the regular set of  $M \setminus K$ .*

*A global description of convex subsets.* It follows from the previous lemma that no “accident” occurs in the map  $\exp : N^1(K) \times (0, R) \rightarrow M$ . Therefore:

- all vertices of  $M_s$  are contained in  $K$ ,
- for each point  $y \in M_s$  outside  $K$ , there is only one geodesic segment minimizing the distance from  $y$  to  $K$ , and it is contained in an edge of  $M_s$ .

The map  $\exp$  defines a homeomorphism from  $\Sigma \times (0, \infty)$ , where  $\Sigma$  is a (non connected) closed surface, to  $M \setminus K$ . Here  $\Sigma$  corresponds to  $N^1K$  so that it is only a  $C^0$  surface. Moreover  $M_s \setminus K$  is the image of  $S \times (0, \infty)$ , where  $S$  is a finite subset of  $\Sigma$ .

The following is a rather direct consequence.

**Lemma A.12.** *Let  $K \subset M$  be a non-empty convex subset. Then  $K$  contains*

- (1) *all vertices of the singular set  $M_s$ ,*
- (2) *all closed geodesics in  $M$ .*

*Proof.* The first point is a direct consequence of Lemma A.11. For the second point we use a “trick” already used before, and define  $u : M \rightarrow \mathbb{R}$  by

$$u(x) = \sinh(d(x, K)) .$$

It is then known that  $u$  satisfies on  $M \setminus K$  the inequality

$$\text{Hess}(u) \geq ug ,$$

where  $g$  is the hyperbolic metric on  $M$ . The reason for this inequality is that, in  $H^3$ , the  $\sinh$  of the distance to a totally geodesic plane satisfies the equality  $\text{Hess}(u) = ug$ , the inequality then follows from a simple local argument using Lemma A.11 and the local convexity of the boundary of  $K$ . Note that the inequality should be understood in a distribution sense if the boundary of  $K$  is not smooth.

If  $\gamma$  is a closed geodesic in  $M$ , parametrized at velocity 1, then it follows that  $(u \circ \gamma)'' \geq u \circ \gamma$  (in a distribution sense), which is clearly impossible by the maximum principle unless  $u \circ \gamma = 0$ . This shows the second point in the lemma.  $\square$



*Convex subsets have non-empty intersection.* We are now ready to obtain the result announced at the beginning of this appendix.

**Lemma A.13.** *Suppose that either  $M_s$  has a vertex or  $\pi_1(M) \neq 0$ . Let  $K, K'$  be two non-empty compact convex subsets of  $M$ , then  $K \cap K' \neq \emptyset$ .*

*Proof.* Suppose first that  $M_s$  has at least one vertex  $v$ , then Lemma A.12 shows that both  $K$  and  $K'$  contain  $v$ , so  $v \in K \cap K'$ .

Suppose now that  $\pi_1(M) \neq 0$ , let  $\gamma$  be a closed curve in a non-trivial element of  $\pi_1(M)$ . Let  $(\gamma_n)_{n \in \mathbb{N}}$  be a minimizing sequence in the homotopy class of  $\gamma$ , that is, a sequence of curves homotopic to  $\gamma$  such that the length of  $\gamma_n$  converges, as  $n \rightarrow \infty$ , to the infimum of the lengths of curves homotopic to  $\gamma$ . It follows from Lemma A.11 – and from the form of the ends of  $M$  – that the  $\gamma_n$  remain at bounded distance from  $K$ . So, after extracting a subsequence,  $\gamma_n$  converges to a closed geodesic  $\gamma_\infty$ . Lemma A.12 shows that  $\gamma_\infty$  is contained in both  $K$  and  $K'$ , and the result follows.  $\square$

It would be useful to weaken the hypothesis of this lemma by supposing only that  $\pi_1(M_r) \neq 0$ . Such an extension might be true, but some care is required. It is quite possible that, if  $\pi_1(M_r) \neq 0$  (and  $M$  contains a non-empty compact convex subset) then  $M_r$  contains a closed geodesic. It is however *not* true that any non-trivial element of  $\pi_1(M_r)$  can be realized as a closed geodesic.

If  $M = V_\alpha$ , for some  $\alpha \in (0, \pi)$ , then Lemma A.13 does not apply: if for instance  $K$  and  $K'$  are each reduced to one point in the singular locus, then  $K \cap K'$  could be empty. There are also of course hyperbolic manifolds (with or without singularities) which do not contain any non-empty compact convex subset.

A more interesting example is obtained from a hyperbolic metric  $h$  with four cone singularities of angle less than  $\pi$  on the sphere  $S^2$ . One can then consider the warped product metric

$$dt^2 + \cosh^2(t)h$$

on  $S^2 \times \mathbb{R}$ , it is easily seen to be a complete hyperbolic metric with four cone singularities along infinite lines. It contains  $S^2 \times \{0\}$  as a compact convex subset. Theorem 1.5 shows that this example can be deformed, by changing the conformal structure at infinity. However,  $\pi_1(S^2 \times \mathbb{R}) = 0$ , so that Lemma A.13 does not apply. It would be desirable to have a more general statement including this example.

*The boundary of the convex core.* Lemma A.13 shows that it is possible to define the convex core  $CC(M)$  of  $M$  as the smallest non-empty convex subset in  $M$ . By the considerations above,  $CC(M)$  contains all the vertices of  $M_s$ , and  $M \setminus CC(M)$  is the disjoint union of “ends”, each of which is homeomorphic to the product of a closed surface by an interval.

**Lemma A.14.** *The boundary of  $CC(M)$  is a surface orthogonal to the singular locus.*

*Proof.* Let  $x \in M_s \cap \partial CC(M)$ . Then  $x$  is contained in a singular arc  $e$  of  $M_s$ , let  $\theta$  be the angle around  $e$ . By construction,  $L_x(CC(M))$  is a subset of  $L_x = S_\theta^2$ . We have seen in Corollary A.7 that

- either  $L_x(CC(M))$  is a closed geodesic,

- or  $\partial L_x(CC(M)) \subset S_\theta^2$  is a locally convex curve and  $L_x$  contains the “south pole”  $p_S$  of  $S_\theta^2$ , i.e., the image in  $S_\theta^2$  of the “south pole” in  $S^2$ .

In the first case  $CC(M)$  is a totally geodesic surface in the neighborhood of  $x$ , and a simple connectedness argument shows that it is everywhere totally geodesic, and thus orthogonal to the singular locus. We therefore consider the second case.

Consider the function  $\phi$  defined on  $S_\theta^2$  as the distance to  $p_S$ .  $\phi(p_S) = 0$ , while  $\phi(p_N) = \pi$ , where  $p_N$  is the “north pole” of  $S_\theta^2$ . Let  $y \in \partial L_x(CC(M))$  be the point where  $\phi$  attains its minimum. Consider the geodesic segment  $\gamma : (-l, l) \rightarrow S_\theta^2$ , parametrized at speed 1, such that  $(\phi \circ \gamma)'(0) = 0$ , where  $l$  is chosen to be maximum under the condition that  $\gamma$  is embedded. Then  $\lim_{-l} \gamma = \lim_l \gamma$ , so that the closure of  $\gamma((-l, l))$  is a closed curve, which is geodesic except at one point. Note that  $\gamma((-l, l))$  is simply the projection to  $S_\theta^2$  of a geodesic segment of length  $2l$  in  $S^2$ . The local convexity of  $\partial L_x$  shows that it remains “under”  $\gamma((-l, l))$ . It follows that:

- If  $\phi(y) > \pi/2$ , then the restriction of  $\phi$  to  $\gamma((-l, l))$  attains a strict maximum at  $y$ , so the restriction of  $\phi$  to  $\partial L_y$  is also maximal at  $y$ , and takes strictly lower values at other points. This clearly contradicts the definition of  $y$ , so this case can be eliminated.
- If  $\phi(y) < \pi/2$ , then the fact that  $\theta < \pi$  implies that the restriction of  $\phi$  to  $\gamma((-l, l))$  is negative, so the restriction of  $\phi$  to  $\partial L_x(CC(M))$  is also negative. This contradicts the definition of  $CC(M)$  as the smallest convex subset in  $M$ , because it would then be possible to reduce  $CC(M)$  by “cutting out” the part above a plane orthogonal to the singular locus but slightly “below”  $x$ , and still get a convex subset of  $M$ .
- If  $\phi(y) = \pi/2$ , then  $\phi$  is identically  $\pi/2$  on  $\gamma((-l, l))$ , so that the restriction of  $\phi$  to  $\partial L_x(CC(M))$  is at most  $\pi/2$ . The definition of  $y$  as the point where  $\phi$  is minimum thus entails that  $\phi$  is identically  $\pi/2$  on  $\partial L_x(CC(M))$ , and this means precisely that  $\partial CC(M)$  is orthogonal to the singular locus at  $x$ .

This argument shows that  $\partial CC(M)$  is orthogonal to the singular locus at  $x$ , as claimed.  $\square$

This has interesting consequences, which can be summed up as follows.

**Lemma A.15.** *The boundary of  $CC(M)$  is a “pleated surface”. Its induced metric is hyperbolic, with cone singularities at the intersection with the singular arcs in  $M_s$ , and the angle at each such cone point is equal to the angle at the corresponding singular arc of  $M_s$ . The surface  $\partial CC(M)$  is “bent” along a measured lamination whose support is disjoint from the cone points.*

*Sketch of the proof.* We do not give complete details of the proof, which is similar to the corresponding situation with no “particle”, as in [30]. The fact that  $\partial CC(M)$  is a pleated surface away from  $M_s$  is a consequence of the fact that it is the boundary of a convex subset of  $M$  without extremal point, as in the non-singular case. Since  $\partial CC(M)$  is orthogonal to the singular arcs, its induced metrics has, at those intersections, cone points with angle equal to the angle at those singular arcs in  $M$ . The same fact also entails that the support of the bending lamination does not contain the singular points (and therefore, since the

angles at the cone points are less than  $\pi$ , the distance from the support of the bending lamination to the cone points is bounded away from 0).  $\square$

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